ISOTROPIC PROBABILITY MEASURES IN INFINITE DIMENSIONAL SPACES

(Inverse Problems/Prior Information/Stochastic Inversion)

George Backus

Institute of Geophysics & Planetary Physics, A-025
University of California, San Diego
La Jolla, CA 92093

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Abstract. Every isotropic probability measure on the space $\mathbb{R}^\infty$ of real sequences $x = (x_1, x_2, \ldots)$ is a convex combination of the measure concentrated at 0 and a member of $I_0(\mathbb{R}^\infty)$, the set of all isotropic probability measures $p_\infty$ on $\mathbb{R}^\infty$ with $p_\infty(\{0\}) = 0$. Each $p_\infty \in I_0(\mathbb{R}^\infty)$ is completely determined by any one of its finite-dimensional marginal distributions $p_n$. Each $p_n$ has a density function $f_n$ with $dp_n(x_1, \ldots, x_n) = dx_1 \cdots dx_n f_n(x_1^2 + \cdots + x_n^2)$. Each $f_n$ is completely monotone in $0 < \xi < \infty$ (hence analytic in the right complex $\xi$ half-plane), and

$$\pi^{n/2} \Gamma(n/2)^{-1} \int_0^\infty d\xi \xi^{n/2-1} f_n(\xi) = 1.$$ 

Every $f$ which satisfies these two conditions is $f_n$ for a unique $p_\infty \in I_0(\mathbb{R}^\infty)$. Hence the equation

$$\pi \int_0^\infty d\xi f_2(\xi) = \int_0^\infty d\mu(t) e^{-t\xi}$$

defines a bijection between $I_0(\mathbb{R}^\infty)$ and the set of all probability measures $\mu$ on $0 \leq t < \infty$. If $p_\infty \in I_0(\mathbb{R}^\infty)$ then $p_\infty(\{x : \sum_{i=1}^\infty x_i^2 < \infty\}) = 0$, so $p_\infty$ is not a "softened" or "fuzzy" version of the inequality $\sum_{i=1}^\infty x_i^2 \leq 1$. If the prior information in a linear inverse problem consists of this inequality and nothing else, stochastic inversion and Bayesian inference are both unsuitable inversion techniques.

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Introduction. Let $R$ be the real numbers, $R^n$ the linear space of all real $n$-tuples, and $R^\infty$ the linear space of all infinite real sequences $x = (x_1, x_2, \ldots)$. Let $P_n : R^\infty \to R^n$ be the projection operator with $P_n(x) = (x_1, \ldots, x_n)$. Let $p_\infty$ be a probability measure on the smallest $\sigma$-ring of subsets of $R^\infty$ which includes all of the cylinder sets $P_n^{-1}(B_n)$, where $B_n$ is an arbitrary Borel subset of $R^n$. Let $p_n$ be the marginal distribution of $p_\infty$ on $R^n$, so $p_n(B_n) = p_\infty(P_n^{-1}(B_n))$ for each $B_n$. A measure on $R^n$ is "isotropic" if it is invariant under all orthogonal transformations of $R^n$. The measure $p_\infty$ will be called isotropic if all its marginal distributions $p_n$ are isotropic. The set of all isotropic probability distributions on $R^\infty$ will be written $I(R^\infty)$. The present note describes all members of $I(R^\infty)$. The result calls into question both stochastic inversion and Bayesian inference, as currently used in many geophysical inverse problems.

Necessary Conditions for Isotropy. Let $0 = (0, 0, \cdots)$ and let $p_\infty^0$ be the member of $I(R^\infty)$ such that $p_\infty^0(\{0\}) = 1$. If $p_\infty \in I(R^\infty)$ and $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, then $\alpha p_\infty + \beta p_\infty^0 \in I(R^\infty)$. Conversely, if $p_\infty \in I(R^\infty)$ and $p_\infty(\{0\}) = \beta$, then $p_\infty = (1-\beta)p_\infty + \beta p_\infty^0$ where $p_\infty \in I(R^\infty)$ and $p_\infty(\{0\}) = 0$. Therefore it is necessary to study only those $p_\infty \in I(R^\infty)$ for which $p_\infty(\{0\}) = 0$. They constitute the subset $I_0(R^\infty)$ of $I(R^\infty)$.

If $p_\infty \in I_0(R^\infty)$, for every $\xi$ in $0 \leq \xi < \infty$ define

$$F_n(\xi) = p_\infty(\{x : x_1^2 + \cdots + x_n^2 > \xi\}).$$

Then $F_n$ is right semi-continuous, and

$$F_n(0) = 1 \quad [2a]$$
$$F_n(\infty) = \lim_{\xi \to \infty} F_n(\xi) = 0. \quad [2b]$$

Also, if $n \leq N$ and $\alpha \leq A$, then

$$0 \leq F_n(A) \leq F_n(\alpha) \leq F_N(\alpha) \leq 1. \quad [2c]$$

Properties sufficient to characterize the members of $I_0(R^\infty)$ are given in

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Theorem 1: Suppose $p_n \in I_0(R^\infty)$ and $F_n$ given by [1]. Then for each integer $n \geq 1$, $F_n(\xi)$ is analytic in the open right half plane of complex $\xi$. There is a function $f_n(\xi)$, also analytic there, such that for every Borel subset $B_n$ of $R^n$

$$p_n(B_n) = \int_{B_n} dx_1 \cdots dx_n f_n(x_1^2 + \cdots + x_n^2). \tag{3a}$$

In particular, if $0 \leq \alpha < \infty$ then

$$F_n(\alpha) = \pi^{n/2} \Gamma(n/2)^{-1} \int_{\alpha}^{\infty} d\xi \xi^{n/2-1} f_n(\xi). \tag{3b}$$

The $f_n$ are related by

$$f_n(\xi) = \int_\xi \frac{d\eta}{(\eta-\xi)^{1/2}} f_{n+1}(\eta) \tag{3c}$$

$$f_{n+1}(\xi) = -\pi^{-1} \frac{\partial}{\partial \xi} \int_\xi \frac{d\eta}{(\eta-\xi)^{1/2}} f_n(\eta) \tag{3d}$$

$$f_n(\xi) = \pi \int_\xi \frac{d\eta}{\xi} f_{n+2}(\eta) \tag{3e}$$

$$f_{n+2}(\xi) = -\pi^{-1} \frac{\partial}{\partial \xi} f_n(\xi) \tag{3f}$$

For every $\beta$ in $0 \leq \beta < \infty$

$$\lim_{n \to \infty} F_n(\beta) = 1. \tag{3g}$$

**PROOF:** Let $S(n-1)$ denote the unit sphere in $R^n$, and let $1S(n-1)$ be its $(n-1)$-dimensional Euclidean content, $2\pi^{n/2} \Gamma(n/2)^{-1}$. Let $1S(n-1) | \phi_n(w)$ be the content of the part of $S(n-1)$ where $x_n^2 \leq 1 - w$. Then

$$\phi_{n+1}(w) = 1 - 1S(n-1) 1S(n) 1^{-1} \int_0^w d\zeta \zeta^{n/2}(1-\zeta)^{-1/2}.$$

Since $p_n$ is the marginal distribution on $R^n$ of $p_{n+1}$ on $R^{n+1}$,

$$F_n(\xi) = -\int_\xi \frac{dF_{n+1}(\eta)}{\xi} \phi_{n+1}(\xi | \eta). \tag{4a}$$

the right side being a Stieltjes integral. For any $\beta$ and $B$ satisfying $\xi < \beta < B$, $\partial_{\eta} \phi_{n+1}(\xi | \eta)$ is continuous in $\beta \leq \eta \leq B$, so integration by parts (1) permits the conclusion
\[
\begin{align*}
\int_{\beta}^{B} dF_{n+1}(\eta) \phi_{n+1}(\xi/\eta) + \int_{\beta}^{B} d\eta F_{n+1}(\eta) \partial_{\eta} \phi_{n+1}(\xi/\eta) \\
= F_{n+1}(B) \phi_{n+1}(\xi/B) - F_{n+1}(\beta) \phi_{n+1}(\xi/\beta).
\end{align*}
\]

Here let \( \beta \to \xi^+ \) and \( B \to \infty \). The integrated parts tend to zero, so the Lebesque bounded convergence theorem permits \([4a]\) to be rewritten

\[
\xi^{-n/2} F_n(\xi) = \int_{-\infty}^{\infty} d\eta \eta^{-(n+1)/2} F_{n+1}(\eta) (\eta - \xi)^{-1/2}.
\]

Iterating this formula once, reversing orders of integration, and invoking the identity

\[
\int_{\xi}^{\zeta} d\eta (\zeta - \eta)^{-1/2} (\eta - \xi)^{-1/2} = \pi
\]

leads to

\[
\xi^{-n/2} F_n(\xi) = (n/2) \int_{\xi}^{\infty} d\zeta \zeta^{-(n+2)/2} F_{n+2}(\zeta).
\]  \[[4b]\]

By induction on \( n \), it follows that \( F_n(\xi) \) is infinitely differentiable in \( 0 < \xi < \infty \). If we define

\[
f_n(\xi) = -\pi^{-n/2} \Gamma(n/2) \zeta^{1-n/2} \partial_{\zeta} F_n(\zeta),
\]  \[[5a]\]

then \( f_n \) is also infinitely differentiable in \( 0 < \xi < \infty \) and \([2b]\) yields \([3b]\). Then \([3a]\) follows by straightforward integration theory. Then the definition of marginal distributions implies

\[
f_n(x_1^2 + \cdots + x_n^2) = \int_{-\infty}^{\infty} dx_{n+1} f_{n+1}(x_1^2 + \cdots + x_{n+1}^2),
\]  \[[5b]\]

which is \([3c]\) with \( \xi = x_1^2 + \cdots + x_n^2, \eta = x_1^2 + \cdots + x_{n+1}^2 \). Also,

\[
f_n(x_1^2 + \cdots + x_n^2) = \int_{-\infty}^{\infty} dx_{n+1} \int_{-\infty}^{\infty} dx_{n+2} f_{n+2}(x_1^2 + \cdots + x_{n+2}^2).
\]  \[[5c]\]

which is \([3e]\). Then \([3f]\) follows from \([3e]\), and \([3d]\) follows from \([3f]\) and \([3c]\) with \( n \) replaced by \( n-1 \). To prove analyticity, note that if \( q \) is an integer \( \geq 0 \) and if \( 0 < \alpha < \beta \), then by Taylor's theorem with remainder

\[
F_2(\alpha) - F_2(\beta) = \sum_{i=1}^{q} \frac{(\beta - \alpha)^i}{i!} (-\partial_{\xi}^i F_2(\beta)) + \frac{1}{q!} \int_{\alpha}^{\beta} d\xi (\xi - \alpha)^q (-\partial_{\xi}^q)^{q+1} F_2(\xi).
\]  \[[6a]\]
But \((-\partial_{\xi})^{i}F_{2} = \pi^{i} f_{2i}\), so by [3b]

\[
\frac{1}{q!} \int_{\alpha}^{\beta} d\xi \xi^{q} (-\partial_{\xi})^{q+1}F_{2}(\xi) = F_{2q+2}(\alpha) - F_{2q+2}(\beta).
\]

[6b]

Hence, the Lebesgue bounded convergence theorem implies that as \(\alpha \to 0\) the integral in [6a] converges to \(1 - F_{2q+2}(\beta)\). Therefore

\[
F_{2q+2}(\beta) - F_{2}(\beta) = \sum_{i=1}^{q} \frac{\beta^{i}}{i!} (-\partial_{\xi})^{i}F_{2}(\beta).
\]

[6c]

All terms in the sum [6c] are nonnegative, and \(F_{2q+2}(\beta) \leq 1\), so the series

\[
\sum_{i=1}^{\infty} \frac{(-\beta)^{i}}{i!} F_{2}^{(i)}(\beta)
\]

[6d]

converges absolutely (here \(F_{2}^{(i)} = \partial_{\xi}^{i} F_{2}\)). Therefore, the power series for \(F_{2}(\xi)\) at \(\xi = \beta\) converges absolutely for all complex \(\xi\) in the closed disk \(|\xi - \beta| \leq \beta\). Since \(\beta\) is arbitrary, \(F_{2}(\xi)\) is analytic for all complex \(\xi\) with positive real part. By [5a], so is \(f_{2}(\xi)\) and then by [3c,d] so is \(f_{n}(\xi)\) for every \(n \geq 1\). Hence so is \(F_{n}(\xi)\) for every \(n \geq 1\). Furthermore, since [6d] converges, Abel's theorem (2) implies that

\[
F_{2}(0) - F_{2}(\beta) = \sum_{i=1}^{\infty} \frac{\beta^{i}}{i!} (-\partial_{\xi})^{i}F_{2}(\beta).
\]

[6e]

Together, [6e], [6c] and [2a] imply [3g].

**COROLLARY 1:** If one of the marginal distributions \(p_{n}\) is known, \(p_{\infty}\) is completely determined.

**COROLLARY 2:** Let \(H(\alpha)\) be the set of \(x\) in \(R^{n}\) with \(\sum_{i=1}^{\infty} x_{i}^{2} < \alpha\). Then \(p_{\infty}(H(\infty)) = 0\). This follows immediately from [3g] and the fact that \(H(\infty)\) is the monotone limit of the sets \(H(\alpha)\) (3).

**Sufficient Conditions for Isotropy.** Let \(M(n)\) be the set of infinitely differentiable real-valued functions \(f\) on the open half-line \(0 < \xi < \infty\) such that

\[
\pi^{n/2} \Gamma(n/2)^{-1} \int_{0}^{\infty} d\xi \xi^{n/2-1} f(\xi) = 1
\]

[7a]
and also for every integer \( q \geq 0 \) and every \( \xi \) in \( 0 < \xi < \infty \)

\[
(-\partial_\xi)^q f(\xi) \geq 0.
\]  

[7b]

Note that if \( P_\infty \in I_0(R^\infty) \) and \( f_n \) comes from \( P_\infty \) via [3a] then \( f_n \in M(n) \). The converse is also true, and to prove it we need

**LEMMA 1:** Suppose \( n \geq 1 \) and \( f \in M(n) \). Then

\[
\lim_{\xi \to \infty} \xi^{n/2} f(\xi) = 0
\]  

[8a]

\[
\lim_{\xi \to 0} \xi^{n/2} f(\xi) = 0
\]  

[8b]

\[
f(\xi) = \int \frac{d\eta}{\xi} \left[-\partial_\xi f(\eta)\right]
\]  

[8c]

\[
(n/2) \int_0^\infty d\xi \xi^{n/2-1} f(\xi) = \int_0^\infty d\xi \xi^{n/2} \left[-\partial_\xi f(\xi)\right]
\]  

[8d]

\[
-\pi^{-1} \partial_\xi f \in M(n+2).
\]  

[8e]

**PROOF:** Let \( m = n/2 - 1 \) and let \( 0 < \alpha < A < \infty \). Integration by parts gives

\[
(m+1) \int_A^\infty d\xi \xi^m f(\xi) = A^{m+1} f(A) - \alpha^{m+1} f(\alpha) + \int_0^\alpha d\xi \xi^{m+1} \left[-\partial_\xi f(\xi)\right].
\]  

[9a]

Fix \( \alpha \). The integral on the right in [9a] increases as \( A \to \infty \) and yet is bounded, so it has a limit.

Therefore \( \lim_{A \to \infty} A^{m+1} f(A) \) exists. By [7a] it cannot be positive, so we have [8a], and hence [8c], and also

\[
(m+1) \int_0^\infty d\xi \xi^m f(\xi) = -\alpha^{m+1} f(\alpha) + \int_0^\alpha d\xi \xi^{m+1} \left[-\partial_\xi f(\xi)\right].
\]  

[9b]

As \( \alpha \) decreases to 0, the integral on the right in [9b] increases, and that on the left has a finite limit, so \( \alpha^{m+1} f(\alpha) \) approaches either \( +\infty \) or a nonnegative limit. Then [7a] requires [8b], and [9b] converges to [8d]. Then [8e] follows from [8d] and [7b].

Now we can prove

**THEOREM 2:** Suppose \( n \) is a nonnegative integer and \( f \in M(n) \). Then there is a \( P_\infty \in I_0(R^\infty) \) whose marginal distribution \( p_n \) on \( R^n \) is given by [3a] with \( f_n = f \).

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PROOF: For every integer \( q \geq 0 \), define \( f_{n+2q}(\xi) = \pi^{-q}(-\partial_\xi)^q f(\xi) \). If \( N-n \) is a nonnegative even integer, induction on \([8c]\) implies

\[
 f_N(x_1^2 + \cdots + x_N^2) = \int dx_{N+1} \int dx_{N+2} f_{N+2}(x_1^2 + \cdots + x_{N+2}^2). \tag{10a}
\]

If \( N-n \) is a nonnegative odd integer, define \( f_N \) from \( f_{N+1} \) via \([3c]\). Then

\[
 f_N(x_1^2 + \cdots + x_N^2) = \int dx_{N+1} f_{N+1}(x_1^2 + \cdots + x_{N+1}^2). \tag{10b}
\]

That \([10b]\) also holds when \( N-n \) is nonnegative and even follows from \([10a]\). Therefore \([10b]\) holds for all \( N \geq n \). Use it inductively to define \( f_N \) for \( 1 \leq N < n \). For \( N=n \), \([7a]\) implies

\[
 \int_{\mathbb{R}^n} \cdots dx_N f_N(x_1^2 + \cdots + x_N^2) = 1. \tag{10c}
\]

and then \([10b]\) implies \([10c]\) for all \( N \geq 1 \). Thus the probability distributions \( p_N \) on \( \mathbb{R}^N \) given by \( f_N \) via \([3a]\) satisfy the Kolmogorov consistency condition. Then the existence of \( p_\infty \) follows from Kolmogorov's Fundamental Theorem \((4)\).

**COROLLARY 1:** If \( f \in M(n) \), \( f(\xi) \) is analytic in the open right half-plane of complex \( \xi \).

**COROLLARY 2:** The equation \( F_2(\xi) = \int_0^\xi dt e^{-t} \) furnishes a bijection between the members of \( I_0(\mathbb{R}^m) \) and the probability measures \( \mu \) on \( 0 \leq t < \infty \).

**PROOF:** Demanding that \( f_2 \in M(2) \) is equivalent to demanding that \( F_2(\xi) \) be completely monotonic on \( 0 \leq \xi < \infty \) \((5)\).

**Examples and Applications.** Setting \( f_2(\xi) = \pi^{-1} e^{-\xi^2} \) gives \( f_n(\xi) = \pi^{-n/2} e^{-\xi^2} \). This \( p_\infty \) is the Gaussian with independent \( x_1, x_2, \ldots \), each having mean 0 and variance 1. Setting \( f_2(\xi) = \pi^{-1} \xi^{\nu-1} e^{-(1+\xi^2)^{\nu-1}} \) with \( 0 < \nu < 1 \) gives a \( p_\infty \) for which \( \lim_{\xi \to 1} f_n(\xi) = \infty \) if \( n \leq 2 \) and also if \( n = 1 \) and \( \frac{1}{2} \leq \nu < 1 \). Thus the densities \( f_n(\xi) \) need not remain finite as \( \xi \to 0 \).

The geophysical application is to inverse theory. An infinite dimensional linear space \( X \) of earth models \( x \) is given, along with a finite number of linear functionals, \( g_j : X \to \mathbb{R} \),

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An observer measures $D$ data $y_i = g_i(x_E) + e_i$ for $i = 1, \ldots, D$. Here $x_E$ is the correct earth model and $e_i$ is the error in observing $y_i$. The observer wants to predict the value of $z = g_{D+1}(x_E)$. Since $\dim X = \infty$, the problem is hopeless unless $g_{D+1}$ is a linear combination of $g_1, \ldots, g_D$, or unless the observer has some prior information about $x_E$ not included among the data (6,7). One common sort of prior information is a quadratic bound on $x_E$, a quadratic form $Q$ on $X$ such that $x_E$ is known to satisfy

$$Q(x_E, x_E) \leq 1.$$  

[11]

Often [11] is a bound on energy content or dissipation rate (8). In stochastic inversion and Bayesian inference, such a bound is often "softened" to a prior personal probability distribution $p_\infty$ on $X$ (8–10). In practice, $X$ is truncated to an $R^n$, and $p_\infty$ is used in the inversion.

To see why this process is questionable, complete $X$ to a Hilbert space with the inner product $x \cdot x' = Q(x, x')$. Let $\xi_1, \xi_2, \ldots$ be an orthonormal basis for $X$, and write $x = \sum_{i=1}^{\infty} x_i \xi_i$. Then $X$ becomes the subset $H(\infty)$ of $R^\infty$ defined in corollary 2 to theorem 1. The prior information [11] can now be written

$$\sum_{i=1}^{\infty} x_i^2 \leq 1.$$  

[12]

If the observer wants to soften [12] to a probability distribution $p_\infty$ without introducing new information not implied by [12], then clearly he should take $p_\infty \in I(R^\infty)$. He is unlikely to assign nonzero probability to 0, so $p_\infty \in I_0(R^\infty)$. But then $p_\infty(X) = 0$ by corollary 2 to theorem 1. Any prior personal probability distribution obtained by softening [12] without adding new information must deny [12] with probability 1.

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References


