ON THE INTERACTION OF SMALL AND LARGE
EDDIES IN TWO DIMENSIONAL TURBULENT FLOWS

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\section*{ABSTRACT}

Our aim in this article is to present some results concerning the interaction of small and large eddies in two dimensional turbulent flows. We show that the amplitude of small structures decays exponentially to a small value and we infer from this a simplified interaction law of small and large eddies. Beside their intrinsic interest for the understanding of the physics of turbulence, these results lead to new numerical schemes which will be studied in a separate work.

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INTRODUCTION

The conventional theory of turbulence in space dimension three implies the existence of a length \( l_d \) which is small in comparison with the macroscopic length \( l_0 \) connected to the geometry, and which is such that the eddies of size less than \( l_d \) are damped by the effect of viscosity and become rapidly small in amplitude; the length \( l_d \) is called the Kolmogorov dissipation length [8]. In space dimension two the situation is similar, but \( l_d \) is replaced by the larger length \( l_\chi \) introduced by Kraichnan [9]. It is one of our aims in this article to derive directly from the Navier-Stokes equations and without any phenomenological consideration a mathematically rigorous proof of this property: the exponential decay of the small eddies toward a small value. Note however that the cut-off size between small and large eddies is much smaller than \( l_\chi \) or even \( l_d \), and this is due in part to the high level of generality allowed here where singular flows can be considered such as those generated by flows in nonsmooth cavities, like the flow in a rectangular cavity. A physical discussion on the necessary cut-off length is presented hereafter.

Our approach is the following one: the Navier-Stokes equations of two dimensional viscous uncompressible flows are written as

\[
\begin{align*}
\frac{\partial u}{\partial t} - v\Delta u + (u \cdot \nabla)u - \nabla \overline{w} &= f \quad \text{in } \Omega \times \mathbb{R}_+ \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times \mathbb{R}_+
\end{align*}
\]

where \( u = u(x,t) = \{u_1, u_2\} \) is the velocity vector, \( \overline{w} = \overline{w}(x,t) \) is the pressure, \( f \) represents volume forces, \( v > 0 \) is the kinematic viscosity. As
usual (0.1)(0.2) are supplemented by boundary conditions which could be for instance

\[ (0.3a) \quad u = 0 \text{ on } \partial \Omega \]

or

\[ (0.3b) \quad u \cdot \nu = 0, \, \nu \times \text{curl} u = 0 \text{ on } \partial \Omega, \, \nu \text{ the unit outward normal on } \partial \Omega, \]

or

\[ (0.3c) \quad \Omega = (0, L_1) \times (0, L_2) \quad \text{and } u, \, \overline{\nu} \text{ are periodic of period } L_i \text{ in the direction } x_i, \, i = 1, 2. \]

Our emphasis here will be on the space periodic case (0.3c), but the other boundary conditions will be considered as well. In all cases (0.1) - (0.3) reduces to an abstract evolution equation for \( u \) in an appropriate Hilbert space \( H \):

\[ (0.4) \quad \frac{du}{dt} + \nu \, Au + B(u) = f. \]

The operator \( A \) linear, self-adjoint unbounded positive in \( H \) with domain \( D(A) \subseteq H \), is the Stokes operator. Since \( A^{-1} \) is compact self adjoint, \( A \) possesses a complete family of eigenvectors \( w_j \) which is orthonormal in \( H \):

\[ Aw_j = \lambda_j w_j, \quad j = 1, 2, \ldots \]

\[ (0.5) \]
\[ 0 < \lambda_1 \leq \lambda_2, \ldots, \lambda_j \to \infty \text{ as } j \to \infty. \]

Of course in the space periodic case (0.3c) the \( w_j \) are directly related to the appropriate sin and cos functions of the Fourier series expansion (see [12]). The operator \( B \) is a quadratic operator; \( B(u) = B(u,u) \), where \( B(\cdot,\cdot) \) is a bilinear compact operator from \( D(A) \) into \( H \).

For fixed \( m \) we denote by \( P = P_m \) the projector in \( H \) onto the space spanned by \( w_1, \ldots, w_m \), and we write \( Q = Q_m = I - P_m \). We set

\[ u = p + q, \quad p = Pu, \quad q = Qu, \]

and we show that, after a transient period, and for various norms, \( p \) is comparable to \( u \), and \( q \) is small in comparison with \( p \) and \( u \) (see Sec. 1).

We then project equation (0.4) on \( PH \) and \( QH \); this yields a coupled system of equations for \( p \) and \( q \):

\[ \frac{dp}{dt} + \nu Ap + PB(p + q) = Pf \]

(0.6)

\[ \frac{dq}{dt} + \nu Aq + QB(p + q) = Qf. \]

(0.7)

Since \( q \) is small in comparison with \( p \) one can speculate that \( B(q,q) \leq B(q) \) is small in comparison with \( B(p,q) \) and \( B(q,p) \) and that these quantities are small in comparison with \( B(p,p) = B(p) \). Also the relaxation time for the linear part of (0.7) of the order of \( (\nu \lambda_{m+1})^{-1} \) is much smaller than that of (0.6) which is of order \( (\nu \lambda_1)^{-1} \). This suggests that an acceptable approximation to (0.7) is given by
This leads us to introduce in $H$ the finite dimensional manifold $M_0$ with equation

\[
\begin{aligned}
q &= \phi_0(p) = (\nu A)^{-1}(Qf - QB(p)) \\
p &= Pu, \quad q = Qu.
\end{aligned}
\]

It is one of our aims to justify this approximation: for large times, i.e., after a sufficiently long transient period, the ratio of $q$ to $u$ is of the order of $\frac{\lambda_1}{\lambda_{m+1}}$ for large $m$, whereas the distance of $q$ to $M_0$, (compared to a quantity of the order of $u$), is of the order of $\frac{\lambda_1}{\lambda_{m+1}}^{3/2}$ for large $m$. The proof of this result appears in Sec. 2. Hence, for large time, an orbit $u(t) = p(t) + q(t)$ corresponding to any solution of (0.4) becomes closer to $M_0$ than to the linear space $q = 0$. We intend in a subsequent work to construct a whole family of explicitly defined manifolds $M_j$ providing better and better approximations to the orbits as $j$ increases\(^{(1)}\). The manifold $M_0$ (as well as the future manifolds $M_j$) plays the role of approximate inertial manifolds to the two dimensional Navier-Stokes equations and constitutes a substitute to them in situations where we cannot prove the existence of such manifolds.

In Sec. 3 we recall and improve significantly a result in [7]: this leads us to introduce a Lipschitz manifold $\Sigma$ of finite dimension like $M_0$,

\(^{(1)}\)C. Foias, O. Manley, and R. Temam, Article in preparation.
to which all the orbits of (0.4) remain eventually at a distance less than 
\[ \exp(-c \frac{\lambda_1}{\lambda_{m+1}}) \]. Hence \( M_0 \) provides a much better approximation than \( M \).

but, on the contrary, the proof of existence is nonconstructive and does not provide an explicit expression like (0.9). It provides nevertheless an interesting complementary aspect. Let us mention also that another type of approximate manifold containing all the stationary solutions has been exhibited by E. Titi\(^{(1)}\).

This article ends with an Appendix providing a technical but totally new method of estimating certain norms of the solutions of an evolution equation like (0.4): taking advantage of the analyticity in time of the solutions, we estimate the domain of analyticity in the complex time plan and using Cauchy's formula, we readily deduce estimates on the derivatives \( \frac{d^k u}{dt^k} \) from the estimates on \( u \) in the domain of analyticity; these estimates on the time derivatives of \( u \) are much sharper than those obtained by real variable methods.

The results presented here were announced in [0]. We intend in a subsequent work to derive approximate manifolds of higher order than \( M_0 \) and to study the three dimensional case.

1. FAST DECAY OF SMALL EDDIES

In Secs. 1.1 and 1.2 we briefly recall the functional setting of the Navier-Stokes equations and some useful estimates. Then in Sec. 1.3 we derive the estimates on the magnitude of the small eddies.

\(^{(1)}\)E. Titi, Article in preparation.
1.1 Preliminaries

As we recalled in the Introduction, the Navier-Stokes equations (0.1)(0.2) associated to one of the boundary conditions (0.3) is equivalent to an evolution equation

\[ \frac{du}{dt} + \nu Au + B(u) = f \]

in an appropriate Hilbert space \( H \). Here \( f \in H, \nu > 0, A \) is a linear self-adjoint positive operator with domain \( D(A) \subset H \), and whose inverse \( A^{-1} \) is compact; we have \( B(u) = B(u,u) \) where \( B(\cdot,\cdot) \) is a bilinear compact operator from \( D(A) \) (endowed with the norm \( |A\cdot| \) into \( H \); \( H \) is a Hilbert subspace of \( L^2(\Omega)^2 \). Its norm and scalar product are denoted \( |\cdot|, (\cdot,\cdot) \) as those of \( L^2(\Omega)^2 \) or \( L^2(\Omega) \); for the details see [11][12].

We recall that for \( u_0 \) given in \( H \) the initial value problem (1.1)(1.2):

\[ u(0) = u_0, \]

possesses a unique solution \( u \) defined for all \( t > 0 \) and such that

\[ u \in C(\mathbb{R}_+;H) \cap L^2(0,T;V), \forall T > 0; \]

here \( V = D(A^{1/2}) \) and the norm \( |A^{1/2}\cdot| = \|\cdot\| \) on \( V \) is equivalent to the \( L^2 \) norm of \( \text{grad } u \). If \( u_0 \in V \) then

\[ u \in C(\mathbb{R}_+;V) \cap L^2(0,T;D(A)), \forall T > 0. \]
In both cases (u_0 \in H or V), u(\cdot) is analytic in t with values in D(A); the domain of analyticity of u in the complex plane \( \mathbb{C}_t \) comprises a band around \( \mathbb{R}_t \) and is described in more details in the Appendix.

It is useful here to reproduce some a priori estimates verified by the solutions u of (1.1)(1.2). Before that we recall some inequalities (continuity properties) concerning B (see [7]): for every u, v, w \in D(A):

\[
|B(u,v)| \leq c_1 \left\{ |u|^{1/2} |u|^{1/2} |v|^{1/2} |Av| \right. \\
\left. + |u|^{1/2} |Au|^{1/2} |v| \right.
\]

\[
|(B(u,v),w)| \leq c_2 \left| u \right|^{1/2} |u|^{1/2} |v| |w|^{1/2},
\]

where \( c_1, c_2 \) like the quantities \( c_i, c_j \), which will appear subsequently, are dimensionless constraints\(^{(1)}\). Also we recall from [1][3] the inequality

\[
|\phi|_{L^\infty(\Omega)^2} \leq c_3 |\phi| \left( 1 + \log \frac{|A\phi|^2}{\lambda |\phi|^2} \right), \quad \forall \phi \in D(A),
\]

from which we deduce that

\[
|B(u,v)| \leq |(u \cdot \nabla)v| \leq \left\{ \begin{array}{ll} |u|_{L^\infty(\Omega)} & |\nabla v| \\
|u|_{L^\infty(\Omega)} & |\nabla |_{L^\infty(\Omega)} 
\end{array} \right.
\]

and using (1.7)

\(^{(1)}\) These constants can be absolute constants or they may depend on the shape of \( \Omega \); by this we mean that they are invariant by translation or homothety of \( \Omega \).
1.2 Behavior of Small Eddies

As mentioned in the Introduction we fix an integer \( m \in \mathbb{N} \) and denote by \( P = P_m \) the projector in \( H \) onto the space spanned by the first \( m \) eigenvectors of \( A, w_1, \ldots, w_m \); we set also \( Q = Q_m = I - P_m \), and for the sake of simplicity

\[
\lambda = \lambda_m, \quad \Lambda = \lambda_{m+1}.
\]

We write \( p = Pu, q = Qu; \) \( p \) represents a superposition of "large eddies" of size larger than \( \lambda_m^{-1/2} \), and \( q \) represents "small eddies" of size smaller than \( \lambda_{m+1}^{-1/2} \). By projecting (1.1) on \( PH \) and \( QH \) we find since \( PA = AP \) and \( QA = AQ \):

\[
\begin{align*}
\frac{dp}{dt} + \nu Ap + PB(p + q) &= Pf \\
\frac{dq}{dt} + \nu Aq + QB(p + q) &= Qf.
\end{align*}
\]

We take the scalar product of (1.10) with \( q \) in \( H \):

\[
\frac{1}{2} \frac{d}{dt} |q|^2 + \nu |q|^2 = (Qf, q) - (B(p + q), q).
\]

Thanks to the orthogonality property
(1.13) \[ (B(\phi,\psi),\psi) = 0, \quad \forall \phi, \psi \in V, \]

the right hand side of (1.11) reduces to

\[ (Qf,q) - (B(p,p),q) - (B(q,p),q). \]

Using (1.6) and Schwarz inequality we majorize it by

\[
|Qf| |q| + c_4 \| p \|^2 |q| \left( 1 + \log \frac{|Ap|^2}{\lambda_1 \| p \|^2} \right)^{1/2} + c_2 |q| \| q \| \| p \|
\]

\[ \leq (\text{since } \| p \| \leq \| u \|) \]

\[ \leq |Qf| |q| + c_4 \| p \|^2 |q| \left( 1 + \log \frac{|Ap|^2}{\lambda_1 \| p \|^2} \right)^{1/2} + c_2 \lambda^{-1/2} |q|^2 \| u \| \]

We denote now by \( M_0 \) (resp. \( M_1, M_2 \)) a bound of \( |u| \) (resp. \( \| u \|, |Au| \)), on the interval of time \( I = (t_0, \infty) \) under consideration

(1.14) \[ M_0 = \sup_{s \in I} |u(s)|, \quad M_1 = \sup_{s \in I} \| u(s) \|, \quad M_2 = \sup_{s \in I} |Au(s)|; \]

we observe that

\[ |Ap|^2 \leq \lambda_m \| p \|^2 = \lambda \| p \|^2 \]

and set

(1.15) \[ L = (1 + \log \frac{\lambda_{m+1}}{\lambda_1}). \]

We obtain
Hence, assuming that \( c_2 \lambda^{-1/2} M_1 \leq v \), i.e.,

\[
\lambda_{m+1} = \Lambda \geq \left( \frac{2c_2 M_1}{v} \right)^2
\]

(1.17) yields

\[
\frac{d}{dt} |q(t)|^2 + \frac{3v}{2} \|q\|^2 \leq \lambda^{-1/2} (|Qf| + (4M_1^2 L^{1/2}) \|q\|)
\]

\[
\leq \frac{v}{2} \|q\|^2 + \frac{1}{v\Lambda} (|Qf|^2 + (c_4 M_1^4 L)).
\]

(1.18)

\[
\frac{d}{dt} |q(t)|^2 + v\Lambda |q(t)|^2 \leq \frac{1}{v\Lambda} (|Qf|^2 + (c_4 M_1^4 L)).
\]

(1.19)

We infer easily from (1.19) that for \( t \geq t_1, t_1, t \in I \):

\[
|q(t)|^2 \leq |q(t_1)|^2 \exp(-v\Lambda(t-t_1)) + \frac{1}{v\Lambda^2} (|Qf|^2 + (c_4 M_1^4 L)).
\]

(1.20)

Before interpreting this inequality, we derive a similar inequality for the \((H^1)_V\) norm. Taking the scalar product of (1.11) with \(Aq\) in \(H\) we find

\[
\frac{1}{2} \frac{d}{dt} \|q\|^2 + v |Aq|^2 = (Qf, Aq) - (B(p + q), Aq).
\]

We expand and use Schwarz inequality and (1.6) - (1.8) to majorize the right
hand side of this equation by

\[ |Qf| |Aq| + c_2 \|p\|^{1/2} |Aq| (\|p\| + \|q\|) + c_4 |q|^{1/2} |Aq|^{3/2} (\|p\| + \|q\|) \]

\[ \leq (\text{with Young's inequality}) \]

\[ \leq \frac{\nu}{2} |Aq|^2 + \frac{1}{\nu} |Qf|^2 + \frac{c_2^2 M_1^4 L}{\nu} + \frac{c_4^2 M_2^4}{\nu^3} \]

Thus,

(1.21) \[ \frac{d}{dt} \|q\|^2 + \nu |Aq|^2 \leq c_3^-(\frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_2^4}{\nu^3}) \]

(1.22) \[ \frac{d}{dt} \|q\|^2 + \nu A \|q\|^2 \leq c_3^-(\frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_2^4}{\nu^3}) \]

and we conclude that

\[ \|q(t)\|^2 \leq \|q(t_1)\|^2 \exp(-\nu \Lambda(t - t_1)) \]

(1.23)

\[ + \frac{c_3^2}{\nu \Lambda} \left( \frac{1}{\nu} |Qf|^2 + \frac{M_1^4 L}{\nu} + \frac{M_2^4}{\nu^3} \right) \]

In (1.20) and (1.23) we can bound \(|q(t_1)|^2\) and \(\|q(t_1)\|^2\) by \(M_0^2\) and \(M_1^2\) respectively. Then after a time depending only on \(M_0\) (or \(M_1\)), \(\nu\) and \(\Lambda = \lambda_{m+1}\), the term involving \(t\) becomes negligible and we obtain
for $t$ large. Alternatively, denoting by $\kappa, \kappa_1, \kappa_2$, some quantities which depend only on the data $v, f, \Omega, \text{ and } M_0, M_1, M_2$, we rewrite (1.24) as

\begin{equation}
\begin{aligned}
|q(t)|^2 &\leq 2 \frac{2}{\nu L^2} (|Qf|^2 + c_2^2 M_1^4 L), \\
\|q(t)\|^2 &\leq \frac{2c_3}{\nu L} (|Qf|^2 + M_1^4 L + \frac{M_0^2 M_1^4}{\nu^2})
\end{aligned}
\end{equation}

(1.25)

Using also the results in the Appendix we conclude the following

**Theorem 1.1:** We assume that $m$ is sufficiently large so that (1.17) holds. Then for any orbit of (1.11), after a time $t_*$ which depends only on the initial value $u(0) = u_0$, the small eddies component of $u, q = Q_m u$, is small in the following sense

\begin{equation}
\begin{aligned}
|q(t)| &\leq \kappa_0 L^{1/2}\delta,  \\
\|q(t)\| &\leq \kappa_1 L^{1/2}\delta^{1/2}, \\
|q(t)| &\leq \kappa_2 L^{1/2}
\end{aligned}
\end{equation}

(1.26)

The first two inequalities in (1.26) follow from (1.25); the third one follows
from (1.25) and the analog of (A.15) for \( q \). The fourth inequality is obtained by writing

\[
\nu Aq = Qf - q - QB(p + q)
\]

\[
|Aq| \leq \frac{1}{\nu} |Qf| + \frac{1}{\nu} |q| + \frac{1}{\nu} |QB(p + q)|
\]

and utilizing (1.5), (1.6), (1.8).

In Sec. 1.3 hereafter we intend to provide a more explicit form of the constants \( \kappa \) in the case of space periodic flows.

1.3 The Space Periodic Case

We first review the well-known a priori estimates on the solutions of (1.1). This will yield more explicit expressions for \( M_0, M_1, M_2 \).

We take the scalar product of (1.1) with \( u \) in \( H \); using the orthogonality property (1.13) we obtain

\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \nu \| u \|^2 = (f,u) \leq |f| |u|
\]

\[
\leq \lambda_1^{-1/2} |f| \| u \|
\]

\[
\leq \frac{\nu}{2} \| u \|^2 + \frac{1}{2\nu\lambda_1} |f|^2
\]

\[\text{(1.26)}\]

(1) Note that \( q \) is analytic in the same region of the complex plane as \( u \).
**Theorem 2.1:** For $t$ sufficiently large, $t \geq t_*$, any orbit of (1.1) remains at a distance in $H$ of $P_\mu H$ of the order of $\kappa L^{1/2}\delta$ and at a distance in $H$ of $M_0$ of the order of $K L^{3/2}$. In the norm of $V$, the corresponding distances are of order $\kappa \delta^{1/2} L^{1/2}$ and $\kappa L\delta$; the constants $\kappa$ depend on the data $v, \lambda_1, |f|$, and $t_*$ depends on these quantities and on $R_0$, when $|u(0)| \leq R_0$.

3. **A NONCONSTRUCTIVE RESULT**

Our aim in this last section is to exhibit a manifold $\Sigma$ which is Lipschitz, has finite dimension, and captures the solutions of (1.1) in a much narrower neighborhood than $M_0$ does. However, the existence of $\Sigma$ is proved in a nonconstructive way, by opposition with the very simple and explicit equation (2.2) available for $M_0$. Secs. 3.1 and 3.2 provide preliminary results and Sec. 3.3 contains the main one.

3.1 **Quotient of Norms**

We consider two solutions $u, v$ of (1.1) and set $w = u - v$:

\[
\begin{align*}
(3.1) & \quad \frac{du}{dt} + v Au + B(u) = f, \quad u(0) = u_0, \\
(3.2) & \quad \frac{dv}{dt} + v Av + B(v) = f, \quad v(0) = v_0, \\
(3.3) & \quad \frac{dw}{dt} + v Aw + B(u,w) + B(w,v) = 0.
\end{align*}
\]

Let $\sigma$ denote the quotient of norms $\frac{\|w\|^2}{|w|^2}$; then
\[
\frac{d\sigma}{dt} = \frac{2((w^*, w)) - 2\|w\|^2}{|w|^4} (w^*, w) = \frac{2}{|w|^2} (w^*, Aw - \sigma w)
\]

\[
= -\frac{2}{|w|^2} (\nu Aw + B(u, w) + B(w, v), Aw - \sigma w).
\]

Since \((Aw, Aw - \sigma w) = |Aw - \sigma w|^2\), we conclude, using (1.5), that

\[
\frac{d\sigma}{dt} + \frac{2\nu}{|w|^2} |Aw - \sigma w|^2 =
\]

\[
= -\frac{2}{|w|^2} (B(u, w) + B(w, v), Aw - \sigma w)
\]

\[
\leq \frac{2}{|w|^2} |Aw - \sigma w| (|B(u, w)| + |B(w, v)|)
\]

\[
\leq \frac{2c_1}{|w|^2} |Aw - \sigma w|^2 (|u|^{1/2} |Au|^{1/2} \|w\| + |w|^{1/2} |Aw|^{1/2} \|v\|)
\]

\[
\leq \frac{\nu}{|w|^2} |Aw - \sigma w|^2 + \frac{2c_1^2}{\nu} (|u| |Au| + \|v\| |Av| \lambda_1^{-1/2})\sigma.
\]

Hence

(3.4) \[
\frac{d\sigma}{dt} + \frac{\nu}{|w|^2} |Aw - \sigma w|^2 \leq \rho \sigma
\]

where

(3.5) \[
\rho = \rho_u + \rho_v, \quad \rho_u = \frac{2c_1^2}{\nu \lambda_1^{1/2}} \|u\| |Au|.
\]

By integration of the differential inequality \(\sigma \leq \rho \sigma\), we find that for

\(t_1 < t < \tau < t_1 + T\)
Now we estimate the integral of $\rho$ in terms of the data; as in (1.16) we assume that on the interval of time under consideration

\begin{equation}
\|u(t)\| \leq M_1, \quad \|v(t)\| \leq M_1.
\end{equation}

With an appropriate value of $M_1$ (3.7) will be valid on some finite interval of time $[0,T]$, or on some interval of time $(t_0,\infty)$, once the orbits have entered the absorbing set.

We have

\[
\int_t^\tau \rho_u \, ds \leq \frac{2c_1}{v\lambda_1} \int_t^\tau \|u\| |Au| \, ds \leq \frac{2c_1^2}{v\lambda_1} M_1 (\tau - t)^{1/2} (\int_t^\tau |Au|^2 \, ds)^{1/2}.
\]

An estimate on $Au$ is obtained by taking the scalar product of (3.1) with $Au$ in $H$:

\[
\frac{d}{dt} \|u\|^2 + 2\nu |Au|^2 = -2(B(u),Au) - 2(f,Au)
\]

\[
\leq 2|B(u)| |Au| + 2|f| |Au|
\]

\[
\leq (\text{with (1.5)})
\]

\[
\leq 2c_1 |u|^{1/2} \|u\| |Au|^{1/2} + 2|f| |Au|.
\]
\[ \leq \nu |Au|^2 + \frac{c^*_1}{\nu^3} |u|^2 \|u\|^4 + \frac{2}{\nu} |f|^2 \]

(3.8) \[
\frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq \frac{1}{\nu} |f|^2 + \frac{c^*_1}{\nu^3 \lambda_1} M_1^6.
\]

Thus,

\[
\int_{t_1}^{t_1+\tau} |Au|^2 ds \leq \frac{\|u(t_1)\|^2}{\nu} + \frac{\tau}{\nu^2} \left( |f|^2 + \frac{c^*_1}{\nu^2 \lambda_1} M_1^6 \right)
\]

(3.9) \[
\int_{t_1}^{t_1+\tau} |Au|^2 ds \leq \frac{M_1^2}{\nu} + \frac{\tau |f|^2}{\nu^2} + \frac{c^*_1 \tau}{\nu^4 \lambda_1} M_1^6
\]

and

\[
\int_{t}^{\tau} \rho u ds \leq \frac{1}{2} (\tau - t)^{1/2} \kappa_3
\]

(3.10) \[
\kappa_3 = \frac{c^*_1}{\nu \lambda_1^{1/2}} M_1 \left( \frac{M_1^2}{\nu} \|f\|^2 + \frac{T M_1^6}{\nu^4 \lambda_1} \right)^{1/2}
\]

Since the estimates on \( \nu \) and \( \rho v \) are the same, we have

(3.11) \[
\int_{t}^{\tau} \rho ds \leq (\tau - t)^{1/2} \kappa_3.
\]

3.2 The squeezing property

The squeezing property is an important property of the solutions of the Navier-Stokes equations which has been introduced in [6]. A stronger form of it, called the strong squeezing property or the cone property was proven in [4] for some other, more strongly dissipative equations. For the two dimensional Navier-Stokes equations, we derive here a form of the squeezing property sharper than in [6].
We take the scalar product of (3.3) with \( w \) in \( H \) and thanks to (1.13), (1.16) we find

\[
\frac{d}{dt} |w|^2 + 2v\|w\|^2 = -2b(w, v, w)
\]

\[
\leq 2c_2 |w| \|w\| \|v\|
\]

\[
\leq v\|w\|^2 + \frac{c_2^2}{v} |w|^2 \|v\|^2
\]

\[
\leq v\|w\|^2 + \frac{c_2^2}{v} M_1^2 |w|^2
\]

\[(3.12) \quad \frac{d}{dt} |w|^2 + \left( v \frac{\|w\|^2}{|w|^2} - \frac{c_2^2}{v} M_1^2 \right) |w|^2 \leq 0.\]

We consider \( t_0, t, 0 < t < t_0 < T \) and write, using (3.6)(3.11)

\[(3.13) \quad \gamma_0 = \frac{\|w(t_0)\|^2}{|w(t_0)|^2} \leq \exp(k^3(t_0 - t)^{1/2}) \frac{\|w(t)\|^2}{|w(t)|^2}.\]

Thus,

\[(3.14) \quad \frac{d}{dt} |w|^2 + \left( v \gamma_0 \exp(-k^3 t_0^{1/2}) - \frac{c_2^2}{v} M_1^2 \right) |w|^2 \leq 0\]

and by integration

\[(3.15) \quad |w(t_0)|^2 \leq |w(0)|^2 \exp(-v \gamma_0 t_0 \exp(-k_3 t_0^{1/2}) + \frac{c_2^2}{v} M_1^2 t_0).\]

Now if \( |Q_m w(t_0)| > |P_m w(t_0)| \), we write
and
\[ \frac{1}{2} \frac{|Q_m w(t_0)|^2}{|Q_m w(t_0)|^2} \geq \frac{\lambda_{m+1}}{2} \]

and
\[ |w(t_0)|^2 \leq |w(0)|^2 \exp(-\nu \lambda_{m+1} \kappa_5 t_0 + \kappa_4 t_0) \]
(3.16)
\[ \kappa_4 = \frac{c_2^2}{\nu} M_1^2, \quad \kappa_5 = \frac{1}{2} \exp(-\kappa_3 t_0^{1/2}). \]

Of course the interval \((0, t_0)\) can be replaced by any interval \((t_1, t_1 + t_0)\) on which the bound (3.7) is valid.

In conclusion (this is the squeezing property), whenever (3.7) is valid on some interval \((t_1, t_1 + t_0)\), then \(w = u - v\) satisfies one of the following conditions:

(3.17a) \[ |Q_m w(t_0 + t_1)| \leq |P_m w(t_0 + t_1)| \]

or

(3.17b) \[ |w(t_0 + t_1)|^2 \leq |w(t_1)|^2 \exp(-\nu \lambda_{m+1} \kappa_5 t_0 + \kappa_4). \]

Since \(\kappa_4, \kappa_5\) are independent of \(m\), the exponential term in (3.17b) can be made arbitrarily small by choosing \(m\) sufficiently large; we will take advantage of this remark in Sec. 3.3.

Of slightly more explicit form of \(\kappa_4, \kappa_5\) can be derived by using the Grashof number \(G = |t|/\nu^2 \lambda_1\) and the Reynolds type number \(R_n = M_1/\nu \lambda_1^{1/2}\). We find \((\tau = t_0)\):
\[ K_3 = c_2 R_n (\nu\lambda_1)^{1/2} (R_n^2 + t_0 \nu\lambda_1 G^2 + t_0 \nu\lambda_1 R_n^6)^{1/2} \]

(3.18) \[ \kappa_4 = c_2 R_n^2 (\nu\lambda_1) \]

\[ \kappa_5 = \frac{1}{2} \exp(-c_3 R_n (\nu\lambda_1 t_0)^{1/2} (R_n^2 + t_0 \nu\lambda_1 G^2 + t_0 \nu\lambda_1 R_n^6)^{1/2}) \].

In the space periodic case we have seen that, for large times, we can take \( M_1 = (2 |f| G)^{1/2} \). Then \( R_n = \sqrt{2} G \) and the above quantities become

\[ \kappa_3 = c_3 (\nu\lambda_1)^{1/2} (G^4 + t_0 \nu\lambda_1 G^4 + t_0 \nu\lambda_1 G^8)^{1/2} \]

(3.19) \[ \kappa_4 = 2c_2 (\nu\lambda_1) G^2 \]

\[ \kappa_5 = \frac{1}{2} \exp(-c_3 (\nu\lambda_1 t_0)^{1/2} (G^4 + t_0 \nu\lambda_1 G^4 + t_0 \nu\lambda_1 G^8)^{1/2}) \].

3.3 The Approximate Manifold

We denote by \( S(t), t > 0 \) the operator in \( H: u_0 + u(t) \), where \( u(\cdot) \) is the unique solution of (1.1) satisfying \( u(0) = u_0 \). The operators \( S(t), t \geq 0 \), form a semigroup in \( H \).

The squeezing property tells us that if \( u(\cdot), v(0) \) are two solutions of (1.1) lying in the ball \( \{ \| \phi \| \leq M_1 \} \), for \( 0 \leq t \leq T \), then at each time \( t \in [0, T] \) and for every \( m \in \mathbb{N} \), we have either

\[ |Q_m (S(t)u_0 - S(t)v_0)| \leq |P_m (S(t)u_0 - S(t)v_0)| \]

or
\[ |S(t)u_0 - S(t)v_0| \leq |u_0 - v_0| \exp\frac{1}{2} (-\nu\lambda m+1\kappa_5 t_0 + \kappa_4 t_0) \]

\[ \kappa_4, \kappa_5 \text{ as above.} \]

Now we choose \( t_0 \in [0,T], \ m \in \mathbb{N} \) and consider a subset \( \sum = \sum (m) \) of \( S(t_0) \{ u_0 \in V, \| u_0 \| \leq M_1 \} \)

which is maximal under the property

\begin{equation}
(3.20) \quad |Q_m (u - v)| \leq |P_m (u - v)|.
\end{equation}

By this we mean that if \( u \in \sum (m) \) then

\[ \{ v \in V, \ v \text{ satisfies (3.20)} \} \supseteq (m). \]

The existence of such a maximal set is easy.

We then apply the squeezing property: whenever \( \| u(s) \| \leq M_1 \), we see that \( S(t_0)u(s) = u(t_0 + s) \) either belongs to \( \sum (m) \), i.e.,

\[ |Q_m (S(t_0) u(s) - S(t_0)\phi)| \leq |P_m (S(t_0)u(s) - S(t_0)\phi)|, \]

for some \( \phi \in V \) such that \( \| \phi \| \leq M_1 \) and \( S(t_0)\phi \in \sum (m) \) or, if not, then for every such \( \phi \)

\[ |S(t_0)u(s) - S(t_0)\phi|^2 \leq |u(s) - \phi|^2 \exp(-\nu\lambda m+1\kappa_5 t_0 + \kappa_4 t_0) \]
In all cases the distance of \( S(t_0)u(s) \) to \( \Sigma (m) \) is bounded by

\[
\frac{2M_1}{\lambda_1^{1/2}} \exp\left( \frac{t_0}{\lambda_1} (\kappa_4 - \nu \lambda_{m+1} \kappa_5) \right).
\]

We can choose \( t_0 = (\nu \lambda_1)^{-1} \) and the bound becomes

\[
\frac{2M_1}{\lambda_1^{1/2}} \exp\left( -\frac{\kappa_5 \lambda_{m+1}}{4 \lambda_1} \right)
\]

provided that

\[
(3.21) \quad \frac{\lambda_{m+1}}{\lambda_1} \geq \frac{2\kappa_4}{\kappa_5 \nu \lambda_1}.
\]

By translation in time \( (t + t - t_*) \), we conclude that once the orbit \( u \) has entered the absorbing set \( \{ |\phi| \leq M_1 \} \), which happens for \( t \geq t_* = t_*(R_0) \) (for \( |u(0)| \leq R_0 \)), the distance of \( S(t)u_0 \) to \( \Sigma (m) \) is bounded by a given quantity \( E \),

\[
(3.22) \quad \text{dist}_H(S(t)u_0, \Sigma (m)) \leq E
\]

provided \( t \geq t_* + (\nu \lambda_1)^{-1} \), and

\[
\exp \left( -\frac{\kappa_5 \lambda_{m+1}}{4 \lambda_1} \right) \leq E,
\]

i.e.,

\[
(3.23) \quad \frac{\lambda_{m+1}}{\lambda_1} \geq -\frac{4}{\kappa_5} \log E.
\]
By definition the set \( \mathcal{S}(m) \) enjoys the property that

\[
|Q_m(u - v)| \leq |P_m(u - v)|, \forall u, v \in M(m).
\]

Hence, \( \mathcal{S}(m) \) is the graph of a Lipschitz function

\[
\psi: P_m \mathcal{S}(m) + Q_m H
\]

\[
|\psi(P_m u) - \psi(P_m v)| \leq |P_m u - P_m v|, \forall P_m u, P_m v \in P_m \mathcal{S}(m).
\]

By the Kirszbaum extension Theorem [16] \( \psi \) can be extended as a Lipschitz function (with the same constant) from \( P_m H \) into \( Q_m H \), that we still denote by \( \psi \). Now \( \psi \) is defined from \( P_m H \) into \( Q_m H \), and its graph is a Lipschitz manifold above all of \( P_m H \).

In conclusion we have proved the following theorem

**Theorem 3.1:** If \( m \) is sufficiently large so that (3.21) is satisfied\(^{(1)}\), then there exists a Lipschitz manifold \( \mathcal{S}(m) \) of dimension \( m \), which enjoys the following property: for an solution \( u(\cdot) \) of (1.1), for \( t \) sufficiently large (\( t \geq t_{**}(R_0, \nu, f, \Omega) \), for \( |u_0| \leq R_0 \), the distance in \( H \) of \( u(t) \) to \( \mathcal{S}(m) \) is majorized by

\[
\frac{2M_1}{\lambda_{1/2}} \exp(-\frac{\kappa_5}{4} \frac{\lambda_{m+1}}{\lambda_1}).
\]

\( ^{(1)} \kappa_5 \) as above with \( t_0 = (\nu \lambda_1)^{-1} \), and \( M_1 \) the radius of an absorbing set in \( V \) for (1.1).
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APPENDIX

Estimates in the Complex Time Plane

It was proved in [6] (see also [12]) that the solutions to the Navier-Stokes equations are analytic in time; we want to show how one can then use Cauchy's formula to get a priori estimates on the time derivatives of the solutions. The main point in the proof is to determine the width of the band of analyticity of the solution around the real axis \( \mathbb{R} \); this will follow as in [6,12] from a priori estimates on the solution in the complex plan.

The complex time is denoted \( \zeta = se^{i\theta} \); \( \mathbb{H}, \mathbb{V}, \mathbb{W}(A) \) are the complexified spaces of \( H, V, D(A) \); \( A, B \) are extended as linear and bilinear operators respectively from \( \mathbb{W}(A) \) into \( \mathbb{H} \):

\[
(A.1) \quad A(u_1 + iu_2) = Au_1 + iAu_2,
\]

\[
(A.2) \quad B(u_1 + iu_2, v_1 + iv_2) = B(u_1, v_1) - B(u_2, v_2) + i[B(u_2, v_1) + B(u_1, v_2)]
\]

\( \Psi u = u_1 + iu_2, \quad v = v_1 + iv_2 \in \mathbb{W}(A) \). The Navier-Stokes equation (1.1) becomes (\( u = u(\zeta) \)):

\[
(A.3) \quad \frac{du}{d\zeta} + vAu + B(u) = f
\]

\[
(A.4) \quad u(0) = u_0^*.
\]
Assuming that \( u_0 \in L^\infty(\mathbb{R}_+; \mathbb{M}) \) and as in (1.14), we denote by \( M_0, M_1 \), the supremum of \( |u(t)| \) and \( \|u(t)\| \), \( t \in \mathbb{R}_+ \). We take the scalar product in \( \mathbb{M} \) of (A.3) with \( Au \); we multiply the resulting equation by \( e^{i\theta} \) and take its real part. This yields

\[
\frac{1}{2} \frac{d}{ds} \|u(e^{i\theta})\|^2 + \nu \cos \theta \|Au(e^{i\theta})\|^2 =
\]

(A.5)

\[
= -\text{Re} \ e^{i\theta} (B(u), Au) - \text{Re} \ e^{i\theta} (f, Au)
\]

\[
\leq |(B(u), Au)| + |f| \|Au\|.
\]

We expand by bilinearity (using (A.2)) and bound the resulting expressions with the help of (1.8):

\[
|(B(u), Au)| \leq c \|u\|^2 (1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2})^{1/2} \|Au\|.
\]

Also

\[
|f| \|Au\| \leq \frac{\nu \cos \theta}{2} \|Au\|^2 + \frac{|f|^2}{2 \nu \cos \theta}.
\]

Hence (with \( u = u(se^{i\theta}) \)):

(A.6)

\[
\frac{d}{ds} \|u\|^2 + \nu \cos \theta |Au|^2 \leq
\]

\[
\leq \frac{|f|^2}{\nu \cos \theta} + c_5 \|u\|^2 |Au| (1 + \log \frac{|Au|^2}{\lambda_1 \|u\|^2})^{1/2}.
\]

We write \( z = \frac{|Au|}{\lambda_1^{1/2} \|u\|} > 1 \) and consider the function
\[ z + \phi(z) = -\frac{\lambda_1 v \cos \theta}{2} z^2 + c_5 \| u \| \lambda^{1/2} z (1 + \log z^2)^{1/2}. \]

By elementary computations\(^{(1)}\)

\[ \phi(z) \leq \frac{c_5^2 \| u \|^2}{2 v \cos \theta} \log \left( \frac{4c_5^2 \| u \|^2}{\lambda_1 v \cos \theta} \right), \text{ for } z \geq 1, \]

and (A.6) yields

\[ \frac{d}{ds} \| u \|^2 + \frac{v \cos \theta}{2} |Au|^2 \leq \frac{|f|^2}{v \cos \theta} + \frac{c_5^2 \| u \|^4}{v \cos \theta} (1 + \log \frac{4c_5^2 \| u \|^2}{\lambda_1 v \cos \theta}). \]

Setting \( y(s) = \frac{(1 + 4c_5^2)}{\lambda_1 v \cos \theta} (|f| + \| u(\theta \omega) \|^2) \) we infer from (A.8) that

\[ \frac{dy}{ds} \leq c_1^1 \lambda_1 v \cos \theta y^2 \log y, \]

where \( c_1^1 \) is an appropriate nondimensional constant. As long as

\[ y(s) \leq 2y_0 = 2y(0), \]

we have

\(^{(1)}\)Looking for the maximum of \(-a^2 z^2 + \beta^2 (1 + \log z^2)\), we find

\[ \beta^2 (1 + \log z^2) \leq a^2 z^2 + \beta^2 \log \frac{\beta^2}{a^2} \]

\[ z \beta (1 + \log z^2)^{1/2} \leq a z^2 + \beta z (\log \frac{\beta^2}{a^2})^{1/2} \]

\[ \leq 2az^2 + \frac{1}{4} \frac{\beta^2}{a} (\log \frac{\beta^2}{a^2}), \]

We then choose \( a = \frac{\lambda_1 v \cos \theta}{4} \), \( \beta = c_5 \| u \| \lambda^{1/2} \).
\[ y' \leq c_1 \lambda_1 v \cos \theta y^2 \log(2y_0) \]

\[ y(s) \leq \frac{y_0}{1 - c_1 \lambda_1 v \cos \theta y \log(2y_0)s}, \]

and this is indeed \( \leq 2y_0 \) as long as \( s \leq T_* \):

\[ T_* = \frac{3}{2c_1 \lambda_1 v \cos \theta y_0 \log(2y_0)}. \]

For \( \| u_0 \| \leq M_1 \), we replace \( T_* \) by

\[ (A.9) \quad T_*(M_1) = \frac{3}{2c_1 \lambda_1 v \cos \theta \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v \cos^2 \theta} \right) \log 2 \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v \cos^2 \theta} \right)} \]

Thus

\[ (A.10) \quad \| u(se^{i\theta}) \|^2 \leq 2(|f| + \| u_0 \|^2) \leq 2(|f| + M_1^2) \]

for

\[ 0 \leq s \leq \frac{3 \cos \theta}{2c_1 \lambda_1 v (G + \frac{M_1^2}{\lambda_1 v^2}) + \log 2 \left( \frac{G}{\cos^2 \theta} + \frac{M_1^2}{\lambda_1 v^2 \cos^2 \theta} \right)} \]

and in particular for
when \( \cos^2 \theta \geq \frac{1}{2} \).

Following the method developed in [6] we conclude that the solution \( u \) of (A.3) (or (1.a)) is analytic in the region

\[
\Delta(u_0) = \{ se^{i\theta}, s < \alpha \cos \theta, \cos \theta > \frac{\sqrt{2}}{2} \}
\]

\[
\alpha = \frac{3}{2} \frac{3 \cos \theta}{2 c_1^2 \lambda_1 \nu(G + \frac{M_1^2}{\lambda_1^2}) + \log 4(G + \frac{M_1^2}{\lambda_1^2})}
\]

which comprises the regions

\[
|\text{Im} \zeta| \leq \text{Re} \zeta, \quad 0 < \text{Re} \zeta \leq \frac{\alpha}{2}
\]

and

\[
|\text{Im} \zeta| \leq \frac{\alpha}{2}, \quad \text{Re} \zeta \geq \frac{\alpha}{2}
\]

At any point \( t \in \mathbb{R}, t > \alpha \), we can apply Cauchy's formula to the circle \( \Gamma \) centered at \( t \) of radius \( \alpha/4 \):

\[
\frac{d^k u(t)}{dt^k} = \frac{k!}{2\pi i} \int_{\Gamma} \frac{u(\zeta)}{(t-\zeta)^{k+1}} \, d\zeta.
\]

Thus,

\[
\sup_{t \geq \alpha} \left| \frac{d^k u(t)}{dt^k} \right| \leq \frac{4^k}{\alpha} \frac{k!}{k!} M_0
\]
Explicit values of $M_0$ and $M_1$ were derived in (1.36) for the two dimensional space periodic case: $M_1 = (2|f|G)^{1/2}(t \geq t_2)$. This yields (assuming $G \geq 1$):

$$
\alpha = \frac{3}{2c_1^2 \lambda_1 \nu (G + 2G^2) \log 4(G + 2G^2)}
$$

and we deduce from (A.15), (A.16) that for $t$ sufficiently large$^{(1)}$

$$
|\frac{d^k u(t)}{dt^k}| \leq c \frac{|f|^{1/2}}{\lambda_1^{1/2}} (|f| \lambda_1)^{k/2} (G^2 \log G)^k
$$

In particular ($k = 1$):

$$
|\frac{du(t)}{dt}| \leq c |f| G^2 \log G
$$

$$
|\frac{d^k u(t)}{dt^k}| \leq c |f|^{1/2} (|f| \lambda_1)^{1/2} G^2 \log G, \quad t \geq T_*.\]

This produces an interesting bound on $|A_u(t)|$ for $t$ large:

---

$^{(1)}$This means as in Theorem 1.1 and elsewhere $t \geq T_*(R_0, \nu, \lambda_1, |f|)$, for $|u_0| \leq R_0$.\]
\[ \nabla Au = f - B(u) - u^- \]

\[ |Au| \leq \frac{1}{\nu} |f| + \frac{c_1}{\nu} |u|^{1/2} \|u\|_1 |Au|^{1/2} + \frac{1}{\nu} |u^-| \]

\[ |Au| \leq \frac{2}{\nu} |f| + \frac{c_1^2}{\nu} |u|^{1/2} \|u\|_1^2 + \frac{2}{\nu} |u^-| \]

\[ \leq c(|f| \lambda_1^{1/2} (G^{1/2} + G + G^{5/2} \log G)) \]

(A.20) \[ |Au(t)| \leq c(|f| \lambda_1^{1/2} G^{5/2} \log G), \quad \text{for } t \geq T_* \]
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Our aim in this article is to present some results concerning the interaction of small and large eddies in two dimensional turbulent flows. We show that the amplitude of small structures decays exponentially to a small value and we infer from this a simplified interaction law of small and large eddies. Beside their intrinsic interest for the understanding of the physics of turbulence, these results lead to new numerical schemes which will be studied in a separate work.