SHAPE DESIGN SENSITIVITY ANALYSIS
AND OPTIMAL DESIGN OF STRUCTURAL SYSTEMS*

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ABSTRACT

The material derivative concept of continuum mechanics and an adjoint variable method of design sensitivity analysis are used to relate variations in structural shape to measures of structural performance. A domain method of shape design sensitivity analysis is used to best utilize the basic character of the finite element method that gives accurate information not on the boundary but in the domain. Implementation of shape design sensitivity analysis using finite element computer codes is discussed. Recent numerical results are used to demonstrate accuracy that can be obtained using the method. Result of design sensitivity analysis is used to carry out design optimization of a built-up structure.

1. INTRODUCTION

A substantial literature has been developed in the field of shape design sensitivity analysis and optimization of structural components [1-3] over the past few years. Contributions to this field have been made using two fundamentally different approaches to structural modeling and analysis. The first approach uses a discretized structural model, based on finite element analysis, and proceeds to carry out shape design sensitivity analysis by controlling finite element node movement and differentiating the algebraic finite element equations [4-6]. The second approach to shape design sensitivity analysis uses an elasticity model of

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the structure and the material derivative method of continuum mechanics to account for changes in shape of the structure [7-13]. Using this approach, expressions for design sensitivity in terms of domain shape change are derived in the continuous setting and evaluated using any available method of structural analysis; e.g., finite element analysis, boundary element analysis, photoelasticity, etc.

Shape design sensitivity analysis for several structural components has been treated in Refs. 2, 9, and 10 where sensitivity information is explicitly expressed as integrals, using integration by parts and boundary and/or interface conditions to obtain identities for transformation of domain integrals to boundary integrals. Numerical calculation of design sensitivity information in terms of the resulting boundary integrals thus requires stresses, strains, and/or normal derivatives of state and adjoint variables on the boundary. However, when the finite element method is used for analysis of built-up structures, the accuracy of numerical results for state and adjoint variables on interface boundaries may not be good [14].

To overcome this difficulty, a domain method of shape design sensitivity analysis is developed in Ref. 15, in which design sensitivity information is expressed as domain integrals, instead of boundary integrals (boundary method). The domain and the boundary methods are analytically equivalent. However, when one uses an approximate numerical method such as finite element analysis, the resulting design sensitivity approximations may give quite different numerical values. Moreover, the domain method offers a remarkable simplification in derivation of shape design sensitivity formulas for built-up structures since interface conditions are not required to obtain shape design sensitivity formulas. In the domain method, numerical evaluation of the sensitivity information is more complicated and inefficient than the result of the boundary method, since the domain method requires integration over the entire domain, whereas the boundary method requires integration over only the variable boundary. To alleviate this problem, a boundary layer of finite elements that vary during the perturbation of the shape of a structural component is introduced in Ref. 16.

In shape design problems, nodal points of the finite element model move as shape changes. In Ref. 17, a method of automatic regridding to account for shape change has been developed using a velocity field in the
domain that obeys the governing deformation equations of the elastic solid.

Using the domain method of Ref. 15 and results of conventional (sizing) design sensitivity analysis theory of Ref. 2, a design component method is developed in Ref. 18 for unified and systematic organization of design sensitivity analysis of built-up structures, with both conventional and shape design variables. That is, conventional and shape design sensitivity formulas for each standard component type can be derived. The result is standard formulas that can be used for design sensitivity analysis of built-up structures, by simply adding contributions from each component. The method gives a systematic organization of computations for design sensitivity analysis that is similar to the way in which computations are organized within a finite element code.

A numerical method has been developed in Ref. 19 to implement the results of the design component method, using the versatility and convenience of existing finite element codes. It is shown in Ref. 19 that calculations can be carried out outside existing finite element codes, using postprocessing data only. Thus, design sensitivity analysis software does not have to be imbedded in an existing finite element code.

The purpose of this paper is to combine these developments to present a unified method of shape design sensitivity analysis and numerical implementation of the method with existing finite element codes. Even though only static response is considered here, the method is also applicable for eigenvalue design sensitivity analysis as shown in Ref. 2.

The design sensitivity analysis method presented here supports optimality criteria method for structural optimization and serves as the foundation for iterative methods of structural optimization using nonlinear programming. At a more practical level, the design sensitivity analysis method can be used to develop an interactive computer-aided design system [2]. A large scale built-up structure is optimized to demonstrate capability of the method.

2. VARIATIONAL FORM OF GOVERNING EQUATIONS

While a substantial library of structural components must be considered to implement the design sensitivity analysis for broad classes of applications, the component library to be considered in this paper is
limited to truss, beam, plate, plane elastic solid, and three dimensional solid components. Even though this is a somewhat restricted class of components, it is general enough that significant applications can be made and practicality of the method can be demonstrated.

In the actual formulation, the truss and beam components, including both bending and torsion of the beam, are incorporated into a single component. Similarly, plate and plane elastic solid components are combined as a single component. To be more specific, the following formation of beam/truss, three dimensional elastic solid, and plate/plane elastic solid components are employed:

A. BEAM/TRUSS

Consider the beam/truss component of Fig. 2.1. The energy bilinear form (internal virtual work) [2] of the component is

\[ a_{\Omega,\Omega}(z,\overline{z}) = \int_{0}^{l} E I \left( \frac{\partial w}{\partial x} \right)^{2} dx + \int_{0}^{l} G J \theta \phi_{x} dx + \int_{0}^{l} h E v \phi_{x} dx \]

where \( w, \theta, \) and \( v \) are two orthogonal lateral displacements, angle of twist, and axial displacement, respectively, and \( z = [w, \theta, v]^{T} \). Throughout this paper, an overbar; e.g., \( \overline{z} \), denotes a virtual displacement. Subscript \( x \) in Eq. 2.1 denotes derivative with respect to \( x \). In Eq. 2.1, \( E, G, I^{1}, I^{2}, J, \) and \( h \) are Young's modulus, shear modulus, two moments of inertia, torsional moment of inertia, and cross-sectional area of the component, respectively. The conventional design variable is \( u = h(x) \) and the shape design variable is the length of the domain \( \Omega = [0, l] \). The load linear form (external virtual work) [2] of the component is
\[
\lambda_{u,\Omega}(\overline{z}) = \int_0^l q_1 w^1 dx + \int_0^l q_2 w^2 dx + \int_0^l r^\sigma dx + \int_0^l f^\sigma dx \tag{2.2}
\]

where \( q^1, q^2, r, \) and \( f \) are two orthogonal lateral loads, twisting moment, and axial load, respectively, as shown in Fig. 2.2 [2]. If there are point loads, a Dirac delta measure can be used for \( q^1, q^2, r, \) and \( f \) in Eq. 2.2 [2].

![Figure 2.2 External Loads For Beam/Truss](image)

The variational equation of the beam/truss component is [2]

\[
a_{u,\Omega}(z,\overline{z}) = \lambda_{u,\Omega}(\overline{z}), \quad \text{for all } \overline{z} \in Z \tag{2.3}
\]

where \( Z \) is the space of kinematically admissible displacement. That is, \( Z \subseteq [H^2(0,l)]^2 \times [H^1(0,l)]^2 \) and elements of \( Z \) satisfy kinematic boundary conditions where \( H^i(0,l) \) is the Sobolev space of order \( i \) [2]. As possible boundary conditions, the beam/truss component can be simply supported, clamped, cantilevered, or clamped-simply supported. It is shown in Ref. 2 that the variational Eq. 2.3 is applicable for all boundary conditions mentioned.

B. THREE DIMENSIONAL ELASTIC SOLID

Consider the three dimensional elastic solid of Fig. 2.3. For plane elastic solid, results of the three dimensional elastic solid may be reduced.

The strain tensor is defined as

\[
\varepsilon^i_j(z) = \frac{1}{2} (z^i_j + z^j_i), \quad i,j = 1,2,3, \quad x \in \Omega \tag{2.4}
\]
Figure 2.3 Three Dimensional Elastic Solid

where $z = [z^1, z^2, z^3]^T$ is displacement field and subscript $i$, $i = 1, 2, 3$, denotes derivatives with respect to variable $x_i$. The stress-strain relation (generalized Hooke's Law) is

$$\sigma^{ij}(z) = \sum_{k, \ell=1}^{3} C^{ijk\ell} \varepsilon^{k\ell}(z), \quad i, j, k, \ell = 1, 2, 3, \quad x \in \Omega \quad [2.5]$$

where $C$ is the elastic modulus tensor, satisfying symmetry relations $C^{ijk\ell} = C^{jik\ell}$ and $C^{ijk\ell} = C^{ij\ell k}$, $i, j, k, \ell = 1, 2, 3$. The energy bilinear form [2] of the three dimensional elastic solid is

$$a_{u,\Omega}(z, \bar{z}) = \iint_{\Omega} \left[ \sum_{i, j=1}^{3} \sigma^{ij}(z) \varepsilon^{ij}(\bar{z}) \right] d\Omega \quad [2.6]$$

Even though shape design variable, which is the shape of the domain $\Omega$, is the only design variable in this case, subscript $u$ is left for general treatment. The load linear form [2] of the three dimensional elastic solid is
\[ U, Q \]

where \( \gamma_0, \gamma', \) and \( \gamma \) are clamped, traction free, and loaded boundaries, respectively, \( f = [f_1, f_2, f_3]^T \) is the body force, and \( T = [T_1, T_2, T_3]^T \) is the traction force.

The variational equation of the three-dimensional elastic solid is \([2]\)

\[ a_{u, \Omega}(z_1, \Omega) = \iint \sum_{i=1}^{3} \int_{r_1}^{r} [f_1 \gamma_i] \, dr + \iint \sum_{i=1}^{3} \int_{r_1}^{r} [T_1 \gamma_i] \, dr \]

where \( r^0, r^1, \) and \( r^2 \) are clamped, traction free, and loaded boundaries, respectively, \( f = [f_1, f_2, f_3]^T \) is the body force, and \( T = [T_1, T_2, T_3]^T \) is the traction force.

For plane elasticity problems in which either all components of stress in the \( x_3 \)-direction are zero or all components of strain in the \( x_3 \)-direction are zero, Eq. 2.8 remains valid, with limits of summation running from 1 to 2 and an appropriate modification of the generalized Hooke's Law of Eq. 2.5.

### C. PLATE/PLANE ELASTIC SOLID

Consider the plate/plane elastic solid component of Fig. 2.4. The energy bilinear form \([2]\) of the component is

\[ a_{u, \Omega}(z, \Omega) = \iint \hat{D}(t) [((w_{11} + \nu w_{22}) w_{11} + (w_{22} + \nu w_{11}) w_{22}]
\]

\[ + 2(1-\nu) w_{12} w_{12}] \, d\Omega + \iint \nu \{ \sum_{i,j=1}^{2} \sigma^{ij}(v) \varepsilon^{ij}(v) \} \, d\Omega \]

where \( z = [w, v^1, v^2]^T \) is the displacement field. In Eq. 2.10, \( \hat{D}(t) = \frac{E t^3}{12(1-\nu^2)} \), \( \nu \), and \( t \) are flexural rigidity, Poisson's ratio, and thickness of the component, respectively. Also, \( \sigma^{ij}(v) \) and \( \varepsilon^{ij}(v) \) are stress and strain due to an in-plane displacement field \( v = [v^1, v^2]^T \), respectively. For this component, the conventional design variable is \( u = t(x) \) and the shape design variable is the shape of the domain \( \Omega \). The load linear form \([2]\) of the component is
where \( q, f = [f^1, f^2]^T \) and \( T = [T^1, T^2]^T \) are lateral load, body force, and traction force, respectively, as shown in Fig. 2.5. As in the beam/truss component, if there are point loads, a Dirac delta measure [2] can be used.

Figure 2.5 External Loads For Plate/Plane Elastic Solid
The variational equation of the plate/plane elastic solid component is [2]

\[ a_u,\Omega(z,\bar{z}) = \lambda_u,\Omega(\bar{z}), \quad \text{for all } \bar{z} \in \mathcal{Z} \]  \hspace{1cm} [2.12]

where \( \mathcal{Z} \subset H^2(\Omega) \times [H^1(\Omega)]^2 \) and elements of \( \mathcal{Z} \) satisfy kinematic boundary conditions. For plane leastic solid, kinematic boundary condition is

\[ \nu^i(x) = 0, \quad i = 1, 2, \quad x \in \Gamma^0 \]  \hspace{1cm} [2.13]

For plate, the boundary can be clamped, simply supported, or free edge. While the calculation may not be as simple as in the case of beam, the variational Eq. 2.12 is valid for all boundary conditions considered [2]. Note that Eqs. 2.3, 2.8, and 2.12, the variational equations for different structural components are all in the same form.

3. MATERIAL DERIVATIVE FOR SHAPE DESIGN SENSITIVITY ANALYSIS

The first step in shape design sensitivity analysis is development of relationships between a variation in shape of a structural component and the resulting variations in functionals that may arise in the shape design problems. Since the shape of domain a structural component occupies is treated as the design variable, it is convenient to think of \( \Omega \) as a continuous medium and utilize the material derivative idea of continuum mechanics. In this section, the definition of material derivative is introduced and several material derivative formulas that will be used in later sections are derived.

Consider a domain \( \Omega \) in one, two, or three dimensions, shown schematically in Fig. 3.1. Suppose that only one parameter \( \tau \) defines the transformation \( T \), as shown in Fig. 3.1. The mapping \( T : x + x_\tau(x), \quad x \in \Omega, \) is given by

\[
\begin{align*}
x_\tau &= T(x,\tau) \\
\Omega_\tau &= T(\Omega,\tau)
\end{align*}
\]  \hspace{1cm} [3.1]
The process of deforming $\Omega$ to $\Omega_T$ by the mapping of Eq. 3.1 may be viewed as a dynamic process of deforming a continuum, with $\tau$ playing the role of time. At the initial time $\tau = 0$, the domain is $\Omega$. Trajectories of points $x \in \Omega$, beginning at $\tau = 0$, can now be followed. The initial point moves to $x_\tau = T(x,\tau)$. Thinking of $\tau$ as time, a design velocity can be defined as

$$V(x_\tau,\tau) = \frac{dx_\tau}{d\tau} = \frac{\partial T(x_\tau,\tau)}{\partial \tau}$$  \hspace{1cm} [3.2]$$

In a neighborhood of $\tau = 0$, under reasonable regularity hypotheses [2],

$$T(x,\tau) = T(x,0) + \tau \frac{\partial T}{\partial \tau}(x,0) + O(\tau^2)$$

$$= x + \tau V(x,0) + O(\tau^2)$$

Ignoring higher order terms,

$$T(x,\tau) = x + \tau V(x)$$  \hspace{1cm} [3.3]$$

where $V(x) \equiv V(x,0)$. In this paper, only the transformation $T$ of Eq. 3.2 will be considered, the geometry of which is shown in Fig. 3.2.

Variations of the domain $\Omega$ by the design velocity field $V(x)$ are denoted as $\Omega_T = T(\Omega,\tau)$ and the boundary of $\Omega_T$ is denoted as $\Gamma_T$. Henceforth in the paper, the term "design velocity" will be referred to simply as "velocity".

Let $\Omega$ be a $C^k$-regular open set; i.e., its boundary $\Gamma$ is closed and bounded and can be locally represented by a $C^k$-function. Let $V(x) \in R^n$ in Eq. 3.2 be a vector defined on a neighborhood $U$ of the closure $\overline{\Omega}$ of $\Omega$.
Figure 3.2 Variation of Domain

and $V(x)$ and its derivatives up to order $k \geq 1$ be continuous. With these hypotheses, it has been shown [20] that for small $\tau$, $T(x,\tau)$ is a homeomorphism (a one-to-one, continuous map with a continuous inverse) from $U$ to $U_T \equiv T(U,\tau)$ and that $T(x,\tau)$ and its inverse mapping $T^{-1}(x,\tau)$ have $C^k$-regularity and $\Omega_T$ has $C^k$-regularity.

Suppose $z_T(x,\tau)$ is a smooth solution of the elasticity equations. Then the mapping $z_T(x,\tau) \equiv z_T(x + \tau V(x))$ is defined on $\Omega$ and $z_T(x,\tau)$ depends on $\tau$ in two ways. First, it is the solution of the boundary-value problem on $\Omega_T$. Second, it is evaluated at a point $x_T$ that moves with $\tau$. The pointwise material derivative (which is shown to exist in Ref. 2) at $x \in \Omega$ is defined as

$$
\dot{z}(x) = \frac{d}{d\tau} z_T(x + \tau V(x)) \big|_{\tau=0} = \lim_{\tau \to 0} \frac{z_T(x + \tau V(x)) - z(x)}{\tau} \tag{3.4}
$$

If $z_T$ has a regular extension to a neighborhood $U_T$ of $\Omega_T$, then

$$
\dot{z}(x) = z'(x) + v z^T V(x) \tag{3.5}
$$

where

$$
z'(x) = \lim_{\tau \to 0} \frac{z_T(x) - z(x)}{\tau} \tag{3.6}
$$

is the partial derivative of $z$. 
One attractive feature of the partial derivative is that, with reasonable smoothness assumptions, it commutes with the derivatives with respect to $x_i$ [2]; i.e.,

$$\left(\frac{\partial z}{\partial x_i}\right)' = \frac{\partial}{\partial x_i} (z'), \ i = 1,2,3$$ \[3.7\]

A pair of technical material derivative formulas that are used throughout the remainder of the paper are summarized in this section. Their proofs are presented in Ref. 2.

**Lemma 3.1:** Let $\psi_1$ be a domain functional, defined as an integral over $\Omega$,

$$\psi_1 = \iint_{\Omega} f_\tau(x_\tau)\ d\Omega_\tau$$ \[3.8\]

where $f_\tau$ is a regular function defined on $\Omega_\tau$. If $\Omega$ has $C^k$-regularity, then the material derivative of $\psi_1$ at $\Omega$ is

$$\psi_1' = \iint_{\Omega} f'_\tau(x)\ d\Omega + \int_{\Gamma} f(x)\ (V^Tn)\ d\Gamma$$ \[3.9\]

or, equivalently,

$$\psi_1' = \iint_{\Omega} [f'_\tau(x) + \nabla f(x)^T V(x) + f(x)\text{div } V(x)]\ d\Omega$$ \[3.10\]

It is interesting and important to note that only the normal component $(V^Tn)$ of the boundary velocity appearing in Eq. 3.9 is needed to account for the effect of domain variation. In fact, it is shown by Theorem 3.5.2 of Ref. 2 that if a general domain functional $\psi$ has a gradient at $\Omega$ and if $\Omega$ has $C^{k+1}$-regularity, then only the normal component $(V^Tn)$ of the velocity field on the boundary is needed for derivative calculations.

In contrast to Eq. 3.9, use of the mathematically equivalent result given in Eq. 3.10 requires that the velocity field $V(x)$ be defined throughout the domain $\Omega$. Of course, it must be consistent with $(V^Tn)$ on $\Gamma$. Nevertheless, there are an infinite number of velocity fields that satisfy this condition, for each of which the result of Eqs. 3.9 and 3.10 must be the same.

Next, consider a functional defined as an integration over $\Gamma_\tau$,
Lemma 3.2: Suppose \( g \) in Eq. 3.11 is a regular function defined on \( \Gamma \). If \( \Omega \) is \( C^{k+1} \) regular, the material derivative of \( \psi_2 \) is

\[
\psi'_2 = \int_\Gamma \left[ g'(x) + (\nabla g^T n + H g(x)) (V^T n) \right] \, d\Gamma
\]

where \( H \) is the curvature of \( \Gamma \) in \( \mathbb{R}^2 \) and twice the mean curvature in \( \mathbb{R}^3 \).

4. ADJOINT VARIABLE FORMULATION OF SHAPE DESIGN SENSITIVITY ANALYSIS

As seen in Section 3, the static response of a structure depends on the shape of the domain. Existence of the material derivative \( \dot{z} \), which is proved in Ref. 2, and material derivative formulas presented in Section 3 are used in this section to derive an adjoint variable method for design sensitivity analysis of several functionals. Since the finite element method is used for numerical analysis of the structural systems in this paper, only the domain method of shape design sensitivity analysis is presented in this section.

The variational equations of several structural components of Eqs. 2.3, 2.8, and 2.12 on a deformed domain, is of the form

\[
\left[ a_{u,\Omega}(z,\tilde{z}) \right]^T = \int_{\Gamma} \tau^T \dot{\tau} \, d\Gamma + \int_{\Omega} (\nabla \tau^T \tau - \dot{\tau} \dot{\tau} - \nabla \tau \dot{\tau} - \dot{\tau} \nabla \tau) \, d\Omega \equiv l_{u,\Omega}(\tilde{z}), \quad \text{for all } \tilde{z} \in Z_{\tau}
\]

where \( Z_{\tau} \) is the space of kinematically admissible displacements on \( \Omega_{\tau} \) and \( c(\cdot,\cdot) \) is a bilinear mapping that is defined by the integrand of Eqs. 2.1, 2.6, and 2.10.

Taking the material derivative of both sides of Eq. 4.1, using Eqs. 3.10 and 3.11 and noting that the partial derivatives with respect to \( \tau \) and \( x \) commute,

\[
\left[ a_{u,\Omega}(z,\tilde{z}) \right]' = a'_{u,\Omega}(z,\tilde{z}) + a_{u,\Omega}(\tilde{z},\tilde{z}) = a'_{u,\Omega}(\tilde{z}), \quad \text{for all } \tilde{z} \in Z
\]
where, using Eq. 3.5,

\[
[a_{u,\Omega}(z,\bar{z})]' = \int_{\Omega} [c(z,\bar{z}') + c(z',\bar{z}) + \nabla c(z,\bar{z})^T V + c(z,\bar{z})\text{div } V] \, d\Omega
\]

\[
= \int_{\Omega} [c(z,\bar{z}' - \nabla \bar{z}^T V) + c(z - \nabla \bar{z}^T V,\bar{z}) + \nabla c(z,\bar{z})^T V
\]

\[
+ c(z,\bar{z})\text{div } V] \, d\Omega
\]

[4.3]

and

\[
E_{u,\nu}'(\bar{z}) = \int_{\Omega} [f^T \bar{z}' + \nu(f^T \bar{z})^T V + f^T \bar{z} \text{div } V] \, d\Omega
\]

\[
+ \int_{\Gamma_2} [T^T \bar{z}' + (\nu(T^T \bar{z})^T n + HT^T \bar{z})(V^T n)] \, d\Gamma
\]

\[
= \int_{\Omega} [f^T (\bar{z}' - \nabla \bar{z}^T V) + \nu(f^T \bar{z})^T V + f^T \bar{z} \text{div } V] \, d\Omega
\]

\[
+ \int_{\Gamma_2} [T^T (\bar{z}' - \nabla \bar{z}^T V) + (\nu(T^T \bar{z})^T n + HT^T \bar{z})(V^T n)] \, d\Gamma
\]

[4.4]

The fact that the partial derivatives of the coefficients, which depend on cross-sectional area and thickness, in the bilinear mapping c(·,·) are zero has been used in Eq. 4.3 and \(f' = T' = 0\) has been used in Eq. 4.4. For boundary variations, it is supposed that the boundary \(\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2\) is varied, except that the curve \(\partial \Gamma_2\) that bounds the loaded surface \(\Gamma_2\) is fixed for three dimensional elastic solid, so the velocity field \(V\) at \(\partial \Gamma_2\) is zero. For the case in which \(\partial \Gamma_2\) is not fixed, variation of the traction term in Eq. 4.1 (given as an integral over \(\Gamma_2\)) gives an additional term that was not discussed in lemmas. For this case, the interested reader is referred to Ref. 21. For plane elastic solid component case, these additional terms will be given in Section 5.

For \(\bar{z}_\tau\), select \(\bar{z}_\tau(x + \tau V(x)) = \bar{z}(x)\); i.e., choose \(\bar{z}\) as constant on the line \(x_\tau = x + \tau V(x)\). Then, since \(H^m(\Omega)\) is preserved by \(T(x,\tau)\) (homeomorphism property noted in Section 3), if \(\bar{z}\) is an arbitrary element of \(H^m(\Omega)\) that satisfies kinematic boundary conditions on \(\Gamma\), \(\bar{z}_\tau\) is an arbitrary element of \(H^m(\Omega_\tau)\) that satisfies kinematic boundary conditions on \(\Gamma_\tau\). In this case, using Eq. 3.5,

\[
\dot{\bar{z}} = \bar{z}' + \nabla \bar{z}^T V = 0
\]

[4.5]
From Eqs. 4.2, 4.3, and 4.4, using Eq. 4.5,

\[
a_{u,v}(z,\bar{z}) = \iint_{\Omega} \left[ \begin{array}{c}
-c(z,\bar{z}^T V) - c(vz^T V,\bar{z}) \\
+ \nabla c(z,\bar{z})^T V + c(z,\bar{z}) \text{div } V
\end{array} \right] d\Omega
\]

and

\[
\ell_{u,v}(\bar{z}) = \iint_{\Omega} \left[ \begin{array}{c}
\bar{z}^T (\nabla^T V) + f^T \text{div } V
\end{array} \right] d\Omega
\]

\[+ \int_{\Gamma} \left[ \begin{array}{c}
-\tau^T (\bar{z}^T V) + (v(T^T z)^T n + HT^T z)(V^T n)
\end{array} \right] d\Gamma
\]

Then, Eq. 4.2 can be rewritten to provide the result

\[
a_{u,v}(\bar{z},\bar{z}) = \ell_{u,v}(\bar{z}) - a_{u,v}(z,\bar{z}), \quad \text{for all } \bar{z} \in \mathcal{Z} \tag{4.8}
\]

Consider a displacement functional that defines the displacement at nodal point \( \hat{x} \in \Omega \)

\[
\psi_1 = z(\hat{x}) = \iint_{\Omega} \delta(x-\hat{x}) z(x) d\Omega \tag{4.9}
\]

where \( \delta(x) \) is the Dirac delta measure at the origin. Taking the first variation of Eq. 4.9, using the material derivative,

\[
\psi_1' = \bar{z}(\hat{x}) = \iint_{\Omega} \delta(x-\hat{x}) \bar{z}(x) d\Omega \tag{4.10}
\]

The objective now is to obtain an explicit expression for \( \psi_1' \) in terms of the velocity field \( V \), which requires eliminating \( \bar{z} \). An adjoint equation is introduced by replacing \( \bar{z} \in \mathcal{Z} \) in Eq. 4.10 by a virtual displacement \( \lambda \in \mathcal{Z} \) and equating terms involving \( \lambda \) to the energy bilinear form, yielding the adjoint equation for the adjoint variable \( \lambda \),

\[
a_{u,\lambda}(\lambda,\bar{z}) = \iint_{\Omega} \delta(x-\hat{x}) \lambda(x) d\Omega, \quad \text{for all } \lambda \in \mathcal{Z} \tag{4.11}
\]

Denote the solution of Eq. 4.11 as \( \lambda^{(1)} \).

To take advantage of the adjoint equation, evaluate Eq. 4.11 at \( \lambda = \bar{z} \), since \( \bar{z} \in \mathcal{Z} \) [2], to obtain the expression
Similarly, evaluate the identity of Eq. 5.8 at $z = \lambda^{(1)}$, since both are in $Z$, to obtain

$$a_{u,\Omega}(\lambda^{(1)}, \hat{z}) = \int \int_{\Omega} \delta(x-x) \hat{z}(x) d\Omega$$  \hspace{1cm}[4.12]$$

Recalling that the energy bilinear form $a_{u,\Omega}(*,*)$ is symmetric in its arguments, the left sides of Eqs. 4.12 and 4.13 are equal, so

$$\int \int_{\Omega} \delta(x-x) \hat{z}(x) d\Omega = \varepsilon'_{u,v}(\lambda^{(1)}) - a'_{u,v}(z,\lambda^{(1)})$$  \hspace{1cm}[4.14]$$

Using Eqs. 4.14, Eq. 4.10 yields

$$\psi_1 = \varepsilon'_{u,v}(\lambda^{(1)}) - a'_{u,v}(z,\lambda^{(1)})$$  \hspace{1cm}[4.15]$$

Explicit expressions of the terms in Eq. 4.15, for each structural component can be obtained using Eqs. 4.6 and 4.7. These explicit expressions will be derived in Section 5. This order of presentation was chosen to show basic idea of the adjoint variable method without complicate derivation of expressions.

Note that evaluation of the design sensitivity formula of Eq. 4.15 requires solution of Eq. 4.1 for $z$. Similarly, Eq. 4.11 must be solved for the adjoint variable $\lambda^{(1)}$. This is an efficient calculation, using finite element analysis, if the boundary-value problem for $z$ has already been solved, requiring only evaluation of the solution of the same set of finite element equations with different right side, called an adjoint load.

Next, consider a locally averaged stress functional over a test volume $\Omega_p$ of the three dimensional elastic solid,

$$\psi_2 = \int \int \int_{\Omega_p} g(\sigma(z)) m_p d\Omega = \frac{\int \int_{\Omega_p} g(\sigma(z)) m_p d\Omega}{\int \int_{\Omega_p} m_p d\Omega}$$  \hspace{1cm}[4.16]$$

where $\sigma$ denotes the stress tensor, $\Omega_p$ is an open set, and $m_p$ is a characteristic function that is constant on $\Omega_p$, zero outside of $\Omega_p$, and
whose integral is 1. Here, \( g \) is assumed to be continuously differentiable with respect to its arguments. Note that \( g(\sigma(z)) \) might involve principal stresses, von Mises failure criterion, or some other material failure criteria. Taking the first variation of Eq. 4.16, using Eq. 3.10 [10],

\[
\psi'_2 = \left( \iiint_{\Omega} (g' + V g^T V + g \text{ div } V) d\Omega \right) d\Omega d\Omega d\Omega
\]

\[
- \iiint_{\Omega} g d\Omega d\Omega d\Omega \text{ div } V d\Omega \right) / \left( \iiint_{\Omega} d\Omega \right)^2
\]

\[
= \iiint_{\Omega} \sum_{i,j=1}^{3} g_{ij}(z) [\sigma_{ij}(z) - \sigma_{ij}(V z^T V)] m_p d\Omega
\]

\[
+ \iiint_{\Omega} \sum_{k=1}^{3} \left( \sum_{i,j=1}^{3} g_{ij}(z) \sigma_{k}(z)V^k \right) m_p d\Omega + \iiint_{\Omega} g \text{ div } \Sigma_{p} d\Omega
\]

\[
- \iiint_{\Omega} g \Sigma_{p} d\Omega \text{ div } V d\Omega
\]

[4.17]

It can be shown that

\[
\sigma_{ij}(V z^T V) = \sum_{k,l=1}^{3} C^{ijk}_{\lambda} (V z^k_{\lambda} V + V z^k_{\lambda} V)
\]

[4.18]

and

\[
\sum_{k=1}^{3} \sigma_{k}(z)V^k = \sum_{k,l=1}^{3} C^{ijk}_{\lambda} (V z^k_{\lambda} V)
\]

[4.19]

Using these results, Eq. 4.17 becomes

\[
\psi'_2 = \iiint_{\Omega} \left( \sum_{i,j=1}^{3} g_{ij}(z) \sigma_{ij}(z) \right) m_p d\Omega
\]

\[
- \iiint_{\Omega} \sum_{i,j=1}^{3} \left( \sum_{k,l=1}^{3} g_{ij}(z) C^{ijk}_{\lambda} (V z^k_{\lambda} V) \right) m_p d\Omega
\]

\[
+ \iiint_{\Omega} g \text{ div } \Sigma_{p} d\Omega - \iiint_{\Omega} gm_p d\Omega \iiint_{\Omega} m_p \text{ div } V d\Omega
\]

[4.20]

As in the displacement functional case, an adjoint equation is introduced by replacing \( z \in Z \) in the term on the right of Eq. 4.20 by a virtual displacement \( \lambda \in \hat{Z} \) and equate the result to the energy bilinear form.
Denote the solution of Eq. 4.21 as $\lambda(2)$. By the same method used for the displacement functional, the sensitivity formula is obtained as

$$
\psi'_2 = \lambda'_{u,v}(\lambda(2)), \quad - a'_{u,v}(z,\lambda(2)) - \iint_{\Omega} \sum_{i,j=1}^{3} \sum_{k,l=1}^{3} g_{ij}(z) C_{ijkl} \sigma_{kl}(V_{x}^{T} V_{y}) \text{div} V \text{div} V \text{div} V \text{div} V \sigma_{ij} \Omega
$$

where explicit expressions of the first two terms in Eq. 4.22 for the three dimensional elastic solid can be obtained using Eq. 2.6, 2.7, 4.6, and 4.7. These explicit expressions will be derived in Section 5. Note that these terms have the same form as those of Eq. 4.15 for the three dimensional elastic solid. The difference is that terms in Eq. 4.15 are evaluated at $\lambda(1)$ and terms in Eq. 4.22 are evaluated at $\lambda(2)$. That is, once the expressions for terms in Eq. 4.15 are derived, they can be used for different functionals.

Finally consider a locally averaged stress functional over a test area $\Omega$ in a plate/plane elastic solid component,

$$
\psi'_3 = \iint_{\Omega} \left[ g^1(t,w_{ij}) + g^2(\sigma(v)) \right] \sigma_{ij} \text{div} V \text{div} V \text{div} V \text{div} V \sigma_{ij} \Omega
$$

where $g^1(t,w_{ij})$ and $g^2(\sigma(v))$ are principal stress, von Mises yield stress, or some other stress measures due to lateral displacement $w$ and in-plane displacement field $v$, respectively. Here, $g^1(t,w_{ij})$ is measured at the extreme fiber and $m_p$ is a characteristic function on that is constant on $\Omega_p$, zero outside $\Omega_p$, and whose integral is 1. Taking the first variation of Eq. 4.23, using Eq. 3.10,

$$
\psi'_3 = \iint_{\Omega} \sum_{i,j=1}^{2} g^1_{wij} \left( \dot{w}_{ij} - (vw^T)_{ij} \right) m_p \text{div} V \text{div} V \text{div} V \text{div} V \sigma_{ij} \Omega
$$

$$
+ \iint_{\Omega} \sum_{i,j=1}^{2} g^2_{ij} \sigma_{ij} \Omega
$$
Define an adjoint equation by replacing $\ddot{w}$ and $\ddot{v}$ in Eq. 4.24 by virtual displacements $\nabla$ and $\nabla$, respectively, and equate terms involving $\nabla$ and $\nabla$ in Eq. 4.24 to the energy bilinear form,

$$
\begin{align*}
- \iiint_{\Omega} \sum_{i,j=1}^{2} \frac{2}{g_{i,j}^2} \left( \sum_{k=1}^{2} \sigma_{i,j}^{k} \left( \nabla v^{k} \right) \right) m_{p} d\Omega \\
+ \iiint_{\Omega} g_{i,j}^{2} \text{div} V m_{p} d\Omega - \iiint_{\Omega} (g_{1}^{2} + g_{2}^{2}) m_{p} d\Omega \iiint_{\Omega} m_{p} \text{div} V d\Omega
\end{align*}
\quad [4.24]
$$

where $\lambda = [\eta, \xi, \tau]^{T}$ is an adjoint variable. Denote the solution of Eq. 4.25 as $\lambda^{(3)}$. By the same method used for the displacement functional, the sensitivity formula is obtained as

$$
\begin{align*}
\psi_{3}^{i} = \phi_{3}^{i}, \psi(\lambda^{(3)}) - \phi_{3}^{i}, \psi(z, \lambda^{(3)}) - \iiint_{\Omega} \sum_{i,j=1}^{2} g_{i,j}^{1} \left( \nabla w \right) \cdot m_{p} d\Omega \\
+ \iiint_{\Omega} \text{div}(g_{1}^{2} V) m_{p} d\Omega - \iiint_{\Omega} \sum_{i,j=1}^{2} \left( \sum_{k=1}^{2} \sigma_{i,j}^{k} \left( \nabla v^{k} \right) \right) m_{p} d\Omega \\
+ \iiint_{\Omega} g_{i,j}^{2} \text{div} V m_{p} d\Omega - \iiint_{\Omega} (g_{1}^{2} + g_{2}^{2}) m_{p} d\Omega \iiint_{\Omega} m_{p} \text{div} V d\Omega
\end{align*}
\quad [4.26]
$$

where explicit expressions of the first two terms in Eq. 4.26 for the plate/plane elastic solid component can be obtained using Eqs. 2.10, 2.11, 4.6 and 4.7. These explicit expressions will be derived in Section 5. As in the displacement functional case, evaluation of the design sensitivity formula of Eq. 4.26 requires solutions $z$ and $\lambda^{(3)}$ of Eqs. 4.1 and 4.25. Design sensitivity information for locally averaged stress functional over a test length $\Omega_{p}$ in a beam/truss component can be derived using the same procedure.
5. SHAPE DESIGN SENSITIVITY ANALYSIS OF STRUCTURAL COMPONENTS

In this section, explicit expressions for terms in Eqs. 4.6 and 4.7 are derived for each structural components by identifying bilinear mapping c(\cdot,\cdot) and loading terms. The result is standard expressions that can be used for design sensitivity analysis of different functionals. These results can also be used for design sensitivity analysis of built-up structures which will be shown in Section 9.

A. BEAM/TRUSS

Using energy bilinear and load linear forms of Eqs. 2.1 and 2.2 for beam/truss component, Eqs. 4.6 and 4.7 become

\[ a'_{u,v}(z,\bar{z}) = \int_0^L \left[ -EI^1 \left[ 3w_{xx}^1 w_{xx} V_x + (w_{xx}^1 + w_{xx}^1) V_{xx} \right] \\
+ EI^1 w_{xx}^1 V_{xx} \right] dx + \int_0^L \left[ -EI^2 \left[ 3w_{xx}^2 w_{xx} V_x \\
+ (w_{xx}^2 + w_{xx}^2) V_{xx} \right] + EI^2 w_{xx}^2 V_{xx} \right] dx \\
+ \int_0^L \left( -GJ_{xx} \theta_x \theta_x V_x + GJ_{xx} \theta_x \theta_x V_{xx} \right) dx \\
+ \int_0^L \left( -hE_{xx} \bar{V}_x V_x + hE_{xx} \bar{V}_x V_{xx} \right) dx \]  \[5.1\]

and

\[ \xi'_{u,v}(\bar{z}) = \int_0^L \left( q_{xx}^1 V_x + q_{xx}^1 V_x \right) dx + \int_0^L \left( q_{xx}^2 V_x + q_{xx}^2 V_x \right) dx \\
+ \int_0^L \left( r_{xx} \bar{\theta} V_x + r_{xx} \bar{\theta} V_x \right) dx + \int_0^L \left( f_x \bar{V}_x + f_x \bar{V}_x \right) dx \]  \[5.2\]

B. THREE DIMENSIONAL ELASTIC SOLID

Using energy bilinear and load linear forms of Eqs. 2.6 and 2.7 for three dimensional elastic solid, Eqs. 4.6 and 4.7 become
\[ a'_{u,v}(z,\bar{z}) = - \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z)\varepsilon^{ij}(\nu z^T v) + \sigma^{ij}(\bar{z})\varepsilon^{ij}(\nu z^T v) \right] d\Omega \\
+ \iiint_{\Omega} \nu \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] v d\Omega + \iiint_{\Omega} \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] \text{div} V d\Omega \]

and

\[ a'_{u,v}(\bar{z}) = \iiint_{\Omega} \sum_{i=1}^{3} \bar{z}^i (\nu f^T_1 V) d\Omega + \iiint_{\Omega} \left[ \sum_{i=1}^{3} f^i z^i \right] \text{div} V d\Omega \\
+ \iiint_{\Omega} \left[ - \sum_{i=1}^{3} T^i (\nu z^T V) + \nu \left[ \sum_{i=1}^{3} T^i z^i \right] T_n + H \left[ \sum_{i=1}^{3} T^i z^i \right] (V^T n) \right] d\Omega \]

It can be verified that

\[ \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\nu z^T v) = \sum_{i,j=1}^{3} \sigma^{ij}(z)(\nu z_j^T V + \nu z_j^T V_j) \]

and

\[ \nu \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] v = \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z)(\nu z_j^T V) + \sigma^{ij}(\bar{z})(\nu z_j^T V) \right] \]

where \( V_j = [v^1_j, v^2_j, v^3_j]^T \). Using the above results, Eqs. 5.3 becomes

\[ a'_{u,v}(z,\bar{z}) = - \iiint_{\Omega} \sum_{i,j=1}^{3} \left[ \sigma^{ij}(z)(\nu z_j^T V_j) + \sigma^{ij}(\bar{z})(\nu z_j^T V_j) \right] d\Omega \\
+ \iiint_{\Omega} \left[ \sum_{i,j=1}^{3} \sigma^{ij}(z)\varepsilon^{ij}(\bar{z}) \right] \text{div} V d\Omega \]

C. PLATE/PLANE ELASTIC SOLID

As in the beam/truss and three dimensional elastic solid components, explicit expressions for terms in Eqs. 4.6 and 4.7 can be obtained using energy bilinear and load linear forms of Eqs. 2.10 and 2.11 for plate/plane elastic solid component. For plane elastic solid component, Eqs. 5.7 and 5.4 remain valid, with limits of summation running from 1 to 2 and an appropriate modification of the generalized Hooke's Law of Eq. 2.5. Equations 4.6 and 4.7 become, for plate/plane elastic solid component,
\[ a_{u,v}(z, \bar{z}) = \int_{\Omega} -\hat{\partial}(t) \left\{ 4(w_{11}\bar{w}_{11}v_{11}^1 + w_{22}\bar{w}_{22}v_{22}^2) + [u(w_{11}\bar{w}_{22} + w_{22}\bar{w}_{11})
\right.
\left. - (w_{11}\bar{w}_{11} + w_{22}\bar{w}_{22}) + 2(1-v)w_{12}\bar{w}_{12}\right\} \text{div } V
\right.
\left. + 2(w_{11}\bar{w}_{12} + w_{12}\bar{w}_{11} + w_{22}\bar{w}_{12} + w_{12}\bar{w}_{22})(v_{12}^1 + v_{12}^2)
\right.
\left. + [w_{11}\bar{w}_{11} + w_{11}\bar{w}_{11} + v(w_{11}\bar{w}_{22} + w_{22}\bar{w}_{11})]v_{11}^1
\right.
\left. + [w_{11}\bar{w}_{22} + w_{22}\bar{w}_{11} + v(w_{11}\bar{w}_{11} + w_{11}\bar{w}_{11})]v_{22}^1
\right.
\left. + [w_{22}\bar{w}_{11} + w_{11}\bar{w}_{22} + v(w_{22}\bar{w}_{22} + w_{22}\bar{w}_{22})]v_{22}^2
\right.
\left. + [-w_{22}\bar{w}_{22} + w_{22}\bar{w}_{22} + v(w_{22}\bar{w}_{11} + w_{11}\bar{w}_{22})]v_{22}^2
\right.
\left. + 2(1-v)[(w_{11}\bar{w}_{12} + w_{12}\bar{w}_{11})v_{12}^1 + (w_{22}\bar{w}_{12} + w_{12}\bar{w}_{22})v_{12}^2]
\right.
\left. + \frac{\varepsilon T^2}{4(1-v^2)} [w_{11}\bar{w}_{11} + v(w_{11}\bar{w}_{22} + w_{22}\bar{w}_{11}) + w_{22}\bar{w}_{22}
\right.
\left. + 2(1-v)w_{12}\bar{w}_{12}\right\} \text{div } V \text{ d}\Omega
\right.
\left. + \int \left\{ -t \sum_{i,j=1}^2 \left[ \sigma^{ij}(\bar{v})(\bar{v}^T V) + \sigma^{ij}(\bar{v})(\bar{v}^T V)
\right.
\right.
\left. + \sigma^{ij}(\bar{v}) \epsilon^{ij}(\bar{v}) \text{div } V \right] + 2 \sum_{i,j=1}^2 \sigma^{ij}(\bar{v}) \epsilon^{ij}(\bar{v}) \text{div } V \right\} \text{ d}\Omega \] [5.8]

and

\[ e_{u,v}(z) = \int_{\Omega} \left[ w(\bar{q}^T V) + qw \text{div } V \right] \text{d}\Omega
\right.
\left. + \int_{\Omega} \left\{ \sum_{i=1}^2 \left[ \bar{v}^i(\bar{v}^i T) + f^i T \right] \text{div } V \right\} \text{d}\Omega
\right.
\left. + \int \left\{ \sum_{i=1}^2 \left[ -T^i(\bar{v}^T V) + [\bar{v}(T^i T) \text{d}T + H T^i \bar{v}^i] \right] \left( V^T n \right) \right\} \text{d}T
\right.
\left. + \sum_{i=1}^2 \left[ T^i \bar{v}^i V \left| p_2 - T^i \bar{v}^i V \left| p_1 \right\right] \right\} \text{d}\Omega \] [5.9]

In Eq. 5.9, H is the curvature of the loaded boundary \( r^2 \) and the last two terms on the right account for corner effects due to movement of points \( p_1 \)
and \( p_2 \) in Fig. 2.5 [21]. In these two terms, the notation \( \mid p_i \) indicates that the terms are evaluated at point \( p_i \) and \( V_T \) is the component of velocity \( V \) tangent to \( \Gamma \), which is positive if it is in a counter-clockwise direction in Fig. 2.5.

Results given in Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9 can be used in Eqs. 4.15, 4.22, and 4.26 for each structural component and each functional. This allows one to develop a modular computer program that will carry out numerical integrations of terms in Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9 using the same shape functions that are employed in finite element analysis codes. The result will then be a general algorithm and numerical method for design sensitivity analysis that can be implemented with existing finite element codes which will be discussed in Section 6.

6. IMPLEMENTATION OF DESIGN SENSITIVITY ANALYSIS WITH EXISTING FINITE ELEMENT CODES

To obtain design sensitivity information, Eqs. 4.1 and 4.11 must be solved for displacement functionals and Eqs. 4.1, 4.21 and 4.25 must be solved for stress functionals. Once the original and adjoint structures are solved, one can integrate Eqs. 4.15, 4.22, and 4.26 numerically to obtain the desired sensitivity information. The finite element method can be viewed as an application of the Galerkin method to Eqs. 4.1, 4.11, 4.21, and 4.25 for an approximate solution of the boundary-value problem. Note that the energy bilinear forms for Eqs. 4.1, 4.11, 4.21, and 4.25 are same. Hence, the adjoint structures of Eqs. 4.11, 4.21, and 4.25 are the same as that of Eq. 4.1, with different adjoint loads. The adjoint load of Eq. 4.11, is a simple unit load at the point \( \hat{x} \) in the positive direction of \( z(\hat{x}) \). To calculate the adjoint load using the load functional on the right side of Eqs. 4.21 and 4.25, one should use the same shape functions that are used in the finite element code. Since \( m_p \) is a characteristic function defined on finite element \( \Omega_p \), numerical integration of the load functional is done on \( \Omega_p \) only and the adjoint equivalent nodal force acts only on the nodal points of \( \Omega_p \).

For numerical implementation with existing finite element codes, one can proceed as in the flow chart of Fig. 6.1. In the beginning, the model is defined by identifying the finite element model, original structural load, design variables, and constraint functionals. In the next step, an
Figure 6.1 Flowchart of Design Sensitivity Calculation Procedure

FE Model Definition
Preprocessing

Original Structural Load

Analysis done by Existing
FE Code

Structural Response

Calculate Adjoint
Load Externally for
each Constraint, using
Shape Functions

Design Sensitivity
Information Calculation
for each Constraint

Adjoint Response
Associated with each
Constraint

Restart
existing finite element code is called to obtain structural response. With the structural response obtained, one calculates an adjoint load, external to the finite element code, using the shape functions of the code. The adjoint load is then input to the finite element code, to obtain an adjoint response for each constraint functional. For adjoint analysis, one can use the multiloading (restart) option of the finite element code, so that only forward and backward substitutions are performed to obtain each adjoint response. Using the original and adjoint structural responses, design sensitivity information is calculated for each constraint functional, by carrying out only numerical integration. This procedure allows one to carry out calculations outside finite element codes, using postprocessing data only. That is, the design sensitivity software does not have to be imbedded into finite element codes. Moreover, the method does not require differentiation of stiffness and mass matrices and the uncertainty of numerical accuracy associated with selection of a finite difference perturbation can be eliminated.

7. NUMERICAL EXAMPLES

Substantial numerical experimentation has been carried out using the material derivative shape design sensitivity analysis formulation, with the boundary method. Good results have been reported [2,23] for a variety of single structural components. These studies have shown that great care must be taken in projecting stress information to the boundary to achieve acceptable design sensitivity accuracy. Higher order elements and extrapolation from Gauss points have been shown to be essential in achieving acceptable accuracy. Substantially inaccurate results have been observed when low order elements are used and elementary boundary projection approaches are employed.

Numerical experimentation with the domain method [15,16,18,22] has indicated consistently good results for structural components, without the requirement for sophisticated elements, clever boundary projection methods, or drastically refined grids. In order to be more quantitative, two examples are discussed to permit numerical comparison.

Consider a plane elastic solid that is composed of two materials of substantially different modulus of elasticity ($E^2/E^1 = 7.65$) and subjected to simple tension, as shown in Fig. 7.1. The finite element configuration,
Dimensions, material properties of each body, and loading conditions are shown in Fig. 7.1. Body $i$ occupies domain $\Omega^i$, $i=1,2$, $AB$ is the interface boundary $\gamma$, and $\Gamma^0$ and $\Gamma^2$ are the clamped and loaded boundaries, respectively. The design variable $b$ controls the position of the interface boundary $\gamma$, while the overall dimensions of the structure are fixed.

The expression for design sensitivity of the von Mises yield stress functional associated with interface boundary movement with the domain method is obtained by simply adding results of Eq. 4.26 for both segments of the structure. For the plane stress interface problem, terms in Eq. 4.26 due to plate bending must be dropped. For the boundary method, design sensitivity computations are carried out in Ref. 10 (Eq. 42) that is analytically equivalent to the result of the domain method.

For numerical computation, the finite element method is used to approximate the state and adjoint equations of Eqs. 4.1 and 4.25, respectively. In order to compute the design sensitivity expressions of Eq. 4.26 one must define a design velocity field $V$ that satisfies regularity properties defined in Refs. 2 and 9, in terms of variations in the design variable $b$. To have a continuous design velocity field, one may define
The finite element model shown in Fig. 7.1 contains 32 elements, 121 nodal points, and 224 degrees-of-freedom. The 8-noded isoparametric element is employed for design sensitivity analysis. For the boundary method, stresses and strains are obtained at Gauss points and extrapolated to the boundary to obtain accurate results on the boundary [23]. Define $q_1$ and $q$ as the functional values for the initial design $b$ and modified design $b + \delta b$, respectively. The ratio $q'/A$ times 100 is used as a measure of accuracy; i.e., 100% means that the predicted change is exactly the same as the actual change. Notice that this accuracy measure will not give meaningful information when $A$ is very small compared to $q'$, because the difference $A$ may lose precision due to the subtraction $q^2 - q^1$.

Numerical results with a 3% design change; i.e., $\delta b = 0.03b$, are shown in Table 7.1 for the boundary method and in Table 7.2 for the domain method. Due to symmetry, sensitivity results for only the lower half of the structure are given. These results indicate that the domain method gives excellent results, whereas accuracy of the boundary method is not acceptable. For elements 22 and 29, the predicted values are less accurate than others. However, the magnitude of actual differences $A\psi$ for those elements are smaller than others, so $A\psi$ may lose precision.

A disadvantage of the domain method is that a velocity field must be defined in the domain and satisfy regularity properties. There is no unique way of defining domain velocity fields for a given normal velocity field ($v^Tn$) on the boundary. Also, numerical evaluation of the sensitivity result of Eq. 4.26 is more complicated than evaluation of Eq. 42 of Ref. 10, since Eq. 4.26 requires integration over the entire domain, whereas Eq. 42 of Ref. 10 requires integration over only the variable boundary. This problem can be alleviated by introducing a boundary layer [16] of finite elements that vary during the perturbation of the shape of

\[ v^1 = \frac{x_1}{b} \delta b \], on $\Omega^1$
\[ v^2 = 0 \]

and
\[ v^1 = \frac{20 - x_1}{20 - b} \], on $\Omega^2$
\[ v^2 = 0 \]

\[ [7.1] \]
\[ [7.2] \]
Table 7.1. Boundary Method for Interface Problem

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<th>El. No.</th>
<th>$\psi^1$</th>
<th>$\psi^2$</th>
<th>$\Delta \psi$</th>
<th>$\psi'$</th>
<th>$(\psi'/\Delta \psi \times 100)%$</th>
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<td>-0.61003</td>
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Table 7.2. Domain Method for Interface Problem

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<th>El. No.</th>
<th>$\psi^1$</th>
<th>$\psi^2$</th>
<th>$\Delta \psi$</th>
<th>$\psi'$</th>
<th>$(\psi'/\Delta \psi \times 100)%$</th>
</tr>
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<tbody>
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</tr>
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<td>0.28671</td>
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<td>-0.61003</td>
<td>-0.58529</td>
<td>95.9</td>
</tr>
</tbody>
</table>

a structural component. This approach is illustrated schematically in Fig. 7.2. The domain $\Omega$ is divided into subdomains $\Omega_1$ and $\Omega_2$, with inner
core $\Omega_1$ held fixed and only boundary layer $\Omega_2$ modified. In this way, the velocity field need be defined only on $\Omega_2$. The thickness of the boundary layer $\Omega_2$ will depend on trade-offs between numerical accuracy and numerical efficiency. In practice, $\Omega_1$ can be a substructure of the finite element model.

To demonstrate feasibility of the boundary layer approach, two examples are solved by the boundary layer approach. The first example is the plane stress interface problem discussed in this section. For a body of given geometry there is a large number of possible boundary-layers, some of which are better than others, from the viewpoint of accuracy and efficiency. It is difficult to estimate the size and location of the best boundary-layers in advance. They can be determined by analyzing the structure and measuring the strain energy density [24].

The boundary-layer is chosen to include elements 13 thru 20 in Fig. 7.1. The design variable $\delta$ for this case is distance between node 51 and node 65 in Fig. 7.1. Consequently, regions outside the boundary-layer remain unchanged. Numerical results with a 3% design change are shown in Table 7.3 for the boundary-layer approach. Due to symmetry, the shape design sensitivity analysis results of the lower half of the structure are shown. Shape design sensitivity analysis results obtained with the boundary-layer approach are excellent, as shown in Table 7.3.
Table 7.3. Boundary Layer Approach for Interface Problem
\((E^2/E^1 = 7.65)\)

<table>
<thead>
<tr>
<th>El. No.</th>
<th>(\psi^1)</th>
<th>(\psi^2)</th>
<th>(\Delta\psi)</th>
<th>(\psi')</th>
<th>((\psi'/\Delta\psi \times 100))%</th>
</tr>
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<tbody>
<tr>
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<td>393.07967</td>
<td>0.06663</td>
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</tr>
</tbody>
</table>

Next, to test validity of the boundary-layer approach, Young's modulus is changed to \(E^1 = 0.2\) MPa and \(E^2 = 100\) MPa for \(\psi^1\) and \(\psi^2\), respectively. In other words, the ratio between \(E^2\) and \(E^1\) is raised to 500, from 7.65, to check a more severe condition. Design sensitivity results are given in Table 7.4. Accuracy of design sensitivity is excellent. For elements 9 and 22, the magnitude of actual change are small, so finite differences may not be accurate. Numerical results obtained with the boundary method given in Ref. 16, indicates that worse results arise if the ratio \(E^2/E^1\) is increased.

Results for the plane stress interface problem clearly indicate that the boundary approach may have considerable difficulty in handling problems with singular characteristics. Accuracy of the boundary approach rapidly deteriorates in the vicinity of a singularity. On the other hand, the boundary-layer approach can give good sensitivity results throughout the domain. Also, in this interface problem, 56% of cpu time is saved by using the boundary-layer approach instead of the domain approach, without sacrificing accuracy of design sensitivity.

Next, the classical fillet shown in Fig. 7.3 is used to study accuracy of the boundary layer approach. The design for this problem is the shape
Table 7.4. Boundary Layer Approach for Interface Problem  
($E_2/E_1 = 500$)

<table>
<thead>
<tr>
<th>El. No.</th>
<th>$\psi^1$</th>
<th>$\psi^2$</th>
<th>$\Delta \psi$</th>
<th>$\psi'$</th>
<th>$(\psi'/\Delta \psi \times 100)$%</th>
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</tr>
<tr>
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</tr>
</tbody>
</table>

Figure 7.3 Fillet

of the varying boundary $\Gamma_1$ between points A and B, without moving these two points. B-spline representation is used for the varying boundary $\Gamma_1$. Due to symmetry, only the upper half of a fillet is analyzed. Dimensions of the structure and applied loads are given in Fig. 7.3. For material property, Young's modulus and Poisson's ratio are 3.0 x $10^7$ psi and $\nu = 0.293$, respectively. The segment $\Gamma_3$ is the center-line of the fillet.
and $r_2$ is the uniformly loaded edge. Sensitivity of von Mises stress averaged over individual finite elements is employed to test accuracy of the boundary-layer approach. The expression for design sensitivity is obtained from Eq. 4.26 with nonzero velocity field on the boundary layer.

The boundary-layer (27% of the total area) shown in Fig. 7.4 is chosen after analyzing the structure and measuring the strain energy density. In Fig. 7.5, a finite element model with optimized boundary profile $r_1$ and 319 elements and 1994 active degrees-of-freedom is shown. The element type used is an 8-node isoparametric element.

![Figure 7.4 Boundary-Layer of Fillet](image)

![Figure 7.5 Finite Element Mesh of Fillet](image)

In Table 7.5, shape design sensitivity results for a fillet with optimized boundary profile $r_1$ (see Fig. 7.5) are given, obtained with 0.1% design perturbation. From Table 7.5, it can be seen that this approach can yield excellent shape design sensitivity results.
Table 7.5. Boundary Layer Approach for Fillet

<table>
<thead>
<tr>
<th>El. No.</th>
<th>( \psi^1 )</th>
<th>( \psi^2 )</th>
<th>( \Delta \psi )</th>
<th>( \psi' )</th>
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<td>102.6</td>
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</table>

8. AUTOMATIC REGRIDDING FOR SHAPE DESIGN

For numerical implementation of shape design sensitivity analysis, one must parameterize the boundary \( \Gamma \) of the domain \( \Omega \). For this purpose, one may use Bezier curves or surfaces [25]. The next step is to develop a general method of defining and computing a velocity field in the domain, in terms of the perturbation of the boundary \( \Gamma \). Moreover, the velocity field must satisfy certain regularity conditions. It is shown in Refs. 2 and 9 that \( C^1 \)-regular and \( C^2 \)-regular velocity fields are sufficient for
shape design sensitivity analysis of truss and elastic solid problems and beam and plate problems, respectively. However, observing Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9, one may relax these regularity conditions. That is, for truss and elastic solid problems, the highest order derivative of the velocity field that appears in Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9 is one. Thus, one may use a C^0-regular velocity field with an integrable first derivative. Similarly, one may use a C^1-regular velocity field with an integrable second derivative for beam and plate problems. Therefore, regularity of the velocity field must be at least at the level of regularity of the displacement field of the structural component considered. This suggests use of displacement shape functions to systematically define the velocity field in the domain. Moreover, one can select a velocity field that obeys the governing equation of the structure. That is, the perturbation of the boundary can be considered as a displacement at the boundary. With no additional external forces and a given displacement at the boundary, one can use the finite element code to find the displacement (domain velocity) field that satisfies the required regularity conditions. Thus

\[ [K](V) = \{f\} \]  

[8.1]

where \([K]\) is the reduced stiffness matrix, \(\{V\}\) is the velocity vector of the nodes of varying domain, and \(\{f\}\) is the unknown ficticious boundary force that produces a perturbation of the boundary. In segmented form, Eq. 8.1 becomes

\[
\begin{bmatrix}
K_{bb} & K_{bd}^T \\
K_{bd} & K_{dd}
\end{bmatrix}
\begin{bmatrix}
\{V_b\} \\
\{V_d\}
\end{bmatrix} = 
\begin{bmatrix}
\{f_b\} \\
\{0\}
\end{bmatrix}
\]  

[8.2]

where \(\{V_b\}\) is the given perturbation of nodes on the boundary, \(\{V_d\}\) is the node velocity vector in the interior of the domain and \(\{f_b\}\) is the ficticious boundary force acting on the varying boundary. Equation for the unknown interior node velocity vector can be obtained from Eq. 8.2 as

\[
[K_{dd}] \{V_d\} = - [K_{bd}] \{V_b\}
\]  

[8.3]
If Bezier curves or surfaces [17] are used for boundary representation, positions of control points are selected as design parameter $b_i$, $i = 1, 2, ..., k$. To use Eqs. 4.15, 4.22, and 4.26 for sensitivity computation, interior node velocity vector $\{V_d\}$ should be expressed in terms of variations of design parameter $\delta b_i$, $i = 1, 2, ..., k$. To obtain this expression, boundary perturbation $\{V_b\}$ should be written in terms of variation $\delta b$ of the design parameter. Once the inverse matrix $[K_{dd}]^{-1}$ is obtained, Eq. 8.3 can be used to express $\{V_d\}$ in terms of the variation $\delta b$ of design parameter. However, this requires large computational effort. To gain computational efficiency, following method is used: first by perturbing a design parameter $b_i$ a unit magnitude, boundary perturbation $\{V_b\}$ can be obtained. Then Eq. 8.3 can be solved to obtain $\{V_d\}$. Using $\{V_d\}$ and displacement shape functions, Eqs. 4.15, 4.22, or 4.26 can be evaluated which gives $\frac{\partial \psi}{\partial b_i}$. This method requires to solve Eq. 8.3 $k$ times. However, much as in the adjoint analysis, this is an efficient calculation, if Eq. 8.3 has already been solved, requiring only evaluation of the solution of the same set of finite element equations with different right side for each unit perturbation of $b_i$, $i = 1, 2, ..., k$. Once design change has been determined using iterative design process, regridding of interior grid points can be carried out using $\{V_d\}$ of Eq. 8.3.

The automatic regridding method presented here can be used with the boundary-layer approach very effectively. That is, for the fixed domain $\Omega_1$, $V_d$ can be set equal to zero and thus reduce the dimension of $[K_{dd}]$ in Eq. 8.3.

To demonstrate feasibility of the method, an engine bearing cap shown in Fig. 8.1 is treated. The engine bearing cap is modeled as a three dimensional elastic solid. Due to symmetry, only the right half of the cap is analyzed. The finite element configuration and loading conditions are shown in Fig. 8.1. The material used is steel with Young's modulus and Poisson's ratio of $E = 1.0 \times 10^7$ psi and $v = 0.3$, respectively. The finite element model shown in Fig. 8.1 contains 82 elements, 768 nodal points, and 2111 degrees-of-freedom. For analysis, ANSYS finite element STIF95 [26], which is a 20-noded isoparametric element, is used.

The design variables for this problem are the shape of the varying surface $\Gamma_1$, distance $C_5$ of clamping bolt center line $AB$, and distance $C_6$ of edge from cap centerline (Fig. 8.2). For surface $\Gamma_1$, a Bezier surface
CLAMPING BOLT FORCE = 14,775 lb.

OIL FILM PRESSURE = 5000 psi

Figure 8.1 Engine Bearing Cap

Figure 8.2 Shape Design Parameters of Engine Bearing Cap

with 4 x 4 control points is used. For simplicity, only $x_2$-coordinates of four control points $C_1$ thru $C_4$ are allowed to be varying. That is, surface $\Gamma_1$ has curvature in the $x_1$-direction only.

The expression for design sensitivity of von Mises stress averaged over individual finite element is obtained from Eq. 4.22. Numerical
computation of design sensitivity information has been carried out using ANSYS finite element code [26] and the computational procedure of Section 6. For computation of domain velocity vector, Eq. 8.2 is solved using ANSYS finite element code.

Numerical results with a 1% design change are shown in Table 8.1 for randomly selected finite elements. Accuracy of design sensitivity is excellent except for elements 5, 22, 36, 56, and 57 where the magnitudes of actual change are small.

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9. DESIGN COMPONENT METHOD FOR BUILT-UP STRUCTURES

In this section, design sensitivity analysis method for built-up structures is presented. Both shape and conventional (sizing) design variables for components of built-up structures are considered. For conventional design sensitivity analysis, distributed parameter structural design sensitivity analysis theory of Ref. 2 is used.

Consider a built-up structure that is made up of \( m > 1 \) structural components that are interconnected by kinematic constraints at their interfaces. Using the principle of virtual work for built-up structures \[2\], one can obtain the variational formulation of the governing equations,

\[
a_{u,\Omega}(z,\overline{z}) = \frac{\partial}{\partial z} \left( u_{i,\Omega}(\overline{z}) \right), \quad \text{for all } \overline{z} \in \Omega \tag{9.1}
\]

where

\[
a_{u,\Omega}(z,\overline{z}) = \sum_{i=1}^{m} a_{i,\Omega}(z,\overline{z}) \tag{9.2}
\]

\[
\frac{\partial}{\partial z} \left( u_{i,\Omega}(\overline{z}) \right) = \sum_{i=1}^{m} \frac{\partial}{\partial \overline{z}} \left( u_{i,\Omega}(\overline{z}) \right) \tag{9.3}
\]
and \( Z \) is the space of kinematically admissible displacements \([2]\), which is defined as the set of displacement fields that satisfy homogeneous boundary conditions and kinematic interface conditions between components. In Eqs. 9.2 and 9.3, \( a_{i,\Omega}^{z} \) and \( \lambda_{i,\Omega}^{z} \) are energy bilinear and load linear forms of component \( i \) with domain \( \Omega^{i} \). Note from Eqs. 9.2 and 9.3, the energy bilinear and load linear forms of Eq. 9.1 are simply summations of corresponding terms from each component. Thus, as will be seen later, the design sensitivity analysis of the built-up structure is a simple additive process.

In this section, design sensitivity information for displacement functional is derived for general built-up structures. Once this is done, extension to locally averaged stress functional can be carried out easily.

Define \( \dot{z} \) as the total variation of \( z \), due to both conventional and shape design changes \([2]\),

\[
\dot{z} = \frac{d}{dt} z(t+\tau V(t), u+\tau \delta u) \bigg|_{\tau=0}
\]

\[
= \frac{d}{dt} z(x, u+\tau \delta u) \bigg|_{\tau=0} + \frac{d}{dt} z(t+\tau V(t), u) \bigg|_{\tau=0}
\]

[9.4]

The first variation of Eq. 9.1 is \([2]\)

\[
a_{\delta u,\Omega}^{z} + a_{u,\Omega}^{z} + a_{u,\Omega}^{z} = \lambda_{\delta u,\Omega}^{z} + \lambda_{u,\Omega}^{z}, \text{ for all } \overline{z} \in Z
\]

[9.5]

where \( \dot{z} = [\dot{w}^{1}, \dot{w}^{2}, \dot{w}, \dot{v}]^{T} \) for beam/truss component, \( \dot{z} = [\dot{z}^{1}, \dot{z}^{2}, \dot{z}^{3}]^{T} \) for three dimensional elastic solid, and \( \dot{z} = [\dot{w}, \dot{v}^{1}, \dot{v}^{2}]^{T} \) for plate/plane elastic solid component. The notation of Eq. 9.5 is chosen to clearly display which variables are held fixed and which vary.

Consider a displacement functional that defines the displacement \( z \) at nodal point \( x \in \Omega^{\Gamma} \)

\[
\psi = \iint_{\Omega^{\Gamma}} \delta(x-x)z(x) \, d\Omega
\]

[9.6]

where \( \delta(x) \) is the Dirac delta measure at the origin. Taking the first variation of Eq. 9.6, one obtains \([2]\)

\[
\psi' = \iint_{\Omega^{\Gamma}} \delta(x-x)\dot{z}(x) \, d\Omega
\]

[9.7]
Define a variational adjoint equation by replacing \( \hat{z} \) in the term on the right of Eq. 9.7 by a virtual displacement \( \lambda \) and equate the result to the energy bilinear form evaluated at the adjoint variable \( \lambda \); i.e.,

\[
a_{u_1,\Omega}(\lambda,\lambda) = \iint_\Omega \epsilon(x-\lambda)\lambda(x) \, d\Omega, \quad \text{for all } \lambda \in \mathbb{R} \tag{9.8}
\]

Denote the solution of Eq. 9.8 as \( \lambda \). Since \( \hat{z} \) satisfies kinematic boundary and interface conditions \([2]\), Eq. 9.8 can be evaluated at \( \lambda = \hat{z} \) and Eq. 9.5 can be evaluated at \( \lambda = \hat{z} \), to obtain

\[
\psi' = \sum_{i=1}^{m} \left[ \frac{\partial}{\partial u_i} \psi_i(\lambda) - a_i' \delta u_i, \lambda \right] + \sum_{i=1}^{m} \left[ \frac{\partial}{\partial u_i} \psi_i(\lambda) - a_i' \delta u_i, \lambda \right] \tag{9.9}
\]

where the first term on the right is due to conventional design variation and the second term is due to shape design variation. Note that Eq. 9.9 is valid for general built-up structures that are composed of \( m > 1 \) structural components. For explicit expressions of the second term on the right of Eq. 9.9, results of Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9 can be used. For the first term on the right of Eq. 9.9, the interested reader is referred to Refs. 2 and 18.

As seen in this derivation, one can systematically organize design sensitivity expressions for built-up structures, using the design component method. Moreover, one can develop a modular computer program that will carry out numerical integration of terms in Eqs. 5.1, 5.2, 5.4, and 5.7 - 5.9, using the same shape functions that are employed by the finite element analysis of the original structure. The result will then be a general algorithm and numerical method for design sensitivity analysis that can be implemented with existing finite element codes as shown in Section 6.

To demonstrate accuracy of the design component method, a truss-beam-plate built-up structure is treated in this section. Consider the truss-beam-plate built-up structure shown in Fig. 9.1. A distributed vertical load \( f(x) \) is applied to the plates. The points supported by the trusses are at the intersections of two crossing beams nearest to the free edges of the structure. No external loads are applied to the truss and beam components. The plates and beams are assumed to be welded together.
Coordinates of intersection points of beams and plates are supposed to be in the mid-planes of the plates and neutral axes of the beams. Beam components have rectangular cross-sections.

Figure 9.1 Truss-Beam-Plate Built-up Structure
The design variables for this built-up structure are thickness $t_{ij}(x)$ of each plate component, width $a_{ij}(x_1)$ and height $c_{ij}(x_1)$ of each longitudinal beam component, width $d_{ij}(x_2)$ and height $b_{ij}(x_2)$ of each transverse beam component, cross-sectional areas $h_k$ ($k=1,16$) of the four 4-bar truss components, and positions $\alpha_i$ ($i=1,4$) and $\beta_j$ ($j=1,4$) of transverse and longitudinal beam components, respectively. The lengths of the trusses are fixed, but they may change their ground positions, and the outside boundary of the entire structure is fixed; i.e., only the locations $\alpha_i$ and $\beta_j$, $i,j=1,4$, of beams are shape variables. Dimensions of the structure and the numbering and spacing of beams in both directions are shown in Fig. 9.1.

For numerical calculations, conventional and shape design sensitivity calculations are carried out separately. For plate components, 12 degree-of-freedom non-conforming rectangular elements [27] are used. For beam components, Hermite cubic shape functions are used. The finite element model used for design sensitivity analysis is shown in Fig. 9.2. Only one quarter of the entire structure is analyzed, due to symmetry. A total of 484 elements, with 1281 degrees-of-freedom, are used to model the built-up structure, including 400 rectangular plate elements, 80 beam elements and 4 truss elements.

For numerical data, Young's modulus and Poisson's ratio are $3.0 \times 10^7$ psi and 0.3, respectively. The overall dimensions are $L_1 \times L_2 = 15$ in. x 15 in. At the nominal design, beam components are located 3 in. apart. Other dimensions of the built-up structure at the nominal design are; uniform thickness $t = 0.1$ in. for plate components, uniform height $h = 0.5$ in. and width $d = 0.15$ in. for beam components, and length $l = 5.364$ in. and cross-sectional area $h = 0.1$ in.$^2$ for truss components. A uniform distributed load $f = 0.1$ lb/in.$^2$ is applied on the plate components.

In Table 9.1, design sensitivity accuracy results are given for several functionals, with 1% uniform change in all conventional design variables except the cross-sectional areas of truss components. Design sensitivity results for displacements, bending stresses $\sigma_{11}$ and $\sigma_{22}$ at the extreme fiber of longitudinal and transverse beam components, and von Mises yield stress

$$ g(\sigma) = (\sigma_{11}^2 + \sigma_{22}^2 + 3\sigma_{12}^2 - 2\sigma_{11}\sigma_{22})^{1/2} $$

[9.10]
at the extreme fiber of plate components are given in Table 9.1. Results given in Table 9.1 show good agreement between predictions $\psi'$ and finite differences $\Delta \psi$ except for von Mises yield stresses on plate elements 177, 358, 380 and 400 which are acceptable but not good. However, note that these elements have low von Mises yield stress and $\Delta \psi$ is small, compared to others, and may not be accurate.

For shape design sensitivity calculations, since the built-up structure is symmetric with respect to the center $C$, the locations $\alpha_i$ and $\beta_j$, $i,j=1,2$, of transverse and longitudinal beams, measured from the center $C$, are taken as design variables.
Table 9.1. Conventional Design Sensitivity of Truss-Beam-Plate Built-Up Structure

(a) Displacement

<table>
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<th>Node No.</th>
<th>$\psi_1$</th>
<th>$\psi_2$</th>
<th>$\Delta \psi$</th>
<th>$\psi'$</th>
<th>$(\psi'/\Delta \psi \times 100)$%</th>
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</thead>
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<td>53</td>
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(b) Bending Stress on Beam Element

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(c) von Mises Stress on Plate Element

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As mentioned in Section 8, for shape design sensitivity calculations, one must define a velocity field that has $C^1$-regularity with its second derivative integrable. The beam components are allowed to move in transverse directions only. Hence, $V^1$ is a function of $x_1$ only and $V^2$ is a function of $x_2$ only. The velocity field in each plate component is

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represented by Hermite cubic functions in each direction. That is, \( V^1(x_1) \) and \( V^2(x_2) \) are represented by Hermite cubic functions. To see the velocity field representation graphically, consider Fig. 9.3, in which the shape functions for \( V^1(x_1) \) are plotted. In Fig. 9.3, \( \delta \alpha_1 \) and \( \delta \alpha_2 \) denote perturbations of locations of transverse beams. From Fig. 9.3, one obtains \( V^1(x_1) = \phi^1(x_1) + \phi^2(x_1) \). That is,

\[
V^1(x_1) = \begin{cases} 
- \frac{2x^2}{3} \left( x - \frac{3\alpha_1}{2} \right) \delta \alpha_1, & 0 < x < \alpha_1 \\
\frac{2(x-\alpha_1)^2}{(\alpha_2-\alpha_1)^3} \left[ (x-\alpha_1) - \frac{3(\alpha_2-\alpha_1)}{2} \right] (\delta \alpha_1 - \delta \alpha_2) + \delta \alpha_1, & \alpha_1 < x < \alpha_2 \\
\frac{2(x-\alpha_2)^2}{(-\frac{1}{2} - \alpha_2)^3} \left[ (x-\alpha_2) - \frac{3(-\frac{1}{2} - \alpha_2)}{2} \right] \delta \alpha_2 + \delta \alpha_2, & \alpha_2 < x < \frac{L_1}{2} 
\end{cases}
\]

[9.11]

and a similar expression for \( V^2(x_2) \).

![Figure 9.3 Shape Functions For The Velocity \( V^1(x_1) \)]

In Table 9.2, design sensitivity accuracy results are given for several functionals, with a 0.25% uniform change in shape design parameters. Sensitivity results for displacements, bending stress for beam components, and von Mises yield stress for plate components, are given in Table 9.2. Results given in Table 9.2 show excellent agreement between predictions \( \psi' \) and the finite differences \( \Delta \psi \).
Table 9.2. Shape Design Sensitivity of Truss-Beam-Plate Built-Up Structure

(a) Displacement

<table>
<thead>
<tr>
<th>Node No.</th>
<th>$\psi^1$</th>
<th>$\psi^2$</th>
<th>$\Delta\psi$</th>
<th>$\psi'$</th>
<th>$(\psi'/\Delta\psi \times 100)%$</th>
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(b) Bending Stress on Beam Element

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(c) von Mises Stress on Plate Element

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### Table 9.2(c) Continued

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10. AN OPTIMIZATION PROBLEM

To demonstrate application of the design sensitivity analysis method for built-up structures in structural design optimization, the truss-beam-
plate built-up structure of Section 9 is optimized using a sparse matrix symbolic factorization technique for iterative structural optimization [28] and Pshenichny's linearization method [29].

For numerical design sensitivity analysis and optimization, design variables are discretized. That is, each plate element has constant thickness and each beam element has constant width and height. Since the built-up structure is symmetric with respect to the center C, thickness $t_i, i = 1, 2, 10$, width $d_i$ and height $b_i, i = 1, 40$, and the locations $a_i, i = 1, 2$ of transverse and longitudinal beams, measured from the center C, are taken as design parameters. Thus, the total number of design parameters is 292.

The optimal design problem of the built-up structure is to minimize weight of the structure, subject to the following constraints:

- Displacement at C: $\psi_1 = z(C) < 0.105$ in.
- Plate element von Mises stress: $\psi_i < 17500$ psi, $i = 2, 211$
- Beam element bending stress: $-70000$ psi $< \psi_i < 70000$ psi, $i = 212, 251$
- Plate thickness: $0.05$ in. $< t_i < 0.25$ in., $i = 1, 210$
- Beam width: $0.075$ in. $< d_i < 0.30$ in., $i = 1, 40$
- Beam height: $0.25$ in. $< b_i < 1.00$ in., $i = 1, 40$
- Beam position: $0 < a_1 < a_2 < 122$

Thus, the total number of inequality constraints is 543.

Same numerical data as in Section 9 is used except weight density is $0.1$ lb/in.$^3$ and uniformly distributed load $f = 17.5$ lb/in.$^2$ is applied to the plate components.

For numerical computation of the shape design sensitivity information, derivatives $I_x, J_x, h_x$ in Eq. 5.1 for beam component and $\psi_t$ in Eq. 5.8 for plate component must be computed. Since each plate element has constant thickness and each beam element has constant width and height, these derivatives are Dirac delta measures and computations of the shape design sensitivity information become complicated.

To avoid this difficulty, the design process is divided into two phases. In Phase I, each plate and beam components (not element) have constant thickness and constant width and height, respectively. Hence in Phase I, the design parameter set includes 6 plate thicknesses, 6 beam heights and widths, and 2 beam locations with total of 20 design parameters. To assign the same design parameter to elements in a component, design variable linking is used in Phase I. Once an optimum
point is reached in Phase I, the design process is switched to Phase II where shape design parameters are fixed and each plate and beam elements are allowed to have different design parameters. Hence the total number of design parameter is 290 in Phase II.

For numerical computation, PRIME-750 and Cray-1S computers are used for design Phases I and II, respectively. The initial and final designs of Phase I is given in Table 10.1. The initial cost is 0.7875 lb and the final cost of Phase I is 0.5894 lb. There are 12 design iterations in

Table 10.1. Initial and Final Designs of Phase I

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<th>Final Design</th>
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Phase I with average CPU time 15986 seconds per iteration on a PRIME-750 computer. It is observed in design Phase I that inner beam stiffeners move inward ($a_1 = 1.4211$ in.) and outer beam stiffeners move outward ($a_2 = 4.5872$ in.). Number of active stress and displacement constraints at the final design of Phase I is 31. Thus, it is necessary to calculate sensitivity information for 31 constraints out of 251.

There are 9 design iterations in Phase II with average CPU time 25.84 seconds per iteration on a Cray-1S computer. Cost function history of Phases I and II is shown in Fig. 10.1. The final cost of Phase II is 0.5388 lb. Number of active stress and displacement constraints at the final design of Phase II is 54. A profile of upper half of the final design is shown in Fig. 10.2.

![Figure 10.1 Cost Function History](image-url)
Figure 10.2 A Profile of the Final Design
REFERENCES