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CONTROL DESIGN FOR ROBUST STABILITY IN LINEAR REGULATORS: APPLICATION TO AEROSPACE FLIGHT CONTROL

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Control Design for Robust Stability in Linear Regulators: Application to Aerospace Flight Control

Final Report for

NASA Langley Research Center

Grant #NAG-1-578

by

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Foreword

This report was prepared by the Department of Eletrical Engineering, The University of Toledo, Toledo, Ohio 43606 under NASA Grant NAG-1-578. The work was performed under the direction of Ms. Carol D. Wieseman and Dr. Jerry R. Newsom of the Aeroservoelasticity Branch of NASA Langley Research Center, Hampton, Va. 23665.

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NOMENCLATURE

R^{α}	= real vector space of dimension
$\rightarrow$	= belongs to
$[\cdot]$	= Eigenvalues of the matrix $[\cdot]$
$[\cdot]$	= singular value of the matrix $[\cdot]$
	= $\lambda([\cdot][\cdot]^T)\}^{1/2}$
$[\cdot]_S$	= symmetric part of a matrix $[\cdot]$
$\|(\cdot)\|$	= modulus of the entry $(\cdot)$
$[\cdot]_m$	= modulus matrix = matrix with modulus entires
$\forall$	= for all
I_α	= $\alpha \times \alpha$ Identity matrix
$\|\|(\cdot)\|\|_S$	= spectral norm of the matrix $(\cdot)$ = $\sigma_{max}(\cdot)$
$\|\|(\cdot)\|\|_F$	= Frobenius norm of the matrix $(\cdot)$ = $(\Sigma(\ )^2_{ij})^{1/2}$
$\|\|(\cdot)\|\|$	= any other norm of the matrix $(\cdot)$

Abstract

Time domain stability robustness analysis and design for linear multivariable uncertain systems with bounded uncertainties is the central theme of the research under the present grant. After reviewing the recently developed upper bounds on the linear, elemental (structured), time varying perturbation of an asymptotically stable linear time invariant regulator, it is shown that it is possible to further improve these bounds by employing state transformations. Then introducing a quantitative measure called the 'stability robustness index', a state feedback control design algorithm is presented for a general linear regulator problem and then specialized to the case of 'modal systems' as well as 'matched systems'. The extension of the algorithm to stochastic systems with Kalman filter as the state estimator is presented. Finally an algorithm for 'robust dynamic compensator' design is presented using Parameter Optimization (PO) procedure. Applications in aircraft control and flexible structure control are presented along with a comparison with other existing methods.

I. INTRODUCTION AND PERSPECTIVE

It is well known that the inevitable presence of modeling errors in the model used for control design invariably limits the performance attainable from the control system designs produced by either classical (frequency domain) or modern (time domain) control theory. It is thus evident that 'robustness' is an extremely desirable (for some applications, even necessary) feature of any feedback control design proposed. 'Robustness' studies of linear systems is the central theme of the present research.

For our present purposes a 'robust' control design is that design which behaves in an 'acceptable' fashion (i.e., satisfactorily meets the system specifications) even in the presence of modeling errors. Since the system specifications could be in terms of stability and/or performance (regulation, time response, etc.) we can conceive two types of robustness, namely, 'Stability Robustness' and 'Performance Robustness'. Limiting our attention in this research to 'parameter errors' as the type of modeling error that may cause instability (or performance degradation) in the system, we formally define 'stability robustness' and 'performance robustness' as follows:

<u>'Stability Robustness'</u>. Maintaining closed loop system stability in the presence of modeling errors, mainly parameter varitions.

<u>'Performance Robustness'</u>: Maintaining satisfactory level of performance (or regulation) in the presence of modeling errors, mainly parameter variations.

Clearly 'stability robustness' is a prerequisite to 'performance robustness'. Hence in this research we concentrate on the aspect of 'stability robustness' while the aspect of 'performance robustness' is addressed in the research sponsored by the Wright Patterson Air Force Base under a separate contract and these details are discussed in ref. [1].

The published literature on the 'robustness' of liner systems can be viewed mainly from two perspectives, namely i) frequency domain analysis and ii) time domain analysis. The main direction of research in frequency domain has been to extend and generalize the well known classical single input single output treatment to the case of multiple input multiple output systems, using the singular value decomposition [2-3]. In the case of frequency domain results, the perturbations are mainly viewed in terms of 'gain' and 'phase' changes [4-5]. The time domain treatment is more or less similar to the frequency domain treatment in spirit but quite different in detail. The time domain treatment is more amenable to treating perturbations in the form of real parameter variations, nonlinearities and external disturbances and also for the physical interpretation of many real life perturbations. This research treats the robustness analysis and design from time domain viewpoint and in particular focuses on the well known Linear Quadratic Regulator problem. In addition, the main tool used is the Lyapunov stability analysis which allows time varying perturbations to be considered in the analysis.

The problem of maintaining the stability of a nominally stable system subject to perturbations has been an active topic of research for quite some time. One factor which clearly influences this type of analysis is the characterization or type of 'perturbation'. Even in the context of nominally stable linear systems, the 'perturbations' can take different forms like linear, nonlinear, time invariant, time varying, structured and unstructured. "Structured perturbations' are those for which bounds on the individual elements of the perturbation matrix are known (or derived) whereas 'unstructured perturbations' are those for which only a norm bound on the perturbation matrix is known (or derived). In this reserch, we focus our attention on linear, time

varying, structured perturbations as affecting a nominally stable linear time invariant system.

With this perspective in mind, the report is organized as follows: Section II briefly reviews the recently developed upper bounds in the linear, time varying, structured (elemental) perturbation of an asymptotically stable linear time invariant system to maintain stability. Then a state transformation technique is presented to further reduce the conservatism of these bounds. Section III is completely devoted to the design of linear full state feedback controllers for robust stability where the algorithms are specialized to 'modal systems' (as in flexible structure examples) and 'matched systems' (where the uncertainty satisfies a special condition called 'matching condition'). The design algorithm is also extended to stochastic systems with state estimate feedback. Section IV addresses the important aspect of designing reduced order dynamic compensators (which have practical implications) with robust stability as an additional constraint to the standard linear quadratic regulator problem. The solution technique involves parameter optimization (PO) concept. The proposed procedures are illustrated with several examples. Finally Section V offers some concluding remarks and explores avenues for future research that needs the continued sponsorship of NASA.

II. ANALYSIS OF STABILITY ROBUSTNESS FOR LINEAR SYSTEMS

In the present day applications of linear systems theory and practice, one of the challenges the designer is faced with is, to be able to guarantee 'acceptable' behavior of the system even in the presence of perturbations. The fundamental 'acceptable' behavior of any control design for linear systems is 'stability' and accordingly one of the important tasks of the designer is to assure stability of the system subject to perturbations.

In particular, as discussed in the introduction, we concentrate on 'parameter uncertainty' as the type of perturbation acting on the system. This section, thus, addresses the analysis of 'stability robustness' of linear systems subject to parameter uncertainty.

2.1 Review of Stability Robustness Bounds in Time Domain

We now briefly review the upper bounds for robust stability available in the literature for the two kinds of perturbation discussed in Section I.

Consider the following linear dynamical system

$$\dot{x}(t) = A(t)\, x\,(t) = [A_0 + E(t)]\, x\,(t) \tag{2.1}$$

where $x(t) \rightarrow R^n$ is the state vector, A_0 is the nxn nominally stable matrix and $E(t)$ is the 'Error' matrix.

2.1.1 Bounds for Unstructured Perturbation (U.P.)

Explicit bounds for robust stability under unstructured perturbations have been reported in refs. [6-8]. In these refs., it is shown that the system of (2.1) is stable if

$$\sigma_{max}[E(t)] < \frac{\sigma_{min}(Q)}{\sigma_{max}(\overline{P})} \equiv \overline{\mu} \tag{2.2a}$$

where $\overline{P}$ satisfies the Lyapunov equation

$$\overline{P}A_o + A_o^T \overline{P} + 2Q = 0 \tag{2.2b}$$

It was shown by Patel and Toda in [8] that $Q = I_n$ maximizes the ratio $\overline{\mu}$ for given A_o. Thus the eventual bound is given by

$$\sigma_{max}(E(t)) < \frac{1}{\sigma_{max}(P)} \equiv \mu_p \tag{2.3a}$$

where P satisfies the Lyapunov equation

$$P\,A_o + A_o^T P + 2I_n = 0 \tag{2.3b}$$

2.1.2 <u>Bounds for Structured Perturbations (S.P.)</u>

In [8], using the bound for unstructured perturbations, a bound for structured perturbation was presented as

$$\varepsilon < \frac{\mu_p}{n} \quad \text{where } E_{ij}(t) < \varepsilon_{ij} \overset{\Delta}{=} \sup_t |E_{ij}(t)|_{max} \text{ and } \varepsilon \overset{\Delta}{=} \max_{i,j} \varepsilon_{ij} \tag{2.4}$$

and μ_p is as defined by (2.3).

Recently, by taking advantage of the structural information of the nominal as well as perturbation matrices, improved measures of stability robustness are presented in [9]-[10] as follows:

The system of (2.1) is asymptotically stable if

$$\varepsilon_{ij} < \frac{1}{\sigma_{max}[P_m U_e]_s} \cdot U_{eij} \equiv \mu_s U_{eij} \equiv \mu_{sij} \tag{2.5a}$$

or $$\varepsilon < \mu_s \tag{2.5b}$$

for all i,j = 1,...,n where P satisfies (2.3b) and

$$U_{eij} \overset{\Delta}{=} \varepsilon_{ij}/\varepsilon \text{ (Thus } 0 \leq U_{eij} \leq 1) \tag{2.5c}$$

It may be noted that U_e can be formed even if one knows only the ratio $\varepsilon_{ij}/\varepsilon$ instead of knowing ε_{ij} (and ε) separately. One suitable choice for the ratio is,

$$U_{eij} = \varepsilon_{ij}/\varepsilon = |A_{oij}|/|A_{oij}|_{max} \qquad (2.5d)$$

for all i,j for which $\varepsilon_{ij} \neq 0$.

Remark 1: From (2.4), it is seen that ε_{ij} are the maximum modulus deviations expected in the individual elements of the nominal matrix A_o. If we denote the matrix Λ as the matrix formed with ε_{ij}, then clearly Δ is the 'majorant' matrix of the actual error matrix E(t). It may be noted that U_e is simply the matrix formed by normalizing the elements of Δ (i.e. ε_{ij}) with respect to the maximum of ε_{ij} (i.e. ε)

$$\text{i.e., } \Delta = \varepsilon\, U_e \text{ (absolute variation).} \qquad (2.6)$$

Thus ε_{ij} here are the absolute variations in A_{oij}. Alternatively one can express Δ in terms of percentage variations with respect to the entries of A_{oij}. Then one can write

$$\Delta = \delta\, A_{om} \text{ (relative (or percentage) variation)} \qquad (2.7)$$

where $A_{omij} = |A_{oij}|$ for all those i,j in which variation is expected and $A_{omij} = 0$ for all those i,j in which there is no variation expected and δ_{ij} are the maximum relative variations with respect to the nominal value of A_{oij} and $\delta = \max_{i,j} \delta_{ij}$. Clearly, one can then get a bound on δ for robust stability as

$$\delta < \frac{1}{\sigma_{max}(P_m A_{om})} \quad \text{where P is the same as in (2.3) and (2.5).}$$

2.2 Reduction in Conservatism by State Transformation:

The proposed stability robustness measures presented in the previous section were basically derived using the Lyapunov stability theorem, which is known to yield conservative results. One 'improvement' obtained in the proposed bounds is the result of exploiting the 'structural' information about the perturbation. Clearly, another avenue available to further reduce the conservatism is to exploit the flexibility available in the construction of the

Lyapunov function used in the analysis. In this section, a method to further reduce the conservatism on the element bounds (for structural perturbation) is proposed by using state transformation. This reduction in conservatism is obtained by exploiting the variance of the 'Lyapunov criterion conservatism' with respect to the basis of the vector space in which the function is constructed. The proposed transformation technique seems to almost always increase the region of guaranteed stability and thus is found to be useful in many engineering applications.

2.2.1 State Transformation and Its Implications on Bounds

It may be easily shown that the linear system (2.1) is stable (or asymptotically stable) if and only if the system

$$\dot{\hat{x}}(t) = \hat{A}(t)\,\hat{x}(t) \tag{2.8a}$$

where

$$\hat{x}(t) = M^{-1}\,x(t),\quad \hat{A}(t) = M^{-1}\,A(t)M \tag{2.8b}$$

and M is a nonsingular time invariant nxn matrix, is stable (or asymptotically stable).

The implication of this result is, of course, important in the proposed analysis. The concept of using state transformation to improve bounds based on a Lyapunov approach has been in use for a long time as given in [11] where Siljak applies this to get bounds on the interconnection parameters in a decentralized control scheme using vector Lyapunov functions. The proposed scheme in this paper is similar to this concept in principle but considerably different in detail when applied to a centralized system with parameter variations. In this context, in what follows, we transform the given perturbed system to a different coordinate frame, derive a stability condition in the new cordinate frame. However realizing that in doing so even the perturbation gets

transformed, we do make an inverse transformation to eventually give a bound on the perturbation in original coordinates and show with the help of examples that it is indeed possible to give improved bounds on the original perturbation, with state transformation as a vehicle than without a transformation.

We now investigte the use of a transformation on the bounds for both unstructured perturbations (U.P.) as well as for structured perturbations (S.P.).

2.2.2 Unstructured Perturbations

Theorem 2.1: The system of (2.1) is guaranteed to be stable if

$$||E(t)||_s = \sigma_{max}[E(t)] < \frac{\hat{\mu}_p}{||M^{-1}||_s ||M||_s} \equiv \mu_p^* \tag{2.9a}$$

where $$\hat{\mu}_p = \frac{1}{\sigma_{max}(\hat{P})} \tag{2.9b}$$

and $\hat{P}$ satisfies

$$\hat{P}\,\hat{A}_o + \hat{A}_o^T\,\hat{P} + 2I_n = 0 \tag{2.9c}$$

and

$$\hat{A}_o = M^{-1}\,A_o M,\ \hat{E}(t) = M^{-1}\,E(t)\,M. \tag{2.9d}$$

Note that $||\hat{E}(t)||_s \leq ||M^{-1}||_s\ ||E(t)||_s\ ||M||_s$ and $\mu_p^* = \hat{\mu}_p/\alpha$ where α is a scalar given as a function of the transformation matrix M. In this case, of course, α is the condition number. Also it is to be noted that the stability condition in transformed coordinates is

$$\sigma_{max}[\hat{E}(t)] < \hat{\mu}_p. \tag{2.10}$$

Thus $\hat{\mu}_p$ is the bound on $||\hat{E}||_s$ whereas μ_p^* is the bound on $||E||_s$ after transformation.

By proper selection of the transformation matrix M it is possible to obtain

$\mu_p^* > \mu_p$ as shown by the following example:

Example 1: Consider the same example considered in [8]. The nominally asymptotically stable matrix A_o is given by

$$Ao = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \text{ with } M = \begin{bmatrix} 0.99964 & -0.28217 \\ 0.0266 & 0.95937 \end{bmatrix}$$

the bounds are obtained as

μ_p	μ_p^*
0.382	0.394

μ_p = Bound before transformation.

μ_p^* = Bound after transformation.

2.2.3 Structured Perturbations

Similar to the unstructured perturbation case, it is possible to use a transformation to get better bounds on the structured perturbation case also. In fact, in the case of a structured perturbation, it may be possible to get higher bounds even with the use of a diagonal transformation. Hence in what follows, we consider a diagonal transformation matrix M for which it is possible to get bound in terms of the elements of M.

Theorem 2.2: Given

$$M = \text{Diag}\,[m_1, m_2, m_3, \ldots, m_n] \tag{2.11}$$

the system of (2.1) is stable if

$$\varepsilon_{ij} < \frac{\hat{\mu}_s}{\max\limits_{i,j} \left| \frac{m_j}{m_i} U_{eij} \right|} U_{eij} \equiv \mu_s^* U_{eij} \tag{2.12a}$$

or

$$\varepsilon < \mu_s^* \tag{2.12b}$$

where

$$\hat{\mu}_s = \frac{1}{\sigma_{max}(\hat{P}_m \hat{U}_e)_s} \tag{2.12c}$$

and $$\hat{P}\hat{A}_o + \hat{A}_o^T \hat{P} + 2I_n = 0 \qquad (2.12d)$$

and $$\hat{U}_{eij} = \hat{\varepsilon}_{ij}/\hat{\varepsilon} \text{ and } \hat{\varepsilon}_{ij} = \left|\frac{m_j}{m_i}\right| \varepsilon_{ij} \qquad (2.12e)$$

As before $\mu_s^* = \hat{\mu}_s/\alpha$ where α is again a function of the transformation matrix elements m_i.

Example 2: As before let

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}. \quad \text{Let } U_e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}; \text{ With } M = \begin{bmatrix} 1 & 0 \\ 0 & 2.2 \end{bmatrix}$$

we get

μ_s	μ_s^*
0.4805	0.6575

μ_s = Bound before transformation.

μ_s^* = Bound after transformation.

The use of a transformation to reduce conservatism of the bound for structured perturbations and its application to design of a robust controller for a VTOL aircraft control problem is presented in [12].

Remark 2: The flow chart for obtaining the bounds by transformation is as follows:

Original Coordinates, x(t) $\|E\|_s < \mu_p$ (U.P.), $\varepsilon < \mu_s$ (S.P.)	$\xrightarrow{\text{Transformation } M}$	Transformed Coordinates, $\hat{x}(t)$ $\|\hat{E}\|_s < \hat{\mu}_p$, $\hat{\varepsilon} < \hat{\mu}_s$	$\xrightarrow{M^{-1}}$	Original Coordinates, x(t) $\|E\|_s < \mu_p^*$, $\varepsilon < \mu_s^*$

Remark 3: The evolution of bounds μ to μ_p to μ_p^* ($\bar{\mu}_s$ to μ_s to μ_s^*) can be summarized as follows:

$\bar{\mu} = \dfrac{\sigma_{min}(Q)}{\sigma_{max}(\bar{P})}$; $\bar{\mu}_s = \dfrac{\sigma_{min}(Q)}{\sigma_{max}(\bar{P}U_e)_s}$ $V = \text{Lyap.funct} = x^T \bar{P} x$ $\bar{P} A_o + A_o^T \bar{P} + 2Q = 0$	$Q \to I_n$ ⟶	$\mu_p = \dfrac{1}{\sigma_{max}(P)}$, $\mu_s = \dfrac{1}{\sigma_{max}(P_m U_e)_s}$ $V = x^T P x$ $P A_o + A_o^T P + 2I_n = 0$	$\hat{x}(t) = M^{-1}x(t)$ ⟶	μ_p^* of (2.9), μ_s^* of (2.12) $V = \hat{x}^T \hat{P} \hat{x}$ $\hat{P} \hat{A}_o + \hat{A}_o^T \hat{P} + 2I_n = 0$

From the above sequence, it is clear that the coordinate frame in which the Lyapunov function is constructed has a significant effect on the bound in relation to the effect of the matrix Q in a given coordinate frame.

2.2.4 Determination of (almost) 'Best' Transformation

As seen from the previous section, in order to get a better (higher) bound, it is crucial to select an appropriate transformation matrix M. Obviously the question arises: How can one find a transformation that gives a better bound than an original one or even the 'best' among all possible choices for the transformation. In this section, we attempt to address this question for the special case of a diagonal transformation to be used in the structured perturbation case.

'Best' Diagonal Transformation for S.P.

Recall from (2.12), the expression for μ_s^*. Without loss of generality, let us look for $m_k > 0$ $(k = 1,2,\ldots,n)$ such that μ_s^* is 'maximized'.

From (2.12), the matrix $\hat{P}$ satisfies

$$\hat{P}(M^{-1}A_oM) + (M^{-1}A_oM)^T \hat{P} = -2I_n \tag{2.13}$$

Since M is diagonal, $M^T = M$ and (2.13) gives

$$(M^{-1}\hat{P}M^{-1})A_o + A_o^T(M^{-1}\hat{P}M^{-1}) = -2(M^{-1})^2 \tag{2.14}$$

Letting

$$P^* \triangleq M^{-1}\hat{P}M^{-1} \quad (\text{i.e. } \hat{p}_{ij} = p^*_{ij}m_i m_j) \tag{2.15}$$

(2.15) becomes

$$P^*A_o + A_o^T P^* = -2(M^{-1})^2 \tag{2.16}$$

The matrix equation (2.16) contains $n(n+1)/2$ scalar equations from which the elements of the matrix P^* can be expressed as functions of m_i. And from (2.15), P_{ij} can then be expressed as functions of m_i. Thus one can express the bound of μ_s^* of (2.12) as a function of m_i. We need to find m_i that maximize μ_s^* by determining the first order derivatives and equating them to zero. However μ_s^* contains the spectral norm of $(\hat{P}_m\hat{U}_e)_s$ which is difficult to express in terms of m_i. Hence, using the fact that $||(\cdot)||_s \leq ||(\cdot)||_F$, we choose to maximize

$$L \triangleq \frac{1}{\sum_{i,j} (\hat{P}_m\hat{U}_e)^2_{sij} \left[\max_{i,j} \left(\frac{m_j}{m_i} U_{eij}\right)\right]^2} \tag{2.17}$$

with respect to m_i, $i = 1,2,\ldots,n$.

The algorithm is best illustrated by a simple example.

<u>Example 3</u>:

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad U_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

For simplicity let us select $M = \text{Diag}[1, m]$.

Carrying out the steps indicated above, we observe that the minimum value of

$$\frac{1}{L} = \sum_{i,j} (\hat{P}_m \hat{U}_e)^2_{sij} \left[\max_{i,j} \left(\frac{m_j}{m_i} U_{eij}\right)\right]^2$$

$$= \hat{p}_{11}^2 + 1/2\ \hat{p}_{12}^2$$

$$= (0.333 + 1.667 \left(\frac{1}{m^2}\right)^2 + \frac{1}{2} \left(\frac{1}{2m}\right)^2$$

occurs at $m \to \infty$ and thus $L_{max} = 3$.

Hence, $L_{max} = 3 \leq \mu_s^* \to \mu_s^* = 3$.

Note that before transformation, $\mu_s = 1.657$. Thus there is an 81% improvement in the bound after transformation.

Application to the Drone Example [5]:

The system matrices for the Drone Lateral Attitude Control system considered in [5] are given by

$$A = \begin{bmatrix} -0.0853 & -0.0001 & -0.9994 & 0.0414 & 0.0000 & 0.1862 \\ -46.8600 & -2.7570 & 0.3896 & 0.0000 & -124.3000 & 128.6000 \\ -0.4248 & -0.0622 & -0.0671 & 0.0000 & -8.7920 & -20.4600 \\ 0.0000 & 1.0000 & 0.0523 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 \end{bmatrix} \tag{2.18a}$$

$$B = \begin{bmatrix} 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 20. & 0. \\ 0. & 20. \end{bmatrix} \tag{2.18b}$$

With a linear state feedback control gain

$$G = \begin{bmatrix} -215.1000 & 4.6650 & 7.8950 & 233.2000 & -6.7080 & 2.5540 \\ -231.5000 & -3.7230 & 7.4530 & -213.5000 & 2.5540 & -6.8690 \end{bmatrix} \tag{2.18c}$$

the closed loop system matrix $\bar{A} = A + BG$ is made asymptotically stable.

Now assuming the element A_{21} to be the uncertain parameter (having a nominal value = -46.86) we get the stability robustness bound on this parameter (using the U_e matrix as $U_{e21} = 1$ and $U_{eij} = 0$ for all other i,j), as

$$\mu_{21} = 2.43 \tag{2.19}$$

However, using the transformation

$$M = \text{Diag}\,[0.005 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \tag{2.20}$$

we get the bound on A_{21} as

$$\overset{*}{}_{21} = 573.46 \tag{2.21}$$

which is clearly a significant improvement.

III. FULL STATE AND STATE ESTIMATE FEEDBACK CONTROL DESIGN FOR ROBUST STABILITY

The foregoing discussion in Section II is basically concerned with the analysis of stability robustness for linear systems. No effort was made to synthesize a controller to achieve stability robustness. In this section, we address this design aspect from a systematic algorithmic point of view. The philosophy behind the proposed procedure is to make use of the perturbation bounds developed in the previous section in a design formulation and give an algorithm to synthesize controllers for robust stability. Towards this direction, a quantitative measure called 'stability robustness index' is introduced and based on this index a design algorithm is presented by which one can pick a controller that possesses good stability robustness property. The algorithm, for given size of perturbation can be used to select the range of control gain for which the system is stability robust or alternatively, for given control gain, can be used to determine the range of the size of allowable perturbations for stability. In this attempt, we first consider the case of full state and state estimate feedback controllers and then investigate the use of reduced order dynamic compensators in Section IV. In this section we also specialize the design algorithm to 'modal systems' as well as 'matched systems'.

3.1 Linear State Feedback Control Design Using Perturbation Bound Analysis

Consider the linear, time invariant system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.1}$$

where x is nx1 state vector, the control u is mx1 and output y (the variables we wish to control) is kx1. The matrix triple (A,B,C) is assumed to be completely controllable and observable. Let the control law be given by

$$u = Gx \tag{3.2}$$

Now let ΔA and ΔB be the perturbation matrices formed by the maximum modulus deviations expected in the individual elements of matrices A and B rspectively. Then one can write

$$\left.\begin{aligned} \Delta A &= \varepsilon_a \, Ue_a \\ \Delta B &= \varepsilon_b \, Ue_b \end{aligned}\right\} \quad \text{Absolute variation} \tag{3.3}$$

where ε_a is the maximum of all deviations in A and ε_b is the maximum of all deviations in B. Then the total perturbation in the linear closed loop system matrix of (3.1) with nominal control $u = Gx$ is given by

$$\Delta = \Delta A + \Delta B G_m = \varepsilon_a \, Ue_a + \varepsilon_b \, Ue_b \, G_m \tag{3.4}$$

Assuming the ratio $\varepsilon_b/\varepsilon_a = \bar{\varepsilon}$ is known, we can extend the main result of section (2.1.2) to the linear state feedback control system of (3.1)-(3.2) and obtain the following design observation.

<u>Design Observation 1</u>:

The perturbed linear system is stable for all perturbations bounded by ε_a and ε_b if

$$\varepsilon_a < \frac{1}{\sigma_{max}[P_m Ue_a + \bar{\varepsilon} \, Ue_b \, G_m]_s} \equiv \mu \tag{3.5a}$$

and $\varepsilon_b < \bar{\varepsilon} \, \mu$ where

$$P(A+BG) + (A+BG)^T P + 2 \, I_n = 0 \tag{3.5b}$$

Alternately, we can write

$$\left.\begin{aligned} \Delta A &= \delta_a \, A_m \\ \Delta B &= \delta_b \, B_m \end{aligned}\right\} \quad \text{Relative variation} \tag{3.6}$$

where $Am_{ij} = |Aij|$ and $Bm_{ij} = |Bij|$ for all those i,j in which variation is expected and $Am_{ij} = 0$, $Bm_{ij} = 0$ for all those i,j in which there is no variation expected. For this situation, assuming $\delta_b/\delta_a = \bar{\delta}$ is known, we get the following bound on δa for

robust stability.

Design Observation 2:

The perturbed linear system is stable for all relative (or percentage) perturbations bounded by δa and δb if

$$\delta_a < \frac{1}{\sigma_{max}[P_m(A_m + B_m G_m)]_s} \equiv \mu_r \tag{3.7}$$

and $\delta b < \bar{\delta}\ \mu_r$ where P satisfies the equation (3.5b).

Stability Robustness Index and Control Design Algorithm:

We now define, as a measure of stability robustness, an index called 'Stability Robustness Index $\beta_{S.R.}$' as follows:

Case a): L.H.S. of (3.5 or 3.7) is known (i.e. checking stability for given perturbation range). For this case

$$\beta_{S.R.} \stackrel{\Delta}{=} \mu - \varepsilon a \text{ (or } \mu_r - \delta_a). \tag{3.8a}$$

Case b): L.H.S. of (3.5 or 3.7) is not known (i.e. specifying the bound). For this case

$$\beta_{S.R.} \stackrel{\Delta}{=} \mu \text{ (or } \mu_r). \tag{3.8b}$$

It is clear from the expressions for μ (3.5), the 'error matrix' (3.4) and $\beta_{S.R.}$ (3.8) that these quantities depend on the control gain G and as the gain G is varied $\beta_{S.R.}$ changes. In order to plot the relationship between $\beta_{S.R.}$ and the gain G, we need a scalar quantitative measure of G. For this, we can either use

$$J_{cn} = ||G||_s = \sigma_{max}(G) \tag{3.9a}$$

or

$$J_{cn} = [\int_0^\infty (u^T u)dt]^{1/2} = [\int_0^\infty x^T G^T G x dt]^{1/2} \tag{3.9b}$$

where J_{cn} denotes a measure of 'nominal control effort'. We use (3.9b).

The variation of $\beta_{S.R.}$ with the control effort J_{cn} is very much dependent on

the perturbation matrices and on the behavior of the Lyapunov solution, which cannot be described analytically in a straightforward way. Assuming stability robustness is the only design objective, the design algorithm basically consists of picking a control gain that maximizes stability robustness ($\beta_{S.R.}$). Specifically the algorithm involves determining the index $\beta_{S.R.}$ and the control effort J_{cn} for different values of the control gain G and plotting these curves. These design curves can then be used to pick a gain that achieves a high $\beta_{S.R.}$. The algorithm thus provides a simple constant gain state feedback control law that is robust from stability point of view. The algorithm, for given perturbations, can be used for selecting the range of control effort for which the system is stability robust or alternatively for given control effort, can be used to detemine the range of allowable perturbations for stability.

The linear control gain G of (3.2) can, of course, be determined in many different ways. In this section, we assume the control gain G to be given by the standard linear quadratic regulator algorithm. Accordingly, we determine G as

$$G = -\frac{1}{\rho_c} R_o^{-1} B^T K \tag{3.10a}$$

where K satisfies the Riccati equation

$$KA + A^TK - KB \frac{R_o^{-1}}{\rho_c} B^TK + \overline{Q} = 0 \tag{3.10b}$$

for a given symmetric positive semi definite matrix $\overline{Q}$ and $R_o = I_m$. Thus ρ_c serves as the design variable.

In other words, in the proposed procedure, we determine the gain by some nominal means and then investigate the robustness of the closed loop system by checking if the gain makes the index $\beta_{S.R.}$ positive for a given ε_a (or ε_b) (case a situation) or by determining $\beta_{S.R.} = \mu$ for given control gain (for case b situation).

Extension to Linear Stochastic Systems With State Estimate Feedback:

We now extend the above analysis to the case of linear stochastic systems. Let us consider a continuous, linear, time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \quad x(0) = x_0 \tag{3.11a}$$

$$y(t) = Cx(t) \tag{3.11b}$$

$$z(t) = \bar{M}x(t) + v(t) \tag{3.11c}$$

where the state vector x is nx1, the control u is mx1, the external disturbance w is qx1, the output y (the variables we wish to control) is kx1, and the measurement vector z is ℓx1. Accordingly the matrix A is of dimension nxn, B is nxm, D is nxq, C is kxn and $\bar{M}$ is ℓxn. The initial condition x(0) is assumed to be a zero-mean, Gaussian random vector with variance X_0, i.e.,

$$E[x(0)] = 0, \quad E[x(0)x^T(0)] = X_0 \tag{3.11d}$$

Similarly the process noise w(t) and the measurement noise v(t) are assumed to be zero-mean white-noise processes with Gaussian distributions having constant covariances, W and V, respectively, i.e.,

$$E[w(t)] = E[v(t)] = 0 \tag{3.11e}$$

$$E\begin{bmatrix} w(t) \\ v(t) \end{bmatrix} [w^T(\tau)v^T(\tau)] = \begin{bmatrix} W & 0 \\ 0 & \rho_e V_0 \end{bmatrix} \tilde{\delta}(t-\tau) \tag{3.11f}$$

where ρ_e is a scalar greater than zero and $V = \rho_e V_0$ and $\tilde{\delta}$ is the dirac delta function and E is the expectation operator.

The state x(t) of the stochastic system is estimated as a function of the measurements, where the state estimator has the following structure

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu + G\hat{z}(t) \tag{3.12a}$$

where

$$\hat{z}(t) = z(t) - \bar{M}\hat{x}(t) \tag{3.12b}$$

is called the measurement residual. For minimum variance requirement, the estimator of Eq. (3.12) is the standard Kalman filter.

We also assume that the matrix pairs [A,B] and [A,D] are completely controllable, and the pairs [A,C] and [A,$\bar{M}$] are completely observable.

For this case of linear stochastic system, we consider the control law given by

$$u = G\hat{x} = \frac{1}{\rho_c} R_0^{-1} B^T K \hat{x} \tag{3.13a}$$

where

$$\dot{\hat{x}} = A\hat{x} + Bu + \hat{G}(z - \bar{M}\hat{x}), \quad \hat{x}(0) = 0 \tag{3.13b}$$

$$= (A + BG - \hat{G}\bar{M})\hat{x} + \hat{G}z \tag{3.13c}$$

$$\hat{G} = \frac{1}{\rho_e} \hat{P}\bar{M}^T V_0^{-1} \tag{3.13d}$$

and $\hat{P}$ and K satisfy the algebraic matrix Riccati equations

$$KA + A^T K - KB \frac{R_0^{-1}}{\rho_c} B^T K + \bar{Q} = 0 \tag{3.13e}$$

$$\hat{P}A^T + A\hat{P} - \hat{P}\bar{M}^T \frac{V_0^{-1}}{\rho_e} \bar{M}\hat{P} + DWD^T = 0 \tag{3.13f}$$

The nominal closed-loop system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BG \\ \hat{G}\bar{M} & \hat{A}_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{G} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \tag{3.14a}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{3.14b}$$

where $\hat{A}_c = A + BG - \hat{G}M$ and the closed loop system matrix

$$\bar{A}_c = \begin{bmatrix} A & BG \\ \hat{G}\bar{M} & \hat{A}c \end{bmatrix} \tag{3.15}$$

is asymptotically stable.

Letting ΔA, ΔB, ΔC, $\Delta\bar{M}$ and ΔD to be the maximum modulus derivations in the system matrices A, B, C, M and D respectively, we can write the total error matrix of the closed loop system as

$$\Delta = \begin{bmatrix} \Delta A & \Delta BGm \\ \hat{G}m\Delta\bar{M} & \Delta\hat{A}c \end{bmatrix} \tag{3.16}$$

and writing $\Delta A = \varepsilon_a Ue_a$, $\Delta B = \varepsilon_b Ue_b$, $\Delta\bar{M} = \varepsilon_m Ue_m$ ... etc. and knowing the ratios $\varepsilon_a/\varepsilon_b$ etc., one can get the stability robustness condition in the same manner as the equations given by (3.5).

<u>Application to the Drone Lateral-Altitude-Control Problem:</u>

The linearized model of the lateral attitude control problem of a drone aircraft, with perturbations in the plant parameters is given by

$$\dot{x} = (A + \Delta A)x + Bu \quad , \quad x(0) = x_o \tag{3.17}$$

The components of the state vector $x \to R^6$ and the control vector $u \to R^2$ are given by

$$x^T = [\beta, \dot{\phi}, \dot{\psi}, \delta 1/20, \delta 2/20]$$

$$u^T = [u_1 \quad u_2] \ , \qquad \begin{array}{l} u_1 = \text{elevon command} = \delta_1 \\ u_2 = \text{rudder command} = \delta_2 \end{array} \tag{3.18}$$

The matrices A and B are given by (2.18).

We assume that the parameters with non zero nominal values in the A matrix are subject to perturbations and thus we take the Ue_a matrix as $Ue_{aij} = |A_{ij}|/|A_{ij}|_{max}$. Accordingly the matrix Ue_a is given by

$$U_{ea} = \begin{bmatrix} 0.0007 & 0.0000 & 0.0078 & 0.0003 & 0.0000 & 0.0014 \\ 0.3644 & 0.0214 & 0.0030 & 0.0000 & 0.9666 & 1.0000 \\ 0.0033 & 0.0005 & 0.0005 & 0.0000 & 0.0684 & 0.1591 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1555 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.1555 \end{bmatrix} \tag{3.19}$$

The linear state feedback control gain is determined using the Riccati based equations of (3.10). For a given control gain (i.e. given ρ_c), the bound μ is calculated. Since ε_a is not known, in this case the stability-robustness index $\beta_{S.R.}$ is simply given by $\beta_{S.R.} = \mu$. The plot of μ with the design variable ρ_c is given in Fig. 1.

From the plot, it is seen that, for this problem, higher the control effort (lesser the ρ_c), higher is the tolerable perturbation for robust stability.

We now extend the algorithm to the stochastic controller case using

$$D = B,\ C = I_6,\ \tilde{Q} = C^T C = I_6,\ R_o = I_2$$

$$W = I_2 \quad V_o = I_2\ ,\ \bar{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.20}$$

The plot of μ vs ρ_c with ρ_e as a parameter and the comparison with the pure state feedback case is given in Fig. 2.

<u>Remark 4</u>: From this plot, it is seen that the bound μ with state estimate feedback is lower than the one with pure state feedback.

<u>Remark 5</u>: For a given ρ_c, the bound μ is higher as the measurement noise covariance is decreased, i.e. as ρ_e is decreased. This appears to be reasonable, because this means that μ becomes higher with better or more accurate measurements.

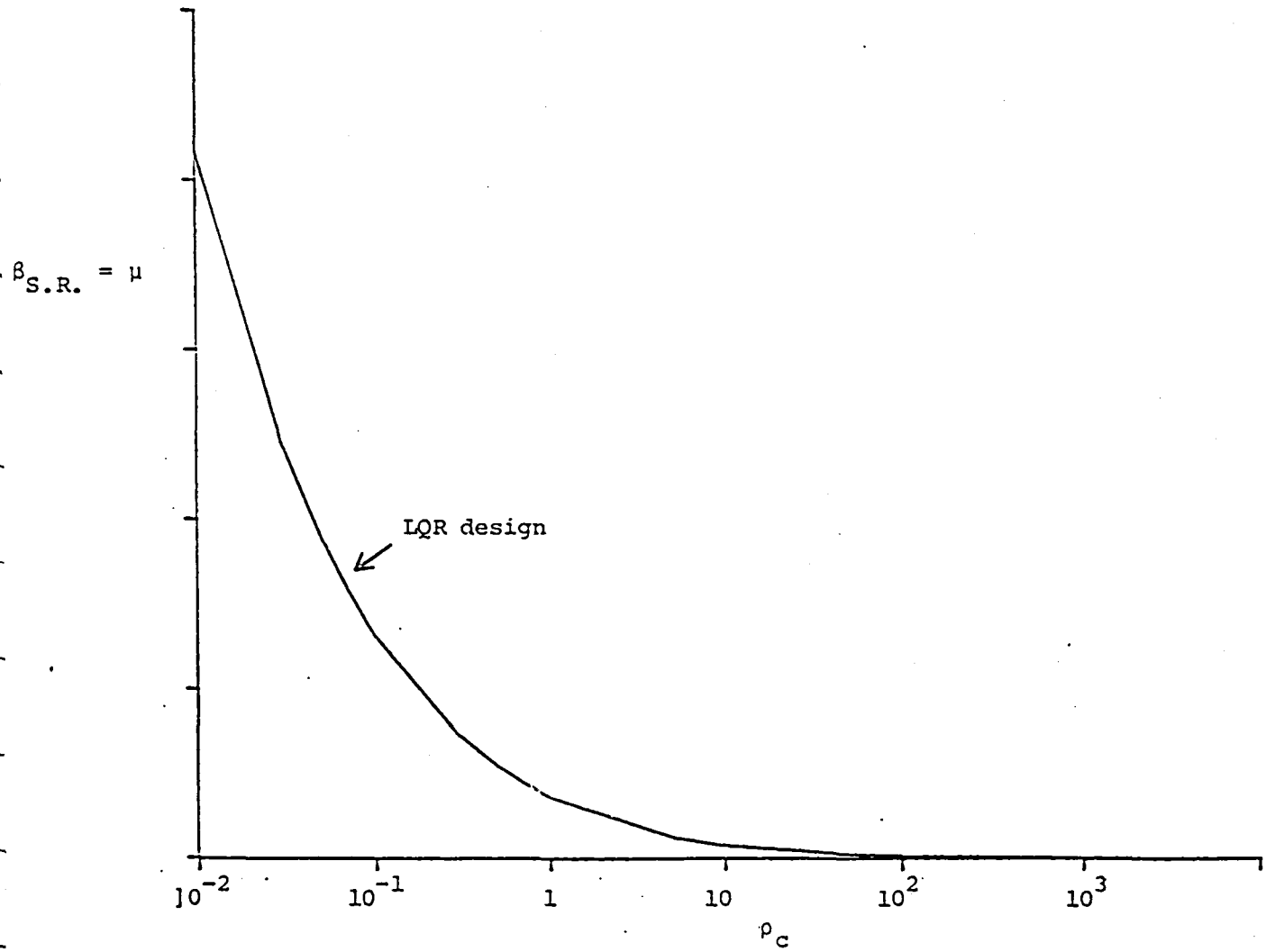

Figure 1. Variation of Bound μ with Control Weighting ρ_c with Full State Linear State Feedback for Drone Example.

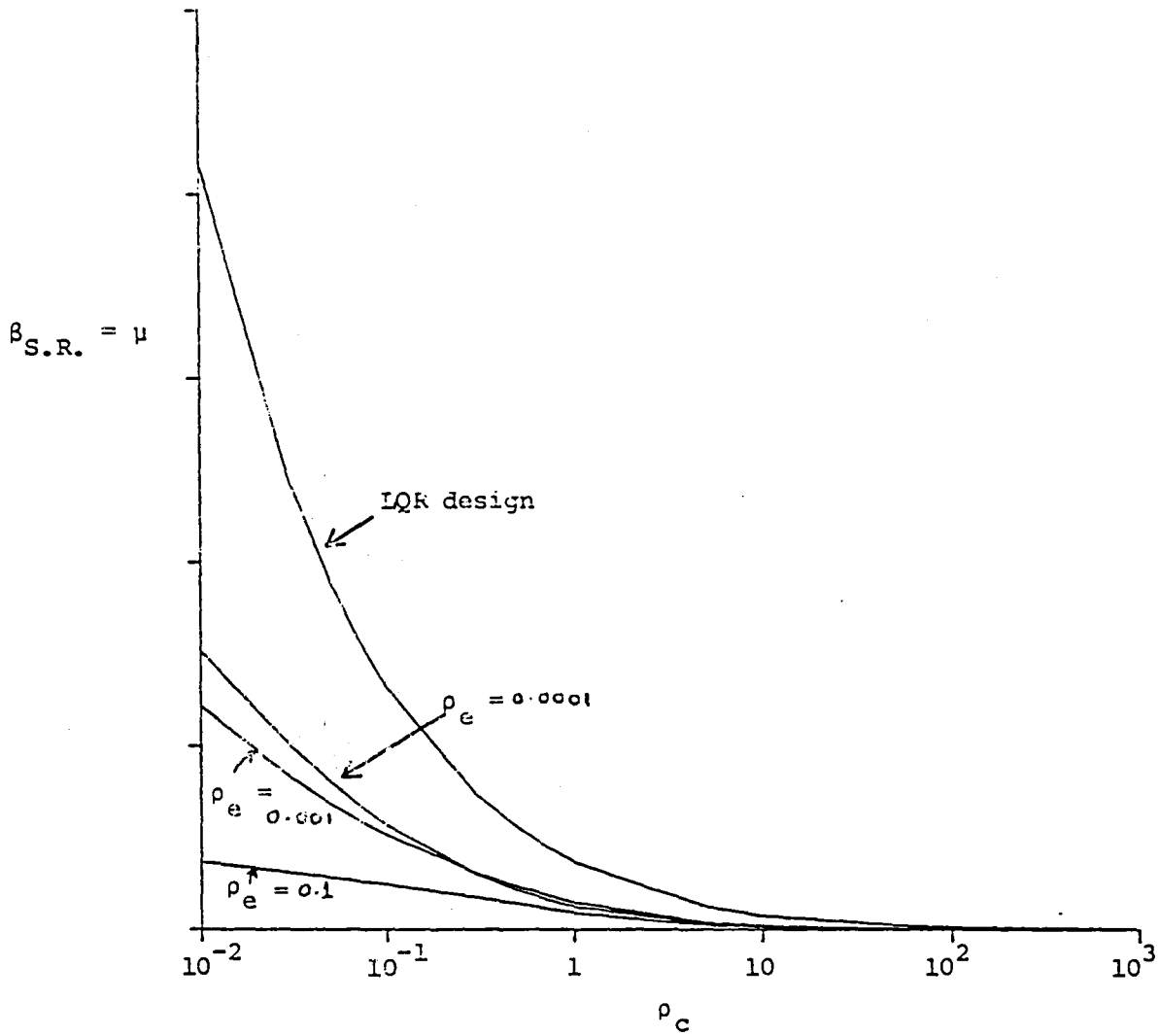

Figure 2. Variation of Bound μ with Control Weighting ρ_c with State Estimate Feedback for Drone Example.

3.2 Linear State Feedback Control for 'Modal Systems'

In this section we apply the robust control design methodology presented in the previous section to 'modal systems'. The evaluation model considered in the Large Space Structure (LSS) control problem constitutes a typical 'modal system' model. We specifically consider the LSS model with vibration suppression of the flexible modes as the control objective. We seek a linear state feedback control that achieves a reasonable trade off between the nominal performance and stability robustness by accommodating the modal uncertainty structure into the design procedure. Towards this direction, the fact that the modal data uncertainty increases with mode number is incorporated in the characterization of uncertainty in LSS model parameters and this uncertainty structure is used to obtain upper bounds for robust stability which are in turn used to get a robust controller.

LSS Models and Nominal Control Design

Consider the standard state space description of LSS evaluation model with N elastic modes:

$$\dot{x} = Ax + Bu \qquad x(0) = x_0;\ x \to R^{n=2N}\ .\ u \to R^m \tag{3.21a}$$

$$y = Cx \qquad y \to R^k \tag{3.21b}$$

where

$$x^T = [x_1^T,\ x_2^T,\ \ldots\ x_N^T];\ x_i = \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} \tag{3.21c}$$

$$A = \text{Block diag. } [\ldots A_{ii} \ldots],\ A_{ii} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix} \tag{3.21d}$$

$$B^T = [B_1^T,\ B_2^T,\ \ldots\ B_N^T];\ B_i = \begin{bmatrix} 0 \\ b_i^T \end{bmatrix} \tag{3.21e}$$

$$C = [C_1\ C_2\ \ldots\ C_N] \tag{3.21f}$$

The performance index for vibration suppression problem may be written as

$$J = \int_0^\infty [(\sum_{i=1}^{N} \omega_i^2 \eta_i^2 + \dot{\eta}_i^2) + \rho u^T u]dt \tag{3.22}$$

which can be written in the form

$$J = \int_0^\infty (y^T Qy + \rho u^T u)dt = \int_0^\infty (x^T C^T QCx + \rho u^T u)dt = J_y + \rho J_u \tag{3.23}$$

where the matrix C of (3.21) is given by

$$C = \text{Block diag. } [\ldots C_i \ldots] \tag{3.24a}$$

and

$$C_i = \begin{bmatrix} \omega_i & 0 \\ 0 & 1 \end{bmatrix} \tag{3.24b}$$

Let the nominal control law be designed by minimizing the performance index of (3.23) which results in [13]

$$u = Gx \tag{3.25a}$$

where

$$G = -\frac{1}{\rho} B^T K \tag{3.25b}$$

$$KA + A^T K - \frac{KBB^T}{\rho} K + C^T QC = 0 \tag{3.25c}$$

The closed loop system matrix

$$\bar{A} = (A + BG) \tag{3.26}$$

is assymptotically stable. In the nominal design situation, an appropriate value for ρ (and hence G) is determined such that a reasonable trade off between J_y and J_u is obtained. However, in LSS models, the parameters of the plant matrix A, namely the modal frequencies and modal damping as well as the parameters of the control distribution matrix B, namely the mode shape slopes at actuator locations are known to be uncertain. It is also known that the uncertainty in these parameters tends to increase with increase in mode number.

Thus with variations ΔA and ΔB in the matrices A and B of (3.21), the nominal control G of (3.25) cannot guarantee stability of the closed loop system. Thus one needs to design a control gain G that guarantees stability for a given range of perturbations ΔA and ΔB. This is done using the design procedure given in the previous section. In other words, the control design algorithm for robust stability consists of picking a control gain (i.e. ρ) that achieves a positive $\beta_{S.R.}$ (for case a) or high value of $\beta_{S.R.}$ (for case b).

The design algorithm involves determining the index $\beta_{S.R.}$ and the costs J_y and J_u for different values of the design parameter ρ and plotting these curves. The algorithm thus provides a simple constant gain state feedback control law (using the standard optimal LQ regulator format) that is robust from stability point of view. The algorithm, for given perturbations, can be used for selecting the range of control weighting (control effort) for which the system is stability robust or alternatively for given control effort, can be used to determine the range of allowable perturbations for stability.

In the next section, we present a specific characterization of uncertainty for LSS models and use the above methodology to design a controller for the Purdue model [14] of a two dimensional LSS.

Characterization of Parameter Uncertainty in L.S.S. Models and Application to Purdue Model

In L.S.S. models having the structure given by (3.21) the uncertainty in the modal parameters such as modal frequency dampings and mode shape slopes at actuator locations tend to increase with increase in mode number. One way of modeling this information in the uncertainty structure is given in the following (specifically we employ the relative variation format of (3.6))

$$\Delta A = \delta_a \begin{bmatrix} 0 & 0 & & & & \\ \otimes_1 & \otimes_1 & & & & \\ & & 0 & 0 & & \\ & & 2\otimes_2 & \otimes_2 & & \\ & & & & 0 & 0 \\ & & & & 3\otimes_3 & 3\otimes_3 \end{bmatrix} \tag{3.27a}$$

$$\Delta B = \delta_b \begin{bmatrix} 0 \\ b_1^T \\ 0 \\ 2b_2^T \\ 0 \\ 3b_3^T \\ \vdots \end{bmatrix} \tag{3.27b}$$

where $\otimes_i$ indicate the nominal entries corresponding to the ith mode. We assume $\delta_a = \delta_b$ which are not known.

With the above proposed uncertainty structure, we apply the robust control design methodology of previous section to the Purdue model [14]. The model used consists of the first five elastic modes. The numerical values of the model are given in ref. [14]. To conserve space the model is not reproduced here.

Since δ_a (and δ_b) are not known, the present design corresponds to case b in which case we pick a control gain that gives high $\beta_{S.R.} = \mu_r$. The plot of μ_r vs. ρ is given in Fig. 3. The robust control gain is the gain corresponding to $\rho = 0.12$.

Figs. 4 and 5 present the variation of μ_r with control effort (i.e. ρ) assuming $\Delta A = 0$ and $\Delta B = 0$ respectively. From these plots it can be concluded that the control effort range available for guaranteed stability for mode shape ($\Delta B \neq 0$, $\Delta A = 0$) variation is limited in comparison to the

range available for model frequency variation. Thus mode shape (slopes at actuator location) variations are more critical from control point of view than modal frequency variations.

3.3 Linear State Feedback Control Design for 'Matched Systems':

In the design procedures presented so far in this report, a linear state feedback control is determined by a chosen method (for example, a Riccati based control gain) and then its robustness property is investigated by computing the tolerable perturbations of the closed loop system formed by this control gain. This control gain is seen to qualify as a 'robust' control gain only if it makes the stability robustness index $\beta_{S.R.}$ positive for a given perturbation range ε (i.e. if $\mu < \varepsilon$). There is no guarantee that there exists a linear state feedback control gain that accommodates a given perturbation range. However it is shown by Thorp and Barmish in [15] that there exists a linear state feedback control that guarantees stability of the perturbed system provided the uncertainty satisfies the so called 'Matching Condition (MC)'. Matching conditions in essence constrain the manner in which the uncertainty is permitted to enter the system dynamics. We denote the systems whose uncertainty structure satisfies Matching Condition as 'Matched Systems'. Assuming uncertainty only in the A matrix of the standard linear state space model given by (3.1), the uncertainty ΔA is said to satisfy the matching condition if there exists a matrix $\bar{D}$ such that

$$\Delta A = B\bar{D}$$

i.e. the uncertainty in A is in the range space of the control distribution matrix B.

In this section, we combine the matching condition assumption with the elemental bound technique presented in the previous sections to design a linear state feedback controller that guarantees stability for the given range of

perturbations. The proposed method possesses theoretical justification only for simple second order systems with single input (at this state of research) but the method is seen to give encouraging results even for higher order systems. The proposed procedure is shown to compare favorably with other existing methods.

Elemental Bounds and Linear Control Design Algorithm:

Consider the simple asymptotically stable second order system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_{21} & -a_{22} \end{bmatrix} x \ , \quad a_{21},\ a_{22} > 0 \tag{3.28}$$

$$= A_c\ x$$

with perturbations in the elements a_{21} and a_{22}. Let $\bar{a} = \text{Max}\ (a_{21}, a_{22})$.

The stability robustness bound μ for this system yields

$$\mu = \frac{1}{\sigma_{max}(P_m U_e)_s} \tag{3.29}$$

where one choice of U_e is

$$U_e = \begin{bmatrix} 0 & 0 \\ a_{21}/\bar{a} & a_{22}/\bar{a} \end{bmatrix} \tag{3.30}$$

and P satisfies the Lyapunov equation $PA_c + A_c^T P + 2I_n = 0$. Another choice for U_e is

$$U_e = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} . \tag{3.31}$$

From the solution of the above Lyapunov equation, one can make the following design observation.

Design Observation 3: The elemental perturbation bound μ of (3.29) increases with increase in the magnitude of the nominal values of the elements a_{21} and a_{22}.

We now utilize the above observation in the context of designing a linear

controller for the simple second order system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_{21} & -a_{22} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.32a}$$

$$= A_c\, x + Bu \tag{3.32b}$$

with again a_{21} and a_{22} being the uncertain parameters, known to vary within a given interval. We can write the perturbation matrix ΔA_c as

$$\Delta A_c = \varepsilon U_e \tag{3.33}$$

where $U_e = \begin{bmatrix} 0 & 0 \\ \varepsilon_{21}/\varepsilon & \varepsilon_{22}/\varepsilon \end{bmatrix}$, $\varepsilon \overset{\Delta}{=} \mathrm{Max}(\varepsilon_{21}, \varepsilon_{22})$, ε_{21} and ε_{22} are the maximum modulus deviations in a_{21} and a_{22} respectively. It is important to realize that for the above system's uncertainty structure, the matching condition [15] is satisfied, i.e.

$$U_e = B\bar{D} \tag{3.34}$$

Hence, according to ref. [15], a linear controller that guarantees stability exists.

With the aid of matching condition (3.34) and design observation 3, we can now present a linear control design algorithm for the system of (3.32), assuming ε_{21}, ε_{22} (and thus ε) are known and that the open loop perturbed system $(A_c + \Delta A_c)$ is unstable.

<u>Linear Control Design Algorithm</u>

<u>Step 1</u>: Determine $\mu_1 = 1/\sigma_{max}(P_{1m}U_e)_s$ (3.35)

where $P_1 A_{c_1} + A_{c_1}^T P_1 + 2I_n = 0$ and $A_{c_1} = A_c$.

<u>Step 2</u>: Let $A_{c_2} = A_{c_1} - \mu_1 U_e$. Note that $|a_{21}|_2 > |a_{22}|_2$. Determine $\mu_2 = 1/\sigma_{max}(P_{2m}U_e)_s$ where $P_2 A_{c2} + A_{c2}^T P_2 + 2I_2 = 0$. Note that $\mu_2 > \mu_1$ by virtue of

design observation 3. Check if $\varepsilon < \mu_2$. If yes proceed to step 4. If not go to step 3.

Step 3: Repeat determining

$$\mu_{i+1} = 1/\sigma_{max}(P_{(i+1)m}U_e)_s \tag{3.36}$$

where $P_{i+1}A_{c_{i+1}} + A_c{}^T_{i+1} P_{i+1} + 2I_2 = 0$

$$A_{c_{i+1}} = A_{c_i} - \mu_i U_e \tag{3.37}$$

and checking if $\varepsilon < \mu_{i+1}$ $i = 3,4,\ldots$

Since $\mu_{i+1} > \mu_i$ for each iteration, for some $i+1 = \alpha$, we will have

$$\varepsilon < \mu_\alpha \tag{3.38}$$

The propagation of μ with each iteration is depicted in Figure 6.

Step 4: Once $\varepsilon < \mu_\alpha$, write

$$A_{c\alpha} = A_c - \sum_{i=1}^{\alpha-1} \mu_i U_e = A_c - \mu_c U_e \tag{3.39a}$$

where

$$\mu_c = \sum_{i=1}^{\alpha-1} \mu_i \ . \tag{3.39b}$$

Also from steps 2 and 3, all A_{c_i} $(i=1,2,\ldots,\alpha)$ as well as $A_{c\alpha} \pm \varepsilon U_e$ are stable. Let

$$A_{c\alpha} = A_c - \mu_c U_e = A_c - \mu_c B\bar{D} = A_c + BG \tag{3.40}$$

where $u = Gx$ is the linear control law we are after. Thus from (3.40), the control gain which guarantees stability of the perturbed closed loop system in the presence of variations of magnitude ε_{21} and ε_{22} in the elements a_{21} and a_{22} is given by

$$G = -\mu_c \bar{D} \tag{3.41}$$

Application Example

We now consider the same example used in [15] and [16] for comparison purposes. The system is given by $\dot{x} = A(q)x + Bu$ where

$$A(q) = \begin{bmatrix} 0 & 1 \\ -2+q_1 & 1+q_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = A_0 + E(q),$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}, \quad \begin{matrix} -1 \le q_1 \le 1 \\ -1 \le q_2 \le 1.5 \end{matrix} \tag{3.42}$$

In order to take advantage of symmetrical perturbation range assumed in our analysis, we take our 'nominal' system matrix and the perturbation matrix as follows:

<u>Nominal</u>: $\dot{x} = A_a x + Bu$ where $A_a = \begin{bmatrix} 0 & 1 \\ -2 & 1.25 \end{bmatrix}$ (3.43)

<u>Perturbation</u>: $\Delta A = \varepsilon U_e$ with $\varepsilon_{11} = 1$, $\varepsilon_{22} = 1.25$ so that

$$U_e = \begin{bmatrix} 0 & 0 \\ 0.8 & 1 \end{bmatrix} \text{ and } \varepsilon = 1.25. \tag{3.44}$$

Since the open loop 'nominal' system is unstable, we first stabilize the 'nominal' system with a control law $u = G_1 x$ where $G_1 = [1.04 \quad -1.45]$ so that the stable nominal open loop system is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -0.959 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \varepsilon = 1.25 \tag{3.45}$$

Now apllying the design algorithm for the system (3.43), it is observed that after seven iterations $\mu_\alpha = 1.2537 > \varepsilon = 1.25$. We obtain

$$\mu_c = \sum_{i=1}^{6} \mu_i = 2.3995, \text{ also } \bar{D} = [0.8 \quad 1] \tag{3.46}$$

Hence the control gain defined by G of (3.41) is given by $G = [-1.9196 \quad -2.3995]$. So the final gain for this example system is given by

$$G_f = G_1 + G = [-0.8986 \quad -3.8495] \tag{3.47}$$

It is to be noted that the final control gain G_f above is with respect to the open loop system matrix A_a (and not A_o). Since the methods compared in [15]-[16] give the final control gain with respect to the matrix A_o, we obtain the control gain obtained by the proposed method with respect to A_o as

$$\begin{aligned} G_c &= [-0.8786 \quad -3.5995] \\ &\sim \\ &= [-0.88 \quad -3.6] \end{aligned} \tag{3.48}$$

Thus G_c is the gain to be used for comparison with other methods.

<u>Comparison</u>: We reproduce the table given in [15] and [16].

Method	$\lvert G_{c_{11}} \rvert$	$\lvert G_{c_{12}} \rvert$	$\lvert\lvert G \rvert\rvert = [\varepsilon(\cdot)2_{ij}]^{1/2}$
GCC (Chang and Peng)	1.36	6.42	6.56
MGCC (Vinkler and Wood)	0.33	3.52	3.53
MC (Thorp and Barmish)	0.67	3.67	3.73
Elemental Perturbation Bound Analysis, EPBA (Yedavalli)	0.88	3.60	3.70

Thus the proposed method fares well in comparison with the other methods. We believe that the proposed method is computationally simpler.

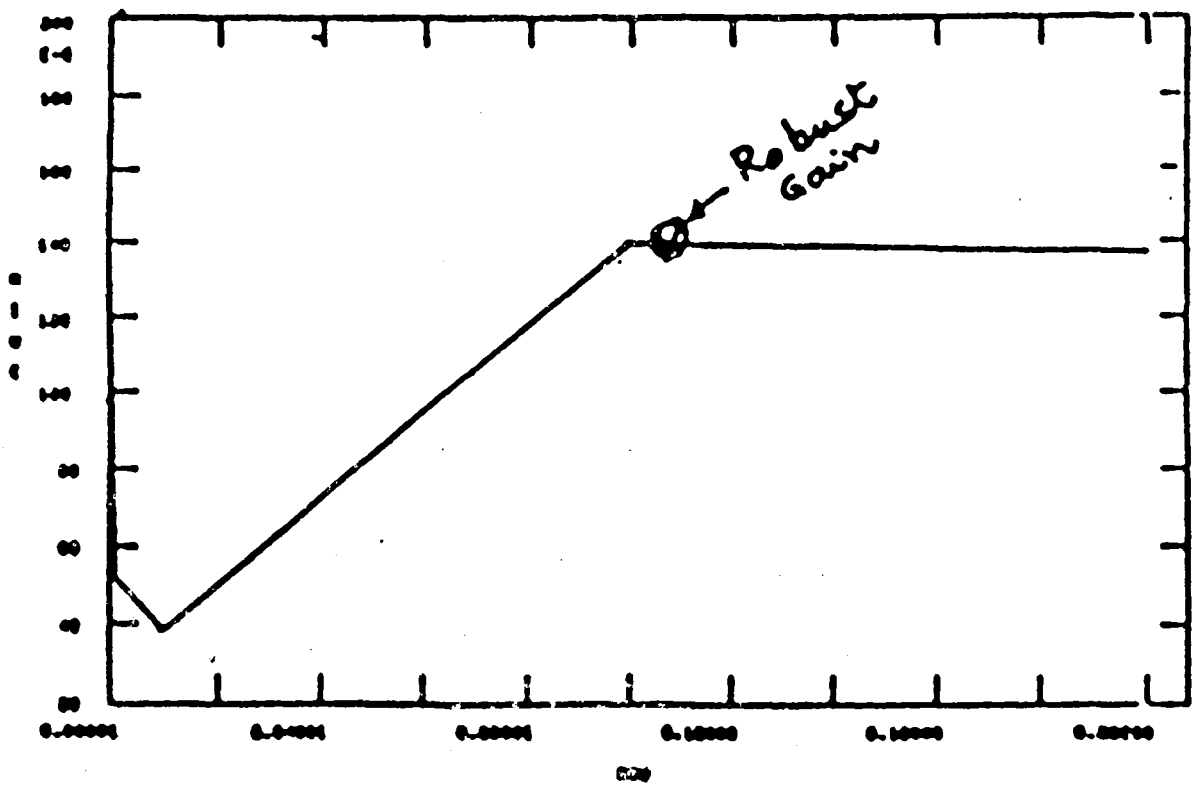

Figure 3. Robust Control Gain Determination for LSS Model.

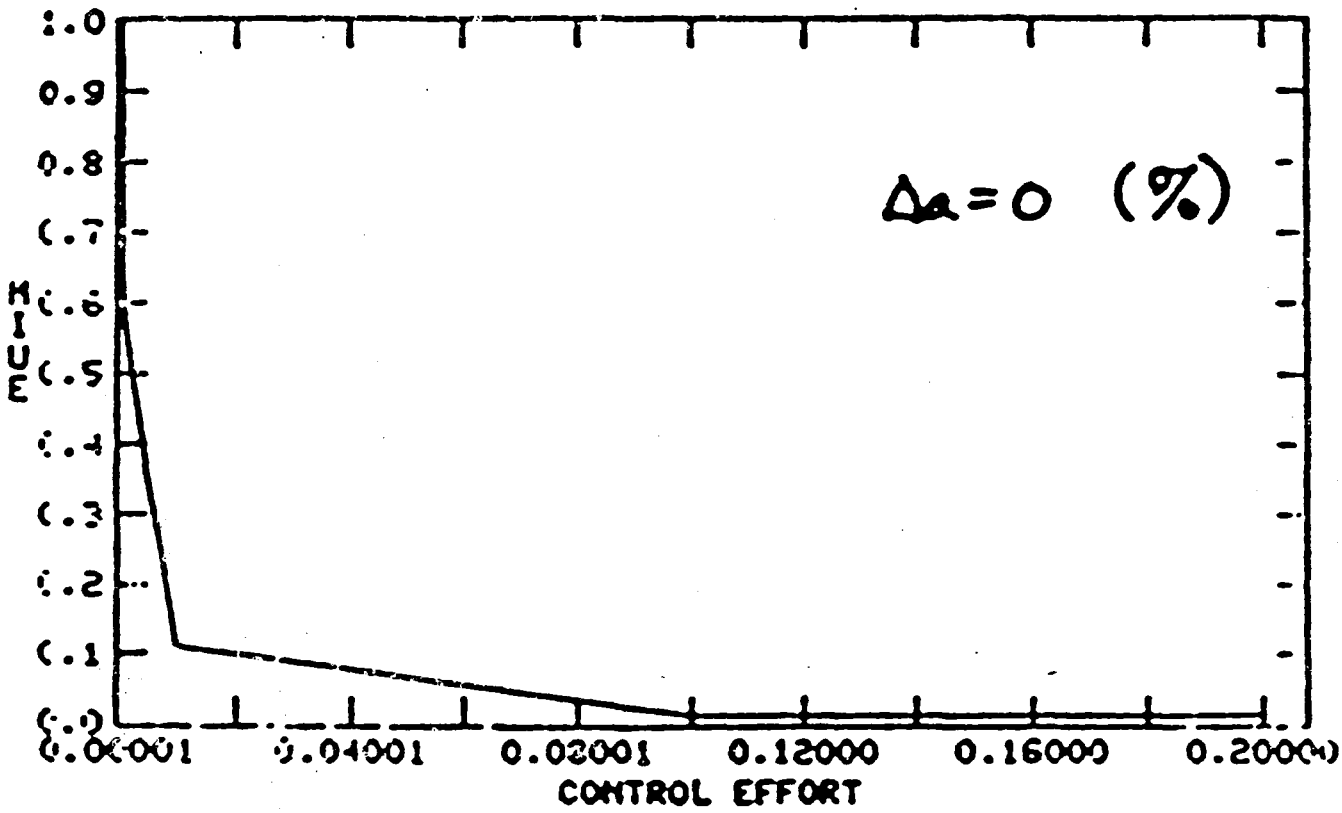

Figure 4. Variation of Bound with Control Effort for LSS Model ($\Delta A = 0$, $\Delta B \neq 0$).

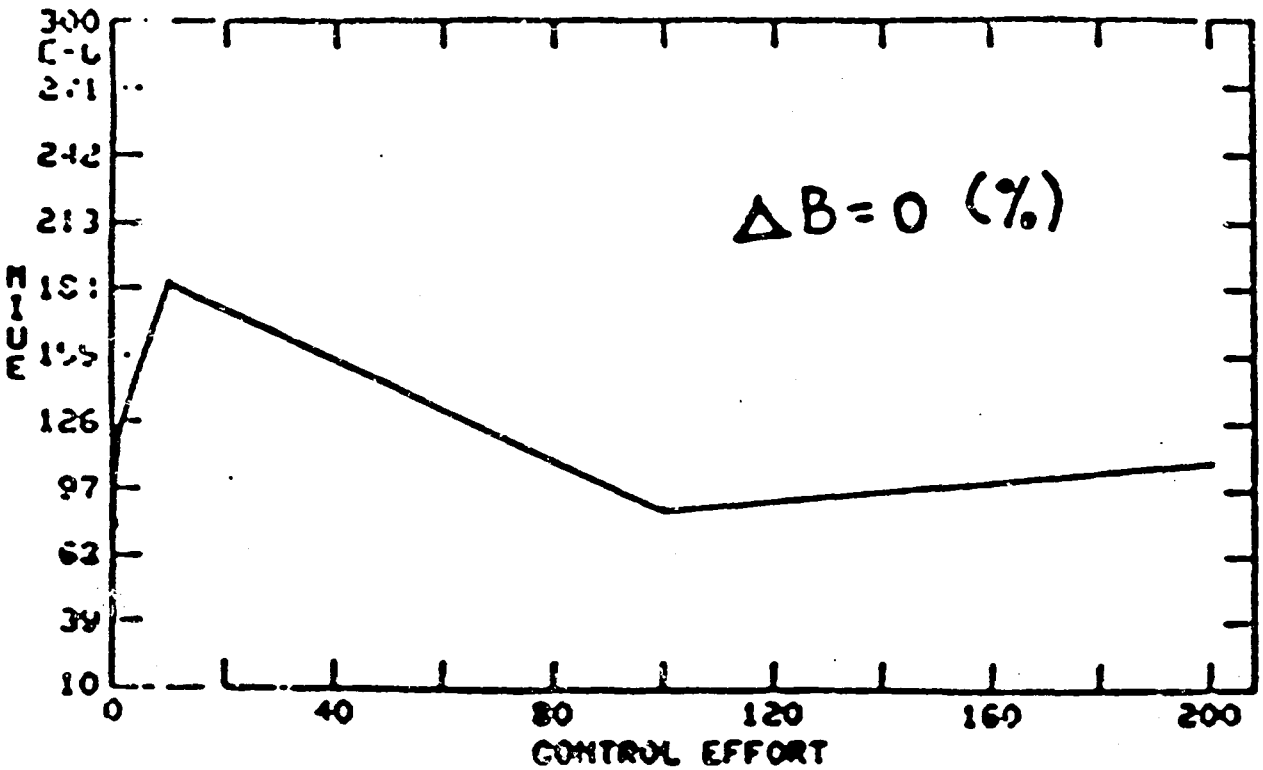

Figure 5. Variation of Bound with Control Effort for LSS Model ($\Delta A \neq 0$, $\Delta B = 0$).

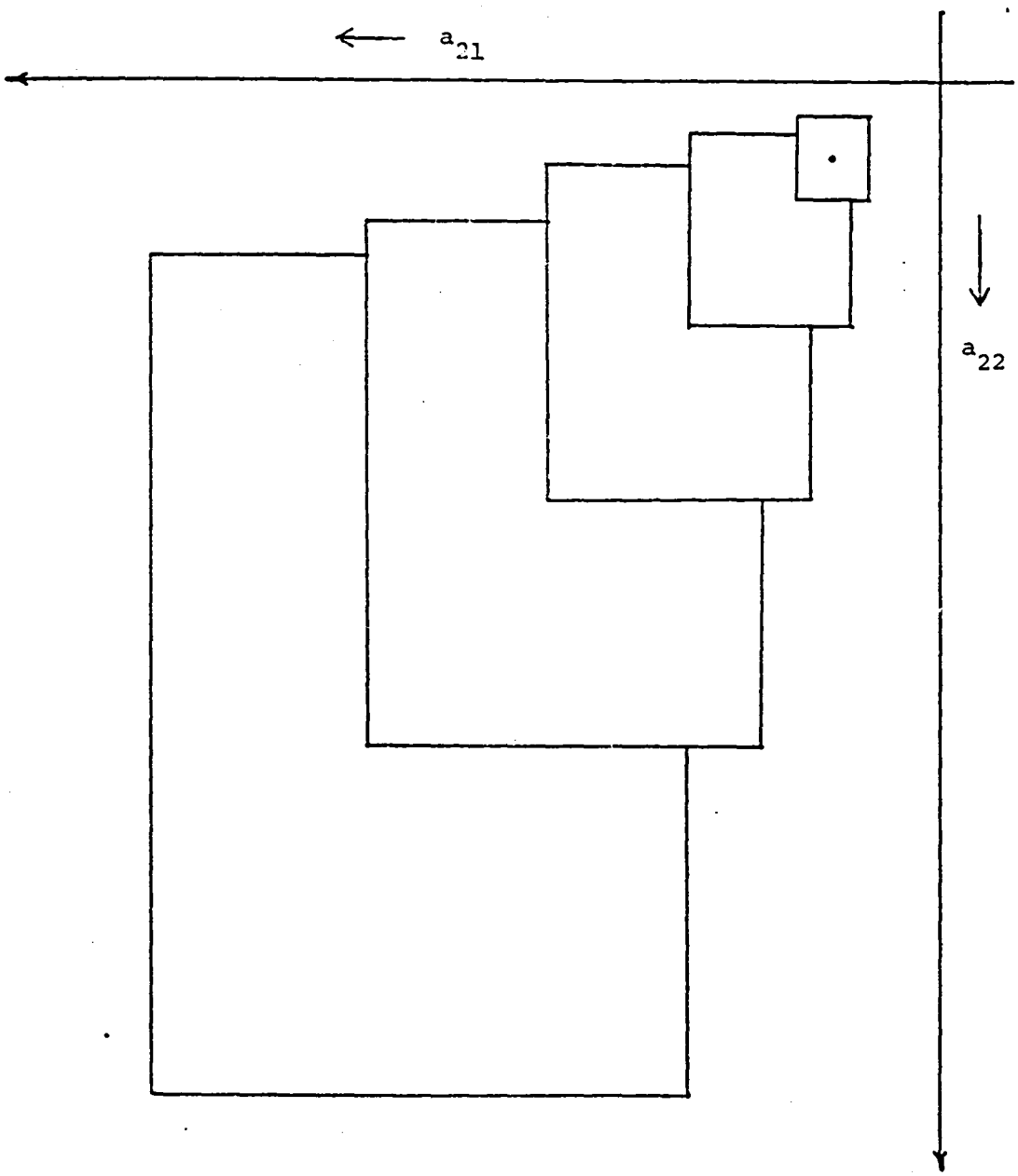

Figure 6. Propagation of μ_i with Each Iteration for Second Order Matched Systems.

IV. REDUCED ORDER DYNAMIC COMPENSATOR DESIGN FOR ROBUST STABILITY

In the previous section, efforts were directed to design a linear full state (and state estimate) feedback controller for robust stability. However, in that treatment, the control gain determination does not directly involve the stability robustness criterion as a design constraint. Instead, for a predetermined linear control gain (obtained by many different nominal methods), the perturbation bound is calculated and in the cases where the parameter perturbation ranges are given, the stability robustness condition is checked (for robust stability). It is also seen that it is possible to guarantee stability for given perturbation ranges only for matched systems. No such guarantee exists for non-matched systems.

In this section, we attempt to solve the control design problem for linear regulators in a more direct and general way by formulating it as a parameter optimization problem. Instead of designing the control gains by nominal means and then checking its stability robustness bounds, we propose to include the stability robustness condition explicitly in the design procedure as a design constant. In addition, we specify the structure of the controller in the form of a linear reduced order dynamic compensator (or given reduced dimension) which operates on the available measurements. In this way the control law is more practically implementable in contrast with the full order linear state feedback which demands the availability of the full state. Of course the problem formulation is such that the full order linear state feedback case comes out as a special case. The proposed formulation is presently given for deterministic systems with the understanding that the treatment for stochastic systems conceptually follows the same lines (with of course considerably different details).

4.1 System Description and Peformance Index for Specification

Consider again the linear time invariant system

$$\dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(0) = x_0$$
$$y(t) = C\,x(t) \tag{4.1}$$
$$z(t) = \bar{M}\,x(t)$$

where the state vector $x \to R^n$, the control vector $u \to R^m$, output vector (the variables we wish to control) $y \to R^k$ and the measurement vector $z \to R^\ell$.

Let the m control vriables in the vector u evolve from an sth order linear dynamical compensator of prescribed structure operating on the ℓ available measurements z,

$$u = \theta_{11}\,\beta + \theta_{12}\,z \qquad \beta \to R^s$$
$$\dot{\beta} = \theta_{21}\,\beta + \theta_{22}\,z \qquad \beta(0) = 0 \tag{4.2}$$

The θ_{ij} are constant gain matrices of appropriate dimensions to be determined according to a criterion discussed later.

The closed loop system is then given by

$$\dot{x}_c = A_c\,x_c \qquad x_c \to R^{n+s}\ ,\ x_c(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \tag{4.3a}$$
$$y_c = C_c\,x_c \qquad y_c \to R^{k+m}$$

where

$$x_c^T = [x^T \quad \beta^T]\ ,\quad y_c^T = [y^T \quad u^T] \tag{4.3b}$$

$$A_c = \begin{bmatrix} A + B\theta_{12}M & B\theta_{11} \\ \theta_{22}\bar{M} & \theta_{21} \end{bmatrix} \tag{4.3c}$$

$$C_c = \begin{bmatrix} C & 0 \\ \theta_{12}\bar{M} & \theta_{11} \end{bmatrix} \tag{4.3d}$$

We assume that the design is such that nominal closed loop system matrix A_c is asymptotically stable. The matrices A_c and C_c can also be expressed as

$$A_c = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_s \end{bmatrix} \theta \begin{bmatrix} 0 & I_s \\ M & 0 \end{bmatrix} \tag{4.4a}$$

$$= \hat{A} + \hat{B}\,\theta\,\hat{M}$$

$$C_c = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_s & 0 \end{bmatrix} \theta\,\hat{M} = \hat{C} + \hat{I}\,\theta\,\hat{M} \tag{4.4b}$$

where

$$\theta \triangleq \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \tag{4.4c}$$

In the nominal linear regulator problem, the gain θ is determined such that the following quadratic performance index is minimized.

$$\underset{\theta}{\text{Min}} \quad \text{W.T.t.} \quad V_1 = \frac{1}{2}\int_0^\infty (y^T Q y + u^T R_1 u + \dot{\beta}^T R_2 \dot{\beta})dt \tag{4.5a}$$

$$= \frac{1}{2}\int_0^\infty (y_c^T \bar{Q} y_c + \dot{\beta}^T R_2 \dot{\beta})dt \quad \text{where } \bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & R_1 \end{bmatrix} \tag{4.5b}$$

The weighting terms on $\dot{\beta}$ and u together penalize the full set of gains θ_{ij} quadratically. The omission of $\dot{\beta}$ term would preclude the gains θ_{21} and θ_{22} from getting weighted in the performance index. Here Q, R_1 and R_2 are symmetric positive definite matrices of appropriate dimensions.

The performance index $\bar{V}_1$ obtained by ignoring the cross coupling terms in V_1, can be written as

$$\bar{V}_1 = \int_0^\infty x_c{}^T\,\tilde{Q}\,x_c\,dt \tag{4.6a}$$

where

$$\tilde{Q} = \text{Block Diag}\,[\{C^TQC + M^T\theta_{22}{}^TR_2\theta_{22}M + M^T\theta_{12}{}^TR_1\theta_{12}M\},$$
$$\{\theta_{21}{}^TR_2\theta_{21} + \theta_{11}{}^TR_1\theta_{11}\}] \tag{4.6b}$$
$$= \hat{Q} + \frac{1}{2}\,[\hat{M}^T\theta^T\hat{R}\theta\hat{M} + T\hat{M}^T\theta^T\hat{R}\theta\hat{M}^T]$$

and

$$T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix},\ \hat{R} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},\ \hat{Q} = \begin{bmatrix} C^TQC & 0 \\ 0 & 0 \end{bmatrix} \tag{4.6c}$$

Assuming the closed loop system matrix Ac to be asymptotically stable, the nominal performance index $\bar{V}_1$ can be expressed as

$$\bar{V}_1 = \int_0^\infty x_c{}^T\,\tilde{Q}\,x_c\,dt = \text{Trace}\,\{P_2\,\underline{X}_0\} \tag{4.7a}$$

where the matrix P_2 satisfies the Lyapunov equation

$$P_2A_c + A_c{}^TP_2 + \tilde{Q} = 0 \tag{4.7b}$$

and the matrix

$$\underline{X}_0 = \begin{bmatrix} x_0x_0{}^T & 0 \\ 0 & 0 \end{bmatrix} \tag{4.7c}$$

As discussed in [17], the dependence of the controller on the initial condition x_0 can be removed by assuming the initial condition x_0 to be a random variable with zero mean and uniformly distributed over a sphere of unit radius thereby expressing x_0 as

$$E(x_0) = 0$$
$$E[x_0x_0{}^T] = \bar{\underline{X}}_0 \tag{4.8}$$

and then modifying the nominal performance index as

$$\bar{V}_1 = \lim_{t\to\infty} \frac{1}{t}\,E\int_0^t (y_c{}^T\,\bar{Q}\,y_c + \dot{\beta}^TR_2\dot{\beta})dt = \text{Trace}\,\{P_2\,\underline{X}_0\} \tag{4.9a}$$

where

$$\underline{\bar{X}}_o = \begin{bmatrix} \underline{\bar{X}}_o & 0 \\ 0 & 0 \end{bmatrix} \text{ and } P_2 \text{ is as given by equation (4.7b).} \qquad (4.9b)$$

Thus the nominal linear regulator problem with specified compensator structure is as follows:

Find θ such that

$$\min_{\theta} \{\text{Trace } P_2 \, \underline{\bar{X}}_o\} \qquad (4.10a)$$

subject to the constraint

$$P_2 A_c + A_c^T P_2 + \tilde{Q} = 0 \qquad (4.10b)$$

where A_c and $\tilde{Q}$ are given by (4.4), (4.6). Note that in A_c and $\tilde{Q}$ all matrices except θ are known.

The above problem formulation is the standard optimal dynamic compensator design formulation discussed in many references [17]-[19]. Our intent now is to include the stability robustness condition also into the problem formulation when the above system matrices are perturbed by finite parameter variations. Let ΔA, ΔB, ΔC and ΔM be the maximum modulus deviations expected in the entries of A, B, C and M respectively. Then, as before, the perturbed closed loop system is given by

$$\dot{x}_{cp} = (A_c + \Delta A_c) x_{cp} = \bar{A}_c \, x_{cp} \qquad (4.11a)$$

$$y_{cp} = C_{cp} \, x_{cp}$$

where

$$\bar{A}_c = \hat{A} + \hat{B}\theta\hat{M} + \Delta\hat{A} + \Delta\hat{B}\theta\Delta M + \Delta\hat{B}\theta\hat{M} + \hat{B}\theta\Delta\hat{M} \qquad (4.11b)$$

Since we are interested only in the stability robustness problem in this research (the performance robustness problem is a separate research topic of its own), our aim is to determine θ such that, in addition to nominal regulation problem

as posed in (4.10), it also maximizes the stability robustness bound μ which arises from the following stability robustness condition. Recall that the perturbed closed loop system matrix $\bar{A}_c$ is stable if

$$[\Delta A_c = \{\Delta\hat{A} + \Delta\hat{B}\theta_m\Delta\hat{M} + \Delta\hat{B}\theta_m\hat{M} + \hat{B}\theta_m\Delta\hat{M}\}]_{ij\ max} < \frac{1}{\sigma_{max}(P_{1m}U_e)_s} \tag{4.12a}$$

where P_1 satisfies

$$P_1A_c + A_c^TP_1 + 2I_{n+s} = 0 \tag{4.12b}$$

and U_e accommodates the structure of the perturbation matrices of the l.h.s. of (4.12a). Of course we can write

$$\Delta A = \varepsilon_a U_{ea},\ \Delta B = \varepsilon_b U_{eb},\ \Delta M = \varepsilon_m U_{em},\ \Delta C = \varepsilon_c U_{ec}$$

and knowing the ratios of ε_a, ε_b and ε_m, we obtain

$$\Delta A_c = \varepsilon_a \left[\begin{pmatrix} U_{ea} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_b}{\varepsilon_a} U_{eb} & 0 \\ 0 & 0 \end{pmatrix} \theta_m \begin{pmatrix} 0 & 0 \\ \frac{\varepsilon_m}{\varepsilon_a} U_{em} & 0 \end{pmatrix} \right.$$

$$\left. + \begin{pmatrix} \frac{\varepsilon_b}{\varepsilon_a} U_{eb} & 0 \\ 0 & 0 \end{pmatrix} \theta_m \begin{pmatrix} 0 & I_s \\ M & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & I_s \end{pmatrix} \theta_m \begin{pmatrix} 0 & 0 \\ \frac{\varepsilon_m}{\varepsilon_a} U_{em} & 0 \end{pmatrix} \right]$$

$$= \varepsilon_a U \tag{4.13}$$

where U is the matrix within the square brackets. Then one can obtain the stability robustness condition as

$$\varepsilon_a < \frac{1}{\sigma_{max}(P_{1m}U)_s} \overset{\Delta}{=} \mu \tag{4.14}$$

Henceforth, we will assume that only ΔA is present for simplicity purposes.

From (4.14), it can be seen that μ is a function of the control gains θ through P_1 and U. Also it may be noted that the problem of maximization of μ with respect to θ can be converted to that of minimixing $\sigma_{max}(P_{1m}U)_s$ with respect to θ subject to the contraint (4.12b).

Thus, we now pose a modified optimization problem by combining the stability robustness condition of (4.14) with the nominal regulation problem of (4.10) as follows:

Find θ such that the performance index

$$V_2 \triangleq [\sigma_{max}(P_{1m}U)_s + \text{Trace } P_2 \underline{X}_o] \tag{4.15a}$$

is minimized subject to the constraints

$$P_1 A_c + A_c^T P_1 + 2I_{n+s} = 0 \tag{4.15b}$$

$$P_2 A_c + A_c^T P_2 + \tilde{Q} = 0 \tag{4.15c}$$

and

$$\text{Re } \{\lambda_i(A_c)\} < 0. \tag{4.15d}$$

<u>Modified Performance-Index:</u>

Note that the above performance index V_2 contains a term involving the maximum singular value as well as a positive matrix P_{1m}. Optimization of an index like the one posed is a formidable task as it is almost computationally and analytically intractable. Hence we intend to modify the performance index such that it becomes more tractable.

Noting that the Frobenius norm of a matrix is always an upper bound on the spectral norm of the matrix, i.e.

$$||(\cdot)||_F \geq \sigma_{max}(\cdot) \tag{4.16}$$

and that

$$\sigma_{max}(\cdot) \geq \sigma_{max}(\cdot)_s \tag{4.17}$$

we propose the following upper bound to be minimized instead of $\sigma_{max}(P_{1m}U)_s$.

Proposition 1:

$$V_r \overset{\Delta}{=} \frac{1}{2} \text{Trace } [P_1TWP_1 + P_1WP_1^T] \geq \sigma_{max}(P_{1m}U)_s \tag{4.18}$$

for some suitable diagonal weighting matrix W.

The diagonal weighting matrix W is such that $W_{ii} = 0$ whenever U_{ij} $(j = 1,2,\ldots, n+s) = 0$ for a given row i and $W_{ii} = w_i$ whenever U_{ij} $(j = 1,2,\ldots,n+s) \neq 0$ for a given row i and any column j. Even though the specification of w_i is crucial in establishing the upper bound property of V_r as in (4.18), it turns out, as seen later in conjunction with the nominal regulation problem, that it is possible to specify the $w_{ii} > 0$ as arbitrary and transfer its implication in the design to another design variable, namely ρ_c, the weighting on the control variable.

We are now in a position to state the problem of finding the 'optimal' dynamic compensator gains θ for robust stability and nominal regulation of a linear regulator.

4.2 Compensator Design by Parameter Optimization Technique

Find θ such that

$$V_3 = \min_{\theta} \{ \frac{1}{2} \text{Trace}[P_1^TWP_1 + P_1WP_1^T] + \text{Trace } P_2 \underline{X}_0\} \tag{4.19a}$$

subject to the constraints

$$P_1A_c + A_c^TP_1 + 2I_{n+s} = 0 \tag{4.19b}$$

$$P_2A_c + A_c^TP_2 + \tilde{Q} = 0 \tag{4.19c}$$

$$\text{Re } \lambda_i[A_c] < 0 \tag{4.19d}$$

where A_c and $\tilde{Q}$ are as in (4.4 and 4.6) and W is given according to the structure of the U matrix.

Solution by Parameter Optimization:

We approach the solution to the above nonlinear (quadratic performance index) programming problem by writing down necessary conditions and investigating the solutions which satisfy them. Using the technique of Lagrange multipliers, we transform the above constrained optimization problem to an unconstrained optimization problem by defining the Hamiltonian. Thus we write

$$\min_{\theta} \{ H \} \tag{4.20}$$

where H is the Hamiltonian given by

$$H = \text{Trace} \ \frac{1}{2} (P_1^T W P_1 + P_1 W P_1^T) + P_2 \underline{\overline{X}}_o + L_1(P_1 A_c + A_c^T P_1 + 2I)$$

$$+ L_2(P_2 A_c + A_c^T P_2 + \tilde{Q})\} \tag{4.21}$$

and L_1 and L_2 are the Lagrange Multiplier matrices.

The first order necessary conditions are:

$$\frac{\partial H}{\partial L_1} = P_1 A_c + A_c^T P_1 + 2I = 0 \tag{4.22a}$$

$$\frac{\partial H}{\partial L_2} = P_2 A_c + A_c^T P_2 + \tilde{Q} = 0 \tag{4.22b}$$

$$\frac{\partial H}{\partial P_1} = L_1^T A_c^T + A_c L_1^T + P_1 W + W P_1 = 0 \tag{4.22c}$$

$$\frac{\partial H}{\partial P_2} = L_2^T A_c^T + A_c L_2^T + \underline{\overline{X}}_o^T = 0 \tag{4.22d}$$

$$\frac{\partial H}{\partial \theta} = 2\hat{B}^T(P_1 L_1 + P_2 L_2)\hat{M}^T + \hat{R}_\theta(\hat{M} L_2 \hat{M}^T + \hat{M}^T L_2^T \hat{M}^T) = 0 \tag{4.22e}$$

In arriving at these conditions, the matrix derivative identities given in [20] are used.

We can get the gain θ computed by simultaneously solving for P_1, P_2, L_1 and L_2 using equations (4.22).

Special Cases:

a) Standard nominal regulation problem with full state feedback:

For this case $W = 0$, $\theta_{11} = \theta_{21} = \theta_{22} = 0$, $M = I$, $L_1 = 0$ and we end up with the standard Algebraic Riccati equation.

b) 'Optimal' state feedback for robust stability:

For this case, $M=I$, $\theta_{11} = \theta_{21} = \theta_{22} = 0$. The gain θ_{12} then is given by

$$\theta_{12} = -R^{-1}B^T(P_1L_1 + P_2L_2)L_2^{-1} \tag{4.23a}$$

c) 'Optimal' measurement feedback for robust stability:

For this case $\theta_{11} = \theta_{21} = \theta_{22} = 0$ and the gain θ_{12} is given by

$$\theta_{12} = -R^{-1}B^T(P_1L_1 + P_2L_2)M^T(ML_2M^T)^{-1} \tag{4.23b}$$

4.3 Example and Discussion of the Results

Consider the simple second order linear time invariant system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a & -0.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ , \ x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.24}$$

where a is the uncertain parameter with nominal value $\bar{a} = 1$

$$y = x$$

$$z = [2 \quad 1]x$$

Let us consider a first order dynamic compensator having the structure

$$\begin{aligned} u &= \theta_{11}\beta + \theta_{12}z \qquad & \beta \in R^1 \\ \dot{\beta} &= \theta_{21}\beta + \theta_{22}z \qquad & \beta(0) = 0 \end{aligned} \tag{4.25}$$

Since $U_{ea} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, we select $W_{11} = W_{33} = 0$ and $W_{22} = 1$ and specify the performance index as

$$V_3 = \text{Trace } L\{ \frac{1}{2} (P_1^T W P_1 + P_1 W P_1^T)\} + P_2 \underline{X}_o \;] \tag{4.26}$$

with $Q = I_2$, $R_1 = 1$, $R_2 = 1$ and P_1 and P_2 satisfying the Lyapunov equations given by (4.22) with Ac being a 3x3 matrix.

With the above weighting matrices and the performance index, the parameter optimization procedure presented before yields the 'optimal' compensator gains to be

$$\theta = \begin{bmatrix} -6.5778 & -1.2255 \\ -4.8419 & 0.3275 \end{bmatrix} \tag{4.27}$$

and the nominal closed loop system matrix A_c is given by

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ -1.45 & -1.72 & -6.57 \\ 0.655 & 0.327 & -4.84 \end{bmatrix} \tag{4.28}$$

The resulting bound μ_{21} on the uncertain parameter a for robust stability is given by

$$\mu_{21} = 1.1644 \tag{4.29}$$

In other words, with the dynamic compensator given by (4.27), the uncertain parameter 'a' can tolerate perturbations up to $\pm$ 1.1644 from its nominal value $\bar{a} = 1$ and still maintain stability. The dynamic compensator found in (4.27) is 'optimal' in the sense that it maximizes (albeit, in an approximate way) the perturbation bound the uncertain parameter can have, to maintain stability with the imposed restrictions on the control effort and nominal regulation as reflected by the weightings Q, R_1 and R_2. Note that the robustness weighting matrix W incorporates the uncertainty

structure (that only a_{22} element is varying) in an explicit way. However, one limitation of the W matrix is that it does not fully reflect the uncertainty structure in the sense that we would use the same W ($w_{22} = 1$ and $w_{11} = w_{33} = 0$) even if both a_{21} and a_{22} were varying. Efforts are underway to prescribe a more versatile performance index (an upper bound on $\sigma_{max}(P_m U_e)_s$) that completely utilizes the structure of U_e. Fortunately, when the uncertinty structure (i.e. U_e matrix) is such that there is only one nonzero entry for each row, then the W matrix completely incorporates the uncertainty structure. Since w_{ii} is arbitrarily specified, one needs to check, after the design is complete, whether the index Trace $\frac{1}{2}\{P_1^T W P_1 + P_1 W P_1^T\}$ is an upper bound on the quantity $\sigma_{max}(P_{1m} U_e)_s$ or not. If it is not, one can either change w_{22} or the control weighting R_1 until this happens.

Measurement feedback:

Using the same weightings and procedure as before, the 'optimal' measurement feedback for the above example, i.e.

$$u = \theta_{12}\, z \tag{4.30}$$

is given by

$$u = 1.3878\, z \tag{4.31}$$

which yields

$$\mu_{21} = 1.278 \tag{4.32}$$

Comparison of 'Robust' State Feedback and 'Nominal' State Feedback

With W = 0 (i.e. no requirement of robust stability) and the weightings $Q = I_2$, $R_1 = \rho R_0 = \rho$ ($R_0 = 1$ and ρ as a design variable), one can get the standard nominal optimal linear regulator state feedback control law given by the solution of the algebraic Riccati equation.

With $w_{11} = 0 = w_{33}$ amd $w_{22} = 1$ and the same control weightings as above, one can get the 'optimal robust' state feedback control law determined by the proposed

parameter optimization (PO) method.

The comparison of 'robust (PO) state feedback control law' vs. the 'nominal state feedback control law' is depicted in Fig. 7 where the perturbation bound μ_{21} is plotted against the nominal control effort $J_{un} = (\int_0^\infty u^T u dt)^{1/2}$.

As anticipated, for a <u>given</u> control effort, the robust control law yields a higher perturbation bound μ_{21} than the nominal control law, indicating the usefulness of the proposed optimization procedure.

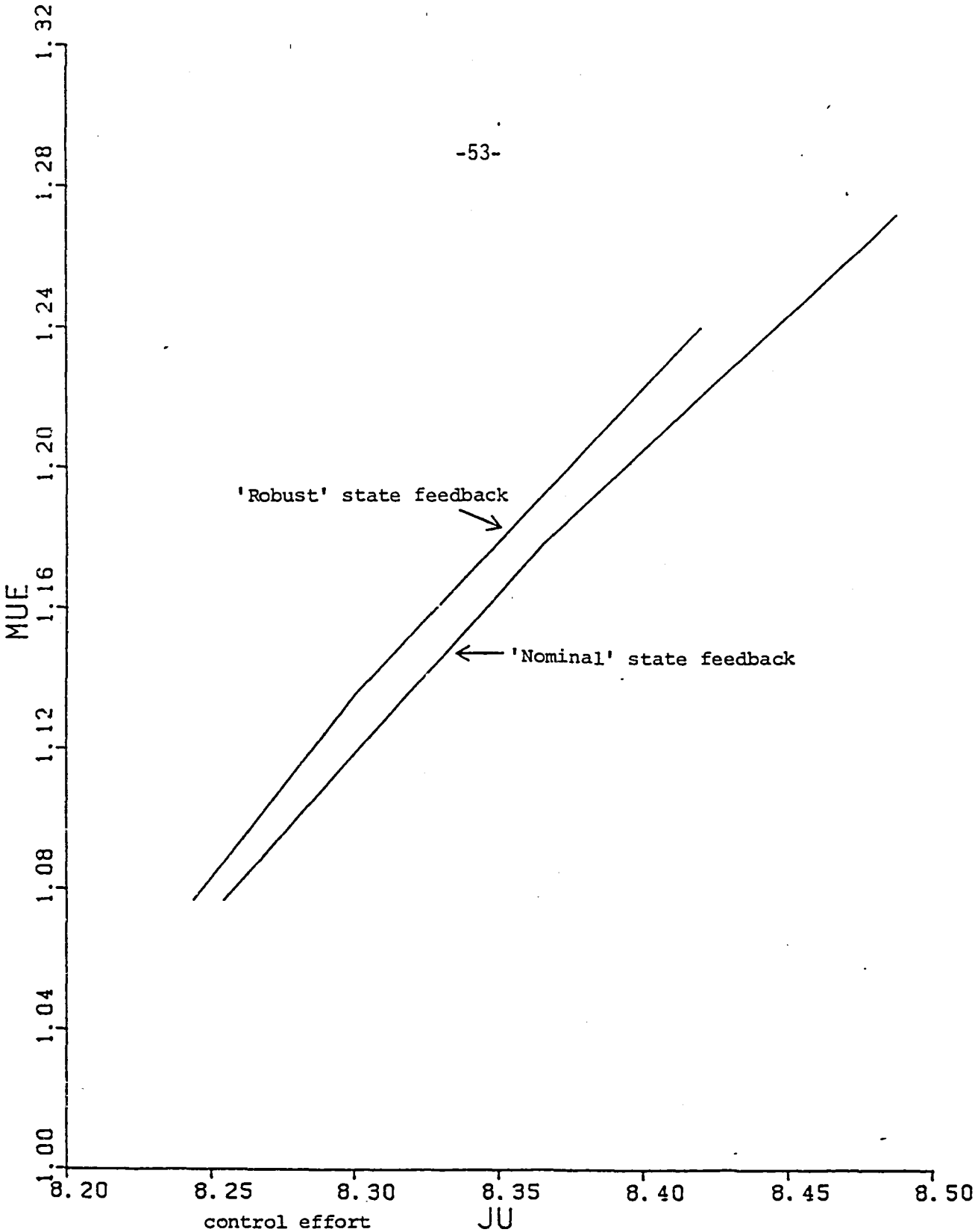

Figure 7. Variations of the Bound μ with Control Effort.

V. CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

5.1 Work in Retrospect

The main theme of the described research under the present grant has been to analyze and synthesize controllers for robust stability for linear time invariant systems subject to linear time varying structured (elemental) perturbations. First the analysis of robustness was considered. The main contribution of the research in this aspect is the reduction of conservatism of the previously developed perturbation bounds for structured (elemental) uncertainty. This is done by employing a state transformation and the improvement of the proposed technique is illustrated with several examples.

Then the aspect of control design is addressed. In this regard, first the case of linear state feedback control is considered. The linear state feedback control is determined by nominal means based on the Riccati equation and the bounds achieved by this control law are computed. The effect of state estimation in the control law (for stochastic systems) on the bounds is illustrated by comparing it with the exact state feedback case. Then the special nature of 'modal systems' (as in Large Space Structure Control example) is incorporated in the uncertainty structure and a linear state feedback control utilizing this special structure is developed. Finally in that section, the conditions under which a linear state feedback control exists (for given perturbation range) is recalled (namely the matching condition) and a design algorithm is presented for determining the linear state feedback control for these 'matched systems' (for simple second order systems at this stage of research).

Section IV comprises the major contribution of the research under this grant in which a design procedure for determining reduced order dynamic compensators for robust stability is presented using the Parameter Optimization (PO) method. The maximization of the perturbation bound is posed as a minimization problem by

specifying an appropriate performance index and the control gains of a compensator of given structure are optimized to minimize the given performance index. The solution method leads to a set of necessary conditions which are then simultaneously solved to obtain the desired gains. The method is illustrated with the help of a simple example.

The publications listed as Refs. [21-25] are the result of this study.

As it normally occurs, another result of this study is that many interesting research topics surfaced for further investigation. These are summarized in the following.

5.2 Avenues for Further Research Which Need the Continued Support of the NASA Langley Research Center

1) The foremost area of research would be to further reduce the conservatism of the perturbation bounds by scaling. Note that a similarity transformation is not a necessary means (but only a sufficient means) to reduce the conservatism. One suggestion is to use positive real transformations.

2) One extension that needs attention is to develop linear state feedback control law for higher order matched systems and then to consider the case of mismatched systems.

3) An area of research would be to extend the development of explicit bounds for structured perturbation to time-invariant perturbations and examine the reduction in conservatism that can be achieved.

4) Another area of interest is to compare the proposed 'Perturbation Bound Analysis' approach to design with other relevant methods like the Guaranteed Cost Control of Chang and Peng [6] and the 'multimodel theory' of Ackermann.

5) It is also of interest to probe the relationship between the perturbation bound and the corresponding degree of stability measured by the real part of the

dominant eigenvalue, i.e. the relationship between perturbation range and eigenvalue displacement.

6) Another aspect for future research would be to extend the "Perturbation Bound Analysis' for actuator-sensor location problems.

7) An area of extreme interest would be to use the perturbation bounds as a criterion for selecting the critical parameters in a system and use this information in model/contoller reduction and develop an algorithm for same and compare it with other relevant schemes.

8) One foremost area of research would be to extend the proposed analysis and design methodology to the case of combined modeling errors such as parameter variation, mode truncation and possibly nonlinearities.

9) There is need for probing into the comparison and contrast of frequency domain results and the proposed time domain results.

10) One immediate application of the developed perturbation bound analysis is in the area of stability analysis and control design for large scale interconnected systems (decentralized control).

11) It is instructive to extend the proposed concepts to the case of combined 'stability robustness' and 'performance robustness' where 'performance' is measured in terms of speed of response, percentage overshoot, damping enhancement, etc.

12) Some interesting application areas are: (i) the vibration control of mechanical systems, (ii) active flutter control in aircraft, (iii) failure mode analysis in turbofan engine control, and (iv) control of robot manipulators.

References

1. Yedavalli, R.K.: "Time Domain Design of Robust Controllers for LQG Regulators: Application to Large Space Structures," AFWAL-TR-85-3093, Dec. 1985.

2. IEEE Trans., special issue on Linear Multi-variable Control Systems, Vol. AC-26, #1, Feb. 1981.

3. Proceedings IEE, Part D on control theory, special issue on Sensitivity and Robustness, 1982.

4. Mukhopadhyay, V. and Newsom, J.R.: "A Multi-loop System Stability Margin Study Using Matrix Singular Values," Journal of Guidance, Control and Dynamics, Vol. 7, Sept-Oct. 1984, pp. 582-587.

5. Newsom, J.R. and Mukhopdhyay, V.: "A Multi-loop Robust Controller Design Study Using Singular Value Gradients," Journal of guidance, Control and Dynamics, Vol. 8, #4, July-Aug. 1985, pp. 514-519.

6. Chang, S.S.L. and Peng, T.K.C.: "Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters," IEEE Trans. on Autom. Contr., Vol. AC-17, Aug. 1972, pp. 472-483.

7. Patel, R.V. and Toda, M. and Sridhar, B.: "Robustness of Linear Quadratic State Feedback Designs in the Presence of System Uncertainty," IEEE Trans., Vol. AC-22, Dec. 1977, pp. 945-949.

8. Patel, R.V. and Toda, M.: "Quantitative Measures of Robustness for Multivariable Systems," Proc. of Joint Auto. Contr. Conf., 1980, TP8-A.

9. Yedavalli, R.K.: "Improved Measures of Stability-Robustness for Linear State Space Models," IEEE Trans. on Autom. Control, Vol. AC-30, June 1985, pp. 577-579.

10. Yedavalli, R.K.: "Perturbation Bounds for Robust Stability in Linear State Space Models," International Journal of Control, 1985, Vol. 42, #6, pp. 1507-1517.

11. Siljak, D.D.: "Large Scale Dynamic Systems: Stability and Structure," New York, North-Holland, 1978.

12. Yedavalli, R.K. and Liang, Z.: "Aircraft Control Design Using Improved Time Domain Stability Robustness Bounds," to appear in Sept-Oct. 1986 issue of the Journal of Guidance, Control and Dynamics.

13. Kwakernaak, H. and Sivan, R.: "Linear Optimal Control Systems, Wiley, Interscience, 1972.

14. Hablani, H.: "Generic Model of a Large Flexible Space Structure for Control Concept Evaluation," Journal of Guidance and Control, Vol. 4, #5, Sept-Oct. 1981, pp. 558-561.

15. Thorp, J.S. and Barmish, B.R.: "On Guaranteed Stability of Uncertain Linear Systems via Linear Control," J. of Optim. Th. and Appl., Vol. 15, #4, Dec. 1981, pp. 559-579.

16. Vinkler, A. and Wood, L.J.: "A Comparison of Several Techniques for Designing Controllers of Uncertain Dynamic Systems," Proc. 1979 Conf. on Decision and Control, San Diego, CA, pp. 31-39.

17. Levine, W.J. and Athans, M.: "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," IEEE Trans. on Autom. Control, Vol. AC-15, #1, Feb. 1970, pp. 44-48.

18. Johnson, T.L. and Athans, M.: "On the Design of Optimal Constrained Dynamic Compensators for Linear Constant Systems," IEEE Trans. on Autom. Control, Vol. ACL-15, Dec. 1970, pp. 658-660.

19. Wieseman, C.D.: "A Method to Stabilize Linear Systems Using Eigenvalue Gradient Information," NASA Technical Paper 2479, Nov. 1985.

20. Athans, M.: "The Matrix Minimum Principle," Journal of Information and Control, Vol. 11, 1968, pp. 592-606.

21. Yedavalli, R.K. and Liang, Z.: "Reduced Conservatism in Testing for Hurwitz Invariance of State Space Models," invited paper at the 1985 Conference on Decision and Control, Proceedings of CDC, pp. 673-678.

22. Yedavalli, R.K. and Liang, Z.: "Reduced Conservatism in Stability Robustness Bounds by State Transformation," to appear in the Sept. 1986 issue of the IEEE Trans. on Automatic Control.

23. Yedavalli, R.K.: "Robust Control Design for the Vibration Suppression of Large Space Structures," paper presented at the 1986 Vibration Damping Workshop II, March 5-7, 1986, Las Vegas.

24. Yedavalli, R.K.: "Linear Control Design for Guaranteed Stability of Uncertain Linear Systems," Proceedings of the 1986 American Control Conference, June 1986, pp. .

25. Yedavalli, R.K.: "Dynamic Compensator Design for the Hurwitz Invariance of Linear Internal Systems," invited paper at the 1986 Conference on Decision and Control, Dec. 10-12, 1986, Athens, Greece.

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