A System of Three-Dimensional Complex Variables

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A SYSTEM OF THREE-DIMENSIONAL COMPLEX VARIABLES

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Abstract. This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.

1. Introduction. Early in the nineteenth century, mathematicians began a search for a “three-dimensional complex number and its algebra” that would be a generalization of the ordinary “two-dimensional” complex number [1, p. 90]. In 1843, William R. Hamilton introduced quaternions (see [1]), an important four-dimensional generalization of complex numbers and variables. Hypercomplex analysis has developed mainly as a further generalization of quaternions and, as such, is often referred to as Clifford analysis. The recent papers [2], [3], [4], [5] supply many references, including early work by Fueter (e.g., [6]). These algebras that generalize quaternions are noncommutative.

S. Bergman [7] has introduced a method based on E. T. Whittaker’s [8, p. 390] general integral solution to Laplace’s equation that provides a certain generalization of analytic functions of one complex variable. However, the present state has been summarized as follows by E. T. Copson [9, p. 207]: “The theory of harmonic functions in two dimensions can be made to depend on the theory of analytic functions of a complex variable, \( z + i y \). There is nothing corresponding to the theory of functions of a complex variable \( z + i y \) in three dimensions. The nearest approach is given by Whittaker’s general solution . . . of Laplace’s equation.”

The elements of the 3-D theory (a commutative algebra) to be described here are direct generalizations of corresponding elements of the classical 2-D theory. Therefore a direct comparison with 2-D is helpful for this description.

2. Basics in Two Dimensions for Comparison. A most important property of analytic functions of an ordinary complex variable is that from them are obtained vector functions \( g \) that are both solenoidal and irrotational. As a result, the components of \( g \) are harmonic functions.

Let \( \mathbf{R} \) denote the set of all real numbers and \( \mathbf{C}_2 \) denote the set of all ordinary complex numbers. The complex variable \( z = x + i y \) in \( \mathbf{C}_2 \) may be written also as \( z = (x, y) = (1, 0)x + (0, 1)y \), which may be interpreted as a vector in \( \mathbf{R}^2 \) with real components \( x, y \) and with basis vectors \( (1, 0) = 1 \) and \( (0, 1) = i \), whose rules of multiplication are: \( 1^2 = 1, \quad 1i = i1 = i, \quad i^2 = -1 \). However, the unit \( (1, 0) = 1 \) as a factor is commonly omitted. If now \( g = \phi_1 + i\phi_2 \), in \( \mathbf{C}_2 \), is defined to be the vector (complex function) whose complex conjugate is an analytic function \( \bar{g} = f(z) = \phi_1 - i\phi_2 \), then the conditions of analyticity for \( \bar{g} = f(z) \) are the Cauchy-Riemann equations: \( \text{div} \, g = \phi_{1x} + \phi_{2y} = 0 \) and \( \text{curl} \, g = \phi_{2x} - \phi_{1y} = 0 \).
(In two dimensions the result of the curl operation is defined as a scalar.) Therefore, \( g \) is solenoidal and irrotational (Sand I).

Any 2-D \( S \) and \( I \) vector may be represented by a complex variable having the same form as \( g \). For example, in 2-D ideal flow with velocity components \( v_1 \) and \( v_2 \) and with velocity potential \( \phi \) and stream function \( \psi \), the velocity vector \( v = v_1 + iv_2 \) and the vector \( g = \phi_1 + i\phi_2 \) (where \( \phi_1 = \phi \) and \( \phi_2 = -\psi \)) are called, respectively, the complex velocity and the complex potential. Both \( v \) and \( g \) are \( S \) and \( I \) vectors, and their respective complex conjugates \( v^* = v_1 - iv_2 \) and \( g^* = \phi_1 - i\phi_2 \) may be represented by analytic functions for which \( w = df/dz \).

3. Definitions and Results in Three Dimensions.

**DEFINITION 1.** Let \( \mathbb{C}_3 \) denote the set of all “three-dimensional (3-D) numbers” of the form \( Z = i x + \delta y + \epsilon z \), in which (i) \( Z \) may be interpreted as a vector with basis vectors \( 1, \delta, \epsilon \) and with components \( x, y, z \) in \( \mathbb{C}_2 \); and (ii) the rules of multiplication are as follows (or other equivalent forms of them):

\[
1^2 = 1, \quad 1\delta = \delta 1 = \delta, \quad 1\epsilon = \epsilon 1 = \epsilon,
\delta^2 = -\frac{1}{2}(1 + i\epsilon), \quad \epsilon^2 = -\frac{1}{2}(1 - i\epsilon), \quad \delta\epsilon = \epsilon\delta = -\frac{1}{2}(i\delta).
\]

The 3-D unit \( 1 \) as a factor may be omitted (as the factor 1 in 2-D is omitted), with \( Z \) written generally as

\[
Z = x + \delta y + \epsilon z = Z_R + iZ_I,
\]
where \( Z_R = x_R + \delta y_R + \epsilon z_R \) and \( Z_I = x_I + \delta y_I + \epsilon z_I \), with \( x_R, y_R, z_R, x_I, y_I, z_I \) real.

**DEFINITION 2.** Let \( \mathbb{C}'_3 \) be a subset of \( \mathbb{C}_3 \) such that, for every element \( Z = x + \delta y + \epsilon z \) in \( \mathbb{C}'_3 \), the components \( x, y, z \) are real.

Then, for \( Z \) in \( \mathbb{C}_3 \), \( Z_R \) and \( Z_I \) are in \( \mathbb{C}'_3 \), and the basis vectors \( 1, \delta, \epsilon \) are in \( \mathbb{C}'_3 \). If \( Z \) is an independent variable, for which values can be prescribed, then one can set \( Z_I = 0 \), so that \( x, y \) and \( z \) are real and \( Z \) is in \( \mathbb{C}'_3 \).

The algebraic properties of these numbers in \( \mathbb{C}_3 \) are developed and discussed in papers by the author to be published. The multiplicative inverse, \( Z^{-1} \), is of special significance. It can be found by setting \( Z^{-1} = a_1 + \delta a_2 + \epsilon a_3 \), where \( a_k \in \mathbb{C}_2 \), and by requiring \( ZZ^{-1} = 1 \).

It is found that there are certain nonzero values of \( Z \) for which \( Z^{-1} \) is not defined, with results including the following:

**THEOREM 3.** For \( Z = x + \delta y + \epsilon z \) in \( \mathbb{C}'_3 \), the domain of definition of \( Z^{-1} \) includes all of the \( \mathbb{R}^3 \) space of \( (x, y, z) \) except the origin and any of the six rays in the plane \( x = 0 \) where \( \vartheta = \tan^{-1}(z/y) - (\ell - 1)x/3 \) for \( \ell = 1, 2, \ldots, 6 \).

**REMARK 4.** The algebra of \( \mathbb{C}_3 \) is a linear algebra of order 3 over the field of ordinary complex numbers, \( \mathbb{C}_2 \). Further, \( \mathbb{C}_3 \) is a commutative ring with unity, and not a field, since, for some nonzero elements \( Z \), the inverse \( Z^{-1} \) is not defined.

Further discussion of \( Z^{-1} \) is beyond the scope of this note, but is included elsewhere.

**DEFINITION 5.** For every \( Z = x + \delta y + \epsilon z \) in \( \mathbb{C}_3 \), denote as the bijugate of \( Z \) the element of \( \mathbb{C}_3 \) given by \( \overline{Z} = \frac{1}{2} x - \delta y - \epsilon z \).

(The bijugate can be defined more generally.) The 3-D bijugate is in some ways analogous to the 2-D conjugate. The similar role in regard to analytic functions will be demonstrated here.
As an analogy to the variables \( z \) and \( g \) in \( C_2 \) described in the previous section, consider the two variables in \( C_3 \):

\[
2 = 2 + 6y + ez \quad \text{and} \quad G = 41 + 642 + 643,
\]

which are also vectors in \( C_3 \). Now let \( G \) be defined to be the vector (3-D complex function) whose bijugate is an analytic function \( \bar{G} = F(Z) = \frac{1}{2} \phi_1 - \delta \phi_2 - \epsilon \phi_3 \). The concepts of function, limit, derivative, and analytic function can be extended, with some care, to the set \( C_3 \). Then, in analogy to the Cauchy-Riemann conditions in two dimensions, the following necessary conditions for the differentiability, and hence analyticity, of \( F(Z) \) are found:

**Theorem 6.** For \( Z \) in some domain \( D_3 \subseteq C_3 \), and \( G \) in \( C_3 \) with components \( \phi_k \) in \( C_2 \) such that \( \bar{G} = F(Z) \), the necessary conditions for analyticity of \( F(Z) \) are:

\[
\begin{align*}
\text{div} G &= \phi_{1x} + \phi_{2y} + \phi_{3z} = 0, \\
\text{curl} G &= 1(\phi_{3y} - \phi_{2z}) + \delta(\phi_{1x} - \phi_{3x}) + \epsilon(\phi_{2x} - \phi_{1y}) = 0,
\end{align*}
\]

along with \( \phi_{1y} - i(\phi_{2z} + \phi_{3y}) = 0 \) and \( \phi_{1z} - i(\phi_{2y} + \phi_{3z}) = 0 \).

Since all the components of the curl must vanish, \( G \) is an \( S \) and \( I \) vector in three dimensions. Further, if we write \( \phi_k = \phi_{kR} + i\phi_{kI} \) and \( G = G_R + iG_I \), with the components \( \phi_{kR} \) of \( G_R \) and components \( \phi_{kI} \) of \( G_I \) real, then \( G_R \) and \( G_I \) are also \( S \) and \( I \) vectors (with the final two equations in Theorem 6 serving to connect the components \( \phi_{kR} \) of \( G_R \) to the components \( \phi_{kI} \) of \( G_I \)). In Theorem 6, \( x, y, \) and \( z \) are independent variables defined generally to be complex, but as independent variables, may be taken to be real (i.e., \( Z \in C_3' \)).

**Corollary 7.** If \( W = v_1 + \delta v_2 + \epsilon v_3 \), in \( C_3 \), is defined to be the vector whose bijugate is the analytic function that is the derivative of \( F(Z) \): \( \bar{W} = V(Z) = dF/dZ = \frac{1}{2}v_1 - \delta v_2 - \epsilon v_3 \), then \( W \) is also an \( S \) and \( I \) vector and

\[
\begin{align*}
v_1 &= \phi_{1x} = -(\phi_{2y} + \phi_{3z}), \\
v_2 &= \phi_{1y} = \phi_{2x} = i(\phi_{2z} + \phi_{3y}), \\
v_3 &= \phi_{1z} = \phi_{3x} = i(\phi_{2y} - \phi_{3z}), \\
\phi_{3y} &= \phi_{2z},
\end{align*}
\]

**Example 8.** For \( Z \) in \( C_3' \) the product \( Z^2 = Z \), with use of the rules of multiplication from Definition 1, is

\[
Z^2 = x^2 - \frac{1}{2}(y^2 + z^2) + \delta(2xy) + \epsilon(2xz) - i\delta(yz) - i\epsilon \frac{1}{2}(y^2 - z^2).
\]

Then for \( F(Z) = Z^2 \), the results are \( \phi_{1R} = 2x^2 - (y^2 + z^2) \), \( \phi_{2R} = -2xy \), \( \phi_{3R} = -2xz \), \( \phi_{1I} = 0 \), \( \phi_{2I} = yz \), \( \phi_{3I} = \frac{1}{2}(y^2 - z^2) \), which are readily seen to satisfy Theorem 6. The two \( S \) and \( I \) vectors \( G_R \) and \( G_I \), with respective Cartesian components \( \phi_{kR} \) and \( \phi_{kI} \), are thus generated by \( F(Z) = Z^2 \).

The (harmonic) components of either \( G_R \) or \( G_I \) can be related to a 3-D velocity potential and general 3-D stream functions, and either \( G_R \) or \( G_I \) can be taken to be a "3-D complex potential," with the corresponding "3-D complex velocity" then being either \( W_R \) or \( W_I \).

A primary result here is that this theoretical structure can be used to generate \( S \) and \( I \) vectors and harmonic functions in three dimensions, as can the Whittaker-Bergman method, but without integration here, as in ordinary analytic-function theory for two dimensions.

Details, proofs, and further results are in [10].

**References**

This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.