Stress Intensity and Crack Displacement for Small Edge Cracks

Thomas W. Orange

Lewis Research Center
Cleveland, Ohio
Summary

The weight function method was used to derive stress intensity factors and crack mouth displacement coefficients for small edge cracks in fracture specimen geometries. Of greatest interest were cracks whose lengths were less than 20 percent of the specimen width. The effects of contact stresses due to point application of loads in bend testing were examined. The results are compared with available solutions and equations from the literature and with unpublished boundary collocation results.

Introduction

The purpose of this effort was to support a research study on the fatigue and fracture behavior of small cracks. Wide-range solutions for stress intensity and crack mouth displacement in edge-crack configurations are available. A “wide-range” solution is one that is considered valid for all possible values of the ratio of crack length to specimen width, that is, from zero to unity. These are typically produced by fitting curves from analytic solutions for cracks in semi-infinite bodies through numerical results for cracks in finite bodies. The accuracy depends upon the numerical results used. However, few numerical results for small cracks are published, and those are not supported by results from alternative methods.

Bend specimens have practical advantages in experimental studies of small cracks. First, a fatigue crack may be initiated from a notch at a fairly high load. Next, the cyclic load is reduced, similar to that in a fatigue-crack threshold test, to produce a truly sharp crack. Then, by machining only one edge of the specimen, most evidence of the prior load history may be removed. Although this appears to be a simple way of producing a small, sharp crack, Timoshenko (ref. 1) has noted the presence of additional stresses due to contact forces in bend testing. It is essential to determine just what influence these stresses have on the stress intensity factor and the crack mouth displacement for small cracks.

In this report, Seewald’s analysis (ref. 2) of the effect of contact stress was applied to the bend specimen configurations of interest. The stress correction terms were evaluated for three-point and four-point bending. The weight function methods of Bueckner (ref. 3) and Rice (ref. 4) were then used to determine the stress intensity and crack mouth displacement coefficients from the corrected stresses. Finally, results were compared with available numerical solutions and wide-range interpolation equations.

Symbols

\[ A_n \quad \text{polynomial coefficient} \]
\[ a \quad \text{crack length} \]
\[ B \quad \text{beam thickness} \]
\[ E \quad \text{modulus of elasticity for plane stress} \]
\[ E' \quad \text{effective modulus} \]
\[ h \quad \text{half-width of beam} \]
\[ K_I \quad \text{opening-mode stress intensity factor} \]
\[ M \quad \text{bending moment (fig. 2)} \]
\[ M(t) \quad \text{weight function} \]
\[ m_1, m_2 \quad \text{coefficients (eq. (4))} \]
\[ P \quad \text{applied load (figs. 1 and 2)} \]
\[ p(t) \quad \text{crack face pressure} \]
\[ Q \quad \text{multiplicative term} \]
\[ t \quad \text{coordinate measured from crack tip toward cracked surface} \]
\[ W \quad \text{specimen width} \]
\[ v \quad \text{crack mouth displacement} \]
\[ z \quad \text{dummy variable} \]
\[ \nu \quad \text{Poisson’s ratio} \]
\[ \rho \quad \text{applied load (figs. 1 and 2)} \]
\[ \sigma_x \quad \text{stress normal to load line} \]
\[ \sigma_{x,0} \quad \text{nominal stress normal to load line} \]
\[ \sigma'_x \quad \text{stress correction term} \]

Method of Analysis

Contact Stresses in Beams

Timoshenko (ref. 1) notes that when a beam is loaded by a concentrated force the stresses are not exactly as given by elementary beam theory. Specifically, the stresses at and normal to the load line are smaller near the surface opposite the load. He presents curves and a few numerical values from
Seewald (ref. 2) to illustrate this trend. These suggest that contact stresses might affect the stress intensity factor and the crack mouth displacement for small cracks.

Seewald (ref. 2) writes the stress normal to the load line as the sum of two terms,

\[ \sigma_x = \sigma_{x,0} + \sigma_x' \]

where \( \sigma_{x,0} \) is calculated by elementary beam theory and \( \sigma_x' \) is the stress correction term. His notation and conventions are shown in figure 1. The stress correction term \( \sigma_x' \) is given by

\[
\sigma_x' = \frac{2P}{B \pi W} \int_0^\infty \left[ \frac{(z \cosh z - \sinh z) \cosh \left( \frac{y}{h} \right) z - \left( \frac{y}{h} \right) z \cosh \left( \frac{y}{h} \right) z \sinh \left( \frac{y}{h} \right) z}{\sinh 2z - 2z} \right] \cos \left( \frac{x}{h} \right) z \, dz
\]

\[
+ \frac{2P}{B \pi W} \int_0^\infty \left[ \frac{(z \sinh z - \cosh z) \sinh \left( \frac{y}{h} \right) z - \left( \frac{y}{h} \right) z \cosh \left( \frac{y}{h} \right) z \sinh \left( \frac{y}{h} \right) z}{\sinh 2z - 2z} \right] \frac{12 \frac{y}{h}}{8z^2} \cos \left( \frac{x}{h} \right) z \, dz
\]

where \( B \) is the beam thickness and \( z \) may be regarded as a dummy variable. The integral was evaluated in piecewise fashion as follows.

For \( 0 \leq z \leq 0.1 \), the hyperbolic functions were replaced by the first two terms of their series expansions; that is,

\[
sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots
\]

(These approximations are accurate to at least five significant figures.) The resulting equation could then be integrated analytically, term by term.

For \( 0.1 \leq z \leq 5 \), the hyperbolic functions were replaced by their exponential equivalents

\[
sinh z = \frac{e^z - e^{-z}}{2} \quad \cosh z = \frac{e^z + e^{-z}}{2}
\]

and the integral was evaluated numerically according to Simpson's rule.

Finally, for \( z \geq 5 \), the large-argument approximation

\[
\sinh z = \cosh z = \frac{e^z}{2}
\]

was invoked (the error is less than 0.5 percent for \( z \geq 5 \)). The result could again be integrated analytically, term by term. The piecewise integrals were summed to give the total integral.

Figure 2 shows the configurations of the specimens analyzed. For the three-point-bend specimen, the integral was evaluated in the crack plane (i.e., at \( x = 0 \)). For the four-point-bend specimen, there are two forces located at a distance \( x = \pm 2W = \pm 4h \) from the crack plane. By symmetry, the effect of each load is identical at the crack plane. By superposition, the effects may be added. Thus, the stress correction term is twice the value of the integral evaluated at \( x/h = 4 \). The discrete calculations are given in table I and plotted in figure 3. Note that the dimensionless stress from elementary beam theory is \( \pm 6 \) at the surfaces \( y/h = \pm 1 \). The stress correction term for three-point bending appears significant, but that for four-point bending is quite small and is later shown to be insignificant. The reaction forces at the ends of the specimens (which were at least as far from the crack plane as the four-point-bend loading forces) were disregarded.

To use these discrete values in the weight function analysis to follow, simple equations were fit to them. A bicubic spline was used for three-point bending while a simple parabola was sufficient for four-point bending. The equations for the dimensionless stress correction terms in the crack plane (\( x = 0 \)) are

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Figure 1.—Notation and conventions of Seewald (ref. 2).
Figure 2.—Configurations analyzed.

\[ \sigma \frac{BW}{P} = 0.2466 + 1.2566 \left( \frac{y}{h} \right) + 0.1338 \left( \frac{y}{h} \right)^2 \]

\[ -0.6092 \left( \frac{y}{h} \right)^3 \quad \text{for } \frac{y}{h} \leq 0 \quad (2a) \]

\[ \sigma \frac{BW}{P} = 0.2466 + 1.2566 \left( \frac{y}{h} \right) + 0.1338 \left( \frac{y}{h} \right)^2 \]

\[ -0.3956 \left( \frac{y}{h} \right)^3 \quad \text{for } \frac{y}{h} \geq 0 \quad (2b) \]

for three-point bending and

\[ \sigma \frac{BW}{P} = 0.00078 - 0.036 \left( \frac{y}{h} \right) - 0.0015 \left( \frac{y}{h} \right)^2 \quad (2c) \]

for four-point bending. All fitted equations are within 0.5 percent of the range of \( \sigma \).

Weight Function Analysis

Bueckner (ref. 3) gives the stress intensity for a general pressure distribution over the face of a crack as

\[ K_i(a) = \sqrt{\frac{2}{\pi}} \int_0^a M(t, a) p(t) \, dt \quad (3) \]

where \( M(t) \) is the weight function, \( a \) is the crack length, \( p(t) \) is the crack face pressure, and \( t \) is the coordinate measured from the crack tip toward the cracked surface. For the case of an edge crack in a strip of unit width, Bueckner approximates the weight function as

TABLE I.—CONTACT STRESS CORRECTION TERM, \( \sigma \frac{BW}{P} \)

[See fig. 1 for notation.]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Distance, } \frac{y}{h} & \text{Three-point bending} & \text{Fitted spline} & \text{Seewald (ref. 2)} & \text{Four-point bending} & \text{Fitted parabola} \\
\hline
1.00 & 1.2400 & 1.2414 & 1.232 & -0.03696 & -0.03572 \\
0.95 & 1.2214 & 1.2220 & -0.03508 & -0.03478 \\
0.90 & 1.1974 & 1.1974 & -0.03310 & -0.03284 \\
0.85 & 1.1684 & 1.1684 & -0.03108 & -0.03090 \\
0.80 & 1.1348 & 1.1330 & -0.02908 & -0.02898 \\
0.70 & 1.0632 & 1.0560 & -0.02514 & -0.02516 \\
0.60 & 0.9620 & 0.9632 & -0.02134 & -0.02136 \\
0.50 & 0.8572 & 0.8588 & 0.856 & -0.01760 & -0.01760 \\
0.40 & 0.7426 & 0.7454 & -0.01394 & -0.01386 \\
0.30 & 0.6234 & 0.6250 & -0.01028 & -0.01016 \\
0.20 & 0.4992 & 0.5002 & -0.00664 & -0.00648 \\
0.10 & 0.3736 & 0.3732 & -0.00298 & -0.00284 \\
0 & 0.2486 & 0.2466 & 0.242 & 0.00066 & 0.00078 \\
-0.10 & 0.1722 & 0.1728 & -0.00430 & -0.00436 \\
-0.20 & 0.0118 & 0.0096 & -0.00794 & -0.00792 \\
-0.30 & -0.0948 & -0.1019 & -0.01154 & -0.01144 \\
-0.40 & -0.1892 & -0.1956 & -0.01510 & -0.01494 \\
-0.50 & -0.2680 & -0.2722 & -0.1860 & -0.1840 \\
-0.60 & -0.3272 & -0.3276 & -0.2204 & -0.2184 \\
-0.70 & -0.3544 & -0.3586 & -0.2544 & -0.2524 \\
-0.80 & -0.3678 & -0.3612 & -0.2878 & -0.2862 \\
-0.85 & -0.3574 & -0.3508 & -0.3044 & -0.3030 \\
-0.90 & -0.3730 & -0.3318 & -0.3210 & -0.3196 \\
-0.93 & -0.3196 & -0.3164 & -0.3308 & -0.3296 \\
-0.95 & -0.3056 & -0.3042 & -0.3376 & -0.3362 \\
-0.97 & -0.2896 & -0.2904 & -0.3442 & -0.3428 \\
-0.98 & -0.2810 & -0.2830 & -0.3476 & -0.3462 \\
-0.99 & -0.2716 & -0.2752 & -0.3510 & -0.3494 \\
-1.00 & -0.2618 & -0.2670 & -0.3544 & -0.3528 \\
\hline
\end{array}
\]
To compute the stress intensity factor, I added the stress from elementary beam theory (evaluated at the crack plane) to the Seewald stress correction term (eq. (1)) and substituted for \( p(t) \) in equation (3). Then I computed the crack mouth displacement by substituting the stress intensity factor and the weight function into equation (5). Since both the stress distribution and Bueckner's expression for the weight function are power series, I integrated equations (3) and (4) directly, term by term. The resulting equations, however, were quite tedious.

The stress distributions involved terms up to the third power in \( a/W \) (depending on the type of loading). The weight function had terms up to the sixth power. When these were multiplied, the final polynomial for the stress intensity factor had terms to the ninth power in \( a/W \). Multiplying the stress intensity factor by the weight function to calculate the crack mouth displacement gave terms to the 15th power. Because such polynomials are too long for practical use, I reduced them in degree by using Chebyshev economization (ref. 5). To do this, I wrote a program on a NASA mainframe computer by using internally developed subroutines. The exact and economized polynomials agreed within less than 0.33 percent over the range 0 \( \leq a/W \leq 0.5 \). Table II lists the coefficients of the polynomials, which are expressed in the general form

\[
P = Q \sum \left[ A_n \left( \frac{a}{W} \right)^n \right]
\]

Here \( P \) is the parameter to be calculated, \( Q \) is a multiplicative term, and \( A_n \) is the polynomial coefficient.

Results and Discussion

The cases of uniform tension and pure bending were examined first. This was done to verify the application of the weight function method. Also, unpublished boundary collocation results for small cracks obtained by Bernard Gross of NASA Lewis were evaluated. The term \( p(t) \) in equation (3) was given a constant value for uniform tension or a linearly varying value for pure bending. Table III lists the values of the stress intensity factor and crack mouth displacement coefficients. Values of wide-range polynomials from the literature and discrete values from numerical analyses are also listed.

Figure 4 shows the case of uniform tension. Over the range 0 \( \leq a/W \leq 0.5 \), the weight function stress intensity and displacement coefficients are within 1.1 and 2.4 percent, respectively, of those from the wide-range equations by Tada (ref. 6) and Koiter (ref. 7). Larger differences in the displacement coefficients were expected, since they depend on the square of the weight function. Numerical results by Keer (ref. 8), using an integral equation method, are within 1 percent for \( a/W \) values as low as 0.05. Published (ref. 9) and
TABLE II.—EXACT AND ECONOMIZED POLYNOMIALS FOR STRESS INTENSITY FACTOR
AND CRACK MOUTH DISPLACEMENT

(Coefficients of polynomials in form \( P = \sum (A_n (a/W)^n) \)).

(a) Coefficients for stress intensity factor

<table>
<thead>
<tr>
<th>Polynomial coefficient</th>
<th>Uniform tension</th>
<th>Pure bending</th>
<th>Four-point bending</th>
<th>Three-point bending</th>
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<td></td>
<td>-124.16</td>
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(b) Coefficients for crack mouth displacement

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<th>Polynomial coefficient</th>
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<th>Four-point bending</th>
<th>Three-point bending</th>
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<td>-796.80</td>
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</tbody>
</table>
unpublished boundary collocation stress intensity factors by Gross are within 2.2 percent at \( a/W = 0.05 \) and within less than 1 percent for \( a/W \geq 0.075 \). His displacement coefficients, however, differ by as much as 8.4 percent.

Gross' convergence criterion was based only on the first term of the Williams stress function, which is proportional to the stress intensity factor. Crack displacements, however, are influenced by higher terms as well. Perhaps his displacements would be more accurate if higher-order terms had been included in the convergence criterion.

Figure 5 shows similar results for pure bending. Stress intensity factor and crack mouth displacement coefficients derived from the weight function approach are within 2.5 percent of the values from the reference equations. Gross' stress intensity factors are within 4 percent at \( a/W = 0.05 \) and within 2.6 percent for \( a/W \geq 0.5 \). Crack mouth displacements show about the same trend as for uniform tension.

The case of four-point bending was analyzed next. The stress intensity factor and the stress intensity factor and the crack mouth displacement coefficients were determined as a percent of the values from the reference equations. Gross' results are

### Table III

**Dimensionless Stress Intensity Factor and Crack Mouth Displacement Coefficients**

<table>
<thead>
<tr>
<th>Relative crack length, ( a/W )</th>
<th>Uniform tension, ( K_{BWW}/p )</th>
<th>Pure bending, ( E'K_{BWW}/6Ma )</th>
<th>Four-point bending, by weight function, ( K_{BWW}/12p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight function</td>
<td>Tada (ref. 6)</td>
<td>Koiter (ref. 7)</td>
<td>Gross (ref. 9)</td>
</tr>
<tr>
<td>0.025</td>
<td>2.989</td>
<td>2.920</td>
<td>2.912</td>
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<td>2.905</td>
</tr>
<tr>
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<td>2.915</td>
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<td>3.525</td>
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<td>6.366</td>
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</table>
Figure 4.—Coefficients for uniform tension.

(a) Stress intensity coefficients.
(b) Crack mouth displacement coefficients.

Figure 5.—Coefficients for pure bending.

(a) Stress intensity coefficients.
(b) Crack mouth displacement coefficients.

Figure 6.—Coefficients for three-point bending.

(a) Stress intensity coefficients.
(b) Crack mouth displacement coefficients.
The most important conclusion from this study is that contact stresses influence the stress intensity factor and crack mouth displacement coefficients for small edge cracks in three-point bending. The effect is small but may be significant in small-crack fracture and fatigue-crack propagation studies.

A second conclusion is that collocation values of the stress intensity factor are accurate for $a/W \geq 0.1$ and values of crack mouth displacement are accurate for $a/W \geq 0.2$. A different convergence criterion may be necessary if the collocation method is to be successful for small cracks. It is also evident that the weight function method is useful and effective for crack analysis.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, September 25, 1987

References

16. Abstract

The weight function method was used to derive stress intensity factors and crack mouth displacement coefficients for small edge cracks (less than 20 percent of the specimen width) in common fracture specimen configurations. Contact stresses due to point application of loads were found to be small but significant for three-point bending and insignificant for four-point bending. The results are compared with available equations and numerical solutions from the literature and with unpublished boundary collocation results.

17. Key Words (Suggested by Author(s))

Elastic deformation; Fracture mechanics; Edge cracks; Stress intensity factors; Crack mouth displacement; Stress analysis; Weight function

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