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A statistical investigation of the single-point pdf of velocity and vorticity based on direct numerical simulations

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Vorticity plays a fundamental role in turbulent flows. However, most closure models presently available do not treat vorticity in an explicit fashion. Hence it is suggested to investigate the dynamics of vorticity in turbulent flows and the effect on single-point closure models. The approach is to use direct numerical simulations of turbulent flows to investigate the pdf of velocity and vorticity.

The pdf of velocity and vorticity is governed by a transport equation, which contains terms describing the dynamical processes of vortex stretching, viscous dissipation, and the effect of fluctuating pressure gradients as conditional fluxes in velocity-vorticity and physical spaces. These fluxes, together with appropriate boundary conditions, determine the evolution of the pdf from an initial state. Analysis of these fluxes shows that they cannot be represented in terms of the single-point pdf only, but require structural information in terms of two-point pdf's or two-point correlations. A direct way of getting information on the conditional fluxes is the statistical evaluation of results obtained from direct numerical simulations of turbulent flows, presently possible only at relatively low Reynolds numbers. This was carried out for a homogeneous shear flow.

Consider a point $(\mathbf{x}, t)$ in a turbulent flow field. Let $v_i(\mathbf{x}, t)$ and $w_i(\mathbf{x}, t)$ be a realization of velocity and vorticity at the chosen point. The quantity (Lundgren 1967)

$$\hat{f} \equiv \delta(\mathbf{v}(\mathbf{x}, t) - \mathbf{V})\delta(\mathbf{w}(\mathbf{x}, t) - \mathbf{W}) \tag{1}$$

denotes the fine-grained pdf of velocity and vorticity at $(\mathbf{x}, t)$. The expectation of $\hat{f}$ is then the pdf at $(\mathbf{x}, t)$,

$$f(\mathbf{V}, \mathbf{W}; \mathbf{x}, t) \equiv < \hat{f} > , \tag{2}$$

where $<>$ denotes an ensemble average. The fine-grained pdf is conserved

$$\partial_t \hat{f} + \frac{\partial}{\partial V_i}(\partial_t v_i \hat{f}) + \frac{\partial}{\partial W_i}(\partial_t w_i \hat{f}) = 0 \ , \tag{3}$$

where $\partial_i = \partial / \partial x_i$ and $\partial_t = \partial / \partial t$. Averaging of (3) and the use of the balances for an incompressible Newtonian fluid, in the absence of body forces, lead to the pdf transport equation:

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$$\partial_t f + V_i \partial_i f = \frac{\partial}{\partial V_i}(< \frac{1}{\rho}\partial_i p | \ldots > f) + \nu \nabla^2 f - \frac{\partial^2}{\partial V_i \partial V_j}(\nu < \partial_k v_i \partial_k v_j | \ldots > f)$$

$$-2\frac{\partial^2}{\partial V_i \partial W_j}(\nu < \partial_k v_i \partial_k w_j | \ldots > f) - \frac{\partial^2}{\partial W_i \partial W_j}(\nu < \partial_k w_i \partial_k w_j | \ldots > f)$$

$$-\frac{\partial^2}{\partial x_j \partial W_i}(V_i W_j f) - \frac{\partial^2}{\partial W_i \partial V_k}(< w_j v_i S_{jk} | \ldots > f)$$

$$-\frac{\partial^2}{\partial W_i \partial W_k}(< w_j v_i \partial_j w_k | \ldots > f) - \frac{\partial}{\partial V_i}(< f_i | \ldots > f)$$

$$-\epsilon_{ijk}\frac{\partial}{\partial W_i}(< \partial_j f_k | \ldots > f) , \tag{4}$$

where $\ldots$ denotes the condition $(\mathbf{v} = \mathbf{V}, \mathbf{w} = \mathbf{W})$ and $S_{ij} \equiv \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ denotes the rate of strain.

This transport equation contains three dynamically different groups of terms. The conditional flux of f due to the fluctuations of the pressure gradient

$$F_i \equiv < \frac{1}{\rho}\partial_i p | \mathbf{v} = \mathbf{V}, \mathbf{w} = \mathbf{W} > \tag{5}$$

acts on f in velocity space only, since the vorticity transport equation does not contain the pressure in explicit form. This implies that

$$\lim_{|\mathbf{V}| \to \infty} F_i f = 0$$

no matter what value $\mathbf{W}$ for vorticity is considered. In this preliminary report, we will focus our attention on F_i.

The conditional fluxes caused by the fluctuating pressure gradient, vortex stretching and viscous stresses are functions of the point $(\mathbf{x}, t)$ and the conditioning variables $(\mathbf{V}, \mathbf{W})$. Hence, they are functions of up to ten independent variables. Consequently, we consider conditional expectations with increasing number of conditions in order to begin with a manageable number of independent variables. Integration of equation (5) over vorticity space leads to

$$\int F_i(\mathbf{V}, \mathbf{W}) f(\mathbf{V}, \mathbf{W}) d\mathbf{W} = f^{\mathbf{V}}(\mathbf{V}) \int F_i(\mathbf{V}, \mathbf{W}) f(\mathbf{W}|\mathbf{V}) d\mathbf{W}$$

and thus

$$F_i^{\mathbf{V}}(\mathbf{V}) f^{\mathbf{V}}(\mathbf{V}) = \int F_i(\mathbf{V}, \mathbf{W}) f(\mathbf{V}, \mathbf{W}) d\mathbf{W} ,$$

where

$$F_i^{\mathbf{V}}(\mathbf{V}; \mathbf{x}, t) = \int F_i(\mathbf{V}, \mathbf{W}; \mathbf{x}, t) f(\mathbf{W}|\mathbf{V}; \mathbf{x}, t) d\mathbf{W}$$

denotes the conditional flux in velocity space irrespective of vorticity and $f(\mathbf{W}|\mathbf{V})$ the conditional pdf of vorticity given velocity. Then,

$$F_i^{\mathbf{V}}(\mathbf{V};\mathbf{x},t) =< \frac{1}{\rho}\partial_i p|\mathbf{v} = \mathbf{V} > . \tag{6}$$

Integration over parts of the velocity space leads furthermore to expectations conditioned with a single variable,

$$F_i^j(V_j;\mathbf{x},t) =< \frac{1}{\rho}\partial_i p|v_j = V_j > . \tag{7}$$

These quantities are most easily accessible to numerical evaluation and they constitute, therefore, the starting point for the investigation.

The direct simulation of a homogeneous linear shear flow carried out by Rogers and Moin (1987) and Rogers, Moin and Reynolds (1986) (case C128U12) was the data base for the evaluation of the pdf's and the conditional fluxes. This was done in three steps: (1) conditioning with one velocity component; (2) conditioning with one vorticity component; (3) conditioning with two variables (two velocity components, one velocity and one vorticity component or two vorticity components).

In the plots that follow, iso-probability contour lines are equally spaced, with high values near the center. The important aspect is the shape of the contour lines, and so the flow-dependent levels and coordinate ranges are omitted. Samples are collected in discrete bins, which produces the rather jagged look to the diagrams. The plotting package used plots the curves over the full range of velocity fluctuation values encountered, ranging from the minimum to the maximum. The abscissa and ordinate range from their minimum to maximum values so their zeros are not exactly in the center of the figure. At each extreme there is only one data point, and hence no conclusion can be drawn on the statistical behavior there. Areas with inadequate statistical sample, and hence highly uncertain values, are shaded.

The joint pdfs of the pressure-gradient component $\partial_1 p$ and one velocity component V_1,V_2, or V_3 are shown in Figures 1–3. The skewed shapes of the iso-probability contours show that $\partial_1 p$ is weakly correlated positively with V_1 and more strongly correlated negatively with V_2; the correlation with V_3 is zero by symmetry. The conditional pdfs of ∂p_1 are shown in Figures 4–6. They show that the conditioned probabilities are of the same form at different values of V_i, with a mean value (expectation) that varies with the conditioning velocity. The conditional expectations $< \partial_1 p|v_i >$ are shown in Figure 7. Note that they are linear in the velocity components over the range of adequate sample, an observation of importance in modeling. Figure 8 shows contours of the expectation of $\partial_1 p$ conditioned on both V_1 and V_2. Note that these contours are straight in the region of adequate sample, consistent with the linear behavior found in Figure 7, and that the rate of change with respect to each velocity component is independent of the velocities.

The joint pdfs of pressure-gradient components and single vorticity components show no discernable correlation between the two. Consequently, the expectation values of the fluctuation pressure gradient, conditioned on the local fluctuation

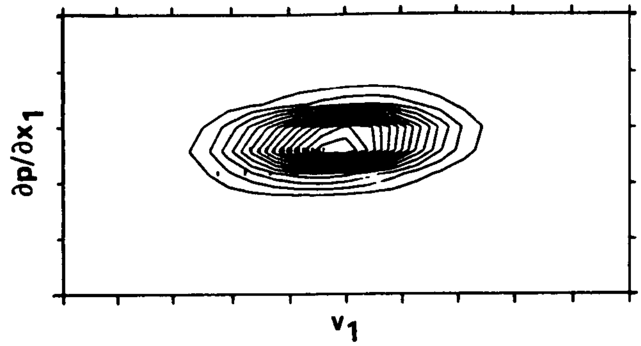

FIGURE 1. Iso-probability lines of $f(v_1, \partial_1 p)$ in a linear shear flow.

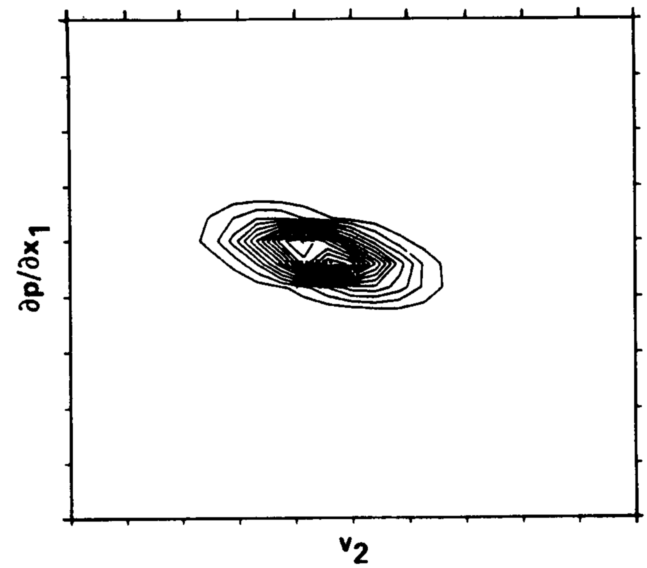

FIGURE 2. Iso-probability lines of $f(v_2, \partial_1 p)$.

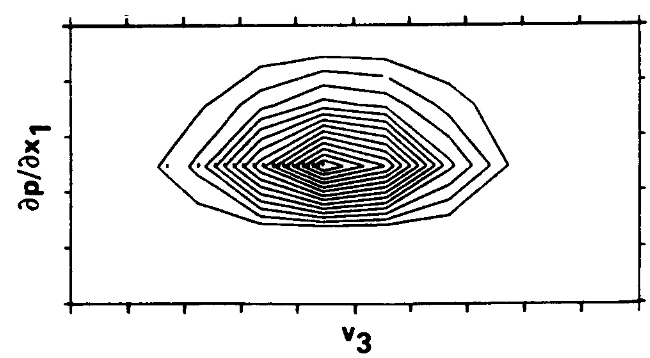

FIGURE 3. Iso-probability lines of $f(v_3, \partial_1 p)$.

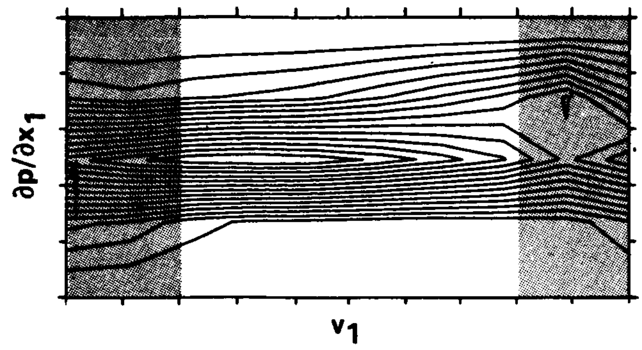

FIGURE 4. Iso-probability lines of the conditional pdf $f(\partial_1 p|v_1)$.

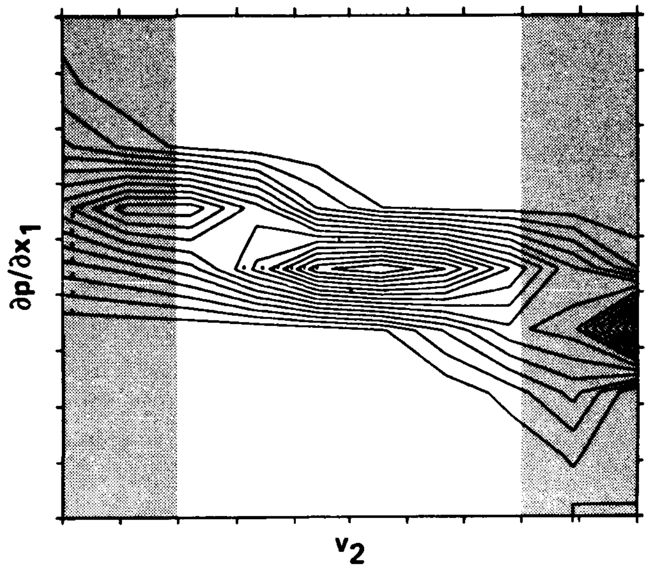

FIGURE 5. Iso-probability lines of the conditional pdf $f(\partial_1 p|v_2)$.

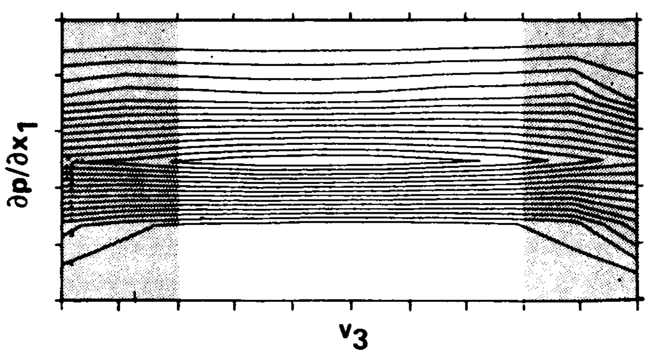

FIGURE 6. Iso-probability lines of the conditional pdf $f(\partial_1 p|v_3)$.

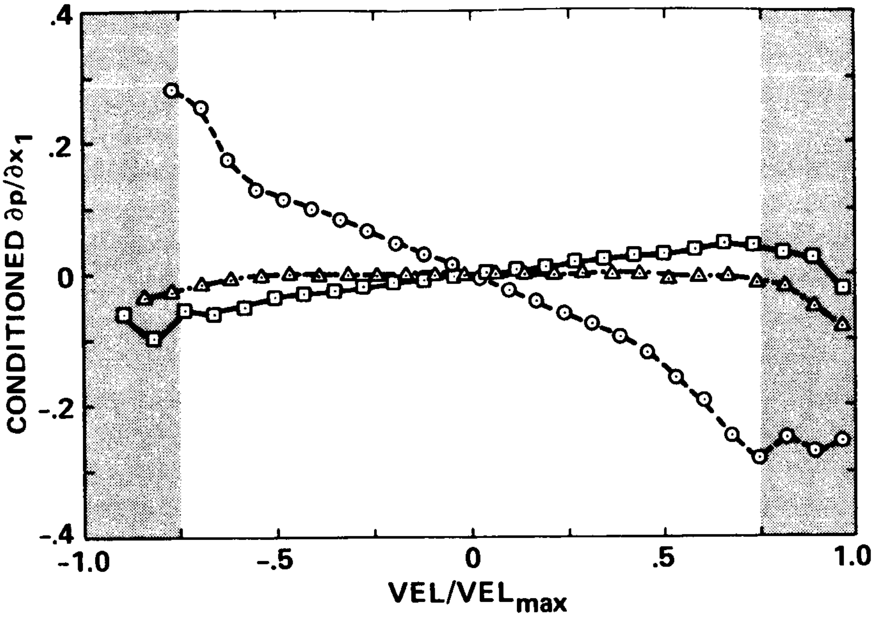

FIGURE 7. Rescaled conditional expectations $< \partial_1 p | v_\alpha >$. —— $\alpha = 1$; - - - - $\alpha = 2$; — · — $\alpha = 3$.

vorticity, are essentially zero. For example, Figure 9 shows the contours of the expectation of the streamwise pressure gradient $\partial_1 p$ conditioned on the velocity component V_2 and the streamwise vorticity component W_1. Note that the expectation is independent of vorticity. Figure 10 shows the expectations of the streamwise pressure gradient, conditioned on the vorticity components W_1 and W_2. The variations at either end are due to inadequate sample, and the flat portion in the middle is at zero, indicating no dependence on the vorticity.

In summary, this preliminary study of homogeneous shear flow has shown that the expectation of the fluctuating pressure gradient, conditioned with a velocity component, is linear in the velocity component, and that the coefficient is independent of velocity and vorticity. In addition, the work shows that the expectation of the pressure gradient, conditioned with a vorticity component, is essentially zero.

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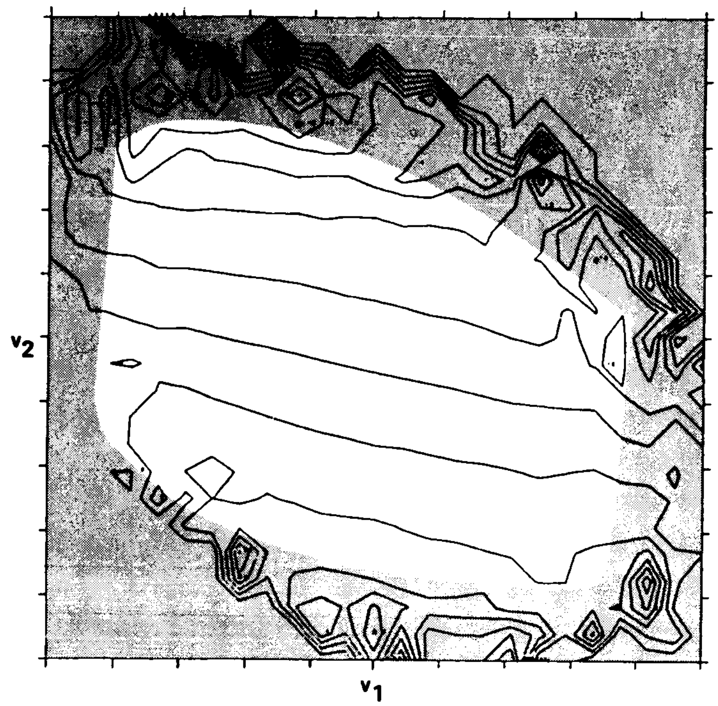

FIGURE 8. Conditional expectation $< \partial_1 p | v_1, v_2 >$

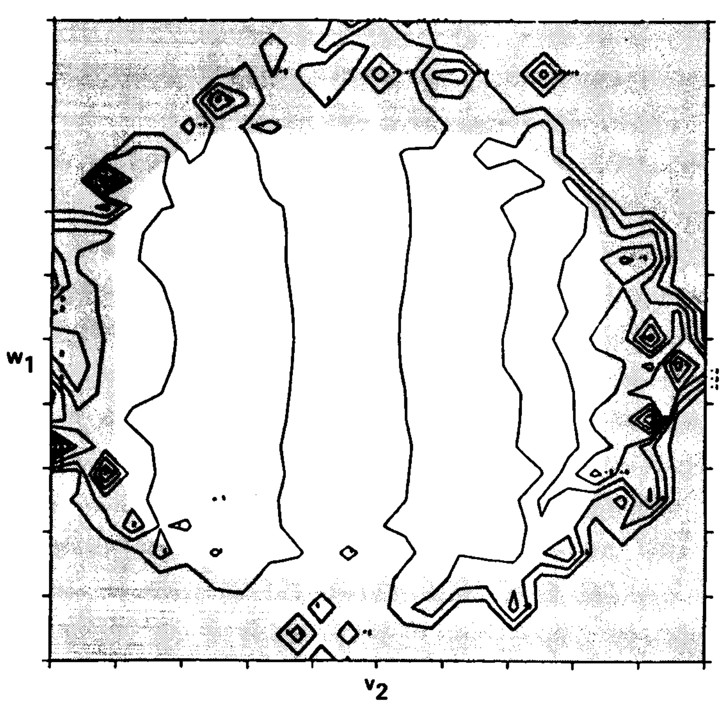

FIGURE 9. Conditional expectation $< \partial_1 p | v_2, w_1 >$

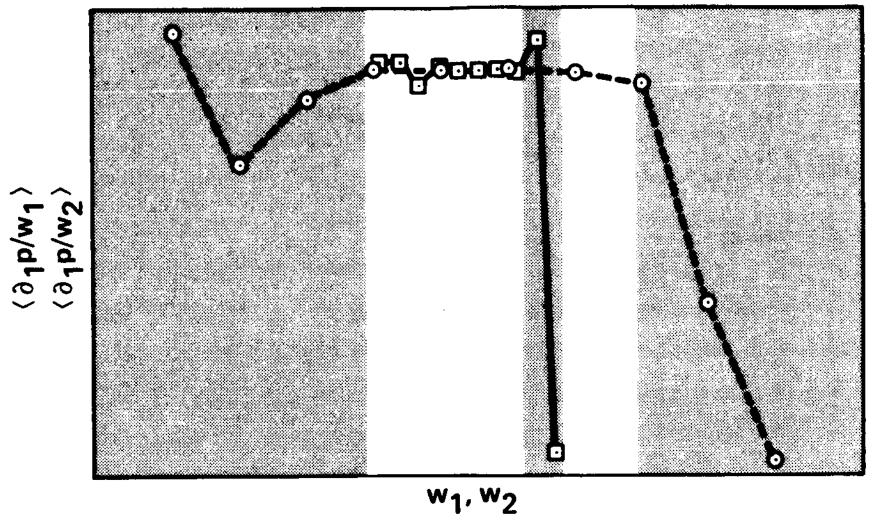

FIGURE 10. Conditional expectations $< \partial_1 p | w_\alpha >$. —— $\alpha = 1$; ---- $\alpha = 2$.