A System of Three-Dimensional Complex Variables

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A SYSTEM OF THREE-DIMENSIONAL COMPLEX VARIABLES

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Abstract. This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.

1. Introduction. Early in the nineteenth century, mathematicians began a search for a "three-dimensional complex number and its algebra" that would be a generalization of the ordinary "two-dimensional" complex number [1, p. 90]. In 1843, William R. Hamilton introduced quaternions (see [1]), an important four-dimensional generalization of complex numbers and variables. Hypercomplex analysis has developed mainly as a further generalization of quaternions and, as such, is often referred to as Clifford analysis. The recent papers [2], [3], [4], [5] supply many references, including early work by Fueter (e.g., [6]). These algebras that generalize quaternions are noncommutative.

S. Bergman [7] has introduced a method based on E. T. Whittaker's [8, p. 390] general integral solution to Laplace's equation that provides a certain generalization of analytic functions of one complex variable. However, the present state has been summarized as follows by E. T. Copson [9, p. 207]: "The theory of harmonic functions in two dimensions can be made to depend on the theory of analytic functions of a complex variable, z + iy. There is nothing corresponding to the theory of functions of a complex variable x + iy in three dimensions. The nearest approach is given by Whittaker's general solution ... of Laplace's equation."

The elements of the 3-D theory (a commutative algebra) to be described here are direct generalizations of corresponding elements of the classical 2-D theory. Therefore a direct comparison with 2-D is helpful for this description.

2. Basics in Two Dimensions for Comparison. A most important property of analytic functions of an ordinary complex variable is that from them are obtained vector functions \( g \) that are both solenoidal and irrotational. As a result, the components of \( g \) are harmonic functions.

Let \( \mathbf{R} \) denote the set of all real numbers and \( \mathbf{C}_2 \) denote the set of all ordinary complex numbers. The complex variable \( z = x + iy \) in \( \mathbf{C}_2 \) may be written also as \( z = (x, y) = (1, 0)x + (0, 1)y \), which may be interpreted as a vector in \( \mathbf{R}^2 \) with real components \( x, y \) and with basis vectors \( (1, 0) = 1 \) and \( (0, 1) = i \), whose rules of multiplication are: \( 1^2 = 1, \, 1i = i1 = i, \, i^2 = -1 \). However, the unit \( (1, 0) = 1 \) as a factor is commonly omitted. If now \( g = \phi_1 + i\phi_2 \), in \( \mathbf{C}_2 \), is defined to be the vector (complex function) whose complex conjugate is an analytic function \( \bar{g} = f(z) = \phi_1 - i\phi_2 \), then the conditions of analyticity for \( \bar{g} = f(z) \) are the Cauchy-Riemann equations: \( \text{div} \, g = \phi_{1x} + \phi_{2y} = 0 \) and \( \text{curl} \, g = \phi_{2x} - \phi_{1y} = 0 \).
(In two dimensions the result of the curl operation is defined as a scalar.) Therefore, $g$ is solenoidal and irrotational (Sand 1).

Any 2-D S and I vector may be represented by a complex variable having the same form as $g$. For example, in 2-D ideal flow with velocity components $v_1$ and $v_2$ and with velocity potential $\phi$ and stream function $\psi$, the velocity vector $v = v_1 + iv_2$ and the vector $g = \phi_1 + i\phi_2$ (where $\phi_1 = \phi$ and $\phi_2 = -\psi$) are called, respectively, the complex velocity and the complex potential. Both $v$ and $g$ are Sand I vectors, and their respective complex conjugates $v = w(z) = v_1 - iv_2$ and $g = f(z) = \phi_1 - i\phi_2$ may be represented by analytic functions for which $w = df/dz$.

3. Definitions and Results in Three Dimensions.

**DEFINITION 1.** Let $C_3$ denote the set of all "three-dimensional (3-D) numbers" of the form $Z = x + \delta y + \epsilon z$, in which (i) $Z$ may be interpreted as a vector with basis vectors $1, \delta, \epsilon$ and with components $x, y, z$ in $C_2$; and (ii) the rules of multiplication are as follows (or other equivalent forms of them):

\[
I^2 = 1, \quad 1\delta = \delta I = \delta, \quad 1\epsilon = \epsilon I = \epsilon, \\
\delta^2 = -\frac{1}{2}(1 + i\epsilon), \quad \epsilon^2 = -\frac{1}{2}(1 - i\epsilon), \quad \delta \epsilon = \epsilon \delta = -\frac{1}{4}(i\delta).
\]

The 3-D unit 1 as a factor may be omitted (as the factor 1 in 2-D is omitted), with $Z$ written generally as

\[
Z = z = x + \delta y + \epsilon z = Z_R + iZ_I,
\]

where $Z_R = x_R + \delta y_R + \epsilon z_R$ and $Z_I = x_I + \delta y_I + \epsilon z_I$, with $x_R, y_R, z_R, x_I, y_I, z_I$ real.

**DEFINITION 2.** Let $C_3'$ be a subset of $C_3$ such that, for every element $Z = x + \delta y + \epsilon z$ in $C_3'$, the components $x, y, z$ are real.

Then, for $Z$ in $C_3$, $Z_R$ and $Z_I$ are in $C_3'$, and the basis vectors $1, \delta, \epsilon$ are in $C_3'$. If $Z$ is an independent variable, for which values can be prescribed, then one can set $Z_I = 0$, so that $x, y$ and $z$ are real and $Z$ is in $C_3'$.

The algebraic properties of these numbers in $C_3$ are developed and discussed in papers by the author to be published. The multiplicative inverse, $Z^{-1}$, is of special significance. It can be found by setting $Z^{-1} = a_1 + i\delta a_2 + \epsilon a_3$, where $a_k \in C_2$, and by requiring $Z Z^{-1} = 1$.

It is found that there are certain nonzero values of $Z$ for which $Z^{-1}$ is not defined, with results including the following:

**THEOREM 3.** For $Z = x + \delta y + \epsilon z$ in $C_3$, the domain of definition of $Z^{-1}$ includes all of the $R^3$ space of $(x, y, z)$ except the origin and any of the six rays in the plane $x = 0$ where $\vartheta = \tan^{-1}(z/y) = (\ell - 1)/3 \varphi$ for $\ell = 1, 2, \ldots, 6$.

**REMARK 4.** The algebra of $C_3$ is a linear algebra of order 3 over the field of ordinary complex numbers, $C_2$. Further, $C_3$ is a commutative ring with unity, and not a field, since, for some nonzero elements $Z$, the inverse $Z^{-1}$ is not defined.

Further discussion of $Z^{-1}$ is beyond the scope of this note, but is included elsewhere.

**DEFINITION 5.** For every $Z = x + \delta y + \epsilon z$ in $C_3$, denote as the bijugate of $Z$ the element of $C_3$ given by $Z = \frac{1}{2}z - \delta y - \epsilon z$.

(The bijugate can be defined more generally.) The 3-D bijugate is in some ways analogous to the 2-D conjugate. The similar role in regard to analytic functions will be demonstrated here.
As an analogy to the variables $z$ and $g$ in $C_2$ described in the previous section, consider the two variables in $C_3$: $Z = x + i y + cz$ and $G = \phi_1 + \delta \phi_2 + \epsilon \phi_3$, which are also vectors in $C_3^2$. Now let $G$ be defined to be the vector (3-D complex function) whose bijugate is an analytic function $\bar{G} = F(Z) = \frac{1}{2} \phi_1 - \delta \phi_2 - \epsilon \phi_3$. The concepts of function, limit, derivative, and analytic function can be extended, with some care, to the set $C_3$. Then, in analogy to the Cauchy-Riemann conditions in two dimensions, the following necessary conditions for the differentiability, and hence analyticity, of $F(Z)$ are found:

**Theorem 6.** For $Z$ in some domain $D_3 \subseteq C_3$, and $G$ in $C_3$ with components $\phi_k$ in $C_2$ such that $\bar{G} = F(Z)$, the necessary conditions for analyticity of $F(Z)$ are:

\[
\begin{align*}
\text{div } G &= \phi_{1x} + \phi_{2y} + \phi_{3z} = 0, \\
\text{curl } G &= \langle \phi_{3y}, -\phi_{2x}, \phi_{1z} \rangle + \delta(\phi_{1x} - \phi_{3z}) + \epsilon(\phi_{2x} - \phi_{1y}) = 0,
\end{align*}
\]

along with $\phi_{1y} - i(\phi_{2z} + \phi_{3y}) = 0$ and $\phi_{1z} - i(\phi_{2y} - \phi_{3z}) = 0$.

Since all the components of the curl must vanish, $G$ is an $S$ and $I$ vector in three dimensions. Further, if we write $\phi_k = \phi_{kR} + i \phi_{kI}$ and $G = G_R + i G_I$, with the components $\phi_{kR}$ of $G_R$ and components $\phi_{kI}$ of $G_I$ real, then $G_R$ and $G_I$ are also $S$ and $I$ vectors (with the final two equations in Theorem 6 serving to connect the components $\phi_{kR}$ of $G_R$ to the components $\phi_{kI}$ of $G_I$). In Theorem 6, $x$, $y$, and $z$ are independent variables defined generally to be complex, but as independent variables, may be taken to be real (i.e., $Z \in C'_3$).

**Corollary 7.** If $W = v_1 + \delta v_2 + \epsilon v_3$, in $C_3$, is defined to be the vector whose bijugate is the analytic function that is the derivative of $F(Z)$: $\bar{W} = V(Z) = dF/dZ = \frac{1}{2} v_1 - \delta v_2 - \epsilon v_3$, then $W$ is also an $S$ and $I$ vector and

\[
\begin{align*}
v_1 &= \phi_{1x} = -\phi_{2y} - \phi_{3z}, \\
v_2 &= \phi_{1y} = \phi_{2x} = i(\phi_{2y} + \phi_{3y}), \\
v_3 &= \phi_{1z} = \phi_{3x} = i(\phi_{2y} - \phi_{3z}), \\
\phi_{3y} &= \phi_{2z},
\end{align*}
\]

**Example 8.** For $Z$ in $C'_3$ the product $Z^2 = Z \bar{Z}$, with use of the rules of multiplication from Definition 1, is $Z^2 = x^2 - \frac{1}{2}(y^2 + z^2) + \delta(2xy) + \epsilon(2xz) - i\delta(yz) - i\epsilon\frac{1}{2}(y^2 - z^2)$. Then for $F(Z) = Z^2$, the results are $\phi_{1R} = 2x^2 - (y^2 + z^2)$, $\phi_{2R} = -2xy$, $\phi_{3R} = -2xz$, $\phi_{1I} = 0$, $\phi_{2I} = yz$, $\phi_{3I} = \frac{1}{2}(y^2 - z^2)$, which are readily seen to satisfy Theorem 6. The two $S$ and $I$ vectors $G_R$ and $G_I$, with respective Cartesian components $\phi_{kR}$ and $\phi_{kI}$, are thus generated by $F(Z) = Z^2$.

The (harmonic) components of either $G_R$ or $G_I$ can be related to a 3-D velocity potential and general 3-D stream functions, and either $G_R$ or $G_I$ can be taken to be a “3-D complex potential,” with the corresponding “3-D complex velocity” then being either $W_R$ or $W_I$.

A primary result here is that this theoretical structure can be used to generate $S$ and $I$ vectors and harmonic functions in three dimensions, as can the Whittaker-Bergman method, but without integration here, as in ordinary analytic-function theory for two dimensions.

Details, proofs, and further results are in [10].

**References**

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