We discuss a nonlinear plasma theory for self-modulation of pulsar radio pulses. A nonlinear Schrödinger equation is derived for strong electromagnetic waves propagating in an electron-positron plasma. The nonlinearities arising from wave intensity induced particle mass variation may excite the modulational instability of circularly and linearly polarized pulsar radiation. The resulting wave envelopes can take the form of periodic wave trains or solitons. These nonlinear stationary waveforms may account for the formation of pulsar microstructures.
SELF-MODULATIONAL FORMATION OF
PULSAR MICROSTRUCTURES

by
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Abstract. We discuss a nonlinear plasma theory for self-modulation of pulsar radio pulses. A nonlinear Schrödinger equation is derived for strong electromagnetic waves propagating in an electron-positron plasma. The nonlinearities arising from wave intensity induced particle mass variation may excite the modulational instability of circularly and linearly polarized pulsar radiation. The resulting wave envelopes can take the form of periodic wave trains or solitons. These nonlinear stationary waveforms may account for the formation of pulsar microstructures.
1. Introduction

Pulsar radio emissions exhibit ultrashort intensity variations within individual pulses on time scales ranging approximately from 1 μs to 1 ms (Cordes, 1979). Since these microstructures (or micropulses) are the most fundamental features of pulsar pulses, they undoubtedly provide important clues concerning the physics of pulsar radio emission and propagation.

Models for pulsar microstructure formation can be divided into two general classes: beaming models and temporal models. In beaming models, the rapid intensity fluctuation is considered to be a geometrical phenomenon caused by the angular sweeping of a nonuniform pulsar pencil beam across the observer's line of sight. For example, Benford (1977) views pulsar microstructures as the sweep of the observer's line of sight across a series of radiating plasma filaments directed along the pulsar magnetic field lines. Ferguson (1981) proposes that quasi-periodic microstructure emission is due to bunches of particles located in many periodically spaced emission regions, perhaps occupying excited locations in a standing plasma wave of very long wavelength. In temporal models, the intensity fluctuation is treated as the result of an actual modulation of the pulsar radiation beam. For example, Cheng (1981) suggests that small oscillations in pulsar polar cap temperature can lead to strong modulation of the outflowing ions and electron-positron fluxes. Harding and Tademaru (1981) treat the temporal modulation of pulsar pulses as they propagate through magnetospheric regions of relativistic velocity shear.

In this paper we discuss a nonlinear plasma theory which may account for temporal modulation of coherent pulsar radio pulses. We demonstrate that the nonlinearities arising from relativistic particle mass variation may excite the
self-modulational instability of strong electromagnetic waves in an electron-positron plasma. The case of a circularly polarized wave is first studied in detail, then the treatment is extended to a linearly polarized wave. In general pulsar radio emissions contain both linear and circular polarization components (Cordes, 1979). The linear polarization originates from intrinsic emission processes such as the two-stream instability (Ruderman and Sutherland, 1975; Asseo et al., 1983), whereas the circular polarization may be converted from the linear polarization via the Faraday rotation effect (Cocke and Pacholczyk, 1980) or the nonuniform magnetic field effect (Hodge, 1982) as the wave propagates through the pulsar magnetosphere.

Strong electromagnetic waves capable of driving plasma particles to relativistic energies have received much attention in connection with the low-frequency pulsar magnetic dipole radiation (e.g., Max, 1973; Kennel et al., 1979; Chian, 1982). Recently Bodo et al. (1981) used the dispersion relation of a strong linearly polarized wave to study the propagation of coherent pulsar radio emissions in the polar cap region. The high radio brightness temperature ($10^{24}$ to $10^{31}$ K, Cordes, 1979) inferred from pulsar observations implies that high-frequency pulsar radio waves can certainly drive the magnetospheric charged particles to relativistic velocities. The intensity of pulsar radio emission in the source region can be measured by a dimensionless, Lorentz-invariant, parameter

$$\nu = \frac{eE}{mc\omega_c} = 5.585 \left( \frac{SAf}{n} \right)^{1/2} \frac{D}{\delta f},$$  \hspace{1cm} (1)
where $S$ = flux density in Jy, $f$ = pulsar emission frequency in MHz, $\Delta f$ = emission bandwidth in MHz, $D$ = pulsar distance in kpc, $\delta$ = size of emission region in $10^6$ cm, and $n$ = index of refraction. Numerical examples of the wave strength parameter are given in Table 1 for two pulsars, PSR 0950 + 08 (Cordes and Hankins, 1979) and PSR 1133 + 16 (Bartel and Hankins, 1982), whose microstructure properties have been well analyzed. In Table 1 the index of refraction is taken to be unity, the observed peak flux densities are used, and the size of emission region $\delta$ is determined by the product $c\tau$, where $c$ = speed of light and $\tau$ = pulsewidth. We expect that, with the exception of the pulsar distance, there may be several order of magnitude variations in most of the parameters in Table 1. Hence it is reasonable to expect that the intensity of pulsar radio emission in the source region lies in the range $10^{-2} \leq \nu \leq 10^3$. For such wave intensities, plasma particles may acquire weakly relativistic to moderately relativistic velocities. This is in contrast to the case of low-frequency pulsar magnetic dipole radiation, for which the particles can reach ultrarelativistic velocities with $\nu \approx 10^{11}$ (Gunn and Ostriker, 1971).

The modulational instability of electromagnetic waves has been extensively studied in connection with laser-plasma interaction for an electron plasma (e.g., Max et al., 1974; Shukla et al., 1977) and an electron-ion plasma (e.g., Berezhiani et al., 1980). While these studies are surely of interest to laser fusion applications, they might not be appropriate for pulsar applications. According to current polar-cap pulsar models (Ruderman and Sutherland, 1975; Arons and Scharlemann, 1979) the pulsar magnetosphere is composed of secondary electrons and positrons resulting from pair production induced by high energy curvature radiation photons emitted by primary positron or electron beams coming from the pulsar surface. In such a magnetosphere, positrons and electrons contribute
equally toward the wave characteristics (Clemmow, 1974; Kennel and Pellat, 1976; Chian, 1980), hence the inclusion of positron dynamics is essential for pulsar radio emission theories. Moreover, the ion acoustic mode is absent in an electron-positron plasma (Tsytovich and Wharton, 1978), therefore the modulational coupling between the high-frequency electromagnetic wave and the low-frequency ion acoustic wave, heretofore considered for an electron-ion plasma, may not be applicable to the pulsar environment.

Astrophysical applications of the modulational instability of electromagnetic waves in a hot plasma have been studied for the situation where the relativistic effect is caused by high plasma temperatures (Pataraya and Melikidze, 1980; Lakhina and Buti, 1981). In the present paper we adopt the model of a cold, unmagnetized plasma in order to single out the effect of relativistic particle mass variation originating from the interaction of electrons and positrons with high intensity waves. Although the magnetic field is certainly not negligible in the pulsar magnetosphere, we postpone the analysis of its effects to another paper in order to investigate the simplest case in detail.

2. Theory

Consider the propagation of a circularly polarized electromagnetic wave in the rest frame of an electron-positron plasma with its vector potential given by

$$A(z,t) = a(z,t)(\hat{x} \cos \omega t + \hat{y} \sin \omega t)$$

In the linear regime the wave is governed by the dispersion relation

$$\omega^2 = c^2 k^2 + \frac{\omega^2}{p}$$

In the equilibrium of pulsar,
where \( \omega_p = \sqrt{2} \omega_{pe} \) is the electron-positron plasma frequency, \( \omega_{pe} = \omega_{pp} = (4\pi N_e e^2/m_e^2)^{1/2} \).

The nonlinear wave propagation is described by the following wave equation

\[
\frac{\partial^2 A}{\partial t^2} - c^2 \frac{\partial^2 A}{\partial z^2} = 4\pi e c (N_e v_e - N_p v_p)
\]  

(4)

where only purely transverse fields are treated, since we may neglect the longitudinal electric field in an electron-positron plasma (Kennel and Pellat, 1976).

The relativistic equations of motion for electrons and positrons are

\[
\frac{\partial}{\partial t} + \gamma \cdot \nabla \gamma = \frac{e}{m_e} (E + \frac{\gamma \times B}{c})
\]  

(5)

where \( \gamma = (1 - v^2/c^2)^{-1/2} \). From (2) and (5) we have

\[
\gamma_e = \frac{e}{m_e c} \frac{A}{(1 + \lambda A^2)^{1/2}} \quad \text{and} \quad \gamma_p = - \frac{e}{m_c c} \frac{A}{(1 + \lambda A^2)^{1/2}}
\]  

(6)

where \( \lambda = e^2/m_e c^4 \). Making use of the quasi-neutrality condition, \( N_e = N_p = N_0 \), appropriate for a circularly polarized wave the nonlinear wave equation becomes

\[
\frac{\partial^2 A}{\partial t^2} - c^2 \frac{\partial^2 A}{\partial z^2} + \omega_p^2 \frac{A}{(1 + \lambda A^2)^{1/2}} = 0
\]  

(7)

Introducing a complex modulational representation

\[
A(z,t) = \frac{1}{2} [g(z,t)e^{-i\omega t} + \text{c.c.}]
\]  

(8)
(7) can be rewritten in terms of the slowly time-varying modulation \( q \) (with \( \frac{1}{\eta} \frac{\partial q}{\partial t} \ll \omega \))

\[
i \frac{\partial q}{\partial t} + \frac{e^2}{2\omega} \frac{\partial^2 q}{\partial z^2} + \frac{\omega}{2} q - \frac{\omega^2}{2\omega} \frac{q}{(1 + \lambda |q|^2)^2} = 0
\]

(9)

where a term proportional to \( \frac{\partial^2 q}{\partial t^2} \) is dropped. It can be shown that (9) admits localized stationary solutions for arbitrary wave amplitudes, however, for the sake of illustration we shall henceforth treat the small amplitude limit of (9).

In the limit \( \lambda |q|^2 \ll 1 \), (9) yields a nonlinear Schrödinger equation

\[
i \frac{\partial q}{\partial t} + p \frac{\partial^2 q}{\partial z^2} + Q|q|^2 q = 0
\]

(10)

with

\[
p = \frac{e^2 \omega}{2\omega} \quad , \quad Q = \frac{e^2 \omega^2}{4\omega m c^2 c^2 c_0^2}
\]

(11)

An additional term, \( Rq \) (where \( R = (\omega^2 - \omega_p^2)/2\omega \)), was removed from (10) by the transformation \( q \rightarrow q \exp(iRt) \). Equation (10) is the desired wave equation for the modulational instability. Alternatively, (10) can be derived by a method introduced by Karpman and Kruskal (1969), by which the coefficients of the nonlinear Schrödinger equation (10) are determined from a weakly nonlinear dispersion relation

\[
\omega = \omega(k, |q|^2)
\]

(12)

with

\[
p = \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2} \quad , \quad Q = - \frac{\partial \omega}{\partial |q|^2}
\]

(13)
The exact nonlinear dispersion relation for a circularly polarized electromagnetic wave in an electron-positron plasma is (e.g., Chiao, 1980)

\[ \omega^2 = c^2k^2 + \frac{\omega_p^2}{(1 + \nu^2)^2} + \frac{\omega_P^2}{(1 + \nu^2)^2} = c^2k^2 + \frac{\omega_p^2}{(1 + \nu^2)^2} \]

where \( \nu^2 = e^2E^2/m_e^2\omega^2c^2 = \lambda A^2 \). Note from (14) that the positron contribution to the wave dispersion is equal to the electron contribution. In the weakly nonlinear limit \( \nu^2 \ll 1 \), (14) reduces to

\[ \omega(k,|q|^2) = (c^2k^2 + \omega_p^2 - \frac{\omega_P^2\lambda}{2}|q|^2)^{1/2} \]

Applying (15) to (13) we obtain precisely the same coefficients as those in (11). It is important to observe that the Kalman-Krushkal method is only valid for the weakly nonlinear regime; it should not be applied directly to a strongly nonlinear dispersion relation such as (14) (see e.g., Durrani et al., 1980). The appropriate modulational wave equation for large wave amplitudes is (9).

Having derived the nonlinear Schrödinger equation, (10), we can now determine whether or not the plane carrier wave (2) is unstable to a low-frequency modulation. It has been established (e.g., Hasegawa, 1975) that the modulational instability can be excited in a medium provided the group dispersion \( P \) and the nonlinear frequency shift \( Q \) are of the same sign, namely \( PQ > 0 \). From (11) we see that this condition is indeed satisfied.

Consider next the dynamics of the modulated wave moving with a group speed \( V = \omega/k \), described by (10). In this case we may assume a finite value \( q_0 \) for the modulation as \( z-Vt \to \pm \infty \), thus in (10) \( |q|^2 \) is replaced by \( |q|^2 - |q_0|^2 \)
The stability of the modulation can be studied by separating \( \varphi \) into two real variables, \( \rho \) and \( \sigma \), representing the real and the imaginary parts of \( \varphi \):

\[
\varphi = \sqrt{\rho(z,t)} e^{i\sigma(z,t)}
\]  
(17)

Substitution of this expression into (16) gives for the real and imaginary parts, respectively,

\[
Q(\rho - \rho_o) + \frac{\rho}{2\rho} \frac{\partial^2 \rho}{\partial z^2} - \frac{\rho}{4\rho^2} \left( \frac{\partial \rho}{\partial z} \right)^2 - \rho_p \frac{\partial \sigma}{\partial z} - \frac{\partial \sigma}{\partial t} = 0
\]  
(18)

\[
\frac{\partial \rho}{\partial t} + 2\rho \frac{\partial}{\partial z} \left( \rho \frac{\partial \sigma}{\partial z} \right) = 0
\]  
(19)

Linearizing (18) and (19)

\[
\rho = \rho_o + \rho_1 e^{i(k_L z - \omega_L t)}
\]  
(20)

\[
\sigma = \sigma_1 e^{i(k_L z - \omega_L t)}
\]  
(21)

we obtain the following dispersion relation for the low-frequency modulation

\[
\omega_L^2 = p^2 k_L^4 - 2PQ \rho_o k_L^2
\]  
(22)
It follows from (22) that the threshold for the modulational instability is

\[ \rho_o > PkL^2/2Q, \]

namely

\[ \nu_o > c \frac{kL}{\rho} \]  \hspace{1cm} (23)

for the circular polarization case and the corresponding maximum growth rate is \( \gamma = Q \rho_o \), namely

\[ \gamma = \omega_p^2 \nu_o^2/4\omega \]  \hspace{1cm} (24)

The instability is purely growing, and therefore non-convective in the rest frame.

As the modulation grows, the instability can evolve to a nonlinear stationary state that results from the balance between nonlinearity and dispersion. The possible stationary solutions are the periodic wave trains (Nishikawa and Liu, 1976)

\[ \rho = \rho_o \text{cn}^2[\frac{Q}{2\rho_o} \rho_o (z - Vt)] \]  \hspace{1cm} (25)

or the solitary waves

\[ \rho = \rho_o \text{sech}^2[\frac{Q}{2\rho_o} \rho_o (z - Vt)] \]  \hspace{1cm} (26)

The solution, (26), is called an envelope soliton since it is the envelope of the wave packet that has the form of a solitary wave. These envelope solitons can be shown to be stable against longitudinal perturbations and mutual collisions.

We now apply the Karpman-Krushkal method to discuss the self-modulational instability of nonlinear linearly polarized electromagnetic waves in an electron-positron plasma. Although the dispersion relation in the strongly nonlinear
limit has been derived by Kennel and Pellat (1976) and Bodo et al. (1981).

In this case, the Karpman-Krushkal method is restricted to the weakly nonlinear regime. The nonlinear dispersion relation for a linearly polarized wave in an electron plasma, in the limit \( v^2 \ll 1 \), is given by (Chian and Clemmow, 1975)

\[
\sigma^2 = 1 - \left( 1 - \frac{3}{8} v^2 \right) \frac{\omega_{\text{pe}}^2}{\omega^2}
\]

(27)

Now, (9) suggests that the dispersion relation for an electron-positron plasma can be obtained from the dispersion relation for an electron plasma by replacing \( \omega_{\text{pe}} \) for \( \sqrt{2} \omega_{\text{pe}} \) (i.e., the electron-positron plasma frequency \( \omega_p \)). Hence the appropriate dispersion relation can be obtained from (27) by replacing \( \omega_{\text{pe}} \) for \( \omega_p \), namely

\[
\omega(k,|q|^2) = (c^2k^2 + \omega_p^2 - \frac{3\omega_p^2}{8} |q|^2)^{\frac{1}{2}}
\]

(28)

A comparison of (28) with (15) shows that, apart from a slight difference in the numerical coefficient, the dispersion relations for linearly and circularly polarized waves are essentially the same. Applying (28) to (13) then gives a nonlinear Schrödinger equation (10) for a linearly polarized wave with

\[
\begin{align*}
  p &= \frac{c^2}{2\omega} \\
  Q &= \frac{3e^2\omega_p^2}{16\omega_m^2c^2}
\end{align*}
\]

(29)

Evidently, \( PQ > 0 \), thus the self-modulational instability of linearly polarized waves in an electron-positron plasma can be excited. The equations describing the evolution of the modulational instability for this case are similar to the circular polarization case with \( P \) and \( Q \) given by (29).
3. Discussion

In the previous section we demonstrated that the self-modulational instability of circularly and linearly polarized electromagnetic waves can be excited in an electron-positron plasma. The theory presented in this paper is compatible with the statistical model for the pulsar signal proposed by Rickett (1975). In Rickett's model the pulsar signal is depicted as an amplitude-modulation \( \alpha(t) \) of a noise-like random process \( n(t) \):

\[
p(t) = \alpha(t) n(t)
\]

where \( p(t) \) is the electric field of the received pulsar signal, \( n(t) \) describes the coherent fast time-varying emission by particle bunches, and \( \alpha(t) \) describes micropulses that vary much more slowly than \( n(t) \). This amplitude-modulated noise model has been found to be consistent with the observed pulsar spectra. According to our theory, as evidenced by (8), the high-frequency coherent pulsar emission with a fast-time scale \( 2\pi/\omega \) can be modulated by a slowly time-varying envelope \( \gamma(z,t) \) due to nonlinearities arising from relativistic particle mass variation. Hence, the resulting modulation envelopes may account for the formation of pulsar microstructures.

We have seen that the end products of the relativistic self-modulational instability may be nonlinear stationary waves of either periodic wave-train type or envelope soliton type. This is consistent with the observed features of pulsar microstructures. Hankins and Boriakoff (1978) showed that the observed pulsar microstructures can be put into two categories, intermittent type or quasi-periodic type, according to their intensity structures. Most micropulses are of the intermittent type, which have bursts of strong emission
interpersed with sections of longitude where the signal returns abruptly to the system noise level. Occasionally, quasi-periodic string of micropulses are detected. The intermittent micropulses can be explained by our theory as a collection of envelope solitons with randomly fluctuating amplitudes, whereas the quasi-periodic micropulses can be explained as periodic wave-trains or a sequence of envelope solitons with little variation in their peak amplitudes.

We now make use of the nonlinear envelope solutions

\[
A(z,t) = \begin{cases} 
A_0 \cosh^{-2}[\frac{Q}{2P} |z - Vt|^2 A_0 (z - Vt)] \\
A_0 \sinh^{-2}[\frac{Q}{2P} |z - Vt|^2 A_0 (z - Vt)]
\end{cases}
\]  

(31)

to estimate the features of microstructures produced by the relativistic self-modulation instability. First we calculate the Lorentz-invariant number \( N \) denoting the number of wave crests in a given modulation. From the observed temporal pulsewidths of pulsar micropulses (~ 1 ms to 1 ms) and the frequency range of pulsar radio emission (~ 100 MHz to 1 GHz) we see that \( N \) varies roughly from \( 10^2 \) to \( 10^6 \). (31) shows that the envelope spatial pulsewidth for the circular polarization case is

\[
\delta = \left| \frac{2P}{Q} \right|^\frac{1}{2} \frac{1}{A_0} = \frac{2c}{\omega_p \nu_0} 
\]  

(32)

Thus, in the rest frame,

\[
N = \frac{2c}{\omega_p \nu_0} \frac{k}{2\pi} = \frac{1}{\pi \nu_0} \left( \frac{k c}{\omega_p} \right) 
\]  

(33)
where $k$ is the wavenumber of high-frequency pulsar radio waves. (33) shows that in order for $N$ to be within the observed range it is required that $kc/\omega_p >> 1$. The weakly nonlinear dispersion relations (15) and (28) indicate that this is possible only if $\omega >> \omega_p$. For example, if $v = 10^{-2}$ and $\omega_p/\omega = 10$, $N \approx 10^3$. Therefore, our theory suggests that self-modulational formation of microstructures takes place in regions of pulsar magnetosphere where the pulsar radio wave frequency is considerably greater than the local plasma frequency. Next we use our theory to calculate the temporal pulsewidth of micropulses. Since (32) is given in the rest frame of secondary particles it is necessary to transform (32) to the observer's frame. Because of time dilation and Doppler shift (Ruderman and Sutherland, 1975) in the observer's frame, $\omega_{pe} \rightarrow 2\gamma_s \omega_{pe}$, where $\gamma_s$ is the relativistic factor associated with the stream velocity $v$ of secondary electrons and positrons. Hence, in the observer's frame, the temporal pulsewidth is

$$\tau = \frac{1}{\sqrt{2} \gamma_s \omega_{pe} v} \quad (34)$$

where the relation $\delta = ct$ is used, since for $\gamma_s >> 1$, the soliton moves with relativistic speeds in the observer's frame. If $\gamma_s = 10^2$, $\omega_{pe} = 10^6 s^{-1}$ (in the rest frame) and $v = 10^{-2}$, then $\tau = 1 \mu s$ which is within the observed range.

The above calculation suggests again that relativistic self-modulational instability is excited in the region of magnetosphere with relatively low plasma densities. Now (34) shows that micropulses of higher intensity have narrower pulsewidths. Such behavior was shown by Bartel et al. (1980) to hold true.
for pulsar subpulses and was suggested by Ferguson (1981) to be true also for
micropulses. It would be interesting for pulsar observers to check this property
across the entire range of microstructure time scales.

In conclusion, we point out that the physical mechanism for formation of
pulsar microstructures discussed in this paper is expected to take place in
most pulsar magnetospheres since the wave intensity induced relativistic effect
responsible for driving the self-modulational instability is general. However,
a quantitative test of a modulational instability theory requires the extension
of the simple model adopted in this paper to include other effects, such as
large wave amplitude, ambient magnetic field, plasma temperature and plasma
inhomogeneity. In the meantime, our simplified analysis indicates that the modu-
lational instability is a promising direction for further investigation.
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References

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TABLE 1

Numerical examples of the wave strength parameter

<table>
<thead>
<tr>
<th>Pulsar</th>
<th>S(Jy)</th>
<th>$f$(MHz)</th>
<th>$\Delta f$(MHz)</th>
<th>D(Kpc)</th>
<th>$\delta(10^8$ cm)</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PSR0950 + 08</td>
<td>850</td>
<td>430</td>
<td>200</td>
<td>0.1</td>
<td>$6 \times 10^{-3}$</td>
<td>89</td>
</tr>
<tr>
<td>PSR1133 + 16</td>
<td>300</td>
<td>1720</td>
<td>10</td>
<td>0.2</td>
<td>$7.5 \times 10^{-4}$</td>
<td>47</td>
</tr>
</tbody>
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