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Approximations of Nonlinear Sytems Having Outputs

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Abstract

For a nonlinear system with output

$$\dot{x} = f(x)$$
$$y = h(x),$$

two types of linearizations about a point x_0 in state space are considered. One is the usual Taylor series approximation, and the other is defined by linearizing the appropriate Lie derivatives of the output with respect to f about x_0. The latter is called the observation model and appears to be quite natural for observation. It is interesting that there is a coordinate system in which these two kinds of linearizations agree. In this coordinate system we introduce a technique to construct an observer.

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Approximations Of Nonlinear Systems Having Outputs

L.R. Hunt* and Renjeng Su**

I. Introduction

A much used technique for dealing with nonlinear systems is to linearize the system about a point or trajectory. That is, we approximate the nonlinear system by a linear system derived using Taylor series expansions about the point or the trajectory.

Given a nonlinear system with nonlinear output

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where f is a C^∞ vector field on $\mathbb{R}^n$ and h(x) is a C^∞ $\mathbb{R}^p$-valued function, $p \geq 1$, what linear approximation about a point x_0 in state space best reflects the local observability properties of system (1)? Under certain generic conditions we construct a linear system of the form

$$\begin{aligned} \dot{x} &= f(x_0) - Ax_0 + Ax \\ y &= h(x_0) - Cx_0 + Cx, \end{aligned} \tag{2}$$

which seems appropriate for use in state estimation. For example if $p = 1$ ((1) is a single output system) and the gradients of h and the first (n-1) Lie derivatives of h with respect to f at x_0 are linearly independent, then A and C in (2) are chosen so that

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$$
\begin{aligned}
dh(x_0) &= C \\
dL_f h(x_0) &= CA \\
&\vdots \\
dL_f^n h(x_0) &= CA^n,
\end{aligned}
$$

where we think of the gradients on the left hand side as n vectors. Here $L_f^k h(x_0)$ denotes the kth Lie derivative of h with respect to f at x_0. The importance of these Lie derivatives for observability is stressed in [4] and [6].

If x_0 is an equilibrium point of f, then A and C can be found by the usual Taylor series approach. However, in general our technique and the Taylor series method yield different A and C matrices. We present an example to show this is the case.

The A and C matrices in the linear approximation (2) of the nonlinear system (1) are defined by nonlinear equations. However, there is a orthogonal linear coordinate change to new coordinates in which the computations of A and C are trivial.

Suppose we take a trajectory $\varphi(t)$ of the equation $\dot{x} = f(x)$ from (1) and let $\varphi(t_0) = x_0$ be a point on the trajectory. If we linearize system (1) about $\varphi(t)$ in the usual way, take appropriate Lie derivatives of the output function $w = y - h(\varphi(t))$ and evaluate these when $t = t_0$, we find that the equations obtained are exactly those used to define matrices A and C in our method. This is one reason why we believe that this linearizing technique will be useful in our automatic flight control research at NASA Ames Research Center.

Because of noise it is obvious that the output function cannot be differentiated on line. However, we hope that system (2) is "sufficiently close" to system (1), that a Luenberger observer built using our A and C matrices will yield satisfying estimates near x_0. However, a problem arises because the Luenberger observer requires choosing a gain K to stabilize the linear part of the difference between the plant and the estimator system. For this stabilization, the usual Taylor series approach about x_0 appears to more appropriate than our observation model. Hence our competing linearizations both have their good and bad points. Fortunately, there is a coordinate system on R^n where the Taylor series linearization and the observation model about x_0 agree. It is in this coordinate system that we shall construct the observer. Practically, we cannot always find the inverse of the exact coordinate transformation, but we can build its linear part about the appropriate point. No claims are made about our approach being guaranteed to work in practice, but it does seem promising. Most problems that arise for our scheme also are evident in any other attempt at nonlinear observation.

For generalized Luenberger observers and mappings of nonlinear systems to other systems for which observers can be constructed we refer to Krener and Isidori [7]. Pioneering research concerning observability of nonlinear systems can be found in Kou, Elliott, and Tarn [6] and Hermann and Krener [4]. Other excellent references appearing in the literature include the recent work of Nijmeijer [9],[10].

This technique for finding approximating linear systems for observability parallels our efforts [2],[5] for establishing linear approximations to the nonlinear control system

$$\dot{x} = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t)). \tag{3}$$

In future research we will add an output function to system (3) and ask for conditions under which the estimation technique introduced in this paper is applicable.

For our work at NASA Ames, trajectories and controls to fly these trajectories are computed in an on board computer. It is impossible to linearize around a trajectory in advance. However, the scheme developed here may prove applicable in this situation.

In section 2 of this paper we give definitions and preliminaries and introduce our linear approximating system (2). Section 3 contains a technique to compute the A and C matrices of (2) and a result relating our approximation to the usual linearization about a trajectory provided that the trajectory is known. We also introduce a technique that can prove useful in constructing an observer. The paper ends with a comparison of the time responses of the outputs of the nonlinear system and the observation model starting at the points x_0

II. Definitions and Preliminaries

Let $a(x)$ be a C^∞ real-valued function on $\mathbb{R}^n$. For our vector field

$$f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

we let

$$\langle da, f \rangle = \frac{\partial a}{\partial x_1} f_1 + \frac{\partial a}{\partial x_2} f_2 + \cdots + \frac{\partial a}{\partial x_n} f_n,$$

the <u>Lie derivative of a with respect to f</u>, denoted by $L_f a(x)$ or $L_f^1 a(x)$. Since <da,f> is real-valued, we can again differentiate with respect to f to obtain $L_f^2 a(x)$. This process can be continued by induction to yield $L_f^3 a(x)$, $L_f^4 a(x)$, etc.

We write the $\mathbb{R}^p$-valued function $y = h(x)$ in (1) as $h = (h_1,h_2,\dots,h_p)$. The following assumption is made for the remainer of this paper. There exist real numbers $\beta_1,\beta_2,\dots,\beta_p \geq 0$ so that $\beta_1+\beta_2+\dots+\beta_n = n$ and the set

$$\{dh_1,dL_fh_1,dL_f^2h_1,\dots,dL_f^{\beta_1-1}h_1,dh_2,dL_fh_2,$$

$$dL_f^2h_2,\dots,dL_f^{\beta_2-1}h_2,\dots,dh_p,dL_fh_p,dL_f^2h_p,\dots,dL_f^{\beta_p-1}h_p\} \tag{4}$$

is linearly independent on $\mathbb{R}^n$ or on an open set in $\mathbb{R}^n$ containing the point x_0 of interest. Since we can renumber $h_1,h_2,\dots,h_p$, no generality is loss in supposing that $\beta_1 \geq \beta_2 \geq \dots \geq \beta_p$, where β_p is the first integer such that $dL_f^{\beta_p-1}h_p$ is linear combination of $dh_1,dh_2,\dots,dh_p,dL_fh_1,dL_fh_2,\dots,dL_fh_p,\dots,dL_f^{\beta_p-1}h_1,\dots,$ $dL_f^{\beta_p-1}h_2,\dots,dL_f^{\beta_p-1}h_{p-1}$, β_{p-1} is the first integer such that $dL_f^{\beta_{p-1}-1}h_{p-1}$ is a linear combination of $dh_1,dh_2,\dots,dh_p$, dL_fh_1, $dL_fh_2,\dots,dL_fh_p,\dots,dL_f^{\beta_{p-1}-1}h_1,dL_f^{\beta_{p-1}-1}h_2,\dots,dL_f^{\beta_{p-1}-1}h_{p-2}$, etc. We assume these linear combinations hold on $\mathbb{R}^n$ or on an open subset of $\mathbb{R}^n$ containing x_0.

Under the above conditions there exists a neighborhood of x_0 in which we can distinguish any two points (see [4] and [7]). This is called strong local observability in [10].

In system (2) we let the rows of the pxn matrix C be

$C_1, C_2, \ldots, C_p$, respectively.

Definition 2.1. Given system (1) and a point x_0 in state space, the linear system (2) is called the observation model at x_0 if A and C are defined by

$$
\begin{aligned}
&C_j = dh_j(x_0), \quad j = 1,2,\ldots,p \\
&C_jA^k = dL_f^k h_j(x_0), \quad j = 1,2,\ldots,p \text{ and } k = 1,2,\ldots,\beta_j.
\end{aligned} \tag{5}
$$

It is easy to show that if x_0 is an equilibrium point of f, then A and C in the observation model agree with A and C computed by the linear parts of Taylor series expansions of f and h about x_0.

We illustrate by an example that the observation model and Taylor series approach may differ for a general point x_0.

Example 2.2. We take the system on $\mathbb{R}^3$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix}
\qquad y = h(x) = \sin x_1, \tag{6}
$$

and find its observation model at the point $x_0 = (\frac{\pi}{4}, 1, 1)$. Computing we have

$$C = dh(x_0) = \left[\frac{\sqrt{2}}{2}, 0, 0\right]$$

$$CA = dL_f h(x_0) = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right]$$

$$CA^2 = dL_f^2 h(x_0) = \left[-\sqrt{2}, -\sqrt{2}, \frac{\sqrt{2}}{2}\right]$$

$$CA^3 = dL_f^3 h(x_0) = \left[-\sqrt{2}, -3\sqrt{2}, -\frac{3\sqrt{2}}{2}\right].$$

Thus

$$C = \left[\frac{\sqrt{2}}{2}, 0, 0\right] \text{ and}$$

(7)
$$A = \begin{bmatrix} -1 & 1 & 0 \\ -3 & -1 & 1 \\ -10 & -6 & -1 \end{bmatrix}$$

Hence the observation model about x_0 is

(8)
$$\begin{aligned} \dot{x} &= f(x_0) - Ax_0 + Ax \\ y &= h(x_0) - Cx_0 + Cx \end{aligned}$$

with A and C as above. If we linearize f and h about x_0 we find

(9)
$$\begin{aligned} \dot{x} &= f(x_0) - \hat{A}x_0 + \hat{A}x \\ y &= h(x_0) - \hat{C}x_0 + \hat{C}x, \end{aligned}$$

with

(10)
$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{C} = \left[\frac{\sqrt{2}}{2}, 0, 0\right].$$

Therefore the two techniques yield different linear sytems (8) and (9).

Returning to equations (6) we compute the output and its first two time derivatives

$$
\begin{aligned}
y &= h(x) = \sin x_1 \\
\dot{y} &= L_f h(x) = (\cos x_1) x_2 \\
\ddot{y} &= L_f^2 h(x) = (-\sin x_1) x_2^2 + (\cos x_1) x_3 .
\end{aligned}
\tag{11}
$$

The inverse function theorem ($dh, dL_f h$, and $dL_f^2 h$ are linearly independent at x_0) implies that we can solve these equations for all x near x_0. Linearizing (11) about x_0 we find

$$
\begin{aligned}
h(x) &\approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x_1 - \frac{\pi}{4}) \\
L_f h(x) &\approx \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x_1 - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}(x_2 - 1) \\
L_f^2 h(x) &\approx -\sqrt{2}(x_1 - \frac{\pi}{4}) - \sqrt{2}(x_2 - 1) + \frac{\sqrt{2}}{2}(x_3 - 1) .
\end{aligned}
\tag{12}
$$

We take our observation model with matrices A and C as in (7) and compute the corresponding time derivatives of the output. This process yields

$$
\begin{aligned}
y &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x_1 - \frac{\pi}{4}) \\
\dot{y} &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x_1 - \frac{\pi}{4}) + \frac{\sqrt{2}}{2}(x_2 - 1) \\
\ddot{y} &= -\sqrt{2}(x_1 - \frac{\pi}{4}) - \sqrt{2}(x_2 - 1) + \frac{\sqrt{2}}{2}(x_3 - 1) ,
\end{aligned}
\tag{13}
$$

exactly as in (12). Therefore the observation model produces the linear part about x_0 of the important equations (11).

Writing equations (9) from the usual Taylor series approach we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}x_2\\ x_3\\ 0\end{bmatrix}$$

$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x_1 - \tfrac{\pi}{4}).$$

Taking the last equation and its first two time derivatives gives us

$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x_1 - \tfrac{\pi}{4})$$

$$\dot{y} = \frac{\sqrt{2}}{2}\, x_2$$

$$\ddot{y} = \frac{\sqrt{2}}{2}\, x_2.$$

Hence $\dot{y}$ and $\ddot{y}$ are not good approximations to the corresponding equations in (12).

Given system (1) and a point x_0 there is a coordinate change on R^n such that in the new coordinates the observation model and the Taylor series linearization agree. Recall that we assumed that the gradients of the functions in (4) are linearly independent. The following coordinate transformations are known in the literature (see [3] and [7]).

We set $T_1 = h_1, T_2 = L_f h_1, T_3 = L_f^2 h_1, \ldots, T_{\beta_1} = L_f^{\beta_1 - 1} h_1, T_{\beta_1+1} = h_2,$ $T_{\beta_1+2} = L_f h_2, \ldots, T_{\beta_1+\beta_2} = L_f^{\beta_2-1} h_2, \ldots, T_{n-\beta_n+1} = h_p, T_{n-\beta_n+2} = L_f h_p, \ldots,$ $T_n = L_f^{\beta_n - 1} h_p$. In these T coordinates system (1) becomes

$$\dot{T}_i = T_{i+1}, \quad i \neq \beta_1, \beta_1 + \beta_2, \ldots, n$$

$$\dot{T}_{\beta_i} = F_i, \quad i = 1,2,\ldots,p,$$

(14) $$y = [T_1, T_{\beta_1+1}, \ldots, T_{n-\beta_n+1}] = [h_1, h_2, \ldots, h_p],$$

with each F_i being a function of $(T_1,T_2,\ldots,T_n)$. Actually the F_i $i = 1,2,\ldots,p$ are functions of $(x_1,x_2,\ldots,x_n)$, but $x_1,x_2,\ldots,x_n$, are functions of $T_1,T_2,\ldots,T_n$ by the inverse function theorem. We rewrite system (14) as

$$\dot{T}_i = f'(T_1,T_2,\ldots,T_n),$$

(15) $$y = [T_1, T_{\beta_1+1}, \ldots, T_{n-\beta_n-1}] = [y_1, y_2, \ldots, y_p]$$

the definition of f' being obvious from (14). If we take the Lie derivatives of y_i with respect to f' through order β_i-1 (i.e. we are taking the time derivatives of y_i through order β_i-1) we find only linear equations. For a point $x_0 = T_0$ in state space, if we start with system (15), then the Taylor series linearization and the observation model produce the same linear system.

The major problem with using this transformation to T coordinates in practice is that is is difficult to solve and invert nonlinear equations. In other words, the major problem is in constructing the inverse of the transformation and the functions $F_i, i = 1,2,\ldots,p$, as they depend on $T_1,T_2,\ldots,T_n$. However, as we show later, it is important that we can build the inverse of the linear part (about a point x_0) of the transformation and the linear part of $F_1,F_2,\ldots,F_p$ about $T(x_0) = T_0$.

We now develop a method to solve the nonlinear equations (5) for A and C.

III. Observation Model Calculations

We consider system (1) and its observation model (2) where A and C are defined in equations (5). The set D is defined as

$$\{dL_f^{\beta_1}h_1(x_0), dL_f^{\beta_1-1}h_1(x_0), \ldots, dL_f^{\beta_2}h_1(x_0), dL_f^{\beta_2}h_2(x_0), dL_f^{\beta_2-1}h_1(x_0),$$
$$dL_f^{\beta_2-1}h_2(x_0), \ldots, dL_f^{\beta_3}h_1(x_0), dL_f^{\beta_3}h_2(x_0), dL_f^{\beta_3}h_3(x_0), dL_f^{\beta_3-1}h_1(x_0),$$
$$dL_f^{\beta_3-1}h_2(x_0), dL_f^{\beta_3-1}h_3(x_0), \ldots, dh_1(x_0), dh_2(x_0), \ldots, dh_p(x_0)\}.$$

Before forming this set checks such as $\beta_1 = \beta_2$ or $\beta_1 > \beta_2$, etc. should be made and no duplications should be included. For example if $\beta_1 = \beta_2$, then D begins with $dL_f^{\beta_1}h_1(x_0), dL_f^{\beta_2}h_2(x_0), dL_f^{\beta_2-1}h_1(x_0), dL_f^{\beta_2-1}h_2(x_0), \ldots,$ provided $\beta_2 > \beta_3$. Our ordering of this set reflects descending superscripts on L_f and ascending subscripts on the h_i. This set D consists of the elements contained in the set described in (4) plus $dL_f^{\beta_1}h_1(x_0), dL_f^{\beta_2}h_2(x_0), \ldots, dL_f^{\beta_p}h_p(x_0)$.

Now we introduce an interesting $(n+p)\times(n+p)$ matrix E. The first row of E is the first element of D followed by p zeroes, the second row of E is the second element of D followed by p zeroes,..., the kth row of E is the kth element of D followed by p zeroes,..., the last row of E consists of $dh_p(x_0)$ and p zeroes.

It is proved in [1] that there are orthogonal coordinate changes on $\mathbb{R}^n$ (actually $\begin{bmatrix}\Lambda & 0\\ 0 & I\end{bmatrix}$ on $\mathbb{R}^{n+p}$ where Λ is orthogonal on $\mathbb{R}^n$ and I is the identity on $\mathbb{R}^p$) that take E to "generalized upper Hessenberg" form (certain adjustments need to be made first). In fact the computer codes to do this are included in [1], and for the single input case (p=1) this is due to Stewart [11]. By an (n+p)x(n+p) generalized upper Hessenberg matrix we mean that all elements below the p^{th} subdiagonal are zero. In addition, if $p > 1$ and a zero appears in the p^{th} subdiagonal and i^{th} row, then all entries in the p^{th} subdiagonal above the i^{th} row are zero. If $(p-1) > 1$, and if a zero appears in the $(p-1)^{th}$ subdiagonal and the j^{th} row, $j \geq i$, then all entries in the $(p-1)^{th}$ subdiagonal above the j^{th} row are also zero. This can be continued if $(p-1) > 2$, etc. Also the last p columns remain zero in our matrix E after the coordinate change.

To calculate A and C as in equations (5), we assume that we start in the coordinate system where the matrix E is generalized upper Hessenberg. Otherwise we can compute the orthogonal coordinate change to move to these desirable coordinates, calculate A and C, and then return to the original coordinates through the transpose (inverse) of the orthogonal change. In this form the zero in the p^{th} subdiagonal is in the same row as $dL_f^{\beta_p}h_p(x_0)$, in the $(p-1)^{th}$ subdiagonal is in the same row as $dL_f^{\beta_{p-1}}h_{p-1}(x_0)$, etc.

We let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \qquad (16)$$

Now, by the form of E (forgetting the last p columns) and the equations (5)

$$C = \begin{bmatrix} 0 & \cdots & 0 & * & \cdots & * & * \\ \vdots & & & & & \vdots & \vdots \\ 0 & 0 & & \cdots & 0 & * & * \\ 0 & 0 & & \cdots & 0 & 0 & * \end{bmatrix}$$

where * represents possible nonzero numbers (the first * in each row must be nonzero by our assumptions on $dh_1(x_0), dh_2(x_0), \ldots, dh_p(x_0)$. The * in the first row of C is in the $(n-p+1)^{\underline{th}}$ column. Then we know our matrix C having rows $C_1, C_2, \ldots, C_p$.

From (5)

$$C_p A = dL_f^1 h_p(x_0) = [0 \ldots 0 * \ldots **],$$

where the first * is nonzero and in the $(n-p+2)^{\underline{th}}$ column. This is

$$[0\ 0\ \ldots\ 0\ 0\ *] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad [0\ \ldots\ 0\ *\ \ldots\ *\ *],$$

and we find $a_{n1}, a_{n2}, \ldots, a_{nn}$ where $a_{n1}, a_{n2}, \ldots, a_{n(n-p+1)}$ are zero. In a similar manner, $C_{p-1}A = dL_f^1 h_{p-1}(x_0)$ yields $a_{(n-1)1}, a_{(n-1)2}, \ldots, a_{(n-1)n}, \ldots,$ $C_1 A = dL_f^1 h_1(x_0)$ yields $a_{(n-p+1)1}, a_{(n-p+1)2}, \ldots, a_{(n-p+1)n}$.

Next, if $\beta_p > 2$,

$$C_p A^2 = (C_p A) A = dL_f^2 h_p(x_0)$$

allows us to find $a_{(n-p)1}, a_{(n-p)2}, \ldots, a_{(n-p)n}$. Continuing in this way we can solve for all entries in A. Viewing $C_jA^k = (C_jA^{k-1})A = ((dL_f^k h_j)(x_0))A$ in solving for A, the rows of the coefficient matrix are the elements in the set (4), which we assumed are linearly independent.

Summarizing our technique, we use orthogonal coordinate changes on $\mathbb{R}^n$ so that solving for our A and C matrices defined in (5) becomes extremely easy.

As mentioned in the introduction, we may not know the trajectory of $\dot{x} = f(x)$ in system (1) in advance so that we can do off-line computations. However, if such a trajectory is known, it is interesting to study the relationship between the usual linear variational equation and the observation model at a point on the trajectory. In fact, the results are quite surprising.

Let $\varphi(t)$ be a solution of $\dot{x} = f(x(t))$. Set $z = x-\varphi(t)$, $w = y-h(\varphi(t))$, $F(t) = \frac{\partial f}{\partial x}(\varphi(t))$, and $H(t) = \frac{\partial h}{\partial x}(\varphi(t))$. Here $\frac{\partial f}{\partial x}$ is the Jacobian matrix and $\frac{\partial h}{\partial x}$ is the pxn matrix with $i^{\underline{th}}$ row $[\frac{\partial h_i}{\partial x_1}, \frac{\partial h_i}{\partial x_2}, \ldots, \frac{\partial h_i}{\partial x_n}]$. Then the linear variational equation along $\varphi(t))$ is

$$\begin{aligned} \dot{z} &= F(t)z \\ w &= H(t)z. \end{aligned} \tag{17}$$

For simplicity we state and prove a theorem for a single output (p=1) system, but all arguments generalize to the multi-output case.

<u>Theorem 3.1</u>. Take the linear variational equation (17) and compute the first (n-1) time derivatives at a point $x_0 = \varphi(t_0)$. Solving the output equation for y and the first (n-1) time derivatives

equation $\dot{y}, \ddot{y} = \dot{y}^{(2)}, \ldots, \dot{y}^{(n-1)}$ respectively, we obtain the same set of equations as taking the observation model (2) (with A and C defined by (5)) and the first (n-1) Lie derivatives of the output.

<u>Proof</u>. The system (17) has output function

$$y - h(\varphi(t)) = w = H(t)z.$$

At $\varphi(t_0) = x_0$ this becomes

$$\begin{aligned} y &= h(x_0) + \frac{\partial h}{\partial x}(x_0)(x-x_0) \\ &= h(x_0) + C(x-x_0), \end{aligned}$$

the use of vector notation being obvious.

We compute

$$\dot{y} - \frac{\partial h(\varphi(t))}{\partial t} = \dot{w} = \frac{\partial H(t)}{\partial t} z + H(t)\dot{z}$$

at time $t_0 (\varphi(t_0) = x_0)$ to find

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x}(x_0) f(x_0) + \frac{\partial^2 h}{\partial x^2}(x_0) f(x_0)(x-x_0) + \frac{\partial h}{\partial x}(x_0) \frac{\partial f}{\partial x}(x_0)(x-x_0) \\ &= dh(x_0) f(x_0) + dL_f^1 h(x_0)(x-x_0) \\ &= Cf(x_0) + CA(x-x_0). \end{aligned}$$

Similarly,

$$\ddot{y} - \frac{\partial^2 h(\varphi(t))}{\partial t^2} = \ddot{w} = \frac{\partial^2 H(t)}{\partial t} z + 2 \frac{\partial H(t)}{\partial t} \dot{z} + H(t)\ddot{z}$$

for $\varphi(t_0) = x_0$ yields

$$\begin{aligned} \ddot{y} &= dL_f^1 h(x_0) f(x_0) + dL_f^2 h(x_0)(x-x_0) \\ &= CAf(x_0) + CA^2(x-x_0). \end{aligned}$$

An induction step shows that

$$\dot{y}^{(k)} = CA^{k-1}f(x_0) + CA^k(x-x_0)$$

for $1 \leq k \leq n-1$.

Suppose we take system (2) and consider the Lie derivatives of the output y with respect to the vector field $f(x_0)-Ax_0+Ax$. We find

$$\begin{aligned} y &= h(x_0) + C(x-x_0) \\ \dot{y} &= Cf(x_0) + CA(x-x_0) \\ \ddot{y} &= CAf(x_0) + CA^2(x-x_0) \end{aligned}$$

and

$$\dot{y}^{(k)} = CA^{k-1}f(x_0) + CA^k(x-x_0),$$

for $1 \leq k \leq n-1$.

Q.E.D.

The computations of Example 2.2 and Theorem 3.1 verify certain advantages of the observation model over the usual Taylor series approach. In fact, if one is interested in state estimation for system (1) using a linear approximation about a point x_0, we have illustrated the importance of the observation model.

At the end of section 2 we introduced a set of coordinate changes $T = (T_1, T_2, \ldots, T_n)$ on $\mathbb{R}^n$ so that the system (1) has a very simple form. Moreover, in these T coordinates, the Taylor series approach and the observation model about a point $T_0 = T(x_0)$ yield the same linear system. We are interested in using the T coordinate changes and system in T coordinates to construct Luenberger observers.

Assume that we know the inverse transformation T^{-1} and the functions $F_1, F_2, \ldots, F_p$ (as functions of $T_1, T_2, \ldots, T_n$) in system (14). We start in x coordinates, but since the outputs of systems (1) and (14) are the same, we suppose that our plant is in the T coordinates. Hence we estimate the state $\hat{T}$ and find estimates $\hat{x}$ in the original coordinates through T^{-1}. We work near x_0 in R^n and let $T(x_0) = T_0$ as above and $x(T_0) = x_0$.

Since Luenberger observers are constructed for linear systems we linearize (14) by a first degree Taylor polynomial about T_0 (this is the same as the observation model about T_0) to obtain

$$\begin{aligned} \dot{T}_i &= T_{i+1} \qquad i \neq \beta_1, \beta_1 + \beta_2, \ldots, n \\ \dot{T}_{\beta_i} &= F_i(x(T_0)) + \frac{\partial F_i}{\partial T}(x(T_0))(T - T_0), \; i = 1,2,\ldots,p, \\ \text{(18)} \qquad y &= [T_1, T_{\beta_1+1}, \ldots, T_{n-\beta_n+1}]. \end{aligned}$$

Because the outputs in (14) are linear, no linearization of the output functions is required.

Set $\hat{T}_0 = T_0$ and define a new system

$$\begin{aligned} \dot{\hat{T}}_i &= \hat{T}_{i+1} + K_i(y^* - \hat{y}^*), \; i \neq \beta_1, \beta_1 + \beta_2, \ldots, n \\ \dot{\hat{T}}_{\beta_i} &= F_i(x(\hat{T}_0)) + \frac{\partial F}{\partial T}(x(\hat{T}_0))(\hat{T} - \hat{T}_0) + K_{\beta_i}(y^* - \hat{y}^*), \; i = 1,2,\ldots,p, \\ \text{(19)} \qquad \hat{y} &= [\hat{T}_1, \hat{T}_{\beta_1+1}, \ldots, \hat{T}_{n-\beta_n+1}], \end{aligned}$$

where * denotes transpose and the K_i are 1xp matrices which remain to be chosen.

Let $\tilde{T} = T - \hat{T}$ and compute

$$\text{(20)} \qquad \dot{\tilde{T}}_i = T_{i+1} - \hat{T}_{i+1} - K_i(y^* - \hat{y}^*), \; i \neq \beta_1, \beta_1 + \beta_2, \ldots, n$$

$$\dot{\tilde{T}}_{\beta_i} = \frac{\partial F_i}{\partial T}(x(T_0))(T - \hat{T}) - K_{\beta_i}(y^* - \hat{y}^*), \; i = 1,2,\ldots,p.$$

By our assumption of linear independence on the set (4), the linear system (18) is observable, and $K_1, K_2, \ldots, K_n$ can be taken to stabilize system (20) about $\tilde{T} = 0$. As mentioned previously, estimates $\hat{x}$ in the original x coordinates can be computed from $T^{-1}(\hat{T})$.

Suppose now that we do not know T^{-1} or $F_1, F_2, \ldots, F_p$ as functions of $T_1, T_2, \ldots, T_n$. It is interesting that we can derive equations (18),(19), and (20) and repeat the above process. However, after obtaining the estimate for $\hat{T}$, we approximate $\hat{x}$ through the universe of the linear part of T about x_0 instead of T^{-1} itself. Recall the definitions of $T_1, T_2, \ldots, T_n$ and $F_1, F_2, \ldots, F_p$ from section 2. Form the set D and the matrix E as in the beginning of this section. Use the orthogonal coordinate changes on $\mathbb{R}^n$ to move E to generalized upper Hessenberg form. Since this coordinate change is orthogonal, when taking its inverse we need only examine its transpose. For this reason we assume that the linear orthogonal coordinate change has been done in system (1).

Thus we have

$$dT_{n-\beta_n+1}(x_0) = dh_p(x_0) = [0,0,\ldots,0,*]$$

$$dT_{n-(\beta_{n-1}+\beta_n)+1}(x_0) = dh_{p-1}(x_0) = [0,0,\ldots,0,*,*]$$

$$\vdots$$

$$dT_1(x_0) = dh_1(x_0) = [0,0,\ldots,*,\ldots,*].$$

$$dT_{n-\beta_n+2} = dL_f h_p(x_0) = [0,0,\ldots,*,\ldots,*]$$

$$\vdots$$

$$dT_{\beta_1} = dL_f^{\beta_1-1} h_1(x_0) = [*,*,\ldots,*] \tag{21}$$

where the first * in each right hand expression is nonzero and these move one place to the left as we move from one line to the next. Solving the equations

$$T_{n-\beta_n+1}(x) = T_{n-\beta_n+1}(x_0) + \frac{\partial T_{n-\beta_n+1}(x_0)}{\partial x_n}(x_n - x_{n0}) + \ldots$$

$$T_{n-(\beta_{n-1}+\beta_n)+1}(x) = T_{n-(\beta_{n-1}+\beta_n)+1}(x) + \frac{\partial T_{n-(\beta_{n-1}+\beta_n)+1}(x_0)}{\partial x_{n-1}} \cdot$$

$$(x_{n-1} - x_{(n-1)0}) + \frac{\partial T_{n-(\beta_{n-1}+\beta_n)+1}(x_0)}{\partial x_n}(x_n - x_{n0}) + \ldots$$

$$\vdots$$

$$T_{\beta_1} = T_{\beta_1}(x_0) + \langle dT_{\beta_1}(x_0), (x - x_0)\rangle + \ldots \tag{22}$$

by ignoring the higher order terms (+...), we find

$$x_n - x_{n0} = (T_{n-\beta_n+1} - T_{n-\beta_n+1}(x_0)) \frac{1}{\frac{\partial T_{n-\beta_n+1}(x_0)}{\partial x_n}}$$

$$x_{n-1} - x_{(n-1)0} = (T_{n-(\beta_{n-1}+\beta_n)+1} - T_{n-(\beta_{n-1}+\beta_n)+1}$$

(23) $$- \frac{\partial T_{n-(\beta_{n-1}+\beta_n)+1}(x_0)}{\partial x_n} (x_n - x_{n0})) \frac{1}{\frac{\partial T_{n-(\beta_{n-1}+\beta_n)+1}(x_0)}{\partial x_{n-1}}}$$

$$\vdots$$

$$x_1 - x_{10} = \ldots .$$

Our notation here is $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$.

Hence we have computed the inverse of the linear part of the transformation T about x_0. We know the $F_i = L_f^{\beta_i} h_i$ in (14) as functions of $x_1, x_2, \ldots, x_n$. By employing equation (23) in the Taylor series of each F_i about x_0 and dropping terms higher than degree 1, we have the linear part of each F_i in the T coordinates. Hence we obtain equations (18), (19), and (20). In this way we can build a linear system and a Luenberger observer which can possibly be useful near x_0. Moreover, the computation above can be implemented on a computer.

In [8], the author considered a control system (3), linearized this system about points as movement along a trajectory occurred, and used the various linearizations to determine control actions. It is our plan to use the observation model in much the same way, but for state estimation.

As mentioned in the introduction, suppose we take a system with controls and output

$$\dot{x} = f(x(t)) + \sum_{i=1}^{m} u_i(t) g_i(x(t))$$

(24) $$y = h(x(t))$$

and desire to build a state estimator to use in controlling the system. What conditions allow us to use an observation model at a point x_0 which is independent of the controls applied? In other words, when can we parallel the method used for a linear system? A Ph.D. student of the first author, Mladen Luksic, is making considerable progress on this problem. He is developing a theory, including application of mathematical topics like the Monge-Ampere equations, and is also working on practical simulation problems.

Suppose we consider the time responses with initial state x_0 of the nonlinear system (1) and the observation model (2), and for simplicity we assume both are single output systems. For (1) the output is

$$y(t) = h(x_0) + \sum_{k=1}^{n+1} L_f^k h(x_0) \frac{t^k}{k!} + 0(|t|^{n+2}),$$

and for (2) we have

$$y(t) = h(x_0) + \sum_{k=1}^{n+1} CA^{k-1} f(x_0) \frac{t^k}{k!} + 0(|t|^{n+2}).$$

But $L_f^k h(x_0) = CA^{k-1} f(x_0)$, $k = 1,2,\ldots,n+1$ since $dL_f^k h(x_0) = CA^k$ [redacted], $k = 0,1,\ldots,n$. Hence these outputs starting when $x=x_0$ have difference $0(|t|^{n+2})$. This can easily be generalized to multioutput systems.

References

[1] H. Ford, L.R. Hunt and R. Su, A simple algorithm for computing canonical forms, Applications of Computers in Mathematics, to appear.

[2] H. Ford, L.R. Hunt, G. Meyer and R. Su, The modified tangent model, submitted.

[3] J.P. Gauthier and G. Bornard, Observability for any u(t) of a class of nonlinear systems, IEEE Trans. on Automat. Contr. 26, No. 4(1981), 922-926.

[4] R. Hermann and A.J. Krener, Nonlinear controllability and observability, IEEE Trans. on Automat. Contr. 22, No. 5(1977), 728-740.

[5] L.R. Hunt and R. Su, Linear approximations of nonlinear systems, 22nd IEEE Conference on Decision and Control, to appear.

[6] S.R. Kou, D.L. Elliott and T.J. Tarn, Observability of nonlinear systems, Infor. Control 22(1973), 89-99.

[7] A.J. Krener and A. Isidori, Linearization by output injection and nonlinear observers, preprint.

[8] G. Meyer, The design of exact nonlinear model followers, Joint Automatic Control Conference (1981), FA3A.

[9] H. Nijmeijer, Observability of a class of nonlinear systems: A geometric approach, preprint.

[10] H. Nijmeijer, Observability of autonomous discrete-time nonlinear systems: A geometric approach, preprint.

[11] G.W. Stewart, Introduction to Matrix Computations, Academic Press, New York, 1973.

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