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Error Assessments of Widely-Used Orbit Error Approximations in Satellite Altimetry

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ABSTRACT

From simulations, the orbit error can be assumed to be a slowly varying sine wave with a predominant wavelength comparable to the earth's circumference. Thus, one can derive analytically the error committed in representing the orbit error along a segment of the satellite ground track by a bias; by a bias and a tilt (linear approximation); or by a bias, tilt and curvature (quadratic approximation). The result clearly agrees with what is obvious intuitively, i.e., (1) the fit is better with more parameters, (2) as the length of the segment increases, the approximation gets worse. But more importantly, it provides a quantitative basis to evaluate the accuracy of past results and, in the future, to select the best approximaton according to the required precision and the efficiency of various approximations.

1. Introduction

Since the launch of GEOS-3, which was followed by SEASAT, GEOSAT and will be followed in the future by the planned TOPEX/POSEIDON and ERS-1 missions, satellite altimetry has become increasingly important as a tool to measure the global sea level. However, its progress has always been hampered by the orbit error problem. Since the orbit error (uncertainty of the altitude of the satellite) can range from 1 m (SEASAT and GEOSAT non-repeat era) to 4 m (GEOSAT exact-repeat era; the situation is expected to improve) and this error translates directly into altimetric sea level uncertainty, ways have to be found to remove it. At places where multiple readings of sea level are obtained (e.g., satellite ground track crossover points and along exact-repeat ground tracks), altimetric sea level changes of over 1 m are common occurences, which can only be attributed to the orbit error.

Along track the orbit error is a very large-scale feature with a predominant spectral peak at the earth's circumference (the so-called once per revolution peak) (e.g., Cutting *et al.*, 1978; March and Williamson, 1980). Hence, one can think of the orbit error as a slowly varying sine wave with a predominant wavelength comparable to the earth's circumference (approximately 40,000 km). If the orbit error along a ground track segment (a fraction of one revolution) is parameterized in terms of a large-scale function, one can solve for these parameters to minimize the altimetric sea level changes at crossover points and along repeat tracks. This is how the orbit error is usually removed. The following large-scale functions have been used to approximate the orbit error: bias, bias and tilt, bias tilt and curvature (i.e., polynomials of the zeroth, first, and second degrees). Exactly how good these approximations are have never been investigated. The residual after removal of the orbit error is an indirect indicator; and certain things are intuitively clear, i.e., the approximation gets better with more parameters, but gets worse as the segment becomes longer. The purpose of this note is to fill in the gap. In the following, it is demonstrated that a rigorous error analysis is not only possible, but it can also be done analytically.

2. Method

Since the orbit error can be thought of as a slowly varying sinusoidal wave train with a dominant wavelength, the whole problem can be set up as a question of how accurately can a zeroth up to second degree polynomial approximate a sine wave over certain interval less than half the dominant wavelength (it will be most unusual to find a segment longer than half a revolution and no approximaton has ever been done using that long a segment anyway, while, in any case, the following derivation is valid for any length). The more complicated derivation involving the quadratic fit is presented below, whereas the results for the much less cumbersome cases of linear and bias-only representation will be stated without proof.

So let the orbit error be represented by a sine wave over a segment from 0 to L with $L \le \pi$, i.e., $A \sin(t + \phi)$, where A is the amplitude and ϕ is the phase. And let $a+bt+ct^2$ be the quadratic best fit over this segment, i.e., a, b, and c minimize the following quantity

$$J = \int_o^L dt\, [a+bt + ct^2 - A \sin(t + \phi)]^2 \ .$$

The strategy is to compute a, b, c, and J (i.e., the square residual) for a particular ϕ, then average J over all possible ϕ, i.e., $o \le \phi \le 2\pi$ and normalize to get the root-mean-square (RMS) relative error. Differentiating J with respect to a, b, c respectively and setting the derivatives to zeros, we find

$$aL + b\frac{L^2}{2} + c\frac{L^3}{3} = \alpha \ , \tag{1}$$

$$a\frac{L^2}{2} + b\frac{L^3}{3} + c\frac{L^4}{4} = \beta \ , \tag{2}$$

$$a\frac{L^3}{3} + b\frac{L^4}{4} + c\frac{L^5}{5} = \gamma \ , \tag{3}$$

where

$$\alpha = A\int_o^L \sin(t+\phi)dt \quad , \tag{4}$$

$$\beta = A\int_o^L t\,\sin(t+\phi)dt \quad , \tag{5}$$

$$\gamma = A\int_o^L t^2\sin(t+\phi)dt \quad . \tag{6}$$

From (1), (2), (3),

$$a = \frac{1}{L^3}\,(9L^2\alpha - 36L\,\beta + 30\gamma) \tag{7}$$

$$b = \frac{12}{L^4}\,(-3L^2\alpha + 16L\,\beta - 15\gamma) \tag{8}$$

$$c = \frac{30}{L^5}\,(L^2\alpha - 6L\,\beta + 6\gamma) \tag{9}$$

Substituting (7), (8), (9) into J, one would get after straightforward but somewhat tedious manipulations

$$J = \frac{1}{L^5}(-9L^4\alpha^2 - 192L^2\,\beta^2 - 180\gamma^2 + 72L^3\alpha\beta - 60L^2\alpha\gamma + 360L\,\beta\gamma)$$
$$+A^2\,[\,\frac{L}{2} - \frac{\sin 2(L+\phi)}{4} + \frac{\sin 2\phi}{4}\,] \; .$$

Let an overbar represent

$$\overline{(\;)} = \frac{1}{2\pi}\int_o^{2\pi} d\phi\,(\;) \; .$$

Then, from (4), (5), (6),

$$\overline{\alpha^2} = A^2(1-\cos L)$$
$$\overline{\alpha\beta} = A^2\frac{L}{2}(1-\cos L)$$
$$\overline{\beta^2} = A^2(1-\cos L - L\sin L - \frac{L^2}{2})$$
$$\overline{\beta\gamma} = A^2\frac{L^2}{2}(L - \sin L)$$
$$\overline{\alpha\gamma} = A^2[(1-\cos L)(\frac{L^2}{2}-2) + L\sin L\,]$$
$$\overline{\gamma^2} = A^2[4(1-\cos L) + 2L^2\cos L - 4L\sin L + \frac{L^4}{2}]$$

Substituting these into $\bar{J}$, one would get after another long process

$$\varepsilon = \frac{\bar{J}/L}{A^2/2} = 1 - \frac{1440}{L^6}(1-\cos L) + \frac{1440}{L^5}\sin L - \frac{144}{L^4}(1+4\cos L) - \frac{96}{L^3}\sin L + \frac{6}{L^2}(\cos L - 3). \tag{10}$$

Note that the mean-square (MS) residual, $\bar{J}$ /L, has been normalized by the MS value of the sine wave, $A^2/2$, to give us the MS relative error. Taking the square root of (10), one would get the RMS relative error. One can check the validity of (10) with two limiting cases. As L goes to infinity, the error should become 100%, which is obviously the case for (10). As L goes to zero, there should be no error. It can be verified that as L goes to zero,

$$\varepsilon = \frac{36}{10!} L^6 + 0(L^8) \ .$$

As a matter of fact, ε vanishes so fast for small L that double precision had to be used on a CRAY machine (over 30 digits of accuracy) to evaluate (10) correctly.

For linear representation, the MS relative error is

$$\varepsilon = 1 - \frac{24}{L^4}(1 - cosL) + \frac{24}{L^3} sinL - \frac{4}{L^2}(2 + \cos L) \tag{11}$$

As L → 0, $\varepsilon = \frac{L^4}{6!} + O(L^6)$. For bias-only representation,

$$\varepsilon = 1 - \frac{2}{L^2}(1 - cosL) \tag{12}$$

3. Results and Discussions

The results represented by (10), (11), and (12) are tabulated and presented in Table 1. L is presented in units of degree (with 360° representing one wavelength), and RMS relative errors are presented in units of percentage points. It is somewhat surprising to find how well a quadratic curve fits a sine wave segment and, in contrast, how poor the bias-only representation really is. To illustrate this point, one commits less error on the average in fitting a 115° sine wave segment with a quadratic curve than fitting a 5° segment with a bias. However, the RMS value can be very deceiving. Near the ends of a long segment being fitted, the fit is usually much worse then the situation near the middle.

Let us try to review some old results with this new perspective starting with the bias-only approximation. Fu and Chelton (1985) have used the bias-only approximation to remove the orbit error in their investigation of the Antarctic Circumpolar Current. They have chosen the bias-only method so as to avoid removing oceanic signal [acutally, a bias-only crossover adjustment can and do remove some oceanic signal, see Tai (1987)]. The bias-only adjustment reduces the crossover difference from 146 cm to 34 cm and to 24 cm if outliers exceeing 60 cm are deleted. The geographical extent is from 40°S to 60°S, which translates to about 30° for the maximum segment length; and from table 1, we get a RMS relative error about 15%, which translates to about 22 cm residual RMS crossover difference from the original 146 cm. This is in good agreement with the 24 cm, since besides the residual orbit error, there are other error sources and the oceanic signal, which can contribute to the residual crossover differences (actually, a contribution of 10 cm from other sources would combine with 22 cm from the residual orbit error to make the total residual 24 cm). But one would have to say this residual crossover difference is dominated by the residual orbit error.

Marsh *et al.* (1982) have applied the bias and tilt crossover adjustment to a 5000 x 5000 km region of the eastern North Pacific. They have been able to reduce the RMS crossover difference from 1 m to 12 cm and to 8 cm if costal areas are deleted (the reason of which they attribute to the noise introduced by measoscale features and nonlinear tides). From table 1, 5000 km corresponds to 45° (i.e., one eighth the wavelength), and one would expect a RMS relative error abut 2.3%. this translates to 2.3 cm residual crossover difference due to the residual orbit error. Thus unlike the previous case, the residual orbit error is of minor importance here. Yet closer inspection of their Figure 2 reveals that crossover differences are generally larger near the boundaries than near the center (less so for the deep sea boundary than the costal boundary), which is compatible with the notion that least-square line fitting is least accurate near the ends.

Rapp (1983), using the bias and tilt method, has undertaken the formidable job of adjusting the entire SEASAT record to remove the orbit error. In his primary adjustment (global in extent), the RMS crossover difference is reduced from 165 cm to 28 cm, which is a reduction to about 17% of the original value. 549 segments are used in this adjustment, of which about 38 are greater than 2300 seconds in duration, which can have RMS relative error over 20% (note that one revolution is 6000 seconds in duration). Withoud further details, one can only say this is in rough agreement with Table 1. However, for segments less than 212 seconds in duration, only the bias is adjusted in Rapp's adjustment, which can cause RMS errors as large as 6.5% versus 0.2% if bias and tilt are used (in retrospect, this is not a good choice).

Cheney *et al.* (1986) have applied the quadratic fit to the first 24 days of GEOSAT in the Pacific between 40°N and 40°S. The RMS crossover difference after adjustment is 8 cm (it is not clear what the value is before adjustment, but typically it is around 1 m). A segment spanning 40°N and 40°S measures (in degrees) 85°, which has a RMS error of only 1% according to Table 1. The residual orbit error is clearly not a significant factor in the residual crossover differences.

It is clear that the quadratic representation is by far the most accurate approximaton, but even it would be in serious error when the segment length is approaching half a wavelength. One can not help wondering why a sinusoidal representation has never been contemplated before, especially when the segment length is approaching half a wavelength, such as in a global adjustment (strictly speaking, the sinusoidal representation is also an approximation, albeit much more accurate than others). There are other reasons to prefer or avoid the sinusoidal representation, which are discussed at length in Tai (1987). There is not a so-called best representation suitable for all situations. It is hoped, however, that equations (10), (11), (12), and Table 1 will assist the effort in choosing the best representation for a particular problem after considering the efficiency and the required accuracy.

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Table 1. RMS relative error (in percents) for bias-only, bias and tilt, and quadratic representations fo the orbit error.

Segment Length (in degrees)	RMS relative error (in percents)			Segment Length (in degrees)	RMS relative error (in percents)		
	bias-only	bias and tilt	quadratic		bias-only	bias and tilt	quadratic
5	2.52	0.028	0.0002	95	45.73	9.851	1.3894
10	5.04	0.113	0.0017	100	47.90	10.868	1.6148
15	7.55	0.255	0.0056	105	50.03	11.929	1.8624
20	10.06	0.453	0.0134	110	52.13	13.031	2.1330
25	12.56	0.708	0.0261	115	54.20	14.173	2.4273
30	15.05	1.018	0.0451	120	56.22	15.353	2.7461
35	17.52	1.383	0.0715	125	58.21	16.570	3.0900
40	19.99	1.804	0.1066	130	60.15	17.823	3.4597
45	22.44	2.279	0.1515	135	62.05	19.109	3.8558
50	24.87	2.807	0.2074	140	63.91	20.428	4.2788
55	27.29	3.389	0.2756	145	65.72	21.777	4.7291
60	29.68	4.023	0.3570	150	67.49	23.155	5.2073
65	32.06	4.709	0.4529	155	69.21	24.560	5.7136
70	34.40	5.445	0.5643	160	70.89	25.990	6.2485
75	36.73	6.231	0.6922	165	72.52	27.444	6.8122
80	39.02	7.066	0.8377	170	74.10	28.920	7.4049
85	41.29	7.948	1.0018	175	75.63	30.416	8.0268
90	43.52	8.877	1.1854	180	77.12	31.929	8.6781