Helical Gears With Circular Arc Teeth: Generation, Geometry, Precision and Adjustment to Errors, Computer Aided Simulation of Conditions of Meshing, and Bearing Contact

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1. Introduction and Principles of Generation

Circular arc helical gears (Wildhaber - Novikov gears) have the following advantages over involute helical gears: (a) there is reduced contacting stresses and (b) the conditions of lubrication are better. The disadvantages of the circular arc helical gears are: (a) higher bending stresses, (b) the sensitivity to the change of center distance and (c) a more complicated shape of the tool. The bending stresses can be reduced by appropriate proportions of tooth elements. The effect of dislocation of the bearing contact due to the change of the gear center distance can be reduced by appropriate relations between the principal curvatures of the gears and may even be compensated technologically. Circular arc gears can be successfully applied in gear trains with limited weight. The success of Westland Helicopter Co. which designed and manufactured these gears is the best evidence of this statement.

The main advantages of the discussed gears - reduced contacting stresses and improved conditions of lubrication - are the result of special conditions of the contact of gear tooth
surfaces and their meshing. Surfaces of the gear teeth contact each other at a point at every instant, instead of a line; the relations between the principal curvatures of surfaces are free of the limitations which exist for gears having line contact of the surfaces; the point of contact (it is the center of the contacting ellipse) moves over the surface along a helix, and it is due to this motion of the contact point and a favorable orientation of the contacting ellipse that the conditions of lubrication are improved substantially.

Consider that shapes \( \Sigma_1 \) and \( \Sigma_2 \) are in contact at point \( M \) (Fig. 1.1); \( \Sigma_1 \) and \( \Sigma_2 \) are the cross-sections of gear tooth surfaces; the instantaneous angular velocity ratio is given by

\[
\frac{m_{12}}{\omega(1)} = \frac{0_2}{0_1} \frac{I}{I}
\]

It is not excluded that \( m_{12} \) is not constant, thus \( m_{12} = f(\phi_1) \) where \( \phi_1 \) is the angle of rotation of gear 1. The derivative \( \frac{df}{d\phi_1} \) is equal to zero if and only if the following equation is satisfied

\[
\frac{\rho_2 - \rho_1}{(\rho_1 - \lambda)(\rho_2 - \lambda)} = \frac{\Delta \rho}{\rho_1^2 - \lambda(\rho_1 + \Delta \rho) + \lambda^2} = \frac{r_1 + r_2}{r_1 r_2 \sin \psi_c} 
\]

Here: \( \rho_2 = C_2 M, \rho_1 = C_1 M \) where \( C_1 \) and \( C_2 \) are the centers of curvatures of shapes \( \Sigma_1 \) and \( \Sigma_2 \), respectively; \( \Delta \rho = \rho_2 - \rho_1 \);
\( \lambda = IM; r_1 = 0_1 I \) and \( r_2 = 0_2 I; \psi_c \) is the angle formed by the shapes normal, \( n \), and line \( m-m \). Equation (1.1) results in that the difference of curvature radii, \( \Delta \rho = \rho_2 - \rho_1 \), depends on \( r_1, r_2, \psi_c, \lambda \), and \( \rho_1 \). Thus, \( \Delta \rho \) is not a free design parameter and we
Fig. 1.1
cannot substantially reduce the contacting stress by minimizing $\Delta \rho$. This obstacle can be overcome if the gears are designed as helical gears and the gear tooth surfaces are in point contact.

Consider that the difference of curvature radii, $\Delta \rho$, provides optimal conditions for contacting stresses, but does not satisfy equation (1.1). If the gear tooth surfaces would be designed as spur or helical gears, whose surfaces are in line contact, then such gears would not be able to transform rotation with the constant angular velocity. But if the gears are designed as helical gears whose surfaces are in point contact, then both requirements - the reduction of contacting stresses and the constancy of gear ratio - can be achieved.

Fig. 1.2a shows a gear tooth surface of a helical gear. This surface may be generated by a planar curve $\Sigma$ in its screw motion about axis 0-0.

Consider two cross-section of the gear tooth surface formed by cutting the surface by two planes, $P_1$ and $P_2$ (Fig. 1.2a,b). Shapes $\Sigma^{(1)}$ and $\Sigma^{(2)}$ lie in planes $P_1$ and $P_2$, respectively. The location and orientation of $\Sigma^{(2)}$ with respect to $\Sigma^{(1)}$ is determined by the axial displacement and rotation of $\Sigma^{(1)}$ in its screw motion while it generates the screw surface of the gear. We assume that in such a screw motion of $\Sigma^{(1)}$, the gear is at rest.

Now, consider that two helical gears are in mesh and their screw surfaces contact each other at point $M$ initially (Fig. 1.2b). The shapes of gears 1 and 2 have a common normal $n^*$ at $M$, which passes through point I - the point of intersection of the
Fig. 1.2
instantaneous axis of rotation I - I with the plane $P_1$. Shape $\Sigma^{(1)}$ of the screw surface of gear 1 will come in contact with the corresponding shape of gear 2 if the gears will be rotated through certain angles about their axes. For instance, shape $\Sigma^{(2)}$ of gear 1 will come in tangency with the mating shape of gear 2 if shape $\Sigma^{(2)}$ takes the position of $\Sigma^{(3)}$. This position can be reached if the gear with its screw surface (thus with the shapes $\Sigma^{(1)}$ and $\Sigma^{(2)}$) is rotated about axis 0-0. Shapes $\Sigma$ and $\Sigma$ have the same orientation but lie in different plane $P_1$ and $P_2$, respectively. In the process of meshing of helical gears with the type of point contact described above, the gear tooth surfaces contact each other at every instant at a point along the line ML, which is parallel to the axes of gear rotations. Line ML is the line of action of gear tooth surfaces.

It is known that a screw surface of a helical gear may be generated by a cylindrical surface $\Sigma_c$ whose generatrix are parallel to plane $\pi$ and form a certain angle with the gear axis (Fig 1.3 a). Plane $\pi$ is the tangent plane to the gear cylinder of radius $r$. While the generating surface $\Sigma_c$ translates with plane $\pi$, with velocity $v$, the gear rotates with angular velocity $\omega$, where $\omega = v \div r$. Plane $\pi$ and the cylinder of radius $r$ are the axodes.

To generate gears having point contact of their surfaces, we have to use two generating cylindrical surfaces, $\Sigma^{(1)}_c$ and $\Sigma^{(2)}_c$ (Fig. 1.3 a), which contact each other along a straight line. While plane $\pi$ translates with velocity $v$, the gears rotate with angular velocities, $\omega^{(1)}$ and $\omega^{(2)}$ respectively (Fig. 1.3 b).
Fig. 1.3
We may imagine that surface \( \Sigma_{C1}^{(1)} \) generates the screw surface of gear 1, and \( \Sigma_{C2}^{(2)} \) generates the screw surface \( \Sigma_{2} \) of gear 2. The surfaces of helical gears, \( \Sigma_{1} \) and \( \Sigma_{2} \), will be in point contact and their line of action will be the line \( ML \) (Fig 1.2 b).

We have to emphasize that surfaces \( \Sigma_{Ci}^{(i)} \) and \( \Sigma_{i} \) (\( i = 1, 2 \)) are in line contact and \( \Sigma_{i} \) is generated as an envelope of the family surfaces \( \Sigma_{Ci}^{(i)} \). Using two different generating surfaces, \( \Sigma_{C1}^{(1)} \) and \( \Sigma_{C2}^{(2)} \), we may generate screw surfaces for both helical gears with a point contact of the gear tooth surfaces, and overcome the limitation of the difference of the curvatures determined by equation (1.1). The described method of generation is the key to the problem of synthesis of helical gears with reduced contacting stresses.

2. Generating Surfaces

Fig. 2.1 shows the normal section of the space of rack cutter \( F \) which generates the tooth of gear 1. The shapes of the rack cutter for each of its sides represent two circular arcs centered at \( C_{F} \) and \( C_{F}^{(F)} \), respectively. The circular arc of radius \( r_{F}^{(F)} \) with center at \( C_{F}^{(F)} \) generates the fillet surface of the gear 1 while the circular arc of radius \( r_{F} \) with center at \( C_{F} \) generates the working surface. Point \( O_{a}^{(F)} \) lies in plane \( \pi \) (Fig. 1.3).

Fig. 2.2 shows the normal section of the tooth of the rack cutter \( P \) which generates the space of gear 2. The shape of the rack cutter for each side represents two circular arcs centered
Fig. 2.1
The circular arc of radius \( \rho_p \) with center at \( C_p \) generates the fillet surface of gear 2 while the circular arc of radius \( \rho_p \) with center at \( C_p \) generates the working surface.

The shapes of the mating rack cutters do not coincide; rather they are in tangency at points \( M_1 \) and \( M_2 \).

We may represent all four circular arcs in the coordinate system \( S_a(x_a, y_a, z_a) \) by the same equations

\[
x_a^{(i)} = \rho_i \sin \theta_i - b_i, \quad y_a^{(i)} = -(\rho_i \cos \theta_i - a_i), \quad z_a^{(i)} = 0 \quad (2.1)
\]

Here: \( \rho_i \) is the radius of the circular arc, \( a_i \) and \( b_i \) are algebraic values which determine the location of the center of the circular arc; \( \theta_i \) is the variable parameter which determines the location of a point on the circular arc (\( \theta_i \) is measured clockwise from the negative axis \( y_a \)); \( P_n \) is the diametral pitch in the normal section; and \( \psi_c \) is the pressure angle. The element proportions of rack cutters \( h_1, h_2, h_3 \) and \( h_4 \) are expressed in terms of the normal diametral pitch, \( P_n \).

It was mentioned above that equations (2.1) represent all four circular arcs - the shapes of both rack cutters. Thus equations

\[
x_a^{(F)} = \rho_F \sin \theta_F - b_F, \quad y_a^{(F)} = -(\rho_F \cos \theta_F - a_F), \quad z_a^{(F)} = 0 \quad (2.2)
\]

represent the circular arc centered at \( C_F \) (Fig. 2.1).

Knowing the normal section of the rack cutter, we may derive
equations of the generating surface using the matrix form of coordinate transformation. Consider that a rack cutter shape is represented in the coordinate system $S_a^{(i)}$ (Fig. 2.3 b) while the coordinate system $S_a^{(i)}$ translates along the line $0_c^{(i)} 0_a^{(i)}$ with respect to $S_c^{(i)}$; $0_c 0_a = u_i$ is a variable parameter. Using the matrix equation

\[
\begin{bmatrix}
 x_c^{(i)} \\
 y_c^{(i)} \\
 z_c^{(i)} \\
 1
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & \sin \lambda_i & \cos \lambda_i & u_i \cos \lambda_i \\
 0 & -\cos \lambda_i & \sin \lambda_i & u_i \sin \lambda_i \\
 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 x_a^{(i)} \\
 y_a^{(i)} \\
 z_a^{(i)} \\
 1
\end{bmatrix}
\]

we obtain ($i = F, P$)

\[
\begin{align*}
 x_c^{(i)} &= \rho_i \sin \theta_i - b_i \\
 y_c^{(i)} &= -(\rho_i \cos \theta_i - a_i) \sin \lambda_i + u_i \cos \lambda_i \\
 z_c^{(i)} &= (\rho_i \cos \theta_i - a_i) \cos \lambda_i + u_i \sin \lambda_i
\end{align*}
\]

In the derivation of equations (2.4), we assumed that $a_i > 0$ and $b_i > 0$. The unit normal to the rack cutter surface is given by the equations

\[
\begin{align*}
 n_c^{(i)} &= \frac{N_c^{(i)}}{|N_c^{(i)}|}, \quad N_c^{(i)} = \frac{\partial x_c^{(i)}}{\partial \theta_1} \times \frac{\partial z_c^{(i)}}{\partial u_1}
\end{align*}
\]

Equations (2.4) and (2.5) yield

\[
[n_c^{(i)}] =
\begin{bmatrix}
 \sin \theta_1 \\
 -\cos \theta_1 \sin \lambda_1 \\
 \cos \theta_1 \cos \lambda_1
\end{bmatrix}
\]

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Fig. 2.3
Consider that coordinate systems \( S_c \) and \( S_{c'} \) coincide. Surfaces \( \Sigma_c \) and \( \Sigma_{c'} \) will be in tangency if the following equations are satisfied

\[
(\mathbf{F}) \quad (\mathbf{P}) \quad (\mathbf{F}) \quad (\mathbf{P}) \quad (\mathbf{F}) \quad (\mathbf{P})
\]
\[
x_c = x_c, \quad y_c = y_c, \quad z_c = z_c \tag{2.7}
\]
\[
(\mathbf{F}) \quad (\mathbf{P}) \quad (\mathbf{F}) \quad (\mathbf{P}) \quad (\mathbf{F}) \quad (\mathbf{P})
\]
\[
n_{xc} = n_{xc}, \quad n_{yc} = n_{yc}, \quad n_{zc} = n_{zc} \tag{2.8}
\]

Equations (2.4), (2.6), (2.7) and (2.8) yield that surfaces \( \Sigma_F \) and \( \Sigma_P \) are in tangency along a straight line a-a (Fig. 1.3, a) if the following conditions are satisfied.

\[
\theta_F = \theta_P = \psi_c, \quad u_F = u_P, \quad \lambda_F = \lambda_P, \quad (\rho_P - \rho_F)\sin\psi_c = b_P - b_F, \tag{2.9}
\]
\[
(\rho_P - \rho_F)\cos\psi_c = a_P - a_F
\]

Here: \( \psi_c \) is the pressure angle.

The normal sections of the gear teeth do not coincide with the corresponding normal sections of the rack cutters. Neglecting this difference we may identify the normal sections of gear teeth with the normal sections of rack cutters. The shapes of the gear teeth in the normal section are shown in Fig. 2.4.

These shapes are in tangency at points \( M_1 \) and \( M_2 \). Considering the two sides of the teeth, we have to consider two pairs of surfaces, \( \Sigma_F \) and \( \Sigma_P \). Each pair of these surfaces is in tangency along a straight line a-a (Fig. 1.3 a) and point \( M_i(i = 1, 2) \) lies on a-a. The shape normals at \( M_1 \) and \( M_2 \) pass through point I which lies on the instantaneous axis of rotation and coincides
with the origins $0_a^{(F)}$ and $0_a^{(P)}$ for the position shown in Fig. 2.1 and Fig. 2.2

3 Tooth Surfaces of Gear 1 and Gear 2

We set up three coordinate systems: $S_c$ and $S_1$ rigidly connected to the rack cutter and gear 1, respectively, and the fixed coordinate system $S_f$ (Fig. 3.1, a). Note that in Fig. 3.1 a, the fixed coordinate system $S_f$ coincides with the auxiliary coordinate system $S_h$.

The derivation of the gear tooth surface $\Sigma_1$ is based on the following considerations: (Here $\Sigma_1$ represents gear 1 tooth surface, see also Appendix I)

The line of contact of the generating surface $\Sigma_c \cap C_1$ with the gear tooth surface $\Sigma_1$ may be determined in the coordinate system $S_c$ by using the following equations:

$$\begin{align*}
    \mathbf{r}_c &= \mathbf{r}_c(u_i, \theta_i) \in C_1 \\
    \mathbf{N}_c \cdot \mathbf{V}^{(c1)} &= f(u_i, \theta_i, \phi_i) = 0
\end{align*}$$

(3.1)

Here: $\mathbf{r}_c(u_i, \theta_i)$ is the vector function which represents in the coordinate system $S_c$, the generating surface; $\mathbf{N}_c$ is the normal to the generating surface; and $\mathbf{V}_c^{(c1)}$ is the relative velocity. The subscript "c" designates that the vector components are represented in the coordinate system $S_c$. In the case of transformation of motions represented in Fig. 3.1 a, the axodes are the plane $\pi$ and the cylinder of radius $r_1$, and $I-I$ is the instantaneous axis of rotation in relative motion. We may derive
Fig. 3.1
the equation of meshing as follows:

\[
\frac{X_C - X_c}{N} = \frac{Y_C - Y_c}{N} = \frac{Z_C - Z_c}{N} \quad (3.2)
\]

Equation (3.2) expresses that the normal to surfaces \( \Sigma_C \) and \( \Sigma_1 \) at their points of contact intersects the instantaneous axis of rotation, \( I-I \). Here

\[ X_c = 0, \quad Y_c = r_1 \phi_1, \quad Z_c = l \]

are the coordinates of \( I-I \).

Equations (3.2), (2.4) and (2.6) yield

\[
(r_1 \phi_1 - u_F \cos \lambda_F - a_F \sin \lambda_F) \sin \theta_F + b_F \cos \theta_F \sin \lambda_F =
\]

\[
f_F(u_F, \theta_F, \phi_1) = 0 \quad (3.3)
\]

Here: \( X_a = -b_F, \quad Y_a = a_F \), are the coordinates of center \( C_F \) (Fig. 2.1)

The equation of meshing (3.3) and equations (2.4) of the generating surface \( \Sigma_C \), considered simultaneously, represent a line on surface \( \Sigma_C \) (line \( L_F \)) which is the line of contact of \( \Sigma_C \) and \( \Sigma_1 \). The location of this line on \( \Sigma_C \) depends on the parameter of motion \( \phi_1 \). In the case of \( b_F = 0 \) equation (3.3) yields that

\[
u_F = \frac{r_1 \phi_1 - a_F \sin \lambda_F}{\cos \lambda_F} \quad (3.4)
\]

for any \( \theta_F \).

Thus the line of contact is a circle of radius \( \rho_F \) (Fig. 3.2 a).
Fig. 3.2
Fig. 3.2 b shows the contact lines for the case with $b_F \neq 0$. It results from equation (3.3) that

$$u_F = b_F \cot \theta_F \tan \lambda_F + \frac{r_1 \phi_1}{\cos \lambda_F} - a_F \tan \lambda_F \quad (3.4)$$

The contact lines approach infinity as $\phi_F$ approaches zero.

Surface $\Sigma_1$ may be determined with the family of contact lines represented in the coordinate system $S_1$. Using the matrix equation

$$\begin{bmatrix} r_1 \end{bmatrix} = \begin{bmatrix} M_1c \end{bmatrix} \begin{bmatrix} r_c^{(F)} \end{bmatrix} = \begin{bmatrix} M_{1f} \end{bmatrix} \begin{bmatrix} M_{fc} \end{bmatrix} \begin{bmatrix} r_c^{(F)} \end{bmatrix} = \begin{bmatrix} x_c^{(F)} \\ y_c^{(F)} \\ z_c^{(F)} \\ 1 \end{bmatrix} \quad (3.5)$$

and equations (2.4) and (3.4), we obtain

$$x_1 = (\rho_F \sin \theta_F - b_F + r_1) \cos \phi_1$$
$$+ (\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \phi_1 \sin \lambda_F \quad (3.6)$$

$$y_1 = (\rho_F \sin \theta_F - b_F + r_1) \sin \phi_1$$
$$- (\rho_F \cos \theta_F - b_F \cot \theta_F) \cos \phi_1 \sin \lambda_F,$$

$$z_1 = \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F$$

Equations (3.6) represent the tooth surfaces of gear 1 with surface coordinates $\theta_F$ and $\phi_1$. To get the normal section of this
surface, we have to cut the gear tooth surface by the plane which is drawn through the axis \(X_1\) perpendicular to the tooth direction in plane \(\pi\) (Fig. 1.3, a). The cutting plane is represented in the coordinate system \(S_1\) by the equation (Fig. 3.3)

\[
y_1 = -z_1 \tan \lambda_F
\]  

(3.7)

Equations (3.6) and (3.7) considered simultaneously yield the following relation between \(\theta_F\) and \(\phi_1\).

\[
A_1 \sin \phi_1 + B_1 \cos \phi_1 + D_1 \phi_1 = E_1
\]  

(3.8)

Here

\[
A_1 = \frac{\rho_F \sin \theta_F - b_F + r_1}{\sin \lambda_F},
\]

\[
B_1 = -\rho_F \cos \theta_F + b_1 \cot \theta_F
\]

\[
D_1 = \frac{\tan^2 \lambda_F}{r_1 \sin \lambda_F}
\]

(3.9)

\[
E_1 = -\rho_F \cos \theta_F + \frac{a_F}{\cos^2 \lambda_F} - b_F \cot \theta_F \tan^2 \lambda_F
\]

Considering that \(\theta_F\) is given, we may determine \(\phi_1\) using equation (3.8). It is easy to verify that \(\phi_1 = 0\) with \(\tan \theta_F = \frac{b_F}{a_F}\).

We may represent the normal section of \(S_1\) in the coordinate system \(S^*\) \((x^*_1, y^*_1, z^*_1)\), where
Fig. 3.3
\[ x_1^* = x_1, \quad y_1^* = y_1 \cos \lambda_F + z_1 \sin \lambda_F, \quad z_1^* = -y_1 \sin \lambda_F + z_1 \cos \lambda_F \]

Equations (3.7) and (3.10) yield

\[ x_1^* = x_1, \quad y_1^* = 0, \quad z_1^* = \frac{z_1}{\cos \lambda_F} \quad (3.11) \]

The sought-for normal section may now be represented by the following equations:

\[ A_1 \sin \phi_1 + B_1 \cos \phi_1 + D_1 \phi_1 = E_1 \]
\[ x_1^* = \sin \lambda_F (A_1 \cos \phi_1 - B_1 \sin \phi_1) \quad (3.12) \]
\[ y_1^* = 0 \]
\[ z_1^* = -E_1 + D_1 \phi_1 \]

Here \( A_1, B_1 \) and \( E_1 \) are functions of \( \theta_F \) (see equations (3.9)). The \( x_1^* \) - axis is the axis of symmetry of the normal section.

Equations similar to (3.12) can be used for the determination of the normal section of the "fillet" surface, but we have to substitute \( \rho_F, \theta_F, b_F \) and \( a_F \) by \( \rho_F^{(f)}, \theta_F^{(f)}, b_F^{(f)} \) and \( a_F^{(f)} \), respectively (Fig. 2.4 and Fig. 2.1). The circular arc \( \overline{DE} \) represents the fillet of the rack cutter in the normal section, points D and E are the points of tangency of this circular arc with the upper and lower parts of the shape of the rack cutter (Fig. 2.1).

Equations (3.12) are of a general nature and they can be used for all cases of the generation of gear 1 with a rack cutter.
having the shape of a circular arc. In particular, these equations may be used in the case of generation of the "fillet" surface of involute gears.

Similarly, we can derive equations of the tooth surface of gear 2. The equation of meshing of the rack cutter \( P \) and gear 2 is given by

\[
(r_2 \phi_2 - u_p \cos \lambda_p - a_p \sin \lambda_p) \sin \theta_p + b_p \cos \theta_p \sin \lambda_p = f_p(u_p, \theta_p, \phi_2) = 0 \tag{3.13}
\]

The line of contact of \( \Sigma^{(P)}_c \) and \( \Sigma_2 \) is represented in \( S_c \) by equations

\[
\begin{align*}
\xi^{(P)}_c &= \rho_p \sin \theta_p - b_p \\
y^{(P)}_c &= -(\rho_p \cos \theta_p - a_p) \sin \lambda_p + b_p \cot \theta_p \sin \lambda_p \\
-z^{(P)}_c &= \rho_p \cos \theta_p \cos \lambda_p - \frac{a_p}{\cos \lambda_p} + b_p \cot \theta_p \tan \lambda_p \sin \lambda_p + r_2 \phi_2 \tan \lambda_p 
\end{align*}
\tag{3.14}
\]

The coordinate transformation from \( S^{(P)}_c \) to \( S_2 \) is represented by the following matrix equation (Fig. 3.1 b):

\[
[r_2] = [M_{2p}] [M_{pc}] [r^{(P)}_c] = \\
\begin{bmatrix}
\cos \phi_2 & \sin \phi_2 & 0 & 0 \\
-\sin \phi_2 & \cos \phi_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & -r_2 \\
0 & 1 & 0 & -r_2 \phi_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\xi^{(P)}_c \\
y^{(P)}_c \\
z^{(P)}_c \\
1 \\
\end{bmatrix}
= 
\]
Equations (3.15) and (3.14) yield

\[ x_2 = (\rho_p \sin \theta_p - b_p - r_2) \cos \phi_2 - (\rho_p \cos \theta_p - b_p \cot \theta_p) \sin \phi_2 \sin \lambda_p \]  

(3.16)

\[ y_2 = -(\rho_p \sin \theta_p - b_p - r_2) \sin \phi_2 - (\rho_p \cos \theta_p - b_p \cot \theta_p) \cos \phi_2 \sin \lambda_p \]

\[ z_2 = \rho_p \cos \theta_p \cos \lambda_p - \frac{a_p}{\cos \lambda_p} + b_p \cot \theta_p \sin \lambda_p \tan \lambda_p + r_2 \phi_2 \tan \lambda_p \]

We will determine the normal section of \( \Sigma_2 \) by cutting the gear tooth surface by the same plane as we cut \( \Sigma_1 \). Considering simultaneously equation (3.16) with the equation

\[ y_2 = -z_2 \tan \lambda_p \]

(3.17)

we get

\[ A_{11} \sin \phi_2 + B_{11} \cos \phi_2 + D_{11} \phi_2 = E_{11} \]

(3.18)

Here:
\[ A_{II} = -\frac{\rho_p \sin \theta_p - b_p - r_2}{\sin \lambda_p} \]
\[ B_{II} = -\rho_p \cos \theta_p + b_p \cot \theta_p \]
\[ D_{II} = \tan^2 \lambda_p \]
\[ E_{II} = -\rho_p \cos \theta_p + \frac{a_p}{\cos^2 \lambda_p} - b_p \cot \theta_p \tan^2 \lambda_p \]

(3.19)

We may represent the sought-for normal section, in the coordinate system \( S_2*(x_2^*, y_2^*, z_2^*) \), whose orientation with respect to \( S_2 \) is similar to the orientation of \( S_1^* \) with respect to \( S_1 \) (Fig. 3.3).

Using equations

\[ x_2^* = x_2, \ y_2^* = 0, \ z_2^* = \frac{z_2}{\cos \lambda_p} \]

(3.20)

which are similar to equations (3.11), we may represent the normal section of \( S_2 \) as follows

\[ A_{II} \sin \phi_2 + B_{II} \cos \phi_2 + D_{II} \phi_2 = E_{II} \]
\[ x_2^* = (-A_{II} \cos \phi_2 + B_{II} \sin \phi_2) \sin \lambda_p \]
\[ y_2^* = 0 \]
\[ z_2^* = -E_{II} + D_{II} \phi_2 \]

(3.21)

Equations similar to (3.21) represent the normal section of the "fillet" surface. To derive these equations, we have to substitute \( \rho_p, \theta_p, b_p \) and \( a_p \) by \( \rho_p(f), \theta_p(f), b_p(f) \), and \( a_p(f) \) in equations (3.19). The normal section of the "fillet" surface of the rack.
cutter P is represented in Fig. 2.2.

4 Principal Curvatures and Directions of Gear Tooth Surfaces

The principal curvatures and directions of two contacting surfaces are necessary to define the size and direction of the contact ellipse at the contact point. If the relations between the principal curvatures and directions of two surfaces which are in mesh are known, the solution of this problem can be significantly simplified. Such relations were worked out first by Dr. F. L. Litvin.

**Step 1:** Principal curvatures and directions of the generating surfaces $\Sigma_F$ and $\Sigma_P$

The rack cutter surface $\Sigma_F$ and $\Sigma_P$ and their unit normals are represented in the coordinate system $S_c$ by equations (2.4) and (2.6), respectively. The principal curvatures and directions for a given surface may be obtained by using Rodrigues' equation [5]:

$$K_{I,II} \mathbf{V}_r = -\dot{\mathbf{n}}_r$$  \hspace{1cm} (4.1)

Here: $K_{I,II}$ are the principal curvatures; $\mathbf{V}_r$ is relative velocity of the point of contact in its motion over the surface, and $\dot{\mathbf{n}}_r$ is the velocity of the tip of the unit normal in the above motion. Equations (2.4) and (2.6) yield the following expressions for the principal directions and curvatures:
\[
\begin{align*}
(1) \quad \frac{d\mathbf{u}_i}{dt} &= 0, \quad \mathbf{v}_i = \frac{V(i)}{V(i)} = \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \sin \lambda_i \\ -\sin \theta_i \cos \lambda_i \end{bmatrix}, \quad \kappa_i = -\frac{1}{\rho_i} \\
(2) \quad \frac{d\theta_i}{dt} &= 0, \quad \mathbf{v}_i = \frac{V(i)}{V(i)} = \begin{bmatrix} 0 \\ \cos \lambda_i \\ \sin \lambda_i \end{bmatrix}, \quad \kappa_{i\perp} = 0, \quad (i = F, P)
\end{align*}
\]

Subscripts I and II designate the two principal directions and curvatures; the unit vectors \( \mathbf{v}_i \) and \( \mathbf{v}_{ii} \) are given in the coordinate system \( S_c \) but they are represented in the coordinate system \( S_f \) by the same matrices. The above unit vectors may also be considered as the unit vectors of axes \( y_t \) and \( z_t \) of the coordinate system \( S_t \) which is rigidly connected to the rack cutter surfaces, \( \Sigma_F \) and \( \Sigma_P \) (Fig. 4.1). The unit vector of the \( x_t \) - axis coincides with the common unit normal to surfaces \( \Sigma_F \) and \( \Sigma_P \) along their line of tangency, axis \( z_t \). Centers \( C_F \) and \( C_P \) are the centers of the principal curvatures \( \kappa_{iF} \) and \( \kappa_{iP} \) (Fig. 4.1 c).

The column matrices \( \mathbf{v}_i \) and \( \mathbf{v}_{ii} \) may be also derived by using the matrix \( [L_{ft}] \) which represents the transformation of direction cosines in transition from \( S_t \) to \( S_f \). Using the drawings of Fig. 4.1 c, Fig. 2.3 and Fig. 3.1, we obtain

\[
[L_{ft}] = [L_{fc}] [L_{ca}] [L_{at}] =
\]
Fig. 4.1
\[
\begin{bmatrix}
\sin\theta_i & \cos\theta_i & 0 \\
-c\sin\lambda_i & \sin\lambda_i & \cos\lambda_i \\
c\cos\lambda_i & -\sin\lambda_i & \sin\lambda_i
\end{bmatrix}, \quad \begin{bmatrix} i_1(t) \\
-1 \end{bmatrix} = \begin{bmatrix} L_{11} \\
0 \\
1 \end{bmatrix},
\]

where: \( i = F, P \) and \( \theta_i \) is the pressure angle (Fig. 4.1 c)

**Step 2: Principal curvatures and directions of \( \Sigma_1 \).**

We may determine the principal curvatures and directions of \( \Sigma_1 \) by using the following equations:

\[
\tan 2\sigma(F1) = \frac{2F(1)}{\kappa_1(F) - \kappa_2(F) + G(1)} \quad (4.5)
\]

\[
\kappa_1 + \kappa_2 = \kappa_1(F) + \kappa_2(F) + S(1) \quad (4.6)
\]

\[
\frac{\kappa_1 - \kappa_2}{\kappa_2} = \frac{\kappa_1(F) - \kappa_2(F) + G(1)}{\cos 2\sigma(F1)} \quad (4.7)
\]

\[
F(1) = \frac{a_{31}(1) a_{32}(1)}{b_3(1) + (v(F1) \cdot i_{II}) a_{31}(1) + (v(F1) \cdot i_{II}) a_{32}(1)} \quad (4.8)
\]

\[
G(1) = \frac{[a_{31}(1)]^2 - [a_{32}(1)]^2}{b_3(1) + (v(F1) \cdot i_{II}) a_{31}(1) + (v(F1) \cdot i_{II}) a_{32}(1)} \quad (4.9)
\]

\[
S(1) = \frac{[a_{31}(1)]^2 + [a_{32}(1)]^2}{b_3(1) + (v(F1) \cdot i_{II}) a_{31}(1) + (v(F1) \cdot i_{II}) a_{32}(1)} \quad (4.10)
\]
All the vectors of equations (4.5) - (4.13) are represented in the coordinate system $S_f$ and the coordinate system $S_h$ coincides with $S_f$ (Fig. 3.1); $\kappa_I^{(1)}$ and $\kappa_{II}^{(1)}$ are the principal curvatures of $\Sigma_1$, $\omega^{(F)}(F)$ is the angle which is formed by vectors $i_I^{(F)}$ and $i_I^{(1)}$ where $i_I^{(F)}$ is the unit vector for the principal direction $I^{(1)}$ on surface $\Sigma_1$; $\kappa_I^{(F)} = -\frac{1}{\rho_F}$ and $\kappa_{II}^{(F)} = 0$ are the principal curvatures of $\Sigma_F$. Let us derive the following auxiliary equations

\[ \begin{bmatrix} \omega^{(F)}(F) \\ \omega^{(F)}(I) \omega^{(1)}(I) \omega^{(1)}(I) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega^{(1)}(I) \end{bmatrix} \]  

(4.14)

Vector $\omega^{(F)} = 0$ because the rack cutter performs translational motion (Fig. 3.1 a)

\[ \begin{bmatrix} n^{(F)} \\ \omega^{(F)}(I) \\ i_I^{(F)} \end{bmatrix} = -\omega^{(1)}(I) \sin \lambda_F \]  

(4.15)

\[ \begin{bmatrix} n^{(F)} \\ \omega^{(F)}(I) \\ i_{II}^{(F)} \end{bmatrix} = -\omega^{(1)}(I) \sin \theta_F \cos \lambda_F \]  

(4.16)

The point of contact of surfaces $\Sigma_1$ and $\Sigma_2$ lies on a straight line which passes through the point whose coordinates are given by
\begin{align*}
x_f^{(1)} &= \rho_F \sin \theta_F - b_F + r_1 \quad (4.17) \\
y_f^{(1)} &= - (\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \quad (4.18) \\
z_f^{(1)} &= \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F \quad (4.19)
\end{align*}

Here \( \theta_F = \theta_p = \theta \) is the pressure angle at the point of contact of surfaces \( \Sigma_1 \) and \( \Sigma_2 \). Equations (4.17) - (4.19) may be derived from equations (1.9) with \( \mu_1 = 0 \) taking into account that the coordinate system \( \Sigma_F \) coincides with \( \Sigma_f \). The transfer velocity of the rack cutter is (Fig. 3.1, a):

\[
\begin{bmatrix}
0 \\
-\omega(1) r_1 \\
0
\end{bmatrix}
\quad (4.20)
\]

The transfer velocity of a point of gear 1 is given by

\[
\begin{bmatrix}
y_f^{(1)} \\
x_f^{(1)} \\
0
\end{bmatrix} = \omega(1) x \begin{bmatrix}
0 \\
-\omega(1) r_f^{(1)} \\
0
\end{bmatrix} = \omega(1) \begin{bmatrix}
y_f^{(1)} \\
-x_f^{(1)} \\
0
\end{bmatrix} =
\]

\[
\begin{bmatrix}
(\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \\
\rho_F \sin \theta_F - b_F + r_1 \\
0
\end{bmatrix}
\quad (4.21)
\]
The sliding velocity is given by

$$v^{(F1)} - v^{(F)} = \omega^{(1)} \left[ (\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \right]$$

(4.22)

$$\rho_F \sin \theta_F - b_F$$

$$0$$

Thus, we obtain (see equations (4.21), (4.2) and (4.3)):

$$v^{(F1)} i_{\text{II}} = \omega^{(1)} \left( \rho_F - \frac{b_F}{\sin \theta_F} \right) \sin \lambda_F$$

(4.23)

$$v^{(F1)} i_{\text{II}} = \omega^{(1)} \left( \rho_F - \frac{b_F}{\sin \theta_F} \right) \sin \theta_F \cos \lambda_F$$

(4.24)

Using equations (4.11), (4.12), (4.15), (4.16), (4.23), (4.24), (4.2), and (4.3), we obtain

$$a_{31} = -\omega^{(1)} \frac{b_F \sin \lambda_F}{\rho_F \sin \theta_F}$$

(4.25)

$$a_{32} = -\omega^{(1)} \sin \theta_F \cos \lambda_F$$

(4.26)

Using equations (4.13), (2.6), (4.20) and (4.21), we get

$$b_3 = -\left( \omega^{(1)} \right)^2 r_1 \sin \theta_F$$

(4.27)

We may now derive the final expressions for $F^{(1)}$, $G^{(1)}$ and $S^{(1)}$ as follows:

$$F^{(1)} = -\frac{b_F \sin \lambda_F \cos \lambda_F}{\rho_F \sin \theta_F A_1}$$

(4.28)
\[ G = \left( \frac{b_F \sin \lambda_F}{\rho_F \sin \theta_F} \right)^2 - \frac{\sin^2 \theta_F \cos^2 \lambda_F}{A_1} \tag{4.29} \]

\[ S = \left( \frac{b_F \sin \lambda_F}{\rho_F \sin \theta_F} \right)^2 + \frac{\sin^2 \theta_F \cos^2 \lambda_F}{A_1} \tag{4.30} \]

Here

\[ A_1 = r_1 \sin \theta_F + \left[ \rho_F - \frac{b_F}{\sin \theta_F} \right] \left[ \frac{b_F \sin^2 \lambda_F}{\rho_F \sin \theta_F} + \sin^2 \theta_F \cos^2 \lambda_F \right] \tag{4.31} \]

Equations (4.5) - (4.7) and (4.25) - (4.31) determine the principal curvatures and directions of surface \( \Sigma_1 \) at the point of contact of surfaces \( \Sigma_1 \) and \( \Sigma_2 \).

**Step 3**: Principal curvatures and directions of \( \Sigma_2 \)

The principal curvatures of the rack cutter \( \Sigma_p \) are:

\[ \kappa_I^{(P)} = -\frac{1}{\rho_p}, \quad \kappa_{II}^{(P)} = 0; \]

the principal directions of \( \Sigma_p \) are the same as of \( \Sigma_F \);

Using similar derivations, we obtain

\[ \omega = \begin{bmatrix} 0 \\ 0 \\ -\omega (2) \end{bmatrix} \tag{4.32} \]

\[ \begin{bmatrix} (P) & (P2) & (P) \\ \omega & i & \omega (2) \end{bmatrix} = \omega \begin{bmatrix} \sin \lambda_p \\ \begin{bmatrix} (P) \\ (P2) \end{bmatrix} = \begin{bmatrix} \sin \lambda_p \\ \begin{bmatrix} (P) \end{bmatrix} \end{bmatrix} \tag{4.33} \]

\[ \begin{bmatrix} (P) & (P2) & (P) \\ \omega & i & \omega (2) \end{bmatrix} = \omega \begin{bmatrix} \sin \theta_p \cos \lambda_p \\ \begin{bmatrix} \sin \theta_p \cos \lambda_p \end{bmatrix} \end{bmatrix} \tag{4.34} \]

\[ x_f = \rho_p \sin \theta_p - b_p - r_2 + C \tag{4.35} \]

\[ y_f = - (\rho_p \cos \theta_p - b_p \cot \theta_p) \sin \lambda_p \tag{4.36} \]
\[ z_f = \rho_P \cos \theta_P \cos \lambda_P - \frac{a_p}{\cos \lambda_P} + b_P \cot \theta_P \sin \lambda_P \tan \lambda_P + r_2 \phi_2 \tan \lambda_P \]

(4.37)

\[ v^{(p)}_{\text{tr}} = \begin{bmatrix} 0 \\ -\omega (2) r_2 \\ 0 \end{bmatrix} \]

(4.38)

\[ v^{(2)}_{\text{tr}} = \omega (2) \times r^{(2)}_f + c \times \omega (2) = -\omega (2) \begin{bmatrix} y_f^{(2)} \\ -x_f^{(2)} + c \\ 0 \end{bmatrix} \]

(4.39)

\[ v^{(p2)} = v^{(p)} - v^{(2)}_{\text{tr}} = -\omega (2) \begin{pmatrix} \rho_P \cos \theta_P - b_P \cot \theta_P \sin \lambda_P \\ \rho_P \sin \theta_P - b_P \\ 0 \end{pmatrix} \]

(4.40)

\[ i^{(p)}_{\text{II}} = -\omega (2) \left( \rho_P - \frac{b_P}{\sin \theta_P} \right) \sin \lambda_P \quad (i^{(p)}_{\text{II}} = i^{(F)}_{\text{II}}) \]

(4.41)

\[ v^{(p2)} \cdot i^{(p)}_{\text{II}} = -\omega (2) \left( \rho_P - \frac{b_P}{\sin \theta_P} \right) \sin \theta_P \cos \lambda_P \quad (i^{(p)}_{\text{II}} = i^{(F)}_{\text{II}}) \]

(4.42)

\[ a_{31} = \begin{bmatrix} (p) & (P2) & (p) \\ \omega & i^{(p)}_{\text{II}} \\ \omega & i^{(p)}_{\text{II}} \end{bmatrix} - \kappa_{\text{I}} (v^{(p2)} \cdot i^{(p)}_{\text{II}}) = -\omega (2) \frac{b_P \sin \lambda_P}{\rho_P \sin \theta_P} \]

(4.43)

\[ a_{32} = \begin{bmatrix} (p) & (P2) & (p) \\ \omega & i^{(p)}_{\text{II}} \\ \omega & i^{(p)}_{\text{II}} \end{bmatrix} - \kappa_{\text{II}} (v^{(p2)} \cdot i^{(p)}_{\text{II}}) = \omega \frac{\sin \theta_P \cos \lambda_P}{P} \]

(4.44)

\[ b_3 = \begin{bmatrix} (p) & (2) & (p) \\ \omega & v_{\text{tr}} \end{bmatrix} - \begin{bmatrix} (p) & (P) & (2) \\ \omega & v_{\text{tr}} \end{bmatrix} = (2)^2 \frac{r_2 \sin \theta_P}{P} \]

(4.45)
The principal curvatures and directions of Σ₂ are determined as follows

\[
\tan 2 \sigma = \frac{2F(2)}{- \frac{1}{\rho_p} + G(2)}
\]

\[
\kappa_I + \kappa_{II} = - \frac{1}{\rho_p} + S(2)
\]

\[
\kappa_I - \kappa_{II} = \frac{1}{\cos 2\sigma F(2)}
\]

**Example 4.1: Principal Curvatures and Directions of Gear Tooth Surfaces**

Given: The rack parameters \( \rho_F = 0.7 \text{ in.} \), \( \rho_P = 0.775 \text{ in.} \) (See Fig. 2.1 and Fig. 2.2); the gear parameters: No. of teeth \( N_1 = 36 \).
12, \( N_2 = 94 \); lead angle \( \lambda_p = 75^\circ \); nominal pressure angle \( \theta^* = 30^\circ \); normal diametral pitch \( p_n = 2 \);

(1) Pinion: By using equations (4.5) – (4.31) we obtain
\[
\begin{align*}
& A_1 = 1.70882, 
& F(1) = -0.02549, 
& G(1) = -0.05648, 
& S(1) = -0.07608, \\
& \text{principal direction } \sigma(F1) = 0.98294^\circ \text{ (Fig. 5.1), and two principal curvatures: } \\
& \kappa_I^{(1)} = -1.49529, \kappa_{II}^{(1)} = -0.00936.
\end{align*}
\]

(2) Gear 2: By using equations (4.32) – (4.52), we obtain
\[
\begin{align*}
& A_2 = 11.98175, 
& F(2) = 0.00429, 
& G(2) = 0.01179, 
& S(2) = 0.01458, \\
& \text{principal direction } \sigma(P2) = -0.19237^\circ \text{ (See Fig. 5.1), and two principal curvatures: } \\
& \kappa_I^{(2)} = -1.27715, \kappa_{II}^{(2)} = 0.00141.
\end{align*}
\]

5. **Contacting Ellipse**

The tangent plane to gear tooth surfaces is formed by axes \( Y_t \) and \( Z_t \) (Fig. 5.1 a). The unit vectors \( i^\text{(F,P)}_I \) and \( i^\text{(F,P)}_{II} \) represent the principal directions of surfaces \( \Sigma_F \) and \( \Sigma_P \) of the rack cutters. Angles \( \sigma(F1) \) and \( \sigma(P2) \), measured counter-clockwise from \( i^\text{(F,P)}_{II} \), determine the principal directions of gear tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \), respectively, with the unit vectors \( i^\text{(1)}_I \) and \( i^\text{(2)}_I \).

Consider that the principal curvatures \( \kappa_I^{(1)}, \kappa_{II}^{(1)}, \kappa_I^{(2)}, \kappa_{II}^{(2)} \) of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are known. Also known are angles \( \sigma(F1) \) and \( \sigma(P2) \). We may then determine the dimensions of the axes of the contacting ellipse with respect to the elastic approach of gear tooth surfaces and the orientation of the contacting ellipse in the tangent plane \( T \). The equations to be used are as follows [5,6]:

\[
\begin{align*}
A = \frac{1}{4} \left[ \kappa_I^{(1)} - \kappa_I^{(2)} - (g_1^2 - 2g_1g_2\cos2\sigma + g_2^2) \right]^{\frac{1}{2}}
\end{align*}
\]
(a)

(b)

CONTACTING ELLIPSE

Fig. 5.1
\[ B = \frac{1}{4} \left[ \kappa^{(1)}_\Sigma - \kappa^{(2)}_\Sigma + (g_1^2 - 2g_1g_2\cos 2\sigma + g_2^2) \right] \]  

\[ a^2 = \left| \frac{\delta}{A} \right|, \quad b^2 = \left| \frac{\delta}{B} \right|; \]

\[ \sin 2\alpha = g_2\sin 2\sigma \]

\[ \cos 2\alpha = g_1 - g_2\cos 2\sigma \]

Here:  
\[ \kappa^{(1)}_\Sigma = \kappa_1^{(1)} + \kappa_\Pi^{(1)}; \quad \kappa^{(2)}_\Sigma = \kappa_1^{(2)} + \kappa_\Pi^{(2)}, \quad g_1 = \kappa_1^{(1)} - \kappa_\Pi^{(1)}, \]

\[ g_2 = \kappa_1^{(2)} - \kappa_\Pi^{(2)}; \quad \sigma = \sigma^{(F2)} - \sigma^{(F1)}; \quad \delta \text{ is} \]

the elastic approach of gear tooth surfaces; \( a \) and \( b \) are the axes of the contacting ellipse and \( \alpha \) is the angle which determines the orientation of contacting ellipse. Angle \( \alpha \) is formed by the \( \eta \) - axis and unit vector \( \hat{z}_I^{(1)} \) and measured counter-clockwise from axis \( \eta \) to \( \hat{z}_I^{(1)} \) (Fig. 5.1 b). Axes \( \eta \) and \( \xi \) are directed along the \( b \)- and \( a \)-axis of the contacting ellipse. The magnitudes of \( a \) and \( b \) are expressed in terms of the elastic approach \( \delta \) which can be obtained from experiments or calculated.

Example 5.1: Dimension and Orientation of Contacting Ellipse

The nominal rack and gear parameters are the same as given in Example 4.1. In Example 4.1, we found \( \sigma^{(F1)} = 0.98294^\circ \), \( \sigma^{(F2)} = -0.19237^\circ \); the two principal curvatures for pinion surface \( \Sigma_1 \) are \( \kappa_1^{(1)} = 1.49529 \) and \( \kappa_{\Pi}^{(1)} = -0.00936 \), and two principal curvatures for gear surface \( \Sigma_2 \) are \( \kappa_{\Pi}^{(2)} = -1.27715 \) and \( \kappa_1^{(2)} = 0.00141 \). By substituting these values into equation (5.1), we obtain \( A = 0.1110 \), \( B = 0.0035 \), \( a = 3.0026 \), \( b = 16.9158\delta \) and \( \alpha = 82.9392^\circ \).
6 Velocity of Motion of the Contacting Ellipse Over Gear Tooth Surface

The velocity of motion of the symmetry center of the contacting ellipse over surface $\Sigma_1$ is represented by the following equations [5,6]:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 &= b_1 \\
    a_{21}x_1 + a_{22}x_2 &= b_2 \\
    a_{31}x_1 + a_{32}x_2 &= b_3
\end{align*}
\]

(6.1)

Here:

\[
\begin{align*}
    a_{11} &= -\kappa^{(1)}_I + \frac{1}{2}(\kappa^{(2)}_\Sigma + g_2\cos 2\sigma) \\
    a_{12} &= a_{21} = kg_2\sin 2\sigma \\
    a_{22} &= -\kappa^{(1)}_{II} + \frac{1}{2} (\kappa^{(2)}_\Sigma - g_2\cos 2\sigma) \\
    a_{31} &= \[\n    a_{32} &= \[\n    b_1 &= \[\n    b_2 &= \[\n    b_3 &= \[\n\end{align*}
\]

\[
\begin{align*}
    b_2 &= -\frac{1}{2}(v^{(12)}_I \cdot i^{(1)}_I) - \frac{1}{2}(v^{(12)}_{II} \cdot i^{(1)}_{II})g_2\sin 2\sigma \\
    b_3 &= -\frac{1}{2}(v^{(12)}_I \cdot i^{(1)}_I)(\kappa^{(2)}_\Sigma - g_2\cos 2\sigma)
\end{align*}
\]

\[
\begin{align*}
    \kappa^{(2)}_\Sigma &= \kappa^{(2)}_I + \kappa^{(2)}_{II} ; g_2 = \kappa^{(2)}_I - \kappa^{(2)}_{II} \\
    x_1 &= v^{(1)}_r \cdot i^{(1)}_I ; x_2 = v^{(1)}_r \cdot i^{(1)}_{II} \\
    \kappa^{(1)}_I \text{ and } \kappa^{(1)}_{II} (i = 1, 2) \text{ are the principal curvatures of surface } \Sigma_i ; \text{ } \sigma \text{ is the angle formed by the unit vectors } i^{(1)}_I \text{ and } i^{(2)}_{II}.
\end{align*}
\]
and measured counter-clockwise from \( \mathbf{i}_I^{(1)} \) to \( \mathbf{i}_I^{(2)} \) (Fig. 5.1 a); \( \mathbf{i}_I^{(1)} \) and \( \mathbf{i}_I^{(2)} \) represent the principal directions of surfaces \( \Sigma_1 \) and \( \Sigma_2 \). \( \mathbf{n} \) is the unit normal to the contacting surfaces represented in the coordinate system \( S_f \) by equations (1.2); \( \omega^{(1)} \) and \( \omega^{(2)} \) are the angular velocities of gear 1 and 2, \( \omega^{(12)} = \omega^{(1)} - \omega^{(2)} \); \( v_{tr}^{(i)} \) is the transfer velocity of the contacting ellipse in the transfer motion, with gear \( i \) (\( i = 1, 2 \)). Here: 

\[
\begin{align*}
\mathbf{v}_{tr}^{(12)} &= \omega^{(1)} \times \mathbf{r}_f^{(1)} + \mathbf{c} \times \omega^{(2)} \\
\mathbf{v}_{tr} &= \omega^{(1)} \times \mathbf{r}_f^{(1)} + \mathbf{c} \times \omega^{(2)}
\end{align*}
\]

where \( \mathbf{r}_f^{(1)} \) is the position vector of the point contact represented by equations (4.17) - (4.19); \( \mathbf{c} = (r_1 + r_2) \mathbf{i} \); \( v^{(12)} = v_{tr}^{(1)} - v_{tr}^{(2)} \) is the sliding velocity. Considering the coordinate system \( S_f \), we have

\[
\begin{align*}
\mathbf{r}_f^{(1)} &= \left[ \begin{array}{c} \rho_F \sin \theta_F - b_F + r_1 \\ - (\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \\ \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F \end{array} \right] \\
\omega_f^{(1)} &= -\omega^{(1)} k \\
\omega_f^{(2)} &= \omega^{(2)} k = \left( \frac{r_1}{r_2} \right) \omega^{(1)} k = m_{21} \omega^{(1)} k
\end{align*}
\]

\[
\begin{align*}
\mathbf{v}_{tr}^{(1)} &= \omega^{(1)} \times \mathbf{r}_f^{(1)} = \omega^{(1)} \begin{bmatrix} y_F^{(1)} \\ -x_f^{(1)} \\ 0 \end{bmatrix} \\
\mathbf{v}_{tr}^{(2)} &= \omega^{(2)} \times \mathbf{r}_f^{(1)} + \mathbf{c} \times \omega^{(2)}
\end{align*}
\]
Using coordinate transformation, we can transform \( \omega_f^{(1)}, \omega_f^{(2)}, v_f^{(1)}, v_f^{(2)}, v_f^{(12)}, \omega_f^{(12)}, \) and \( n_f^{(1)} \) from coordinate system \( S_f \) to \( S_q \) by using Fig. 2.3, Fig. 3.1, Fig. 4.1 c and Fig. 6.1 a. We obtain:

\[
[L_{qf}] = [L_{qt}][L_{ta}][L_{ac}][L_{cf}]
\]

\[
= \begin{bmatrix}
\sin^\theta_F & -\sin^\lambda_F \cos^\theta_F \\
\cos^\theta_F \cos^\lambda_F & \sin^\theta_F \cos^\lambda_F + \sin^\theta_F \sin^\lambda_F \sin^\theta_F + \sin^\lambda_F \cos^\theta_F \\
\cos^\theta_F \sin^\lambda_F & -\sin^\theta_F \sin^\lambda_F + \cos^\theta_F \cos^\lambda_F
\end{bmatrix}
\]

\[L_{cf} = \begin{bmatrix}
\cos^\theta_F & -\sin^\theta_F \\
\sin^\theta_F & \cos^\theta_F
\end{bmatrix}
\]

\[L_{ac} = \begin{bmatrix}
\cos^\lambda_F & -\sin^\lambda_F \\
\sin^\lambda_F & \cos^\lambda_F
\end{bmatrix}
\]

\[L_{ta} = \begin{bmatrix}
\cos^\theta_F & -\sin^\theta_F \\
\sin^\theta_F & \cos^\theta_F
\end{bmatrix}
\]

\[L_{qt} = \begin{bmatrix}
\cos^\theta_F & -\sin^\theta_F \\
\sin^\theta_F & \cos^\theta_F
\end{bmatrix}
\]

(6.10)
Fig. 6.1
\[
\omega^{(1)} = \begin{bmatrix} L_{qf} \end{bmatrix} \begin{bmatrix} \omega_f^{(1)} \end{bmatrix}
\]

\[
= -\omega^{(1)} \begin{bmatrix}
\cos \theta_F \cos \lambda_F \\
-\sin \theta_F \cos \sigma(F,1) \cos \lambda_F + \sin \sigma(F,1) \sin \lambda_F \\
\sin \theta_F \sin \sigma(F,1) \cos \lambda_F + \cos \sigma(F,1) \sin \lambda_F
\end{bmatrix}
\]

\[
(6.11)
\]

\[
\omega^{(2)} = \begin{bmatrix} L_{qf} \end{bmatrix} \begin{bmatrix} \omega_f^{(2)} \end{bmatrix}
\]

\[
= m_{21} \omega \begin{bmatrix}
\cos \theta_F \cos \lambda_F \\
-\sin \theta_F \cos \sigma(F,1) \cos \lambda_F + \sin \sigma(F,1) \sin \lambda_F \\
\sin \theta_F \sin \sigma(F,1) \cos \lambda_F + \cos \sigma(F,1) \sin \lambda_F
\end{bmatrix}
\]

\[
(6.12)
\]

\[
v_{tr(q)}^{(1)} = \begin{bmatrix} L_{qf} \end{bmatrix} \begin{bmatrix} v_{tr(f)}^{(1)} \end{bmatrix}
\]

\[
= \begin{bmatrix}
y_f^{(1)} \sin \theta_F + x_f^{(1)} \sin \lambda_F \cos \theta_F \\
y_f^{(1)} \cos \theta_F \cos \sigma(F,1) - x_f^{(1)} \sin \theta_F \cos \sigma(F,1) \sin \lambda_F + \sin \sigma(F,1) \cos \lambda_F \\
y_f^{(1)} \cos \theta_F \sin \sigma(F,1) - x_f^{(1)} \sin \theta_F \sin \sigma(F,1) \sin \lambda_F + \cos \sigma(F,1) \cos \lambda_F
\end{bmatrix}
\]

\[
(6.13)
\]

\[
v_{tr(q)}^{(2)} = \begin{bmatrix} L_{qf} \end{bmatrix} \begin{bmatrix} v_{tr(f)}^{(2)} \end{bmatrix}
\]

\[
= m_{21} \omega \begin{bmatrix}
y_f^{(1)} \sin \theta_F - (x_f^{(1)} - c) \sin \lambda_F \cos \theta_F \\
y_f^{(1)} \cos \theta_F \cos \sigma(F,1) + (x_f^{(1)} - c) \{\sin \theta_F \cos \sigma(F,1) \sin \lambda_F + \sin \sigma(F,1) \cos \lambda_F \}
\end{bmatrix}
\]

\[
(6.14)
\]

\[
v^{(12)} = \begin{bmatrix} L_{qf} \end{bmatrix} \begin{bmatrix} v_f^{(12)} \end{bmatrix}
\]

\[
= m_{21} \omega \begin{bmatrix}
y_f^{(1)} (1 + m_{21}) \sin \theta_F - \{x_f^{(1)} (1 + m_{21}) \} \sin \lambda_F \cos \theta_F \\
y_f^{(1)} (1 + m_{21}) \cos \theta_F \cos \sigma(F,1) + \{x_f^{(1)} (1 + m_{21}) \} \{\sin \theta_F \cos \sigma(F,1) \sin \lambda_F + \sin \sigma(F,1) \cos \lambda_F \}
\end{bmatrix}
\]

\[
(6.15)
\]
\[ \omega_{q}^{(12)} = \left[ L_{qf} \right] \omega_{f}^{(12)} \]
\[
= -\omega^{(1)} \begin{bmatrix}
(1 + m_{21})\cos \theta_{F} \cos \lambda_{F} \\
(1 + m_{21})(-\sin \theta_{F}\cos \phi_{1}\cos \lambda_{F} + \sin \phi_{1}\sin \lambda_{F}) \\
(1 + m_{21})(\sin \theta_{F}\sin \phi_{1}\cos \lambda_{F} + \cos \phi_{1}\sin \lambda_{F})
\end{bmatrix}
\] (6.16)

\[ n_{q}^{(1)} = \left[ L_{qf} \right] n_{f}^{(1)} \]
\[
= \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\] (6.17)

Also, we may represent the unit vectors \( i_{I}^{(1)} \) and \( i_{II}^{(1)} \) of the principal directions of surface \( \Sigma_{1} \) in coordinate system \( S_{q} \) as follows: (Fig. 6.1b):

\[ i_{I}^{(1)} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \] (6.18)

\[ i_{II}^{(1)} = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \] (6.19)

\[ \begin{bmatrix}
\omega_{q}^{(12)} \\
i_{I}^{(1)} \\
i_{II}^{(1)}
\end{bmatrix} = n_{q}^{(1)} \cdot (\omega_{q}^{(12)} \times i_{I}^{(1)}) = -\omega_{qz}
\]
\[
= \omega^{(1)} (1 + m_{21})(\sin \theta_{F}\sin \phi_{1}\cos \lambda_{F} + \cos \phi_{1}\sin \lambda_{F})
\] (6.20)

\[ \begin{bmatrix}
\omega_{q}^{(12)} \\
i_{I}^{(1)} \\
i_{II}^{(1)}
\end{bmatrix} = n_{q}^{(1)} \cdot (\omega_{q}^{(12)} \times i_{I}^{(1)}) = \omega_{yq}
\]
\[
= -\omega^{(1)} (1 + m_{21})(-\sin \theta_{F}\cos \phi_{1}\cos \lambda_{F} + \sin \phi_{1}\sin \lambda_{F})
\] (6.21)
\[ y^{(12)} \cdot i^{(1)}_I = \omega^{(1)} \left\{ [-x_f^{(1)}(1 + m_{21}) + m_{21}c] [\sin \theta_F \cos \sigma^{(F,1)} \sin \lambda_F + \sin \sigma^{(F,1)} \cos \lambda_F] \right. \\
+ y_f^{(1)}(1 + m_{21}) \cos \theta_F \cos \sigma^{(F,1)} \right\} \]  
(6.22)

\[ y^{(12)} \cdot i^{(1)}_{II} = \omega^{(1)} \left\{ [-x_f^{(1)}(1 + m_{21}) + m_{21}c] [\sin \theta_F \sin \sigma^{(F,1)} \sin \lambda_F + \cos \sigma^{(F,1)} \cos \lambda_F] \right. \\
- y_f^{(1)}(1 + m_{21}) \cos \theta_F \sin \sigma^{(F,1)} \right\} \]  
(6.23)

\[ \begin{bmatrix} n^{(1)}_q \cdot \omega^{(1)} \times y^{(1)}_t \end{bmatrix} = m_{21} \left( \omega^{(1)} \right)^2 \left[ -y_f^{(1)} \cos \theta_F \sin \lambda_F + x_f^{(1)} \sin \theta_F \right] \]  
(6.24)

\[ \begin{bmatrix} n^{(1)}_q \cdot \omega^{(1)} \times y^{(2)}_t \end{bmatrix} = m_{21} \left( \omega^{(1)} \right)^2 \left[ -y_f^{(1)} \cos \theta_F \sin \lambda_F + (x_f^{(1)} - c) \sin \theta_F \right] \]  
(6.25)

An easier method of deriving equations (6.20) - (6.25) is to consider the tangent plane in Fig. 4.1a and the two unit vectors \( i^{(1)}_I \) and \( i^{(1)}_{II} \) along the two principal directions of gear surface \( E_1 \).

The projections of \( i^{(1)}_I \) and \( i^{(1)}_{II} \) along axis \( X_t, Y_t \) and \( Z_t \) are expressed as:

\[ i^{(1)}_I = \begin{bmatrix} 0 \\ \cos \sigma^{(F,1)} \\ \sin \sigma^{(F,1)} \end{bmatrix} \]  
(6.26)

\[ i^{(1)}_{II} = \begin{bmatrix} 0 \\ -\sin \sigma^{(F,1)} \\ \cos \sigma^{(F,1)} \end{bmatrix} \]  
(6.27)

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using the matrix transformation of direction cosines in transition from $S_t$ to $S_f$ (Fig. 4.1 c), we obtain

$$L_{ft} = L_{fc}L_{ca}L_{at}$$

$$= \begin{bmatrix}
\sin \theta_F & \cos \theta_F & 0 \\
-cos \theta_F \sin \lambda_F & \sin \theta_F \sin \lambda_F & \cos \lambda_F \\
\cos \theta_F \cos \lambda_F & -\sin \theta_F \cos \lambda_F & \sin \lambda_F
\end{bmatrix}$$

(6.28)

$$i^{(1)}_{I} = L_{ft} \begin{bmatrix} 0 \\ \cos \sigma(F,1) \\ \sin \sigma(F,1) \end{bmatrix}$$

$$= \begin{bmatrix}
\cos \theta_F \cos \sigma(F,1) \\
\sin \theta_F \sin \lambda_F \cos \sigma(F,1) + \cos \lambda_F \sin \sigma(F,1) \\
\sin \theta_F \cos \lambda_F \sin \sigma(F,1) + \sin \lambda_F \cos \sigma(F,1)
\end{bmatrix}$$

(6.29)

$$i^{(1)}_{II} = L_{ft} \begin{bmatrix} 0 \\ -\sin \sigma(F,1) \\ \cos \sigma(F,1) \end{bmatrix}$$

$$= \begin{bmatrix}
-cos \theta_F \sin \sigma(F,1) \\
-sin \theta_F \sin \lambda_F \sin \sigma(F,1) + \cos \lambda_F \cos \sigma(F,1) \\
\sin \theta_F \cos \lambda_F \sin \sigma(F,1) + \sin \lambda_F \cos \sigma(F,1)
\end{bmatrix}$$

(6.30)
\[ v(12) = v(1) - v(2) \]  
\[ \sim_{tr} \]
\[ = \omega(1) x r_f(1) - (\omega(2) x r_f(1)) + c x \omega(2) \]

where:  
\[ c = (r_1 + r_2) i \]

\[ r_f(1) = [M_f c][r_c(F)] \]

\[
\begin{bmatrix}
1 & 0 & 0 & r_1 \\
0 & 1 & 0 & -r_1 \phi_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_c(F) \\
y_c(F) \\
z_c(F) \\
1
\end{bmatrix}
\]

\[ = \begin{bmatrix}
\rho_F \sin \theta_F - b_F + r_1 \\
-(\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \\
\rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F \\
\end{bmatrix} \]

\[ \omega(1) = -\omega(1)_k \]
\[ \omega(2) = \omega(2)_k \]
\[ = m_{21} \omega(1)_k \]

where:  
\[ m_{21} = \frac{\omega(2)}{\omega(1)} = \frac{r_1}{r_2} \]

\[ v(1)_{tr} = \omega(1) x r_f(1) \]

\[
\begin{bmatrix}
(\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \\
-\omega(1) \rho_F \sin \theta_F - b_F + r_1 \\
0
\end{bmatrix}
\]

\[ v(2)_{tr} = \omega(2) x r_f(1) + c x \omega(2) \]

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\[
\begin{bmatrix}
i \\
j \\
k \\
0 \\
x_f^{(1)} \\
y_f^{(1)} \\
z_f^{(1)}
\end{bmatrix}
= \begin{bmatrix}
i \\
j \\
k \\
0 \\
c \\
0 \\
\omega(2)
\end{bmatrix}
\]
\[
= m_{21} \omega(1) \begin{bmatrix}
-y_f^{(1)} \\
x_f^{(1)} - c \\
0
\end{bmatrix}
\]

\[
\gamma_i^{(12)} = \gamma_i(1) - \gamma_i(2)
\]
\[
= \gamma_i \begin{bmatrix}
(1 + m_{21})y_f^{(1)} \\
m_{21}c - (1 + m_{21})x_f^{(1)} \\
0
\end{bmatrix}
\]

\[
\omega_i^{(12)} = \omega_i(1) - \omega_i(2)
\]
\[
= \omega_i \begin{bmatrix}
0 \\
0 \\
-(1 + m_{21})
\end{bmatrix}
\]

\[
\hat{n}_f^{(1)} = \left[ L_{fc} \right] \left[ n_c^{(F)} \right]
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\sin \theta_F \\
\cos \theta_F \sin \lambda_F \\
\cos \theta_F \cos \lambda_F
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sin \theta_F \\
-\cos \theta_F \sin \lambda_F \\
\cos \theta_F \cos \lambda_F
\end{bmatrix}
\]

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Comparing the expressions in equations (6.20) - (6.25) and (6.40) - (6.45) we obtained the same results. Substituting these expressions into equation (6.1), we get the coefficients $a_{31}$, $a_{32}$
b₁, b₂ and b₃. Equations (6.1) represent a system of three linear equations with two unknowns:

\[
\begin{align*}
  x_1 &= v_r^{(1)} \cdot i^{(1)}, \\
  x_2 &= v_r^{(1)} \cdot i^{(12)}
\end{align*}
\] (6.46)

where \(v_r^{(1)}\) is the velocity of motion over the gear tooth surface \(\Sigma_1\). These equations provide a unique solution for \(x_1\) and \(x_2\) if the following condition is observed [5,6]:

\[
\begin{vmatrix}
  a_{11} & a_{12} & b_1 \\
  a_{21} & a_{22} & b_2 \\
  a_{31} & a_{32} & b_3
\end{vmatrix} = 0
\] (6.47)

It is assumed that surfaces \(\Sigma_1\) and \(\Sigma_2\) are in point contact at every instant. Using any two equations of the system equations (6.1) we may determine \(x_1\) and \(x_2\) and thus \(v_r^{(1)} = (x_1^{(2)} + x_2^{(2)})^{\frac{1}{2}}\).

An alternative method of deriving the relative velocity \(v_r^{(1)}\) at the point of contact \(M\) is:

**Step 1** The point of contact of surface \(\Sigma_1\) and \(\Sigma_2\) lies on a straight line which passes through the following equations:

\[
\begin{align*}
  x_f^{(1)} &= \rho_F \sin \theta_F - b_F + r_1 \\
  y_f^{(1)} &= -(\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F \\
  z_f^{(1)} &= \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi \tan \lambda_F
\end{align*}
\] (6.48)
Here, \( \theta_F = \theta_P = \theta = 30^\circ \) is the pressure angle at the point of contact of surface \( \Sigma_1 \) and \( \Sigma_2 \).

**Step 2** The transfer velocity \( v^{(1)}_{tr} \) of a point on gear 1 is:

\[
v^{(1)}_{tr} = \omega^{(1)} \times r^{(1)}_f
\]

\[
= \omega^{(1)} \begin{bmatrix}
y^{(1)}_f \\
x^{(1)}_f \\
0
\end{bmatrix}
\]

**Step 3** The direction of absolute velocity \( v^{(1)}_{abs} \) of point M of gear 1 is parallel to the axes of gear rotation \( Z^{(1)}_f \). Hence

\[
v^{(1)}_{abs} = \frac{dz^{(1)}_f}{dt}
\]

\[
= (- \rho_F \sin \theta_F \cos \lambda_F - \frac{b_F}{\sin^2 \theta_F} \tan \lambda_F \sin \lambda_F) \frac{d\theta_F}{dt} + (r_1 \tan \lambda_F) \frac{d\phi_1}{dt}
\]

Here: \( \frac{d\theta_F}{dt} = 0 \) and \( \frac{d\phi_1}{dt} = \omega^{(1)} \). Therefore, we have:

\[
v^{(1)}_{abs} = \begin{bmatrix}
0 \\
0 \\
r_1 \omega^{(1)} \tan \lambda_F
\end{bmatrix}
\]

**Step 4** The relative velocity \( v^{(1)}_r \) of point M of gear 1 is: \( v^{(1)}_r \) is tangent to the helix of gear 1.)
where \( v(\cdot) \) is the relative velocity of the contact point \( M \) of gear 2; \( v^{(12)} \) is the sliding velocity expressed in equation (6.7).

Example 6.1: Relative Velocity of Motion of the Contacting Ellipse Over the Gear Tooth Surface

The rack cutter and gear nominal parameters are the same as given in Example 4.1.

(1) We may determine the relative velocity in the motion of the contacting ellipse over the gear tooth surface \( \Sigma_1 \) by using equations (6.1) - (6.46). Then we obtain \( X_1 \) and \( X_2 \), thus

\[
v^{(1)}_r = \left( X_1^2 + X_2^2 \right)^{1/2} = 12.06574 \, \omega^{(1)}
\]

and \( \sigma = 16.1479^\circ \) is the angle formed by \( v^{(1)}_r \) and axis \( z_f^{(1)} \).
(2) By using equations (6.48) - (6.52) to solve \( \nu_r^{(1)} \), we obtain

\[
\nu_r^{(1)} = \nu_r^{(1)} - \nu_r^{(1)} = 12.06706 \omega^{(1)}
\]

(3) From equation (6.54), we have

\[
\nu_r^{(2)} = 11.99259 \omega^{(1)}
\]

and \( \beta = 14.8674^\circ \) is the angle formed by \( \nu_r^{(2)} \) and axis \( Z_f^{(2)} \).

Note: Due to a lot of computational procedures and matrix transformation for approach (1), there is a small difference between the approaches (1) and (2), and approach (2) is better than approach (1).

7 Computer Aided Simulation of Conditions of Meshing

We simulated the conditions of meshing of gears, which have some errors, using the equations of continuous tangency of gear tooth surfaces. We set up four coordinate systems: \( S_1 \) and \( S_2 \), rapidly connected to the gears and \( S_h \) and \( S_f \), rigidly connected to the frames. By using the coordinate transformations from \( S_1 \) via \( S_h \) to \( S_f \), we may represent the equations of the surfaces \( \Sigma_i \) \((i = 1, 2) \) and its surface normal in coordinate system \( S_f \).

The conditions of continuous tangency of gear tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are represented by the following equations [5, 6]:

\[
r_f^{(1)}(\theta_p, \phi_1, \mu_1) = r_f^{(2)}(\theta_p, \phi_2, \mu_2)
\]

(7.1)

\[
n_f^{(1)}(\theta_p, \mu_1) = n_f^{(2)}(\theta_p, \mu_2)
\]

(7.2)
Equation (7.1) expresses that surfaces $\Sigma_1$ and $\Sigma_2$ have a common point determined with the position vectors $\mathbf{r}_f^{(1)}$ and $\mathbf{r}_f^{(2)}$. Equation (7.2) indicates that surfaces $\Sigma_1$ and $\Sigma_2$ have a common unit normal at their common point. Equations (7.1) and (7.2) when considered simultaneously yield a system of only five independent equations, since $|n_f^{(1)}| = |n_f^{(2)}| = 1$. These five equations relate six unknowns: $\theta_F$, $\phi_1^i$, $\phi_2^i$, $\theta_p$, $\phi_2^i$, and thus one of these unknowns may be considered as a variable.

8 Influence of Manufacturing and Assembly Errors, and Adjustment of Gears to the Errors

(i) Change of Axes Distance

Fig. 8.1a and Fig. 8.1b show that the operating center distance $C'$ is not equal to the sum of the radic of pitch cylinders in this case; Thus $C' \neq r_1 + r_2$. Considering the gear tooth surface $\Sigma_1$ and its unit normal $n_1$, and gear tooth surface $\Sigma_2$ and its unit normal $n_2$ are represented in the coordinate systems $S_1$ and $S_2$, respectively. We may represent $\Sigma_i$ and $n_i$ ($i = 1, 2$) in the coordinate system $S_f$ using the following matrix equations:

$$
[r_f^{(i)}] = [M_{f1}] [r_i]
$$

(8.1)

where (Fig. 8.1):

$$
[M_{f1}] = \begin{bmatrix}
\cos \phi_1^i & \sin \phi_1^i & 0 & 0 \\
-sin \phi_1^i & \cos \phi_1^i & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

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Fig. 8.1
\[
\begin{bmatrix}
M_{f2}
\end{bmatrix} =
\begin{bmatrix}
\cos \phi_2' & -\sin \phi_2' & 0 & C' \\
\sin \phi_2' & \cos \phi_2' & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
\mathbf{n}_{fi}^{(i)}
\end{bmatrix} = \begin{bmatrix}
L_{fi}
\end{bmatrix} \begin{bmatrix}
\mathbf{n}_i
\end{bmatrix}
\]

where:
\[
\begin{bmatrix}
L_{f1}
\end{bmatrix} =
\begin{bmatrix}
\cos \phi_1' & \sin \phi_1' & 0 \\
-\sin \phi_1' & \cos \phi_1' & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
L_{f2}
\end{bmatrix} =
\begin{bmatrix}
\cos \phi_2' & -\sin \phi_2' & 0 \\
\sin \phi_2' & \cos \phi_2' & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\(\phi_1'\) and \(\phi_2'\) are the angles of rotation of the gear in mesh with the mating gear, while \(\phi_1\) and \(\phi_2\) are the angles of rotation of gear 1 and gear 2 in mesh with the corresponding rack cutter.

Using equations (8.1), (8.2) or (I.9) - (I.14) and (7.1), (7.2) yield the following procedure for computations:

**Step 1:** Using equations \(n_{zf}^{(1)} = n_{zf}^{(2)}\), we obtain

\[
\cos \theta_F \cos \lambda_F = \cos \theta_P \cos \lambda_P
\]

(8.3)
Equation (8.3) with $\lambda_F = \lambda_p = \lambda$ yields that

$$\theta_F = \theta_p = \theta$$ \hspace{1cm} (8.4)

Step 2: Using equations $n_{y_f}^{(1)} = n_{y_f}^{(2)}$, $y_f^{(1)} = y_f^{(2)}$ and $x_f^{(1)} = x_f^{(2)}$, we obtain the following system of three equations in three unknowns $(\theta, \mu_1$ and $\mu_2)$:

\[
\sin\theta \sin\mu_1 - \cos\theta \sin\lambda \cos\mu_1 = -\sin\theta \sin\mu_2 - \cos\theta \sin\lambda \cos\mu_2 \hspace{1cm} (8.5) \\
\left(\rho_F \sin\theta - b_F\right) \left(\sin\theta \sin\mu_1 - \cos\theta \sin\lambda \cos\mu_1\right) + r_1 \sin\theta \sin\mu_1 = \\
\left(\rho_F \sin\theta - b_F\right) \left(\sin\theta \sin\mu_2 - \cos\theta \sin\lambda \cos\mu_2\right) + r_2 \sin\theta \sin\mu_2 \hspace{1cm} (8.6) \\
\left(\rho_p \sin\theta - b_p\right) \left(\sin\theta \cos\mu_1 + \cos\theta \sin\sin\mu_1\right) + r_1 \sin\theta \cos\mu_1 = \\
\left(\rho_p \sin\theta - b_p\right) \left(\sin\theta \cos\mu_2 + \cos\theta \sin\sin\mu_2\right) - r_2 \sin\theta \cos\mu_2 + \\
C' \sin\theta \hspace{1cm} (8.7)
\]

Where: $C' = r_1 + r_2 + \Delta C$ and $\Delta C$ is the change of center distance.

The solution to these equations for $\theta$, $\mu_1$ and $\mu_2$ provides constant values whose magnitude depends on the operating center distance $C'$ only (the change of the center distance, $\Delta C$). The location of the center of the contacting ellipse is determined by $\theta \left(\Delta C\right)$. Thus, the bearing contact also depends on $\Delta C$.

We may check up the solution to equations (8.5), (8.6) and (8.7) using the equation $n_{x_f}^{(1)} = n_{x_f}^{(2)}$ which yields

$$\sin\theta \cos\mu_1 + \cos\theta \sin\lambda \sin\mu_1 = \sin\theta \cos\mu_2 - \cos\theta \sin\lambda \sin\mu_2$$

Step 3: Knowing $\theta$, we may determine the relation between para-
meters $\phi_1$ and $\phi_2$ using equation $z_f^{(1)} = z_f^{(2)}$ which yields

$$
\rho F \cos \theta \cos \lambda - \frac{a_F}{\cos \lambda} + b_F \cot \theta \tan \lambda \sin \lambda + r_1 \phi_1 \tan \lambda =
$$

$$
\rho p \cos \theta \cos \lambda - \frac{a_p}{\cos \lambda} + b_p \cot \theta \tan \lambda \sin \lambda + r_2 \phi_2 \tan \lambda
$$

Equation (8.9) provides a linear function which relates $\phi_1$ and $\phi_2$ since $\theta$ is constant.

**Step 4:** It is easy to prove that since $\theta, \mu_1$ and $\mu_2$ have constant values, the angular velocity ratio for the gears does not depend on the center-distance. The proof is based on the following considerations: (i) Equation (8.9) with $\theta = \text{const}$ yields that $r_1 d\phi_1 = r_2 d\phi_2$ and $\frac{d\phi_1}{d\phi_2} = \frac{r_2}{r_1}$; (ii) Since $\mu_1 = \phi_1 - \phi_1'$ and $\mu_2 = \phi_2 - \phi_2'$ are constant, we obtain that $d\phi_1 = d\phi_1'$, $d\phi_2 = d\phi_2'$, and

$$
m_{12} = \frac{\omega(1)}{\omega(2)} = \frac{d\phi_1'}{d\phi_2'} = \frac{r_2}{r_1}
$$

(8.10)

**Step 5:** It is evident that since $\theta, \mu_1$ and $\mu_2$ have constant values, the line of action of the gear tooth surfaces represents, in the fixed coordinate system $S_f$, a straight line which is parallel to the $z_f$ - axis. We may determine the coordinates $x_f^{(i)}$ and $y_f^{(i)}$ ($i = 1, 2$) of the line of action using equations (I.9) or (I.12) (see Appendix I). The location of the instantaneous point of contact on the line of action may be represented as a function of $\phi_1'$:
Step 6: We may also derive an approximate equation which relates $\theta$ and the change of the center-distance, $\Delta C$. Since $\mu_1$ and $\mu_2$ are small, we assume $\cos \mu_1 = 1$ and $\sin \mu_1 = 0$ in equation (8.7). We then obtain

$$z_f^{(1)} = \rho_F \cos \theta \cos \lambda - \frac{a_F}{\cos \lambda} + b_F \cot \theta \tan \lambda \sin \lambda + r_1 (\mu_1 + \phi') \tan \lambda$$  

(8.11)

$$\rho_F \sin \theta - b_F + r_1 = \rho_P \sin \theta - b_P - r_2 + C'$$  

(8.12)

where $C' = r_1 + r_2 + \Delta C$

Equation (8.12) yields

$$\sin \theta = \frac{\Delta C + b_F - b_P}{\rho_F - \rho_P}$$  

(8.13)

The nominal value of $\theta^\circ$ which corresponds to the theoretical value of the center distance $C$, where $C = r_1 + r_2$, is given by

$$\sin \theta^\circ = \frac{b_F^\circ - b_P^\circ}{\rho_F^\circ - \rho_P^\circ}$$  

(8.14)

Compensation for the Location of Bearing Contact Induced by $\Delta C$

The sensitivity of the gears to the change of center distance, $\Delta C$, may be reduced by increasing the difference $|\rho_F - \rho_P|$. However, this results in the increase of contacting stresses.

The dislocation of the bearing contact may be compensated for by refinishing of one of the gears (preferably the pinion)
with new tool settings.

Consider that $\theta^\circ$ is the nominal value for the pressure angle; $b_F^o$ and $b_P^o$ are the nominal values for the machine settings and $\rho_F^o, \rho_P^o$ are the nominal values for the radii of circular arcs. These parameters are related by equation (8.14). The location of the bearing contact won't be changed if the pinion is refinished with a new tool setting $b_F^F$ determined as follows (see equation (8.13):

\[ \sin\theta^\circ = \frac{\Delta C + b_F^F - b_P^o}{\rho_F^o - \rho_P^o} \]  

(8.15)

\[ b_F^F = b_F^o - \Delta C \]  

(8.16)

Change of Machine-Tool Settings $b_F^F$ and $b_P^o$

The change of machine-tool settings $b_F^F$ and $b_P^o$ causes: (i) the change of gear tooth thickness and backlash between the mating teeth, and (ii) the dislocation of the bearing contact. The most dangerous result is the dislocation of the bearing contact.

Using similar principles of investigation, we may represent the new value of the pressure angle which corresponds to the changed machine-tool settings by using the following equation

\[ \sin\theta = \frac{b_F^F - b_P^o}{\rho_F^o - \rho_P^o} \]  

(8.17)
Here: \( b_F \) and \( b_P \) are the changed settings; \( b_F \neq b_F^0 \), \( b_P \neq b_P^0 \), where \( b_F^0 \) and \( b_P^0 \) are the nominal machine-settings; \( \theta \neq \theta^0 \) is the new pressure angle.

We may compensate for the dislocation of the bearing contact making \( \theta = \theta^0 \). This can be achieved by refinishing of the pinion with a corrected setting \( \Delta b_F \). Similar to equation (8.15) we obtain

\[
\sin \theta^0 = \frac{b_F - b_P^0 + \Delta b_F}{\rho_F^0 - \rho_P^0}
\]

(ii) Misalignment of Crossed Axes of Gear Rotation

Consider that the axis of rotation of gear 1 is not parallel to the axis of rotation of gear 2 and form an angle \( \Delta \gamma \) (Fig. 8.2). The coordinate transformation from \( S_h \) to \( S_f \) is represented by the matrix equations

\[
\begin{bmatrix} r_f^{(1)} \end{bmatrix} = [M_{fh}] \begin{bmatrix} r_h^{(1)} \end{bmatrix}, \quad \begin{bmatrix} n_f^{(1)} \end{bmatrix} = [L_{fh}] \begin{bmatrix} n_h^{(1)} \end{bmatrix}
\]

(8.19)

where:

\[
[M_{fh}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & \cos \Delta \gamma & \sin \Delta \gamma & 0 & 0 \\
0 & -\sin \Delta \gamma & \cos \Delta \gamma & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \end{bmatrix}; \quad [L_{fr}] = \begin{bmatrix} 1 & 0 & 0 \\
0 & \cos \Delta \gamma & \sin \Delta \gamma \\
0 & -\sin \Delta \gamma & \cos \Delta \gamma \\
0 & 0 & 0 \end{bmatrix}
\]

Using equations (8.19), (1.9)-(1.12) and (7.1), (7.2), we may represent the tangency of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) for crossed misaligned
Fig. 8.2
gears as follows:

\[ A_2 \cos \mu_2 - B_2 \sin \mu_2 + C = A_1 \cos \mu_1 + B_1 \sin \mu_1 \quad (8.20) \]

\[-A_2 \sin \mu_2 - B_2 \cos \mu_2 = (A_1 \sin \mu_1 - B_1 \cos \mu_1) \cos \Delta \gamma +
\]
\[(\rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F) \sin \Delta \gamma \quad (8.21)\]

(see expressions (I.11) and (I.14) in Appendix I)

\[
\rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F) \cos \Delta \gamma
\]
\[
= -(A_1 \sin \mu_1 - B_1 \cos \mu_1) \sin \Delta \gamma + (\rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} +
\]
\[b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F) \cos \Delta \gamma \quad (8.22)\]

\[\sin \theta_F \cos \mu_2 - \cos \theta_F \sin \lambda_F \sin \mu_2 = \sin \theta_F \cos \mu_1 + \cos \theta_F \sin \lambda_F \sin \mu_1 \quad (8.23)\]

\[-\sin \theta_F \sin \mu_2 - \cos \theta_F \sin \lambda_F \cos \mu_2 = (\sin \theta_F \sin \mu_1 - \cos \theta_F \sin \lambda_F \cos \mu_1) \cos \Delta \gamma + \]
\[\cos \theta_F \cos \lambda_F \sin \Delta \gamma \quad (8.24)\]

\[\cos \theta_F \cos \lambda_F = -(\sin \theta_F \sin \mu_1 - \cos \theta_F \sin \lambda_F \cos \mu_1) \sin \Delta \gamma + \]
\[\cos \theta_F \cos \lambda_F \cos \Delta \gamma \quad (8.25)\]

Equations (8.20) - (8.25) form a system of five independent equations in six unknowns: \( \theta_F, \theta_p, \mu_1, \mu_2, \phi_1 \) and \( \phi_2 \). We remind that only two equations from equation system (8.23) - (8.25) are independent since \( n_f^3 = 1 \) and \( n_f^4 = 1 \).

The computational procedure is as follows: (i) We consider equations (8.20), (8.21), (8.24) and (8.25) which form a system of 4 equations in five unknowns: \( \theta_F, \theta_p, \mu_1, \mu_2 \) and \( \phi_1 \). Fixing in \( \phi_1 \) we may obtain the solutions by \( \theta_F(\phi_1), \theta_p(\phi_1), \mu_1(\phi_1) \) and \( \mu_2(\phi_1) \);

(ii) Using equation (8.22) we obtain \( \phi_2(\phi_1) \); (iii) Then, using the equations

\[ \phi'_1 = \phi_1 - \mu_1, \quad \phi'_2 = \phi_2 - \mu_2 \quad (8.26) \]
we can obtain the relation between the angles $\phi_2'$ and $\phi_1'$ of gear rotation. Function $\phi_2'(\phi_1')$ is a non-linear function and its deviation from the linear function is given by

$$\Delta \phi_2'(\phi_1') = \phi_2'(\phi_1') - \frac{N_1}{N_2} \phi_1'$$ (8.27)

Here: $\Delta \phi_2'(\phi_1')$ represents the kinematical errors of the gear train and $\theta_F(\phi_1')$ and $\theta_p(\phi_1')$ represent the change of location of the bearing contact induced by the misalignment of gear axes.

(iii) Misalignment of Intersected Axes of Gear Rotation

In the case of intersected axes of gear rotation, two axes form an angle $\Delta \gamma$ (Fig. 8.3). The coordinate transformation from $S_h$ to $S_f$ is represented by the matrix equations:

$$
\begin{bmatrix}
    r_f^{(1)} \\
    n_f^{(1)}
\end{bmatrix} =
\begin{bmatrix}
    M_{fh} & L_{fh}
\end{bmatrix}
\begin{bmatrix}
    r_h^{(1)} \\
    n_h^{(1)}
\end{bmatrix}
$$ (8.28)

where

$$M_{fh} =
\begin{bmatrix}
    \cos\Delta\gamma & 0 & -\sin\Delta\gamma & 0 \\
    0 & 1 & 0 & 0 \\
    \sin\Delta\gamma & 0 & \cos\Delta\gamma & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}$$
Fig. 8.3
\[
[L_{fh}] = \begin{bmatrix}
\cos \Delta \gamma & 0 & -\sin \Delta \gamma \\
0 & 1 & 0 \\
\sin \Delta \gamma & 0 & \cos \Delta \gamma
\end{bmatrix}
\]

using equations (8.28), (1.9) - (1.12) and (7.1), (7.2), we may represent the tangency of surface \( \Sigma_1 \) and \( \Sigma_2 \) for intersected misaligned gear axes as follows:

\[
A_2 \cos \mu_2 - B_2 \sin \mu_2 + C = (A_1 \cos \mu_1 + B_1 \sin \mu_1) \cos \Delta \gamma - \left( \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F \right) \sin \Delta \gamma
\]

(8.29)

\[
-A_2 \sin \mu_2 - B_2 \cos \mu_2 = A_1 \sin \mu_1 - B_1 \cos \mu_1
\]

(8.30)

\[
\rho_p \cos \theta_p \cos \lambda_p - \frac{a_p}{\cos \lambda_p} + b_p \cot \theta_p \sin \lambda_p \tan \lambda_p + r_2 \phi_2 \tan \lambda_p = (A_1 \cos \mu_1 + B_1 \sin \mu_1) \sin \Delta \gamma + \left( \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \right) \sin \lambda_F + r_1 \phi_1 \tan \lambda_F \cos \Delta \gamma
\]

(8.31)

\[
\sin \theta_p \cos \mu_2 - \cos \theta_p \sin \lambda_p \sin \mu_2 = (\sin \theta_p \cos \mu_1 + \cos \theta_p \sin \lambda_p \sin \mu_1)
\]

\[
\cos \Delta \gamma - \cos \theta_F \cos \lambda_F \sin \Delta \gamma
\]

(8.32)

\[
-\sin \theta_p \sin \mu_2 - \cos \theta_p \sin \lambda_p \cos \mu_2 = \sin \theta_p \sin \mu_1 - \cos \theta_p \sin \lambda_p \cos \mu_1
\]

(8.33)

\[
\cos \theta_p \cos \lambda_p = (\sin \theta_p \cos \mu_1 + \cos \theta_p \sin \lambda_p \sin \mu_1) \sin \Delta \gamma + \cos \theta_F \cos \lambda_F \cos \Delta \gamma
\]

(8.34)

Equations (8.29) - (8.34) form a system of five independent equations in six unknowns: \( \theta_p, \theta_F, \mu_1, \mu_2, \phi_1 \) and \( \phi_2 \). Only two equations from equation system (8.32) - (8.34) are independent. The computational procedure is the same as we discussed before.

Compensation for the Location of Bearing Contact Induced by the Gear Misalignment
The dislocation of the bearing contact induced by misalignment of the axes of gear rotation may be compensated for by the change of the lead angle $\lambda_F$ (or $\lambda_P$). This can be done technologically by refinishing of the pinion.

Example 8.1: The Influence of Change of Axes Distance

Given: the rack parameters (see Fig. 2.1 and Fig. 2.2); the gear parameters: No. of teeth $N_1 = 12$, $N_2 = 94$; lead angle $\lambda_F = \lambda_P = 75^\circ$ nominal pressure angle $\theta^o = 30^o$; normal diametral pitch $P_n = 2$; nominal axes distance $C = 27.43482$ in.; change of axes distance $\Delta C = 0.021$ in. Due to the change of axes distance the new value of the pressure angle $\theta$ is: (i) $\theta = 12.82082^o$ (exact solution provided by equation system (8.5) - (8.7)); (ii) $\theta = 12.70903^o$ (approximate solution provided by equation (8.13))

The compensation for the dislocation of bearing contact is achieved by the new machine setting $b_F = b_F^o - 0.021$ in. which provides $\theta = \theta^o = 30^o$ although $C = C^o + \Delta C$.

Example 8.2: The Influence of Misalignment of Crossed Gear Axes

The rack and gear nominal parameters are the same as shown in Example 8.1. The misalignment of crossed gear axes is given by $\Delta Y = 0.1^o$ (Fig. 8.2). The kinematical errors $\Delta \phi_2^i$ and the change of $\theta_F$ and $\theta_P$ are given in Table 1.

The compensation of kinematical errors is achieved with the change of the lead angle of the pinion $\lambda_F = 75.10^o$ ($\Delta \lambda_F = 0.10^o$). The kinematical errors after compensation are given in Table 2.
Table 1. Kinematical Errors

<table>
<thead>
<tr>
<th>φ</th>
<th>θ_F (°)</th>
<th>θ_P (°)</th>
<th>Δϕ'(_2) (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20°</td>
<td>32.2603°</td>
<td>31.6606°</td>
<td>59.88&quot;</td>
</tr>
<tr>
<td>-10°</td>
<td>32.2610°</td>
<td>31.6613°</td>
<td>29.94&quot;</td>
</tr>
<tr>
<td>0°</td>
<td>32.2613°</td>
<td>31.6616°</td>
<td>0.00&quot;</td>
</tr>
<tr>
<td>10°</td>
<td>32.2613°</td>
<td>31.6615°</td>
<td>-29.94&quot;</td>
</tr>
<tr>
<td>20°</td>
<td>32.2609°</td>
<td>31.6611°</td>
<td>-59.89&quot;</td>
</tr>
</tbody>
</table>

Table 2. Compensated Kinematical Errors

<table>
<thead>
<tr>
<th>φ</th>
<th>θ_F (°)</th>
<th>θ_P (°)</th>
<th>Δϕ'(_2) (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20°</td>
<td>29.9988°</td>
<td>29.9989°</td>
<td>-0.00&quot;</td>
</tr>
<tr>
<td>-10°</td>
<td>29.9996°</td>
<td>29.9996°</td>
<td>-0.00&quot;</td>
</tr>
<tr>
<td>0°</td>
<td>29.9999°</td>
<td>29.9999°</td>
<td>0.00&quot;</td>
</tr>
<tr>
<td>10°</td>
<td>30.0000°</td>
<td>29.9995°</td>
<td>-0.00&quot;</td>
</tr>
<tr>
<td>20°</td>
<td>29.9996°</td>
<td>29.9995°</td>
<td>-0.00&quot;</td>
</tr>
</tbody>
</table>

By using the proposed method of compensation we could reduce substantially the kinematical errors induced by the misalignment of crossed axes of gear rotation (kinematical errors approach zero).

Example 8.3: The Influence of Misalignment of Intersected Gear Axes

The rack and gear nominal parameters are the same as shown in Example 8.1. The misalignment of intersected gear axes is given by Δγ = 0.1° (Fig. 8.3). The kinematical errors Δϕ'\(_2\) and the change of θ_F and θ_P are given in Table 3. The compensation of kinematical errors is achieved with the change of the lead angle of the pinion λ_F = 75.06° (Δλ_F = 0.06°). The kinematical errors with compensation are given in Table 4.

Table 3. Kinematical Errors

<table>
<thead>
<tr>
<th>φ</th>
<th>θ_F (°)</th>
<th>θ_P (°)</th>
<th>Δϕ'(_2) (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20°</td>
<td>38.1358°</td>
<td>37.7508°</td>
<td>42.57&quot;</td>
</tr>
<tr>
<td>-10°</td>
<td>34.7987°</td>
<td>34.4138°</td>
<td>19.98&quot;</td>
</tr>
<tr>
<td>0°</td>
<td>31.5911°</td>
<td>31.2063°</td>
<td>-0.00&quot;</td>
</tr>
<tr>
<td>10°</td>
<td>28.4910°</td>
<td>28.1062°</td>
<td>-17.65&quot;</td>
</tr>
<tr>
<td>20°</td>
<td>25.4817°</td>
<td>25.0969°</td>
<td>-33.19&quot;</td>
</tr>
</tbody>
</table>

Table 4. Compensated Kinematical Errors

<table>
<thead>
<tr>
<th>φ</th>
<th>θ_F (°)</th>
<th>θ_P (°)</th>
<th>Δϕ'(_2) (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20°</td>
<td>37.1275°</td>
<td>37.0376°</td>
<td>5.17&quot;</td>
</tr>
<tr>
<td>-10°</td>
<td>33.5327°</td>
<td>33.4847°</td>
<td>1.21&quot;</td>
</tr>
<tr>
<td>0°</td>
<td>29.9883°</td>
<td>29.9898°</td>
<td>0.00&quot;</td>
</tr>
<tr>
<td>10°</td>
<td>26.4191°</td>
<td>26.4823°</td>
<td>1.30&quot;</td>
</tr>
<tr>
<td>20°</td>
<td>22.6873°</td>
<td>22.8334°</td>
<td>4.98&quot;</td>
</tr>
</tbody>
</table>
By using the proposed method of compensation, we could reduce the kinematical errors induced by the misalignment of intersected axes of gear rotation.

9. Computer Aided Simulation of Bearing Contact (With Computer Graphics)

We simulated the bearing contact of gears by setting up two coordinate systems: $S_1$ and $S_2$, rigidly connected to the gear 1 (pinion) and gear 2, respectively (Fig. 3.1). Due to the computer graphics system, the figures showed in this section are two dimensional computer graphics.

Fig. 9.1 showed the normal crossed section of gear 1 (pinion) in coordinate system $S_1$ (Fig. 3.1 a), there are 12 teeth on the gear 1. We simulated gear 1 by considering the equations (3.6) - (3.12), the $x_1$ and $y_1$ axes are the axes of symmetry of the normal section. As we discussed in Chapter 3, it is important to mention that the normal section of the "fillet" of gear 1 can be simulated by using the same equations which we simulated the normal section of the working part and substituted $\rho_F$, $\theta_F$, $b_F$ and $a_F$ by $\rho_{F}^{(f)}$, $\theta_{F}^{(f)}$, $b_{F}^{(f)}$, and $a_{F}^{(f)}$ in equations (3.6) - (3.12).

Fig. 9.2 showed the normal cross section of gear 2 in coordinate system $S_2$ (Fig. 3.1 b), there are 94 teeth on the
FILE NAME: G12P

PLOT OF GEAR 1

Fig. 9.1
gear 2. We simulated gear 2 by considering the equations (3.16) - (3.21), the $x_2$ and $y_2$ axes are the axes of symmetry of this normal section. As we mentioned above, the normal section of the "fillet" of gear 2 can be simulated by using the same equations which we simulated the normal section of the working part, and then substituted $\rho_{P}, \theta_{P}, b_{P}$ and $a_{P}$ by $\rho^{(f)}_{P}, \theta^{(f)}_{P}, b^{(f)}_{P}$ and $a^{(f)}_{P}$ in equations (3.16) - (3.21).

Fig. 9.3 showed the front view of gear 1 and the orientation of contacting ellipse of bearing contact when the center distance did not change. It should be mentioned that the size of contacting ellipse showed in Fig. 9.3 depended on the value of elastic approach $\delta$. Also, the contacting ellipses showed here was a side view (the projection on $x - z$ plane). Fig. 9.4 showed the same case for the gear 2.

Fig. 9.5 showed the bearing contact of gear 1 due to an increased of center distance 0.02 inches. Fig. 9.6 showed the same case for the gear 2. From these two figures, we found that the size and magnitudes of two axes of contacting ellipse are changed a lot, this prove that circular arc helical gears are very sensitive to the change of center-distance.

Fig. 9.7 showed the bearing contact of gear 1 due to the misalignment of crossed axes of gear rotation for 1.0 degree. The size and magnitudes of two axes of contacting ellipse were not changed significantly. Fig. 9.8 showed the same case for the gear 2. Fig. 9.9 showed the kinematical errors due to the misalignment of crossed axes of gear rotation for 1.0 degree with and without
FILE NAME: G1CC (TS0)

BEARING CONTACT OF GEAR 1 DUE TO CHANGE OF CENTER DISTANCE = 0.0 INCH

Fig. 9.3
FILE NAME: G2CC (TS0)

BEARING CONTACT OF GEAR 2 DUE TO
CHANGE OF CENTER DISTANCE = 0.0 INCH

Fig. 9.4
Fig. 9.5
FILE NAME: G2CC (TS0)

BEARING CONTACT OF GEAR 2 DUE TO CHANGE OF CENTER DISTANCE = 0.02 INCH

Fig. 9.6
FILE NAME: G2MA (TS0)

BEARING CONTACT OF GEAR 2
DUE TO AXIAL MISALIGNMENT = 1.0 DEG

Fig. 9.8
KINEMATICAL ERRORS DUE TO MISALIGNMENT OF AXIS = 1.0 DEG

WITHOUT COMPENSATION:

WITH COMPENSATION:

(LAMDA = 75.937 DEG)

Fig. 9.9
compensation. From this, we prove that using the proposed method of compensation discussed in Chapter 8, we can cause the kinematical errors to approach zero.

10. Conclusion

The authors have presented a method of generation of tooth surfaces for circular arc helical gears, derived the basic equations which represent the geometry of gears, and proposed a computer aided method for simulation of conditions of meshing and of the bearing contact for these gears. The sensitivity of the gears to the change of center-distance, machine-tool settings and to the misalignment of axes of gear rotation have been investigated. A technological technique for the compensation of the dislocation of the bearing contact induced by the above errors have been proposed.
11. References


*The new edition (in English), revised and completed, sponsored by NASA, is in press.
12. Appendix I  Gear Tooth Surfaces

Gear 1 Tooth Surface. Substituting subscript "i" by "F" in equations (2.4) and (2.6) and taking into account that \( b_F > 0 \), we obtain:

\[
\begin{bmatrix}
\rho_F \sin \theta_F - b_F \\
-(\rho_F \cos \theta_F - a_F) \sin \lambda_F + u_F \cos \lambda_F \\
(\rho_F \cos \theta_F - a_F) \cos \lambda_F + u_F \sin \lambda_F \\
1
\end{bmatrix}
\]

Equations (I.1) and (I.2) represent the generating surface \( \Sigma_F \) and the unit normal to this surface. We may derive the equation of meshing using equations (I.1), (I.2) and (3.2) with

\[
X^{(F)}_c = 0, \quad Y^{(F)}_c = r_1 \phi_1, \quad Z^{(F)}_c = \lambda
\]

where \( X^{(F)}_c, Y^{(F)}_c \) and \( Z^{(F)}_c \) are coordinates of the point of intersection of the normal to \( \Sigma_F \) and the instantaneous axis of rotation, I-I (Fig. 3.1, a). We then obtain

\[
f_F(u_F, \theta_F, \phi_1) = (r_1 \phi_1 - u_F \cos \lambda_F - a_F \sin \lambda_F) \sin \theta_F + b_F \cos \theta_F \sin \lambda_F = 0
\]
Equation of meshing (1.4) yields
\[ u_F = \frac{r_1\phi_1 - a_F\sin\lambda_F}{\cos\lambda_F} + b_F\cot\theta_F\tan\lambda_F \]  
\[(I.5)\]

Equations (I.1) and (I.5) when considered simultaneously represent a family of contacting lines on surface \( \Gamma_F \). Eliminating \( u_F \), we may represent this family of lines of contact as follows:

\[
\begin{bmatrix}
  x_c^{(F)} \\
  y_c^{(F)} \\
  z_c^{(F)}
\end{bmatrix} =
\begin{bmatrix}
  \rho_F\sin\theta_F - b_F \\
  (\rho_F\sin\theta_F - b_F)\cot\theta_F\sin\lambda_F + r_1\phi_1 \\
  (\rho_F\sin\theta_F + b_F\tan^2\lambda_F)\cot\theta_F\cos\lambda_F - \frac{a_F}{\cos\lambda_F} + r_1\phi_1\tan\lambda_F
\end{bmatrix}
\]
\[(I.6)\]

Using equations (I.6) and the coordinate transformation from \( S^{(F)}_c \) to \( S_1 \) we obtain

\[
\begin{align*}
x_1 &= (\rho_F\sin\theta_F - b_F + r_1)\cos\phi_1 + (\rho_F\cos\theta_F - b_F\cot\theta_F)\sin\phi_1\sin\lambda_F \\
y_1 &= (\rho_F\sin\theta_F - b_F + r_1)\sin\phi_1 - (\rho_F\cos\theta_F - b_F\cot\theta_F)\cos\phi_1\sin\lambda_F \\
z_1 &= \rho_F\cos\theta_F\cos\lambda_F - \frac{a_F}{\cos\lambda_F} + b_F\cot\theta_F\tan\lambda_F\sin\lambda_F + r_1\phi_1\tan\lambda_F
\end{align*}
\]
\[(I.7)\]

The surface unit normal is given by
\[
[n_1] = \begin{bmatrix}
  \sin\theta_F\cos\phi_1 + \cos\theta_F\sin\lambda_F\sin\phi_1 \\
  \sin\theta_F\sin\phi_1 - \cos\theta_F\sin\lambda_F\cos\phi_1 \\
  \cos\theta_F\cos\lambda_F
\end{bmatrix}
\]
\[(I.8)\]

Using the coordinate transformation from \( S_1 \) to \( S_h \) we obtain
\[
\begin{align*}
    x_h^{(1)} &= A_1 \cos \mu_1 + B_1 \sin \mu_1 \\
    y_h^{(1)} &= A_1 \sin \mu_1 - B_1 \cos \mu_1 \\
    z_h^{(1)} &= \rho_F \cos \theta_F \cos \lambda_F - \frac{a_F}{\cos \lambda_F} + b_F \cot \theta_F \tan \lambda_F \sin \lambda_F + r_1 \phi_1 \tan \lambda_F
\end{align*}
\]  

(I.9)

\[
\begin{bmatrix}
    \sin \theta_F \cos \mu_1 + \cos \theta_F \sin \lambda_F \sin \mu_1 \\
    \sin \theta_F \sin \mu_1 - \cos \theta_F \sin \lambda_F \cos \mu_1 \\
    \cos \theta_F \cos \lambda_F
\end{bmatrix}
\]

[I.10]

Here:

\[
A_1(\theta_F) = \rho_F \sin \theta_F - b_F + r_1, \quad B_1(\theta_F) = (\rho_F \cos \theta_F - b_F \cot \theta_F) \sin \lambda_F,
\]

and \(\mu_1 = \phi_1 - \phi_1^1\)  

(I.11)

Equations (I.9) and (I.10) with a fixed value for \(\phi_1^1\), represent in the coordinate system \(S_h\), surface \(\Sigma_1\) and the unit normal to \(\Sigma_1\). These equations with different values for \(\phi_1^1\), represent in \(S_h\), a family of surfaces \(\Sigma_1\) and the unit normals to these surfaces.

The derivation of equations for gear 2 surface \(\Sigma_2\) and its unit normal is based on similar considerations. We may represent these equations in \(S_f\) as follows:

\[
\begin{align*}
    x_f^{(2)} &= A_2 \cos \mu_2 - B_2 \sin \mu_2 + C \\
    y_f^{(2)} &= -A_2 \sin \mu_2 - B_2 \cos \mu_2 \\
    z_f^{(2)} &= \rho_p \cos \theta_p \cos \lambda_p - \frac{a_p}{\cos \lambda_p} + b_p \cot \theta_p \sin \lambda_p \tan \lambda_p + r_2 \phi_2 \tan \lambda_p
\end{align*}
\]

(I.12)
\[
\begin{bmatrix}
\sin\theta_p\cos\mu_2 - \cos\theta_p\sin\mu_p\sin\mu_2 \\
-\sin\theta_p\sin\mu_2 - \cos\theta_p\sin\lambda_p\cos\mu_2 \\
\cos\theta_p\cos\lambda_p
\end{bmatrix}
\]

\[\text{(I.13)}\]

Here:

\[A_2(\theta_p) = \rho_p\sin\theta_p - b_p - r_2, \quad B_2(\theta_p) = (\rho_p\cos\theta_p - b_p\cot\theta_p)\sin\lambda_p,\]

and \(\mu_2 = \phi_2 - \phi'_2\)

\[\text{(I.14)}\]

The nominal value of the center distance is \(C = r_1 + r_2\).
List of Symbols

(Note: i = 1,2; d = F, P)

a
Half the length of major axis of contacting ellipse.

a_d
Algebraic values which determine the location of the center of the circular arc.

a_{31}^{(1)}
Auxiliary function defined in Eq. (4.11)

a_{32}^{(1)}
" " Eq. (4.12)

a_{31}^{(2)}
" " Eq. (4.43)

a_{32}^{(2)}
" " Eq. (4.44)

A
" " Eq. (5.1)

A_1
" " Eq. (3.9)

A_{II}
" " Eq. (3.19)

b
Half the length of minor axis of contacting ellipse.

b_d
a parameter of tool setting

b_0^d
Nominal value for the machine settings

b_{3}^{(1)}
Auxiliary function defined in Eq. (4.13)

b_{3}^{(2)}
" " Eq. (4.45)

B
" " Eq. (5.1)

B_1
" " Eq. (3.9)

B_{II}
" " Eq. (3.19)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_F$</td>
<td>center of working part of circular arc rack cutter F</td>
</tr>
<tr>
<td>$C_P^{(f)}$</td>
<td>center of the fillet of circular arc rack cutter F</td>
</tr>
<tr>
<td>$C_P$</td>
<td>center of working part of circular arc rack cutter P</td>
</tr>
<tr>
<td>$C_P^{(f)}$</td>
<td>center of the fillet of circular arc rack cutter P</td>
</tr>
<tr>
<td>$D_I$</td>
<td>Auxiliary function defined in Eq. (3.9)</td>
</tr>
<tr>
<td>$D_{II}$</td>
<td>&quot; &quot; Eq. (3.19)</td>
</tr>
<tr>
<td>$E_I$</td>
<td>&quot; &quot; Eq. (3.9)</td>
</tr>
<tr>
<td>$E_{II}$</td>
<td>&quot; &quot; Eq. (3.19)</td>
</tr>
<tr>
<td>$F^{(i)}$</td>
<td>Auxiliary function defined in Eq. (4.9), Eq. (4.46) to compute the principal directions of surface $\Sigma_i$</td>
</tr>
<tr>
<td>$g_1 = K_I^{(1)} - K_{II}^{(1)}$</td>
<td>Auxiliary function defined in Eq. (5.1) to determine the size of contacting ellipse</td>
</tr>
<tr>
<td>$g_2 = K_I^{(2)} - K_{II}^{(2)}$</td>
<td>Auxiliary function defined in Eq. (5.1) to determine the size of contacting ellipse</td>
</tr>
<tr>
<td>$G^{(i)}$</td>
<td>Auxiliary function defined in Eq. (4.9), Eq. (4.47) to compute the principal curvatures of surface $\Sigma_i$</td>
</tr>
<tr>
<td>$i_{II}^{(d)}, i_{II}^{(d)}$</td>
<td>unit vectors along principal direction of surface $\Sigma_d$</td>
</tr>
<tr>
<td>$K_I^{(i)}, K_{II}^{(i)}$</td>
<td>principal curvatures of surface $\Sigma_i$</td>
</tr>
</tbody>
</table>
\begin{align*}
\kappa^{(1)}_I &= \kappa^{(1)}_I + \kappa^{(1)}_{II} \quad \text{Auxiliary function defined in Eq. (5.1)} \\
\kappa^{(2)}_I &= \kappa^{(2)}_I + \kappa^{(2)}_{II} \quad \text{Auxiliary function defined in Eq. (5.1)} \\
\left[ L_{ij} \right] &= \text{projection transformation matrix; transformation from } S_j \text{ to } S_i \\
M &= \text{point of contact of tooth surface} \\
\left[ M_{ij} \right] &= \text{coordinate transformation matrix; transformation from } S_j \text{ to } S_i \\
\hat{n}^{(d)} &= \text{surface } d \text{ unit normal} \\
\hat{n}^{(i)} &= \text{relative velocity of the tip of the unit normal vector } \hat{n}_i \\
N^{(d)} &= \text{surface } d \text{ normal vector} \\
P_n &= \text{Diametral pitch in normal section} \\
r_c &= \text{position vector represented in the coordinate system } S_c \\
r_i &= \text{Pitch radius of gear } i \\
r_i^{(u_i, \theta_i)} &= \text{surface } \Sigma_i \text{ position vector with surface coordinates } (u_i, \theta_i) \\
S_f &= \text{coordinate system rigidly connected with frame} \\
S_h &= \text{Auxiliary coordinate system } h \\
S_i(x_i, y_i, z_i) &= \text{coordinate system rigidly connected with gear } i \\
s^{(i)} &= \text{Auxiliary function defined in Eq. (4.10), Eq. (4.48)}
\end{align*}
\( u_d \)  
\[ \text{generating surface coordinate} \]

\( v^{(1)}_{\text{abs}} \)  
\[ \text{Absolute velocity of the point on the surface } \Sigma_i \]

\( v^{(12)}_{\text{rel}} \)  
\[ \text{Relative velocity represented in coordinate system } S_f \text{ of a contact point on surface } \Sigma_1 \text{ with respect to contact point on surface } \Sigma_2. \]

\( v^{(i)}_{\text{rel}} \)  
\[ \text{Relative velocity of contact point on surface } \Sigma_i \]

\( v^{(i)}_{\text{tr}} \)  
\[ \text{Transfer velocity of contact point on surface } \Sigma_i \]

\( v^{(1)}_{\text{tr}} \)  
\[ \text{Transfer velocities of points on surface } \Sigma_i \text{ in coordinate system } 1. \]

\( v^{(21)}_{\text{rel}} \)  
\[ \text{Relative velocity of point 2 with respect to point 1} \left( v^{(21)}_{\text{rel}} = v^{(2)}_{\text{rel}} - v^{(1)}_{\text{rel}} \right) \]

\( \alpha \)  
\[ \text{Angle of the orientation of contacting ellipse measured from axis } \eta \text{ to the unit vector } \xi_1 \]

\( \Delta C \)  
\[ \text{Change of center distance (inches)} \]

\( \Delta \gamma \)  
\[ \text{misalignment of gear rotation axes} \]

\( \delta \)  
\[ \text{Approach of surface } \Sigma_1 \text{ and } \Sigma_2 \]

\( \theta^o \)  
\[ \text{Nominal value of the pressure angle} \]

\( \theta_d \)  
\[ \text{pressure angle of gear } d \]

\( \theta_i \)  
\[ \text{variable parameter which determines the location of a point on circular arc gear } i \]

\( \lambda_i \)  
\[ \text{helical gear } i \text{ lead angle} \]
\[ \mu_1 = \phi_1 - \phi'_1 \] Auxiliary function

\[ \mu_2 = \phi_2 - \phi'_2 \] Auxiliary function

\[ \rho_d \] Radius of working part of circular arc rack cutter \( d \)

\[ \rho_d^o \] Nominal value for the radius of circular arc

\[ \rho_d(f) \] Radius of fillet of circular arc rack cutter \( d \)

\[ \Sigma_d \] generating surface \( d \)

\[ \Sigma_i \] generated surface of pinion and gear

\[ \sigma \] Angle form by principal direction of two surfaces measured from \( i_I^{(1)} \) to \( i_I^{(2)} \) and positive angle for counterclockwise

\[ \sigma(F1) \] Angle measured from \( i_I^{(F)} \) to the unit vector \( i_I^{(1)} \); positive if counterclockwise

\[ \sigma(P2) \] Angle measured from \( i_I^{(P)} \) to the unit vector \( i_I^{(2)} \); positive if counterclockwise

\[ \phi_i \] gear \( i \) rotation angle in mesh with the corresponding rack cutter

\[ \phi'_i \] gear \( i \) rotation angle in mesh with the mating gear

\[ \Delta\phi'_2 \] kinematical error function defined in Eq. (8.27)

\[ \psi_c \] pressure angle

\[ \omega_{(i)} \] gear \( i \) angular velocity
# Helical Gears With Circular Arc Teeth: Generation, Geometry, Precision and Adjustment to Errors, Computer Aided Simulation of Conditions of Meshing, and Bearing Contact

**Authors:** Faydor L. Litvin and Chung-Biau Tsay

**Abstract**

The authors have proposed a method for generation of circular arc helical gears which is based on application of standard equipment, worked out all aspects of the geometry of gears, proposed methods for the computer aided simulation of conditions of meshing and bearing contact, investigated the influence of manufacturing and assembly errors, and proposed methods for the adjustment of gears to these errors. The results of computer aided solutions are illustrated with computer graphics.

**Key Words (Suggested by Author(s))**

- Gears
- Circular arc gear teeth
- Gear transmission error
- Machine design
- Kinematics

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