NASA CR-181,683

NASA Contractor Report 181683

ICASE REPORT NO. 88-41

NASA-CR-181683
19880016773

ICASE

SEMI-DISCRETE APPROXIMATIONS TO NONLINEAR SYSTEMS OF CONSERVATION LAWS; CONSISTENCY AND L^{∞}-STABILITY IMPLY CONVERGENCE

Eitan Tadmor

LIBRARY COPY

AUG 8 1988

LANGLEY RESEARCH CENTER
LIBRARY NASA
HAMPTON, VIRGINIA

Contract No. NAS1-18107
July 1988

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

NF00900

SEMI-DISCRETE APPROXIMATIONS TO NONLINEAR SYSTEMS OF CONSERVATION LAWS; CONSISTENCY AND L^∞– STABILITY IMPLY CONVERGENCE

Eitan Tadmor
Raymond and Beverly Sackler, Faculty of Exact Sciences
School of Mathematical Sciences, Tel-Aviv University
and
Institute for Computer Applications in Science and Engineering

ABSTRACT

We develop a convergence theory for semi-discrete approximations to nonlinear systems of conservation laws. We show, by a series of scalar counterexamples, that consistency with the conservation law alone does not guarantee convergence. Instead, we introduce a notion of consistency which takes into account both the conservation law and its augmenting entropy condition. In this context, we conclude that consistency and L^∞-stability guarantee for a "relevant" class of admissible entropy functions, that their entropy production rate belong to a compact subset of $H^{-1}_{\text{loc}}(x,t)$. One can use now compensated compactness arguments in order to turn this conclusion into a convergence proof. The current state of the art for these arguments includes the scalar and a wide class of 2×2 systems of conservation laws.

We study the general framework of the vanishing viscosity method as an effective way to meet our consistency and L^∞-stability requirements. We show how this method is utilized to enforce consistency and L^∞-stability for scalar conservation laws. In this context, we prove under the appropriate assumptions (L^∞-bounds), the convergence of finite-difference approximations (e.g., the high-resolution TVD and UNO methods), finite-element approximations (e.g., the Streamline-Diffusion methods) and spectral and pseudospectral approximations (e.g., the Spectral Viscosity methods).

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665. Additional support was provided by U.S.-Israel BSF Grant No. 85-00346, and by NSF Grant No. DMS85-03294 while in residence at the University of California, Los Angeles, CA 90024.

N88-26157 #

TABLE OF CONTENTS

1.	Introduction	1
2.	The Necessity of Entropy Dissipation	9
3.	Consistency with the Entropy Condition	13
4.	Consistency of the Scalar Vanishing Viscosity Method	20
5.	L^∞-Stability of the Scalar Vanishing Viscosity Method	40
6.	Consistency and L^∞-Stability imply Convergence	47
7.	Appendix: Consistency with the Conservation Law	53
	References	58

1. INTRODUCTION

We consider 2π-periodic initial-value problems, which consist of the one-dimensional system of conservation laws

$$\frac{\partial}{\partial t}[u(x,t)] + \frac{\partial}{\partial x}[f(u(x,t))] = 0, \tag{1.1a}$$

together with an augmenting entropy condition, which requires that for <u>all</u> convex entropy pairs $(U(u), F(u) \equiv \int^u < U'(w), f'(w)dw >)$ the following entropy inequality holds [7]

$$\frac{\partial}{\partial t}[U(u(x,t))] + \frac{\partial}{\partial x}[F(u(x,t))] \leq 0. \tag{1.1b}$$

We want to solve this problem by a semi-discrete algorithm. To this end one associates with a large parameter N (or a small parameter $\Delta x \equiv \frac{2\pi}{2N+1}$, depending on the point of view),

I. A finite $(2N+1)$-dimensional space, ϕ_N, spanned by 2π-periodic basis functions, $\{\varphi_k(x)\}_{k=-N}^{N}$,

<u>and</u>

II. A possibly nonlinear, f-dependent, spatial discretization operator, P_N^f,

$$P_N^f : L^\infty[0, 2\pi] \to \Phi_N,$$

such that $P_N^f w(x)$ is an appropriate Φ_N-approximation of $f(w(x))$; here, Φ_N denotes the $(2N+1)$-space spanned by the 2π-periodic <u>primitives</u> of $\{\varphi_k(x)\}_{k=-N}^{N}$.
The exact solution of (1.1a), (1.1b), $u(x,t)$, is then approximated by a ϕ_N-element, $u_N(x,t)$,

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t)\varphi_k(x), \tag{1.2}$$

which is determined by the following procedure:

Starting with prescribed (possibly pre-processed) initial-data $u_N(x,0)$ in ϕ_N, we let $u_N(x,t)$ evolve later in time according to the $(2N+1)$-dimensional approximate model

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N^f u_N(x,t)] = 0. \tag{1.3}$$

In what sense does (1.3) approximate (1.1a)? Let us rewrite (1.3) in the form

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[f(u_N(x,t))] = \frac{\partial}{\partial x}[f(u_N(x,t)) - P_N^f u_N(x,t)]. \tag{1.4}$$

The expression inside the right brackets is the discrete local error, $E_N^f u_N(x,t)$,

$$E_N^f u_N(x,t) \equiv f(u_N(x,t)) - P_N^f u_N(x,t), \tag{1.5}$$

which reflects the amount by which $u_N(x,t)$ fails to satisfy (1.1a). Its size may serve us as a measure for the order of accuracy of (1.3).

Definition 1.1: (Order of accuracy) The approximation (1.3) is accurate of order $s > 0$ with the conservation law (1.1a), if there exist constants, $C_j, j = 0, 1, \cdots, s-1$, such that for any $r, 0 < r \leq s$, and for all $w_N(x)$ in ϕ_N, the following estimate holds [1]

$$\|E_N^f w_N(x)\| \leq \frac{1}{N^r}\left[C_0 \cdot \|\frac{\partial^r}{\partial x^r} w_N(x)\| + \sum_{j=1}^{r-1} C_j\right]. \tag{1.6}$$

Here $C_j, j = 0, 1, \cdots$, are constants which may depend on the L^∞-bounds of $w_N(x)$ and its first j-derivatives, but otherwise are independent of N.

Remark: In a similar manner one can define accuracy of nonintegral orders $s > 0$.

We note that (1.6) is a refinement of the usual definition of accuracy, in that here, the local error on the left and the highest derivative involved on the right are weighted by the L^2 rather than the usual L^∞-norm. This then implies that s-order accurate approximations are, in particular, consistent with the conservation law (1.1a), in the sense of

Definition 1.2: (Consistency with the conservation law) The approximation (1.3) is consistent with conservation law (1.1a), if for all $w_N(x)$ in ϕ_N which are uniformly bounded w.r.t. N, there exists a vanishing sequence, $c_N \to 0$, such that the following estimate holds

$$\|E_N^f w_N(x)\| \leq c_N \|\frac{\partial}{\partial x} w_N(x)\| . \tag{1.7}$$

Indeed, an s-order accurate approximation satisfies the consistency requirements, (1.7), with $c_N = \frac{C_0}{N}$.

Remark: In the generic case, the discrete local error is upper-bounded by

$$\|E_N^f w_N(x)\| \leq \frac{Const.}{N} \|\frac{\partial}{\partial x} f(w_N(x))\| , \tag{1.8}$$

with some universal constant $Const.$ Hence (1.7) holds with $c_N = Const_0 \cdot \frac{1}{N}$, where $Const_0 \sim \|A \equiv f'(w_N(x))\|_{L^\infty(x)}$.

The first of the two ingredients involved in discretizing the conservation law (1.1a), is the choice of the finite $(2N+1)$-space ϕ_N. A very convenient choice in this respect is

[1] We use $< \cdot, \cdot >$ and $|\cdot|$ to denote the usual Euclidean vector inner product and norm. Similar notations are used for vector functions, e.g., $(\cdot,\cdot) = \int_0^{2\pi} < \cdot, \cdot > dx$ and $\|\cdot\|^2 = (\cdot,\cdot)$ for spatial vector functions and $(\cdot,\cdot) = \int_0^T \int_0^{2\pi} < \cdot, \cdot > dx\, dt$, $\|\cdot\|^2 = (\cdot,\cdot)$ for $L^2_{\text{loc}}(x,t)$ space-time integration.

the space of N-trigonometric polynomials, π_N, spanned by $\{e^{ikx}\}_{k=-N}^{N}$. In this case, π_N coincides with the space spanned by its 2π-periodic primitives, and they consist of spatially smooth approximants of the form

$$u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx}. \tag{1.9}$$

The principal raison d'etre for the π_N space is contained in its isometry with the $(2N+1)$-space of Fourier coefficients $\{\hat{u}_k(t)\}_{k=-N}^{N}$. Spectral and pseudospectral methods are the canonical examples for π_N-discretizations of the conservation law (1.1a).

Example 1.3: Spectral Methods. Denote by $S_N w(x)$ the spectral-Fourier projection of $w(x)$ into π_N

$$S_N w(x) = \sum_{k=-N}^{N} \hat{w}(k) e^{ikx}, \qquad \hat{w}(k) = \frac{1}{2\pi}\int_0^{2\pi} w(x) e^{-ikx} dx. \tag{1.10}$$

The approximation (1.3) with $P_N^f u_N(\cdot,t) = S_N f(u_N(\cdot,t))$, amounts to the spectral method for the trigonometric approximant, $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx}$, which reads

$$\frac{d}{dt}\hat{u}_k(t) + ik\hat{f}_k(t) = 0, \qquad \hat{f}_k(t) = \frac{1}{2\pi}\int_0^{2\pi} f(u_N(x,t)) e^{-ikx} dx, \qquad |k| \le N. \tag{1.11}$$

In Section 7 we prove that the spectral method (1.11) is accurate of any order $s > 0$, i.e., it is "infinitely-order" or spectrally accurate with the conservation law (1.1a).

Example 1.4: Pseudospectral Methods. Denote by $\psi_N w(x)$ the ψdospectral-Fourier projection of $w(x)$, which interpolates $w(x)$ at the $2N+1$ equidistant collocation points $x_{\nu+\theta} \equiv (\nu+\theta)\Delta x$, $\nu = 0, 1, \cdots, 2N$, with fixed $0 \le \theta < 1$ and $\Delta x = \frac{2\pi}{2N+1}$,

$$\psi_N w(x) = \sum_{k=-N}^{N} \hat{w}(k) e^{ikx}, \qquad \hat{w}(k) = \frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N} w(x_{\nu+\theta}) e^{-ikx_{\nu+\theta}}. \tag{1.12}$$

The approximation (1.3) with $P_N^f u_N(\cdot,t) = \psi_N f(u_N(\cdot,t))$, amounts to the pseudospectral method for the trigonometric approximant, $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx}$, which reads

$$\frac{d}{dt}\hat{u}_k(t) + ik\hat{f}_k(t) = 0, \qquad \hat{f}_k(t) = \frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N} f(u_N(x_{\nu+\theta},t)) e^{-ikx_{\nu+\theta}}, \qquad |k| \le N. \tag{1.13}$$

In Section 7 we prove that the ψdospectral method (1.13) is accurate of any order $s > \frac{1}{2}$, i.e., it is "infinitely-order" or spectrally accurate with the conservation law (1.1a).

Finite-difference and finite-element methods are usually interpreted as evolution discretizations of (1.1a) in terms of piecewise-constant or piecewise-linear approximants. As

we shall now see, these methods can be equally well interpreted as π_N-discretizations of (1.1a), governing the evolution of trigonometric approximants instead of piecewise-constant or piecewise-linear ones. Let us turn to

Conservative Methods. We deal with discrete approximations of (1.1a) which admit the conservative form [9]

$$\frac{d}{dt}u_N(x_\nu,t)+\frac{1}{\Delta x}[h^f_{\nu+\frac{1}{2}}-h^f_{\nu-\frac{1}{2}}]=0,\qquad x_\nu=\nu\Delta x,\qquad \nu=0,1,\cdots,2N. \tag{1.14}$$

Here $h^f_{\nu+\frac{1}{2}}\equiv h^f(u_N(x_{\nu-p+1},t),\cdots,u_N(x_{\nu+p},t))$ is the Lipschitz continuous numerical flux which is consistent with the differential one

$$h^f(u_N(x,t),\cdots,u_N(x,t))\equiv f(u_N(x,t)). \tag{1.15}$$

In Section 7, we prove that arbitrary conservative schemes of this form are at least first-order accurate approximations of the conservation law (1.1a). In order to interpret such schemes within the π_N-framework (1.3)(1.9), let us introduce the π_N-polynomial, $H_N(x,t)=\sum_{k=-N}^{N}\hat{H}_k(t)e^{ikx}$, which interpolates the numerical flux values at the grid points $x_{\nu+\frac{1}{2}}=(\nu+\frac{1}{2})\Delta x,\ \nu=0,1,\cdots,2N$,

$$H_N(x=x_{\nu+\frac{1}{2}},t)=h^f_{\nu+\frac{1}{2}},\qquad \hat{H}_k(t)=\frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N}h^f_{\nu+\frac{1}{2}}e^{-ikx_{\nu+\frac{1}{2}}}, \tag{1.16a}$$

and then define $P^f_N u_N(x,t)$ as the sliding average of this interpolant

$$P^f_N u_N(x,t)=\overline{H_N(x)},\qquad \overline{H_N(x)}\equiv\frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}H_N(\xi)d\xi. \tag{1.16b}$$

By applying such averaging plus summation by parts to $H_N(x,t)$, we end up with the (possibly nonlinear) spatial operator, P^f_N, which approximates $f(u_N(x,t))$ from π_N via

$$P^f_N u_N(x,t)=\sum_{k=-N}^{N}\hat{p}_k(t)e^{ikx},\qquad \hat{p}_k(t)=\frac{1}{2\pi ik}\sum_{\nu=0}^{2N}[h^f_{\nu+\frac{1}{2}}-h^f_{\nu-\frac{1}{2}}]e^{-ikx_\nu}. \tag{1.17}$$

We note that due to conservation, $\hat{p}_0(t)$, is well-defined as $\hat{p}_0(t)\equiv 0$. The approximation (1.3) with P^f_N defined in (1.17) amounts to

$$\frac{d}{dt}\hat{u}_k(t)+\frac{1}{2\pi}\sum_{\nu=0}^{2N}[h^f_{\nu+\frac{1}{2}}-h^f_{\nu-\frac{1}{2}}]e^{ikx_\nu}=0,\qquad u_N(x_\nu,t)=\sum_{k=-N}^{N}\hat{u}_k(t)e^{ikx_\nu}, \tag{1.18}$$

which by the inverse discrete Fourier transform is equivalent with (1.14).

In this manner one can imbed any finite-difference or finite-element method as an evolution scheme in π_N.

An instructive example for this process is provided by

Example 1.5: Centered Finite-Difference Methods. Consider the standard centered finite-difference method

$$\frac{d}{dt}u_N(x_\nu,t)+\frac{1}{2\Delta x}[f(u_N(x_{\nu+1},t))-f(u_N(x_{\nu-1},t))]=0. \tag{1.19}$$

The discrete Fourier coefficients, $\hat{p}_k(t)$ of $P_N^f u_N(x,t)$ in (1.17), equal

$$\begin{aligned}\hat{p}_k(t) &= \tfrac{1}{2\pi ik}\textstyle\sum_{\nu=0}^{2N}\tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t))-f(u_N(x_{\nu-1},t))]e^{-ikx_\nu}\equiv\\ &\equiv \tfrac{sink\Delta x}{k\Delta x}\cdot\tfrac{\Delta x}{2\pi}\textstyle\sum_{\nu=0}^{2N}f(u_N(x_\nu,t))e^{-ikx_\nu}.\end{aligned} \tag{1.20}$$

In Section 7 we prove that the finite-difference method (1.19) satisfies the expected second-order accuracy requirement (1.6). We note that the numerical flux associated with (1.19),

$$h^f_{\nu+\frac{1}{2}}=\frac{1}{2}[f(u_N(x_\nu,t))+f(u_N(x_{\nu+1},t))], \tag{1.21}$$

depends linearly on the gridvalues of $f(u_N(\cdot,t))$; consequently we end up with a corresponding discretization operator, P_N^f, which operates linearly on $f(u_N(\cdot,t))$, i.e.,

$$P_N^f u_N(\cdot,t)=FD_{\Delta x}f(u_N(\cdot,t)), \tag{1.22a}$$

where

$$FD_{\Delta x}w(x)=\sum_{k=-N}^{N}\hat{w}(k)e^{ikx},\qquad \hat{w}(k)=\frac{sink\Delta x}{k\Delta x}\cdot\frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N}w(x_\nu)e^{-ikx_\nu}. \tag{1.22b}$$

The fact that the finite-difference operator, $FD_{\Delta x}$, turns out to be a smoothed version of the ψdospectral projection ψ_N in (1.12),

$$FD_{\Delta x}w(x)=\frac{1}{2\Delta x}\int_{x-\Delta x}^{x+\Delta x}\psi_N(w(\xi))d\xi \tag{1.22c}$$

is typical to all standard linear differencing methods, consult [19]. In other cases, the resulting operator P_N^f in (1.17) may depend on the flux f in a more intricate way as shown by

Example 1.6: Finite-Element Methods. Let $(U_*(u),F_*(u))$ be any preferred entropy pair associated with (1.1a). Define the entropy variables

$$v=U_*'(u), \tag{1.23}$$

and note that the strict convexity of $U_*(u)$ enables to uniquely invert (1.23), $u=u(v)$.

The finite-element approximation of (1.1a), based on piecewise-linear elements in the entropy variables v reads [18, Section 5] [21, Section 4],

$$\frac{d}{dt}\left[\frac{1}{6}u_N(x_{\nu-1},t)+\frac{4}{6}u_N(x_\nu,t)+\frac{1}{6}u_N(x_{\nu+1},t)\right]+\frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}}-h^*_{\nu-\frac{1}{2}}]=0\ , \tag{1.24a}$$

and the mass lumped version of this yields

$$\frac{d}{dt}u_N(x_\nu,t)+\frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}}-h^*_{\nu-\frac{1}{2}}]=0. \tag{1.24b}$$

Here $h^*_{\nu+\frac{1}{2}}$ is the numerical flux given by

$$h^*_{\nu+\frac{1}{2}}=\int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f(u(v_{\nu+\frac{1}{2}}(\xi))d\xi, \tag{1.25a}$$

where

$$v_{\nu+\frac{1}{2}}(\xi)\equiv(\frac{1}{2}-\xi)U'_*(u_N(x_\nu,t))+(\frac{1}{2}+\xi)U'_*(u_N(x_{\nu+1},t)). \tag{1.25b}$$

We recall that in the case of standard linear finite-difference methods, the numerical flux depends solely on the <u>gridvalues</u> $f(u_N(x_\nu,t))$. In contrast, the current finite-element numerical flux in (1.25a), (1.25b), $h^*_{\nu+\frac{1}{2}}$, depend on <u>all</u> the intermediate values $f(u(v(\xi)))$. This fact is reflected in the corresponding discretization operator, P^f_N, which according to (1.16a), (1.16b) is given by

$$P^f_N u_N(\cdot,t)=FE_{\Delta x}f(u_N(\cdot,t))\ , \tag{1.26a}$$

with

$$FE_{\Delta x}w(x)=\frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}}H^*_N(\xi)d\xi\ ,\qquad H^*_N(x=x_{\nu+\frac{1}{2}})=\int_{\xi=-1/2}^{1/2}w(v_{\nu+\frac{1}{2}}(\xi))d\xi\ . \tag{1.26b}$$

<u>Remark</u>: In both cases of the finite-difference and finite-element methods (1.22), (1.26), the spatial discretization, P^f_N, operates <u>linearly</u> on the flux f, i.e.,

$$P^{\alpha_1 f_1+\alpha_2 f_2}_N=\alpha_1 P^{f_1}_N+\alpha_2 P^{f_2}_N. \tag{1.27}$$

The more modern shock-capturing techniques employ numerical fluxes, $h^f_{\nu+\frac{1}{2}}$, which depend on the data, $u_N(x,t)$ and f, in an essentially nonlinear manner [3]. These methods can be also interpreted as evolution schemes in π_N using the discretization recipe (1.16a), (1.16b); in such essentially nonlinear cases, however, the resulting discretization operator, P^f_N, need not satisfy the linearity property (1.27). Our foregoing discussion equally applies to linear methods as well as the essentially nonlinear ones.

The examples considered so far, were formulated within the trigonometric π_N-framework. We close this section with an example which shows how the finite-difference/element methods (1.24), can be equivalently formulated within the canonical piecewise-linear framework.

Example 1.7: Finite-Difference/Element Methods revisited. Let ϕ_N be the space of piecewise-linear grid functions, $w_N(x)$,

$$w_N(x) = \sum_{k=-N}^{N} w_N(x_k)\Lambda_k(x) \ , \tag{1.28a}$$

spanned by the 2π-periodic 'hat' functions, $\varphi_k(x) = \Lambda_k(x)$, which are centered at $x_k = k\Delta x$,

$$\Lambda_k(x) = \frac{1}{\Delta x}\min([x] - x_{k-1}, x_{k+1} - [x]) \quad , \qquad [x] = x[\mathrm{mod}\, 2\pi] \ . \tag{1.28b}$$

This family of basis functions is not orthogonal – the corresponding mass matrix M, $M_{ij} \equiv (\Lambda_i(x), \Lambda_j(x))$, is given by the circulant matrix

$$M = \Delta x C \ , \qquad C = \mathrm{Circulant}\left(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right) \ . \tag{1.29}$$

In order to interpret (1.24) within the current piecewise-linear framework, we introduce the finite-element discretization operator, $FE_{\Delta x}$, which maps $L^\infty[-\pi, \pi]$ into the space Φ_N of piecewise-quadratic B-spline gridfunctions,

$$FE_{\Delta x} w(x) = \sum_{k=-N}^{N} \overline{w}_{k+\frac{1}{2}} \Omega_{k+\frac{1}{2}}(x). \tag{1.30a}$$

Here, $\Omega_{k+\frac{1}{2}}(x)$ stands for the 2π-periodic "bell-shaped" functions supported on $[x_{k-1}, x_{k+2}]$,

$$\Omega'_{k+\frac{1}{2}}(x) = \frac{1}{\Delta x}[\Lambda_k(x) - \Lambda_{k+1}(x)] \ , \tag{1.30b}$$

and $\overline{w} \equiv \{\overline{w}_{k+\frac{1}{2}}\}_{k=-N}^{N}$ is the vector of cell averages

$$\overline{w}_{k+\frac{1}{2}} = \overline{w}(x_{k+\frac{1}{2}}) \equiv \frac{1}{\Delta x}\int_{x_k}^{x_{k+1}} w(x)dx \ . \tag{1.30c}$$

Differentiating (1.30a) we obtain, in view of (1.30b) and (1.29),

$$\frac{\partial}{\partial x} FE_{\Delta x} w(x) = \sum_{k=-N}^{N} \frac{1}{\Delta x}[\overline{w}_{k+\frac{1}{2}} - \overline{w}_{k-\frac{1}{2}}]\Lambda_k(x) \ . \tag{1.31}$$

Hence approximation (1.3) with $P_N^f u_N(\cdot, t) = FE_{\Delta x} f(u_N(\cdot, t))$ reads, at the collocation points $x = x_\nu$,

$$\frac{d}{dt} u_N(x_\nu, t) + \frac{1}{\Delta x}[\overline{f}_{\nu+\frac{1}{2}} - \overline{f}_{\nu-\frac{1}{2}}] = 0 \ ; \tag{1.32a}$$

since $u_N(\cdot,t)$ is piecewise-linear we have

$$\overline{f}_{\nu+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_\nu}^{x_{\nu+1}} f(u_N(x,t))dx = \int_{\xi=-1/2}^{1/2} f(u_{\nu+\frac{1}{2}}(\xi))d\xi = h^*_{\nu+\frac{1}{2}} \ , \tag{1.32b}$$

and we recover the mass lumped version of the finite-element method (1.24b), with $U_*(u) = \frac{1}{2}u^2$ as the "preferred" entropy function,

$$\frac{d}{dt}u_N(x_\nu,t) + \frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}] = 0 \ . \tag{1.33}$$

Similarly, the finite-element method (1.24b) with no mass lumping corresponds to

$$FE_{\Delta x}w(x) = \sum_{k=-M}^{N} \tilde{w}_{k+\frac{1}{2}}\Omega_{k+\frac{1}{2}}(x) \ , \tag{1.34a}$$

where $\Delta\tilde{w} \equiv \{\tilde{w}_{k+\frac{1}{2}} - \tilde{w}_{k-\frac{1}{2}}\}_{k=-N}^{N}$ is a vector solution of[2]

$$C\cdot\Delta\tilde{w} = \Delta\overline{w} \quad , \qquad \Delta\overline{w} \equiv \{\overline{w}_{k+\frac{1}{2}} - \overline{w}_{k-\frac{1}{2}}\}_{k=-N}^{N} \ . \tag{1.34b}$$

In this case, (1.31) is replaced by

$$\frac{\partial}{\partial x}FE_{\Delta x}w(x) = \sum_{k=-N}^{N}\frac{1}{\Delta x}[C^{-1}\cdot\Delta\overline{w}]_k\Lambda_k(x) \ , \tag{1.35}$$

and approximation (1.3) with $P_N^f u_N(\cdot,t) = FE_{\Delta x}f(u_N(\cdot,t))$ recovers the finite-element method (1.24a)

$$\frac{d}{dt}\left[\frac{1}{6}u_N(x_{\nu-1},t) + \frac{4}{6}u_N(x_\nu,t) + \frac{1}{6}u_N(x_{\nu+1},t)\right] + \frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}] = 0 \ . \tag{1.36}$$

[2]Since the vector of ones is an eigenvector of C, a conservative solution of (1.34b) exists.

2. THE NECESSITY OF ENTROPY DISSIPATION

The consistency condition (1.7) guarantees that as N tends to infinity, the approximation (1.3) approaches the conservation law (1.1a). The question whether the approximate solution, $u_N(x,t)$, approaches the corresponding conservative solution, $u(x,t)$, is the question of convergence. The following scalar counterexamples–one for each of the discretizations methods mentioned above, show that the solutions of consistent approximations may fail to converge to the appropriate conservative solutions.

Counterexample 2.1: [20]. The spectral-Fourier approximation (1.11) of the scalar equation (1.1a) reads

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[S_N f(u_N(x,t))] = 0. \tag{2.1}$$

Multiplying this by $u_N(x,t)$ and integrating over the 2π-period, we obtain that u_N–being orthogonal to $\frac{\partial}{\partial x}[(I - S_N)f(u_N(x,t))]$, satisfies

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u_N^2(x,t)dx = -\int_0^{2\pi} u_N(x,t)\frac{\partial}{\partial x}[f(u_N(x,t))]dx = -\int^{u_N(x,t)} uf'(u)du|_{x=0}^{x=2\pi} = 0. \tag{2.2}$$

Thus, the total quadratic entropy, $U(u) = \frac{1}{2}u^2$, is globally conserved in time

$$\frac{1}{2}\int_0^{2\pi} u_N^2(x,t)dx \equiv \sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x = \sum_{\nu=0}^{2N} U(u_N(x_\nu,0))\Delta x, \qquad U(u) = \frac{1}{2}u^2, \tag{2.3}$$

which in turn yields the existence of a weak $L^2(x)$-limit $\overline{u}(x,t) = w\lim_{N\to\infty} u_N(x,t)$. Yet, $\overline{u}(x,t)$ cannot be the entropy solution of a genuinely nonlinear (GNL) Equation (1.1a) where $f''(\cdot) \neq 0$. Otherwise, $S_N f(u_N(x,t))$ and therefore $f(u_N(x,t))$ should tend weakly to $f(\overline{u}(x,t))$; consequently, since $f(u)$ is GNL, $\overline{u}(x,t) = s\lim_{N\to\infty} u_N(x,t)$ which by (2.3) should satisfy $\frac{1}{2}\int_0^{2\pi}\overline{u}^2(x,t)dx = \frac{1}{2}\int_0^{2\pi}\overline{u}^2(x,0)dx$. But this is incompatible with the entropy condition (1.1b) if $\overline{u}(x,t)$ contains shock discontinuities.

Counterexample 2.2: The pseudospectral approximation (1.13) of the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial(e^u)}{\partial x} = 0, \tag{2.4}$$

reads

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[\psi_N e^{u_N(x,t)}] = 0. \tag{2.5}$$

Multiply this by $\psi_N e^{u_N(x,t)}$ and integrate over the 2π-period: since the trapezoidal rule is exact with integration of the $2N$-trigonometric polynomial obtained from the second brackets, we have

$$\frac{d}{dt}\sum_{\nu=0}^{2N} e^{u_N(x_\nu,t)}\Delta x = -\int_0^{2\pi}\frac{\partial}{\partial x}[\frac{1}{2}(\psi_N e^{u_N(x,t)})^2]dx = 0. \tag{2.6}$$

Thus, the total exponential entropy, $U(u) = e^u$, is globally conserved in time

$$\sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x = \sum_{\nu=0}^{2N} U(u_N(x_\nu,0))\Delta x, \qquad U(u) = e^u. \tag{2.7}$$

Hence, if $u_N(x,t)$ converges (even weakly) to a discontinuous weak solution $\overline{u}(x,t)$ of (1.1a), then $\psi_N e^{u_N(x,t)}$ tends (at least weakly) to $e^{\overline{u}(x,t)}$. Consequently, (2.7) would imply the global entropy conservation of $\int_0^{2\pi} e^{\overline{u}(x,t)}dx$ in time, which rules out the possibility of $\overline{u}(x,t)$ being the unique entropy solution of (1.1a).

Counterexample 2.3: [8]. The centered-difference approximation (1.19) of the scalar conservation law (2.4), yields the nonlinear completely integrable system [5],[12]

$$\frac{d}{dt}u_N(x_\nu,t) + \frac{1}{2\Delta x}[e^{u_N(x_{\nu+1},t)} - e^{u_N(x_{\nu-1},t)}] = 0. \tag{2.8}$$

Multiplying (2.8) by $e^{u_N(x_\nu,t)}$ we obtain

$$\frac{d}{dt}[e^{u_N(x_\nu,t)}] + \frac{1}{\Delta x}[F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}] = 0, \tag{2.9a}$$

where the consistent entropy flux, $F_{\nu+\frac{1}{2}} = F(u_N(x_\nu,t), u_N(x_{\nu+1},t))$, is given by

$$F_{\nu+\frac{1}{2}} = \frac{1}{2}e^{[u_N(x_\nu,t)+u_N(x_{\nu+1},t)]}. \tag{2.9b}$$

In particular, this implies a global entropy conservation of the exponential entropy, $U(u) = e^u$,

$$\sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x = \sum_{\nu=0}^{2N} U(u_N(x_\nu,0))\Delta x, \qquad U(u) = e^u. \tag{2.10}$$

It follows from the Lax-Wendroff theorem [9] that $u_N(x,t)$ <u>cannot</u> converge boundedly a.e. to <u>any</u> function $u(x,t)$, for otherwise $u \equiv u(x,t)$ would be an entropy conservative weak solution of (2.4), satisfying the entropy <u>equality</u>

$$\frac{\partial}{\partial t}[e^u] + \frac{\partial}{\partial x}[\frac{1}{2}e^{2u}] = 0; \tag{2.11}$$

this is incompatible with the entropy condition (1.1b), once the initial data $u(x,0) = \lim_{N\to\infty} u_N(x,0)$ permit discontinuous solution $u(x,t)$. Moreover, the bounded solutions of (2.8) cannot even converge weakly to the entropy solution, $\overline{u}(x,t)$, of (2.4). Otherwise, $\frac{\partial}{\partial x}FD_{\Delta x}e^{u_N(x,t)}$ – and hence by (1.22c) also $\psi_N e^{u_N(x,t)}$ — should tend weakly to $e^{\overline{u}(x,t)}$, which in view of (2.10) leads to the same contradiction we had in the ψ-dospectral Counterexample 2.2.

Counterexample 2.4: [18]. The scalar finite-element approximation (1.24b) of (1.1a), induced by the quadratic entropy, $U_*(u) = \frac{1}{2}u^2$, reads

$$\text{(2.12}a\text{)} \qquad \frac{d}{dt}u_N(x_\nu,t) + \frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}] = 0, \qquad h^*_{\nu+\frac{1}{2}} = \int_{\xi=-\frac{1}{2}}^{\frac{1}{2}} f(u_{\nu+\frac{1}{2}}(\xi))d\xi,$$

where $u_{\nu+\frac{1}{2}}(\xi)$ abbreviates

$$\text{(2.12}b\text{)} \qquad u_{\nu+\frac{1}{2}}(\xi) = \frac{1}{2}[u_N(x_\nu,t) + u_N(x_{\nu+1},t)] + \xi[u_N(x_{\nu+1},t) - u_N(x_\nu,t)].$$

Multiplying this by $u_N(x_\nu,t)$ we obtain after rearrangement

(2.13a)

$$\frac{1}{2}\frac{d}{dt}[u_N(x_\nu,t)]^2 + \frac{1}{\Delta x}u_N(x_\nu,t)[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}] \equiv \frac{1}{2}\frac{d}{dt}[u_N(x_\nu,t)]^2 + \frac{1}{\Delta x}[F^*_{\nu+\frac{1}{2}} - F^*_{\nu+\frac{1}{2}}] = 0,$$

where the consistent entropy flux $F^*_{\nu+\frac{1}{2}}$ is given by [18]

$$\text{(2.13}b\text{)} \quad F^*_{\nu+\frac{1}{2}} = \frac{1}{2}[u_N(x_\nu,t) + u_N(x_{\nu+1},t)]h^*_{\nu+\frac{1}{2}} - \frac{1}{2}[\int^{u_N(x_\nu,t)} f(u)du + \int^{u_N(x_{\nu+1},t)} f(u)du].$$

In particular, this implies a global quadratic entropy conservation in time

$$\text{(2.14)} \quad \int_0^{2\pi} U(u_N(x,t))dx \equiv \sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x = \sum_{\nu=0}^{2N} U(u_N(x_\nu,0))\Delta x, \qquad U(u) = \frac{1}{2}u^2.$$

Hence $u_N(x,t)$ cannot converge boundedly a.e. to any function $u(x,t)$, which otherwise, by the Lax-Wendroff theorem, would be an entropy conservative solution of (1.1a). Moreover, by the same argument as before, bounded solutions, $u_N(x,t)$, cannot even converge weakly to any conservative solution of the GNL equation (1.1a), for otherwise the convergence should be strong–a contradiction.

The essential ingredient behind the failure of convergence demonstrated in counterexamples 2.1-2.4, is the lack of entropy dissipation. Namely, in each case we have found an entropy function, $U(u)$, such that the total amount of entropy $\sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x$ is conserved in time

$$\text{(2.15)} \qquad \sum_{\nu=0}^{2N} U(u_N(x_\nu,t))\Delta x = \sum_{\nu=0}^{2N} U(u_N(x_\nu,0))\Delta x.$$

This rules out (even weak) convergence to the entropy solution of our problem. In fact, in the finite-difference and finite-element cases, we found a local (cellwise) entropy conservation, consult (2.9), (2.13), which prevents the existence of any strong limit as well.

We conclude that some sort of global entropy dissipation is necessary for an consistent discrete approximation of (1.1a) to converge, for otherwise, the lack of such entropy dissipation is inconsistent with the augmenting entropy condition (1.1b).

3. CONSISTENCY WITH THE ENTROPY CONDITION

Let $U(u)$ be a strictly convex entropy function (we assume that the system (1.1a) is equipped with at least one such entropy function). Multiplying (1.4) by $U'(u_N(x,t))$ we have

$$\frac{\partial}{\partial t}[U(u_N(x,t))] + \frac{\partial}{\partial x}[F(u_N(x,t))] =< U'(u_N(x,t)),\ \frac{\partial}{\partial x}E_N^f u_N(x,t) > . \tag{3.1}$$

A possible attempt to define consistency with the entropy condition would be to require that the righthand-side of (3.1) is nonpositive for <u>all</u> U's associated with (1.1a), in agreement with (1.1b). This means that we prohibit any <u>local</u> entropy production by our approximation. In this context one is led to the concept of E-schemes [13], [17], [15]. We recall that in the scalar case, E-schemes are convergent in view of their Total-Variation-Diminishing (TVD) property [13], [17], [15], and that in the GNL case of strictly hyperbolic 2×2 systems, E-schemes are convergent provided their solutions remain with sufficiently small variation [1, p.33]. Unfortunately, such an E-consistency requirement which prohibits any local entropy producing waves, restricts our approximation to first-order accuracy [13], [15]. As noted by DiPerna, however, one can allow entropy producing waves to be mixed with entropy dissipating waves and still retain convergence, as long as the sum of the <u>total</u> amount of entropy production, $\|U_{\text{prod}}(u_N(x,t))\|_{L^1_{\text{loc}}(t)}$, where

$$U_{\text{prod}}(w_N(x)) \equiv -\int_0^{2\pi} < \frac{\partial}{\partial x}U'(w_N(x)), E_N^f w_N(x) >^- dx\ , \tag{3.2a}$$

plus the total amount of entropy dissipation, $\|U_{\text{diss}}(u_N(x,t)\|_{L^1_{\text{loc}}(t)}$, where

$$U_{\text{diss}}(w_N(x)) \equiv \int_0^{2\pi} < \frac{\partial}{\partial x}U'(w_N(x)), E_N^f w_N(x) >^+ dx\ , \tag{3.2b}$$

remains uniformly bounded w.r.t. N. To this end, let us pick <u>one</u> strictly convex entropy function, say $U_*(u)$, and integrate by parts (3.1) with $U(u) = U_*(u)$, obtaining

$$\frac{d}{dt}\int_0^{2\pi} U_*(u_N(x,t))dx + \left(\frac{\partial}{\partial x}U_*'(u_N(x,t)), E_N^f u_N(x,t)\right) = 0\ . \tag{3.3}$$

The second term on the left is the amount of entropy dissipation rate. The counterexamples given in the previous section show that in order to agree with the entropy condition (1.1b), this amount of entropy dissipation rate should stay larger than a sufficiently small lower bound. In our next definition we precisely quantify the size of such lower bound. We make

Definition 3.1: (<u>Consistency with the entropy condition</u>). The approximation (1.3) is <u>consistent with the entropy condition</u> (1.1b) w.r.t. the "relevant" class $\mathcal{U}$, if for <u>all</u> $w_N(x)$ in ϕ_N which are uniformly bounded w.r.t. N, there exist

(i) at least one "preferred" entropy function, $U_*(u)$,

(ii) positive sequences, $e_N \downarrow 0$ and $0 < \eta_N < \eta_\infty$,

(iii) a nonempty "relevant" class, $\mathcal{U}$, of strictly convex entropy functions,

such that for all U's in $\mathcal{U}$ the following estimate holds

$$(3.4) \qquad \frac{1}{e_N}\|E_N^f w_N(x)\|^2 + \frac{1}{\eta_N}U_{\text{prod}}(w_N(x)) \leq \left(\frac{\partial}{\partial x}U_*'(w_N(x)), E_N^f w_N(x)\right) + \|w_N(x)\|^2 .$$

Remark (on the entropy consistency definition): The essence of the Definition 3.1 lies in the so called "relevant" class $\mathcal{U}$. As we shall see later on, consistency in the sense of Definition 3.1 $+L^\infty$ stability implies convergence to a limit solution of scalar and some 2×2 approximations (3.1). This limit solution agrees with the entropy inequalities (1.1b), precisely for those entropy pairs $(U(u), F(u))$ with $U(u)$ belonging to $\mathcal{U}$. Thus, Definition 3.1 yields consistency w.r.t. the entropy functions in the class $\mathcal{U}$, which motivates its description as the "relevant" class. Clearly, the larger the size of $\mathcal{U}$ is, the sharper convergence results can be deduced.

An immediate consequence from this kind of entropy consistency definition is

Lemma 3.2: *If the approximation (1.3) is consistent with the entropy condition (1.1b), then it is also consistent with the conservation law (1.1a).*

Proof: Using the Cauchy-Schwartz inequality to upper bound the first term on the right of (3.4), we obtain

$$(3.5) \qquad \frac{1}{e_N}\|E_N^f w_N(x)\|^2 \leq U_\infty'' \cdot \frac{e_N}{2}\|\frac{\partial}{\partial x}w_N(x)\|^2 + \frac{1}{2e_N}\|E_N^f w_N(x)\|^2 + \|w_N(x)\|^2 ,$$

with $U_\infty'' \equiv \|U''(w_N(x))\|^2_{L^\infty(x)}$. Multiplying (3.5) by $2e_N$ we find

$$(3.6) \qquad \|E_N^f w_N(x)\|^2 \leq U_\infty'' \cdot e_N^2\|\frac{\partial}{\partial x}w_N(x)\|^2 + 2e_N\|w_N(x)\|^2 .$$

Finally, thanks to conservation we can restrict our attention to $w_N(x)$ with zero mean, $\frac{1}{2\pi}\int_0^{2\pi} w_N(x)dx = 0$, for which we have the Poincaré inequality

$$(3.7) \qquad \|w_N(x)\|^2 \leq \|\frac{\partial}{\partial x}w_N(x)\|^2 .$$

The consistency estimate (1.7) with $c_N = \sqrt{U_\infty'' e_N^2 + 2e_N} \longrightarrow 0$ follows from (3.6) and (3.7). □

Thus, Definition 3.1 implies consistency with both parts of the initial-value problem (1.1a, (1.1b) – the conservation law (1.1a) and its augmenting entropy condition (1.1b). We shall therefore refer to approximations which fulfill Definition 3.1 as consistent with the initial-value problem (1.1a), (1.1b), or simply as consistent approximations.

We observe that our consistency requirement (3.4) places a rather weak restriction on the approximation (1.3), in that it allows for a mixture of entropy producing and entropy dissipating waves. Instead, it aligns with the necessary condition for convergence discussed already in Section 2, which requires some sort of global entropy dissipation. Indeed, the total mass of entropy production plus the total mass of entropy dissipation remains uniformly bounded in this case, as told by the essential

Lemma 3.3: *Consider the approximation (1.3) which is consistent with the initial-value problem (1.1a), (1.1b), and assume that it is L^∞-stable, i.e., that there exists a constant M_∞ (independent of N), such that*

$$\|u_N(x,t)\|_{L^\infty_{loc}(x,t)} \leq M_\infty. \tag{3.8}$$

Then there exist constants (independent of N), such that

I. The local error, $E^f_N u_N(x,t)$, satisfies

$$\|E^f_N u_N(x,t)\|_{L^2_{loc}(x,t)} \leq Const_I \cdot \sqrt{e_N} \to 0. \tag{3.9}$$

II. For all entropy functions $U(u)$ in $\mathcal{U}$, we have

$$\int_{t=0}^{T} U_{\text{prod}}(u_N(x,t))dt \leq Const_{II} \cdot \eta_N \ . \tag{3.10}$$

III. For all entropy functions $U(u)$ in $\mathcal{U}$, we have

$$\| < \frac{\partial}{\partial x} U'(u_N(x,t), E^f_N u_N(x,t) > \|_{L^1_{loc}(x,t)} \leq Const_{III} \ . \tag{3.11}$$

Proof: Applying the entropy consistency estimate (3.4) to $w_N(\cdot) = u_N(\cdot,t)$, we find after temporal integration that for any $U(u)$ in $\mathcal{U}$ the following estimate holds

$$\begin{aligned} &\tfrac{1}{e_N} \textstyle\int_{t=0}^{T} \|E^f_N u_N(x,t)\|^2 dt + \tfrac{1}{\eta_N} \textstyle\int_{t=0}^{T} U_{\text{prod}}(u_N(x,t))dt \leq \\ &\leq \textstyle\int_{t=0}^{T} \left(\tfrac{\partial}{\partial x} U'_*(u_N(x,t)), E^f_N u_N(x,t)\right) dt + \textstyle\int_{t=0}^{T} \|u_N(x,t)\|^2 dt \ . \end{aligned} \tag{3.14}$$

To upper bound the righthand-side of (3.14), we integrate (3.1) by parts, obtaining that for <u>all</u> entropy functions $U(u)$, we have

$$(3.15)\qquad \int_{t=0}^{T}\left(\frac{\partial}{\partial x}U'(u_N(x,t)),E_N^f u_N(x,t)\right)dt=\int_{x=0}^{2\pi}U(u_N(x,t))dx\Big|_{t=T}^{t=0}\le 2U_\infty\ .$$

Using this with $U(u)=U_*(u)$, implies that the righthand-side of (3.14) does not exceed

$$(3.16)\qquad 2U_{*\infty}+T\cdot M_\infty^2\ ,\qquad U_{*\infty}\equiv\|U_*(u_N(x,t))\|_{L^\infty_{\rm loc}}(x,t)\ .$$

Hence (3.9) and (3.10) follow with $Const_I^2=Const_{II}=2U_{*\infty}+TM_\infty^2$. Finally, according to (3.15) we have for <u>all</u> entropy functions, $U(u)$,

(3.17)

$$\int_{t=0}^{T}U_{\rm diss}(u_N(x,t))dt-\int_{t=0}^{T}U_{\rm prod}(u_N(x,t))dt\equiv\int_{t=0}^{T}\left(\frac{\partial}{\partial x}U'(u_N(x,t)),E_N^f u_N(x,t)\right)dt\le 2U_\infty\ ,$$

and hence (3.11) follows with $Const_{III}=2[U_\infty+Const_{II}\cdot\eta_\infty]$,

$$\|<\tfrac{\partial}{\partial x}U'(u_N(x,t)),E_N^f u_N(x,t)>\|_{L^1_{\rm loc}(x,t)}\ \equiv\int_{t=0}^{T}[U_{\rm diss}(u_N(x,t))+U_{\rm prod}(u_N(x,t))]dt\le$$

$$\le 2[U_\infty+\int_{t=0}^{T}U_{\rm prod}(u_N(x,t))dt]\le Const_{III}\ .$$

<u>Remarks</u>:

1. The estimate (3.9) shows that the local error of consistent schemes, $E_N^f u_N(x,t)$, tends to zero <u>independently</u> whether the underlying solutions are smooth are not.

2. The estimate (3.10) shows that for <u>any</u> entropy function $U(u)$ in the "relevant" class $\mathcal{U}$, the total mass of its production remains bounded (by $Const_{II}\cdot\eta_\infty$). Moreover, the total mass of its entropy production tends to zero in case

$$(3.19)\qquad \int_{t=0}^{T}U_{\rm prod}(u_N(x,t))dt\le Const_{II}\cdot\eta_N\longrightarrow 0\ .$$

3. According to the first remark

$$(3.20a)\qquad F_N(u_N(x,t))\equiv F(u_N(x,t))-<U'(u_N(x,t)),E_N^f u_N(x,t)>\ ,$$

is a numerical entropy flux which is consistent with the differential one

$$(3.20b)\qquad \|F(u_N(x,t))-F_N(u_N(x,t))\|\le\|U'(u_N(x,t))\|_{L^\infty_{\rm loc}(x,t)}\cdot\|E_N^f u_N(x,t)\|\longrightarrow 0\ .$$

The estimate (3.11) shows that for <u>any</u> entropy function $U(u)$ in the "relevant" class $\mathcal{U}$, the total mass of its production + dissipation, $\|\frac{\partial}{\partial t}[U(u_N(x,t)]+\frac{\partial}{\partial x}[F_N(u_N(x,t))]\|_{L^1_{\rm loc}(x,t)}$

remains bounded, as asserted. This agrees with DiPerna observation [1, p.39] that the control on the total entropy production is independent of the choice of the preferred entropy function $U_*(u)$.

4. Let us restrict our attention to consistent approximations of the scalar initial-value problem (1.1a), (1.1b), and apply Gronwall's inequality to (3.3) with $U_*(u) = \frac{1}{2}u^2$,

$$\frac{1}{2}\frac{d}{dt}\int_{x=0}^{2\pi} u_N^2(x,t)dx = -\left(\frac{\partial}{\partial x}u_N(x,t), E_N^f u_N(x,t)\right)_{L^2(x)} \leq \int_{x=0}^{2\pi} u_N^2(x,t)dx \ . \tag{3.21}$$

We conclude that independently of the L^∞-stability assumption (3.8), we have

$$\|u_N(x,t)\|_{L^2(x)} \leq M_2 \ . \tag{3.22}$$

In case $U_*(u)$ is a strictly convex entropy function, the same conclusion also holds for systems of conservation laws, by appealing to (3.10) with the quadratic part of $U_*(u)$.

The intricate point in our entropy consistency Definition 3.1, is the size of the "relevant" class $\mathcal{U}$. The optimal situation of course, occurs when $\mathcal{U} \equiv \mathcal{U}_{\text{all}}$ which includes all the admissible strictly convex functions associated with (1.1a). In this respect, it would be useful to have a sufficient consistency criterion which guarantees the optimality of $\mathcal{U}$ by checking only one preferred entropy function, say $U_*(u)$. Our next result provides us with such a convenient criterion.

Lemma 3.4: (Sufficient criterion for consistency). *Assume that for all $w_N(x)$ in ϕ_N which are uniformly bounded w.r.t. N, there exist (at least) one strictly convex entropy function, $U_*(u)$, and vanishing sequences, $c_N \downarrow 0$ and $d_N \downarrow 0$, such that the following two estimates hold:*

A.(Consistency with the conservation law)

$$\|E_N^f w_N(x)\| \leq c_N \|\frac{\partial}{\partial x}w_N(x)\| \ , \tag{3.23a}$$

B. ("Enough" entropy dissipation)

$$d_N \|\frac{\partial}{\partial x}w_N(x)\|^2 \leq \left(\frac{\partial}{\partial x}U_*'(w_N(x)), E_N^f w_N(x)\right) + \|w_N(x)\|^2 \ . \tag{3.23b}$$

Then the approximation (1.3) *is consistent w.r.t. all admissible entropy functions, $\mathcal{U} = \mathcal{U}_{\text{all}}$, provided*

$$\frac{c_N}{d_N} \leq Const. \tag{3.24}$$

Proof: By Cauchy-Schwartz inequality and (3.23a) we have for <u>all</u> entropy functions, $U(u)$,

$$U_{\text{prod}}(w_N(x)) \le U''_\infty \cdot c_N \|\frac{\partial}{\partial x} w_N(x)\|^2 \ , \qquad U''_\infty = \|U''(w_N(x))\|_{L^\infty(x)} \ . \tag{3.25}$$

Take $e_N = 2Const.^2 d_N \downarrow 0$ and $\eta_N = 2\frac{c_N}{d_N} U''_\infty \le \eta_\infty \equiv 2Const.U''_\infty$; then using (3.23a), (3.25) and (3.23b) we obtain for <u>all</u> strictly convex entropies, $U(u)$,

$$\begin{aligned}
\tfrac{1}{e_N}\|E_N^f w_N(x)\|^2 + \tfrac{1}{\eta_N} U_{\text{prod}}(w_N(x)) &\le \tfrac{1}{2Const.^2 d_N} c_N^2 \left\|\tfrac{\partial}{\partial x} w_N(x)\right\|^2 + \tfrac{d_N}{2c_N U''_\infty} U''_\infty c_N \left\|\tfrac{\partial}{\partial x} w_N(x)\right\|^2 \le \\
&\le \tfrac{1}{2}[d_N + d_N] \left\|\tfrac{\partial}{\partial x} w_N(x)\right\|^2 \le \\
&\le \left(\tfrac{\partial}{\partial x} U'_*(w_N(x)), E_N^f w_N(x)\right) + \|w_N(x)\|^2 \ . \qquad \square
\end{aligned}$$

<u>Remarks</u>:

1. The sufficient consistency condition provided in Lemma 3.4 is sharp, in the sense that if (3.23) holds, and if instead of (3.24) we have

$$\frac{c_N}{d_N} \longrightarrow 0 \ , \tag{3.26}$$

then approximation (1.3) does <u>not</u> converge. Indeed, if a limit solution of (1.3) exists in this case, then it is necessarily U_*-entropy conservative, for (3.23), (3.26) imply that both the U_*-entropy production <u>and</u> entropy dissipation tend to zero. Put differently, approximation (1.3) is <u>not</u> L^∞-stable in such case, for otherwise it would contradict convergence along the lines of the scalar counterexamples in Section 2.

2. The entropy dissipation estimate, (3.23b), with no "lower-order" term $\|w_N(x)\|^2$,

$$d_N \|\frac{\partial}{\partial x} w_N(x)\|^2 \le \left(\frac{\partial}{\partial x} U'_*(w_N(x)), E_N^f w_N(x)\right) \ , \tag{3.27}$$

can be viewed as an <u>accretivity</u> condition on the discrete local error E_N^f. However, it is worthwhile noting that such accretive approximations are restricted to first-order accuracy due to the following argument:

3. By the weighted Cauchy-Schwartz inequality, the righthand-side of (3.27) is smaller than

$$U''_{*\infty} \|\frac{\partial}{\partial x} w_N(x)\| \cdot \|E_N^f w_N(x)\| \ ,$$

and hence (3.27) implies

$$\frac{d_N}{U''_{*\infty}} \|\frac{\partial}{\partial x} w_N(x)\| \le \|E_N^f w_N(x)\| \ . \tag{3.28}$$

The inequality (3.24) implies that $d_N \geq Const. \cdot \frac{1}{N}$. Consequently, the inequality (3.28) shows that approximations which satisfy the entropy dissipation estimate, (3.27), are restricted to first-order accuracy. Instead, we adopt here the weaker entropy consistency estimate, (3.4), which, roughly speaking, requires the total entropy dissipation minus production to be proportional to the local discrete error $E_N^f w_N(x)$. This will enable us to deal with higher (than first) order approximations of (1.1a), including the modern high-resolution finite-difference schemes [2],[3],[14],[15].

4. CONSISTENCY OF THE SCALAR VANISHING VISCOSITY METHOD

The vanishing viscosity method is one of the most effective ways to provide affirmative answers to the questions of consistency and L^∞-stability, in the context of nonlinear systems of conservation laws. To this end, one starts from a given basic approximation, P_N^f, which is consistent with the conservation law (1.1a)

$$\|E_N^f w_N(x)\| \le c_N \|\frac{\partial}{\partial x} w_N(x)\| \tag{4.1a}$$

Such basic approximation is then appended with a so called vanishing viscosity approximation, $-\varepsilon_N Q_N \frac{\partial}{\partial x}$, which results in the modified viscous approximation

$$\tilde{P}_N^f = P_N^f - \varepsilon_N Q_N \frac{\partial}{\partial x}. \tag{4.1b}$$

Here $\varepsilon_N \downarrow 0$ is the vanishing viscosity amplitude, and $Q_N : L^\infty[0, 2\pi] \to \Phi_N$, is a possibly nonlinear, f dependent, spatially bounded operator

$$\|Q_N \frac{\partial}{\partial x} w_N\| \le Q_\infty \|\frac{\partial}{\partial x} w_N(x)\| \ . \tag{4.2}$$

The boundedness of Q_N implies that $-\varepsilon_N Q_N \frac{\partial}{\partial x}$ is consistent with the zero flux, i.e.,

$$\| - \varepsilon_N Q_N \frac{\partial}{\partial x} w_N(x)\| \le Q_\infty \varepsilon_N \|\frac{\partial}{\partial x} w_N(x)\| \ , \qquad Q_\infty \varepsilon_N \to 0 \ ;$$

consequently, the discrete local error of the modified viscous approximation (4.1b),

$$\tilde{E}_N^f w_N(x) \equiv f(w_N(x)) - \tilde{P}_N^f w_N(x) = E_N^f w_N(x) + \varepsilon_N Q_N \frac{\partial}{\partial x} w_N(x), \tag{4.3a}$$

satisfies, by (4.1a) and (4.2),

$$\|\tilde{E}_N^f w_N(x)\| \le \tilde{c}_N \|\frac{\partial}{\partial x} w_N(x)\| \ , \qquad \tilde{c}_N \equiv c_N + Q_\infty \varepsilon_N \to 0 \ . \tag{4.3b}$$

In this manner, the vanishing viscosity method retains the consistency with the conservation law, (1.1a), of its underlying basic approximation.

There are various ways to tune the viscosity parameters so as to guarantee the full consistency of the viscous discretization (4.1b). Namely, we seek for ε_N and Q_N so that $\tilde{P}_N^f = P_N^f - \varepsilon_N Q_N \frac{\partial}{\partial x}$, which remains a consistent discretization of the conservation law (1.1a), gains in addition, the consistency with the entropy condition (1.1b). Taking advantage of the scalar case, we make the canonical choice of $U_*(u) = \frac{1}{2}u^2$ as our preferred entropy function for the sufficient entropy consistency condition stated in Lemma 3.4, which is then simplified to

Theorem 4.1: *Let P_N^f be a consistent discretization of the scalar conservation law (1.10)*

$$\|E_N^f w_N(x)\| \le c_N \|\frac{\partial}{\partial x} w_N(x)\| \ , \qquad c_N \to 0 \ . \tag{4.4}$$

Assume that there exists a vanishing sequence, $\tilde{d}_N \downarrow 0$, such that for uniformly bounded $w_N(x)$ in ϕ_N,

$$\tilde{d}_N \|\frac{\partial}{\partial x} w_N(x)\|^2 \le \left(\frac{\partial}{\partial x} w_N(x), \tilde{E}_N^f w_N(x)\right) + \|w_N(x)\|^2 \ . \tag{4.5}$$

Then we have

(i) The vanishing viscosity method, $\tilde{P}_N^f = P_N^f - \varepsilon Q_N \frac{\partial}{\partial x}$, is consistent w.r.t. all strictly convex entropy functions, $\mathcal{U} = \mathcal{U}_{\text{all}}$, provided

$$\frac{\tilde{c}_N}{\tilde{d}_N} \le Const. \ , \qquad \tilde{c}_N \equiv c_N + Q_\infty \varepsilon_N \ . \tag{4.6}$$

(ii) The total amount of entropy produced by the basic approximation, P_N^f,

$$\int_{t=0}^{T} U_{\text{prod}}(u_N(x,t))dt \equiv \int_{t=0}^{T}\int_{x=0}^{2\pi} - < \frac{\partial}{\partial x} U'(u_N(x,t)), E_N^f u_N(x,t) >^- dx\,dt \ , \tag{4.7a}$$

tends to zero, provided

$$\frac{c_N}{\tilde{d}_N} \longrightarrow 0 \ . \tag{4.7b}$$

<u>Remark</u>: The second part of Theorem 4.1 shows that due to the presence of vanishing viscosity parametrized according to (4.7b), the entropy produced by the basic approximation tends to zero. Indeed, by (4.4), (4.5), (3.15) and (3.22) we have

$$\begin{aligned} -\int_{t=0}^{T}\int_{x=0}^{2\pi} < \tfrac{\partial}{\partial x} U'(u_N(x,t)), E_N^f u_N(x,t) >^- dx\,dt \ &\le U''_\infty \cdot c_N \left\|\tfrac{\partial}{\partial x} u_N(x,t)\right\|^2 \le \\ &\le U''_\infty \cdot \tfrac{c_N}{d_N}(T+1)M_2^2 \longrightarrow 0 \ . \end{aligned} \tag{4.8}$$

The following consequences of Theorem 4.1 are at the heart of the matter.

Corollary 4.2: *Let P_N^f be a consistent approximation of the scalar conservation law (1.1a),*

$$\|E_N^f w_N(x)\| \le c_N \|\frac{\partial}{\partial x} w_N(x)\| \ . \tag{4.9}$$

Assume that for all $w_N(x)$ in Φ_N, there exists a positive constant (independent of N), $Const. > 0$, such that,

$$\varepsilon_N \left(\frac{\partial}{\partial x} w_N(x),\ Q_N \frac{\partial}{\partial x} w_N(x) \right) + \|w_N(x)\|^2 \geq Const.\varepsilon_N \|\frac{\partial}{\partial x} w_N(x)\|^2 \ . \tag{4.10}$$

Then, the resulting vanishing viscosity method

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N^f u_N(x,t)] = \varepsilon_N \frac{\partial}{\partial x}[Q_N \frac{\partial}{\partial x} u_N(x,t)] \tag{4.11}$$

satisfies

(i) It is a consistent approximation of the scalar initial-value problem (1.1a), (1.1b), w.r.t. all strictly convex entropies, $\mathcal{U} = \mathcal{U}_{\text{all}}$, provided

$$\frac{c_N}{\varepsilon_N} \leq Const_0 < Const. \ . \tag{4.12a}$$

(ii) The total amount of entropy produced by its basic approximation in (4.7a) tends to zero, provided

$$\frac{c_N}{\varepsilon_N} \longrightarrow 0 \ . \tag{4.12b}$$

Proof: By (4.9), and (4.12a) or (4.12b) with sufficiently large N, we have

$$\begin{aligned} \left(\tfrac{\partial}{\partial x} w_N(x), E_N^f w_N(x)\right) &\geq -\|\tfrac{\partial}{\partial x} w_N(x)\| \cdot \|E_N^f w_N(x)\| \geq -c_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2 \\ &\geq -Const.\theta\varepsilon_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2 \ , \qquad 0 < \theta = \tfrac{Const_0}{Const.} < 1. \end{aligned}$$

Hence, (4.5) is fulfilled with $\tilde{d}_N = Const.(1-\theta)\varepsilon_N$, for

$$\begin{aligned} \tilde{d}_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2 &= Const.\varepsilon_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2 - Const.\theta\varepsilon_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2 \leq \\ &\leq \left[\varepsilon_N \left(\tfrac{\partial}{\partial x} w_N(x), Q_N \tfrac{\partial}{\partial x} w_N(x)\right) + \|w_N(x)\|^2\right] + \left(\tfrac{\partial}{\partial x} w_N(x), E_N^f w_N(x)\right) = \\ &= \left(\tfrac{\partial}{\partial x} w_N(x), \tilde{E}_N^f w_N(x)\right) + \|w_N(x)\|^2 \ . \end{aligned} \tag{4.13}$$

Now, with this choice of $\tilde{d}_N$ we have

$$\frac{\tilde{c}_N}{\tilde{d}_N} = \frac{1}{Const.} \frac{c_N + Q_\infty \varepsilon_N}{(1-\theta)\varepsilon_N} \leq \frac{\theta}{1-\theta} + \frac{Q_\infty}{Const.(1-\theta)} \ ;$$

moreover, (4.12a) implies

$$\frac{c_N}{\tilde{d}_N} = \frac{1}{Const.} \frac{c_N}{(1-\theta)\varepsilon_N} \downarrow 0 ,$$

and the result follows from Theorem 4.1. □

The first part of Corollary 4.2 guarantees consistency, provided the amount of entropy produced by the basic approximation, P_N^f, is dominated by the amount of entropy dissipated by $\tilde{P}_N^f = P_N^f - \varepsilon_N Q_N \frac{\partial}{\partial x}$. In case the basic approximation was entropy conservative to begin with, then we can do with even less vanishing viscosity as told by

Corollary 4.3: *Let P_N^f be a consistent approximation of the scalar conservation law (1.1a), which is (quadratic) entropy conservative, i.e.,*

$$\left(\frac{\partial}{\partial x} w_N(x), E_N^f w_N(x)\right) = 0 . \tag{4.14}$$

Assume that (4.10) holds. Then the resulting vanishing viscosity method (4.11) is a consistent approximation of the initial-value problem (1.1a), (1.1b) w.r.t. all strictly convex entropies, $\mathcal{U} = \mathcal{U}_{\text{all}}$, provided there exists an arbitrary positive constant (independent of N), $Const_0 > 0$, such that

$$\frac{c_N}{\varepsilon_N} \leq Const_0 , \qquad Const_0 > 0 . \tag{4.15}$$

Proof: Choosing $\tilde{d}_N = Const.\varepsilon_N \downarrow 0$, then (4.14) and (4.10) yield

$$\begin{aligned}\left(\tfrac{\partial}{\partial x} w_N(x), \tilde{E}_N^f w_N(x)\right) + \|w_N(x)\|^2 &= \varepsilon_N \left(\tfrac{\partial}{\partial x} w_N(x), Q_N \tfrac{\partial}{\partial x} w_N(x)\right) + \|w_N(x)\|^2 \geq \\ &\geq \tilde{d}_N \|\tfrac{\partial}{\partial x} w_N(x)\|^2\end{aligned}$$

and the result follows from Theorem 4.1 since

$$\frac{\tilde{c}_N}{\tilde{d}_N} = \frac{c_N + Q_\infty \varepsilon_N}{Const.\varepsilon_N} \leq \frac{1}{Const.}(Const_0 + Q_\infty) . \qquad \square$$

Corollaries 4.2 and 4.3 show that the vanishing viscosity method enables to gain entropy consistency while retaining consistency with the conservation law. It is possible however, that this method gains the entropy consistency at the expense of lowering the order of accuracy of the underlying discretization P_N^f. The vanishing viscosity method should be carefully parameterized in order to retain both the entropy consistency as well as the original order of accuracy. An extreme situation in this respect is provided by the "infinitely-order" accurate spectral and pseudospectral methods, which bring us to

The Spectral Viscosity (SV) Method [20] [10]. We consider spectral vanishing viscosity modifications, defined in terms of a convolution with a π_N-kernel, $Q_N(x)$,

$$-\varepsilon_N Q_N \frac{\partial}{\partial x} w_N(x) = -\varepsilon_N Q_N(x) * \frac{\partial}{\partial x} w_N(x), \tag{4.16a}$$

where $Q_N(x) \equiv Q_N(w_N(x), x)$ is a possibly nonlinear viscosity kernel of the form

$$Q_N(x) = \sum_{k=-N}^{N} \hat{Q}_k e^{ikx}, \qquad 0 \le \hat{Q}_k \le Q_\infty. \tag{4.16b}$$

Appending this either to the spectral ($P_N \equiv S_N$) or ψdospectral ($P_N \equiv \psi_N$) approximation of (1.1a), results in the SV method

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N f(u_N(x,t))] = \varepsilon_N \frac{\partial}{\partial x}[Q_N(x,t) * \frac{\partial}{\partial x} u_N(x,t)]; \tag{4.17}$$

it can be efficiently implemented in the Fourier space as, see (1.11) or (1.13),

$$\frac{d}{dt}\hat{u}_k(t) + ik\hat{f}_k(t) = -\varepsilon_N k^2 \hat{Q}_k(t)\hat{u}_k(t). \tag{4.18}$$

According to Theorems 7.1 and 7.3, the spectral and pseudospectral methods are first-order accurate with (1.1a), i.e., they satisfy (4.1a) with $c_N = Const_0 \cdot \frac{1}{N}$, where $Const_0 = \left(1 + \frac{\pi}{\sqrt{3}}\right) \|f'(u_N(x,t))\|_{L^\infty_{\mathrm{loc}}(x,t)}$. Appealing to Corollaries 4.2 and 4.3 we arrive at

Theorem 4.4: (Consistency of the SV Methods) *The following SV methods are consistent approximations of the initial-value problem (1.1a), (1.1b) w.r.t. all strictly convex entropies $U(u)$:*

I. *The spectral viscosity approximation*

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[S_N f(u_N(x,t))] = \varepsilon_N \frac{\partial}{\partial x}[Q_N(x,t) * \frac{\partial}{\partial x} u_N(x,t)], \tag{4.19a}$$

with spectral viscosity parameters satisfying

$$\hat{Q}_k(t) \ge \left(Const. - \frac{1}{\varepsilon_N k^2}\right)^+, \quad \frac{1}{N} \le \varepsilon_N \downarrow 0, \qquad Const. > 0\ . \tag{4.19b}$$

II. *The ψdospectral viscosity approximation*

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[\psi_N f(u_N(x,t))] = \varepsilon_N \frac{\partial}{\partial x}[Q_N(x,t) * \frac{\partial}{\partial x} u_N(x,t)], \tag{4.20a}$$

with spectral viscosity parameters satisfying (here $Const_0 = \left(1 + \frac{\pi}{\sqrt{3}}\right) \|f'\|_{L^\infty_{\mathrm{loc}}(x,t)}$)

$$\hat{Q}_k(t) \ge \left(Const. - \frac{1}{\varepsilon_N k^2}\right)^+, \qquad \frac{1}{N} \le \varepsilon_N \downarrow 0\ , \qquad Const. > Const_0\ . \tag{4.20b}$$

III. *The total amount of entropy produced by the basic spectral approximation (where $P_N = S_N$), or the ψdospectral one (where $P_N = \psi_N$), tends to zero,*

$$(4.21a) \qquad -\int_{t=0}^{T}\int_{x=0}^{2\pi} < \frac{\partial}{\partial x}U'(u_N(x,t)), [I-P_N]f(u_N(x,t)) >^{-} dx\,dt \longrightarrow 0 \ ,$$

with spectral viscosity parameters satisfying

$$(4.21b) \qquad \hat{Q}_k(t) \geq \left(Const. - \frac{1}{\varepsilon_N k^2}\right)^{+} \ , \quad \frac{1}{N\varepsilon_N} \longrightarrow 0 \ .$$

Proof: According to Corollaries 4.2 and 4.3, consistency follows if (4.10) holds, and by Fourier transform this is equivalent with

$$(4.22) \qquad \varepsilon_N k^2 \hat{Q}_k(t) + 1 \geq Const.\varepsilon_N k^2 \ .$$

In the spectral case there is no entropy production

$$\left(\frac{\partial}{\partial x}w_N(x), [I-S_N]f(w_N(x))\right) = 0$$

and the first part of the theorem follows from Corollary 4.3. In the ψdospectral case we apply Corollary 4.2(i) and the second part of the theorem follows. Finally, the third part of the theorem follows from Corollary 4.2(ii). □

Example 4.5: The following choice of spectral viscosity for the SV method (4.17),

$$(4.23a) \qquad \varepsilon_N \frac{\partial}{\partial x}[Q_N * \frac{\partial}{\partial x}u_N(x,t)] = -\frac{1}{N^\beta}\sum_{|k|=m_N}^{N} k^2 \hat{Q}_k(t)\hat{u}_k(t)e^{ikx}, \qquad m_N^2 = \frac{N^\beta}{Const.},$$

where

$$(4.23b) \qquad \hat{Q}_k(t) \geq Const. - \frac{N^\beta}{k^2}, \qquad |k| \geq m_N \ , \qquad 0 < \beta \leq 1 \ , \quad Const. > Const_0 \ ,$$

demonstrates a consistent approximation of (1.1a), (1.1b) w.r.t. all strictly convex entropies, which <u>retains</u> the "infinite-order" accuracy of the underlying spectral or pseudospectral method. Moreover, if $0 < \beta < 1$, then the entropy produced by the basic spectral or ψdospectral approximation tends to zero, when augmented with (4.23a), (4.23b). For example, the SV approximation

$$(4.24) \qquad \frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N f(u_N(x,t))] = -Const.\frac{1}{N}\sum_{|k|=\sqrt{N}}^{N} k^2 \frac{(k-\sqrt{N})^2}{N-\sqrt{N}}\hat{u}_k(t),$$

is 'infinitely-order' accurate approximation of (1.1a), and at the same time it is consistent with (1.1a), (1.1b).

Remark: Let us rewrite the SV method (4.17) in the form

(4.25)
$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[f(u_N(x,t))] = \frac{\partial}{\partial x}[(I-P_N)f(u_N(x,t))] + \varepsilon_N \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x} u_N(x,t)\right].$$

This highlights the fact that the SV approximation has two sources of errors: the first term on the right is the error committed due to spectral discretization; the second term is due to the presence of spectral viscosity. The viscosity parameterization in (4.19b), (4.20b) guarantees that the local discretization error is dominated by the dissipative spectral viscosity error.

Next, we turn to study the vanishing viscosity method in the context of

Finite-Difference and Finite-Element Methods. We consider conservative approximations of (1.1a), whose numerical flux $h^f_{\nu+\frac{1}{2}}$, is given by (abbreviating $\Delta u_{\nu+\frac{1}{2}}(t) \equiv u_N(x_{\nu+1},t) - u_N(x_\nu,t)$)

$$h^f_{\nu+\frac{1}{2}} = \frac{1}{2}[f(u_N(x_\nu,t)) + f(u_N(x_{\nu+1},t))] - \frac{1}{2}Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t)\ . \tag{4.26}$$

The conservative approximation (1.14) then recast into the form

$$\begin{aligned} \tfrac{d}{dt}u_N(x_\nu,t) \quad &+ \tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t))] = \\ &= \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]\ , \end{aligned} \tag{4.27}$$

which reveals the role $Q_{\nu+\frac{1}{2}}$ play as the numerical viscosity coefficients in (4.27). Simple linear examples are provided by centered finite-difference method (1.19), where

$$Q_{\nu+\frac{1}{2}} \equiv 0\ , \tag{4.28}$$

and the piecewise-linear finite element method (2.12), whose numerical flux

$$h^*_{\nu+\frac{1}{2}} = \frac{1}{2}[f(u_N(x_\nu,t)) + f(u_N(x_{\nu+1},t))] - \frac{1}{2}Q^*_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t)\ , \tag{4.29a}$$

corresponds to the viscosity coefficient, $Q_{\nu+\frac{1}{2}} = Q^*_{\nu+\frac{1}{2}}$, consult [18], [21]

$$Q^*_{\nu+\frac{1}{2}} = 2\int_{\xi=-1/2}^{1/2} \xi f'(u_{\nu+\frac{1}{2}}(\xi))d\xi \equiv \int_{\xi=-1/2}^{1/2}\left(\frac{1}{4}-\xi^2\right) f''(u_{\nu+\frac{1}{2}}(\xi))d\xi \cdot \Delta u_{\nu+\frac{1}{2}}(t)\ . \tag{4.29b}$$

Once more, we would like to emphasize that our present treatment of the vanishing viscosity method allows for nonlinear viscosity coefficients, $Q_{\nu+\frac{1}{2}}$, which may depend on the computed solution $Q_{\nu+\frac{1}{2}} \equiv Q_{\nu+\frac{1}{2}}(x,t) = Q_{\nu+\frac{1}{2}}(u_N(x,t),x,t;f)$. In particular, this includes the essentially nonlinear features which characterize the modern nonoscillatory finite-difference schemes [3].

In order to interpret (4.27) within the vanishing viscosity framework (4.1a), (4.1b), the following viscosity operator $Q_N : L^\infty[0,2\pi] \to \pi_N$ is introduced: We construct the π_N-polynomial

$$K_N(x) = K_N(x;Q) = \sum_{k=-N}^{N} \hat{K}_k e^{ikx} ,$$

which interpolates the Q-weighted cell averages, $\overline{w}(x) \equiv \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} w(\xi)d\xi$ of $w(x)$,

$$(4.30a) \qquad K_N(x = x_{\nu+\frac{1}{2}}) = Q_{\nu+\frac{1}{2}}\overline{w}(x_{\nu+\frac{1}{2}}) , \qquad \hat{K}_k = \frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N} Q_{\nu+\frac{1}{2}}\overline{w}(x_{\nu+\frac{1}{2}})e^{-ikx_{\nu+\frac{1}{2}}} ,$$

and then define $Q_N w(x)$ as the sliding average of this interpolant

$$(4.30b) \qquad Q_N w(x) = \overline{K}_N(x;Q) , \qquad \overline{K}_N(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} K_N(\xi;Q)d\xi .$$

By applying such averaging plus summation by parts to $K_N(x,t)$ we find, in analogy with (1.17),

$$(4.31a) \qquad Q_N u_N(x,t) = \sum_{k=-N}^{N} \hat{q}_k(t)e^{ikx} ,$$

where

$$(4.31b) \qquad \hat{q}_k(t) = \frac{1}{2\pi ik}\sum_{\nu=0}^{2N}[Q_{\nu+\frac{1}{2}}\overline{u}_N(x_{\nu+\frac{1}{2}},t) - Q_{\nu-\frac{1}{2}}\overline{u}_N(x_{\nu-\frac{1}{2}},t)]e^{-ikx_\nu} .$$

Now, since $\overline{\frac{\partial u_N}{\partial x}}(x = x_{\nu+\frac{1}{2}}) \equiv \frac{1}{\Delta x}\Delta u_{\nu+\frac{1}{2}}(t)$, we have

$$(4.32) \qquad Q_N\widehat{\frac{\partial}{\partial x}}u_N(k,t) = \frac{1}{2\pi ik}\sum_{\nu=0}^{2N}\frac{1}{\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]e^{-ikx_\nu} ,$$

and we conclude that $\frac{1}{2}\Delta x\frac{\partial}{\partial x}Q_N\frac{\partial}{\partial x}u_N(x,t)$ is the π_N-polynomial which interpolates at the equidistant point $x_\nu = \nu\Delta x$, the viscous part of (4.27),

$$(4.33) \qquad \frac{1}{2}\Delta x\frac{\partial}{\partial x}Q_N\frac{\partial}{\partial x}u_N(x = x_\nu,t) = \frac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t)] .$$

Thus, the conservative approximation (4.27) can be viewed as the vanishing viscosity method (4.1), which is based on the consistent centered finite-difference approximation (1.19) and augmented by the vanishing viscosity approximation (4.33),

$$\tilde{P}_N^f = FD_{\Delta x}f(\cdot) - \varepsilon_N Q_N \frac{\partial}{\partial x}\,, \qquad \varepsilon_N \equiv \frac{\Delta x}{2}\,. \tag{4.34}$$

The representation of a given approximation as a vanishing viscosity method is in general not unique: one can assign <u>any</u> part of it to serve as a basic approximation (as long as it is consistent with the conservation law), and then consider the rest as the vanishing viscosity contribution. For example, we can represent (4.27) as a vanishing viscosity method which is based on the finite-element approximation, $P_N^f = FE_{\Delta x}f(\cdot)$ in (1.26). To this end we use (4.29a) to rewrite (4.27) in the equivalent form

$$\frac{d}{dt}u_N(x_\nu,t) + \frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}] = \frac{1}{2\Delta x}[D_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - D_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]\,, \tag{4.35a}$$

where

$$D_{\nu+\frac{1}{2}} \equiv Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}}\,, \tag{4.35b}$$

are the <u>numerical dissipation coefficients</u> of (4.27). Now, by introducing the corresponding viscosity operator $D_N : L^\infty[0,2\pi] \to \pi_N$,

$$D_N w(x) = \overline{K}_N(x;D)\,, \quad K_N(x = x_{\nu+\frac{1}{2}};D) = D_{\nu+\frac{1}{2}}\overline{w}(x_{\nu+\frac{1}{2}})\,, \tag{4.36}$$

we conclude that (4.35) is associated with the (modified) discretization operator

$$\tilde{P}_N^f = FE_{\Delta x}f(\cdot) - \varepsilon_N D_N \frac{\partial}{\partial x}\,, \qquad \varepsilon_N \equiv \frac{\Delta x}{2}\,. \tag{4.37}$$

Which representation should we prefer? Corollary 4.3 suggests that in order to obtain sharp consistency estimates for a given approximation, the key step lies in isolating an <u>entropy conservative</u> part of it as a basic approximation, and considering the rest as the vanishing viscosity contribution. We therefore prefer (4.37) over (4.34) to make

Theorem 4.5: *The finite-difference/element approximation*

$$\begin{aligned} \tfrac{d}{dt}u_N(x_\nu,t) \;&+\tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t))] = \\ &= \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]\,, \end{aligned} \tag{4.38a}$$

with viscosity coefficients, $Q_{\nu+\frac{1}{2}}$, satisfying

$$0 < Q^*_{\nu+\frac{1}{2}} + Const. \le Q_{\nu+\frac{1}{2}} \le Q_\infty\,, \qquad Const. > 0\,, \tag{4.38b}$$

is a consistent approximation of the scalar initial-value problem (1.1a), (1.1b) w.r.t. all strictly convex entropies.

Proof: We first verify the (quadratic) entropy conservation of the basic finite-element approximation, $P_N^f = FE_{\Delta x}f(\cdot)$. By the exactness of the trapezoidal rule for π_{2N}-polynomials and the identity (2.13a), we have

$$\begin{aligned}\left(w_N(x), \tfrac{\partial}{\partial x}FE_{\Delta x}f(w_N(x))\right) &= \textstyle\sum_{\nu=0}^{2N} w_N(x_\nu)\frac{1}{\Delta x}[h^*_{\nu+\frac{1}{2}} - h^*_{\nu-\frac{1}{2}}]\cdot\Delta x = \\ &= \textstyle\sum_{\nu=0}^{2N}[F^*_{\nu+\frac{1}{2}} - F^*_{\nu-\frac{1}{2}}] = 0\ ,\end{aligned}$$

and since $< w_N(x), \frac{\partial}{\partial x}f(w_N(x)) >= \frac{\partial}{\partial x}F(w_N(x))$ is a perfect derivative,

$$\left(w_N(x), \frac{\partial}{\partial x}f(w_N(x))\right) = 0\ .$$

Integrating by parts the difference between the last two equalities, we conclude

$$\left(\frac{\partial}{\partial x}w_N(x), E_N^f w_N(x)\right) = \left(\frac{\partial}{\partial x}w_N(x), [I - FE_{\Delta x}]f(w_N(x))\right) = 0\ . \tag{4.39}$$

This reaffirms the quadratic entropy conservation, (4.14), of the basic finite-element approximation (2.12), which was already indicated (with different terminology) in Counterexample 2.4. According to Theorem 7.5, the finite-difference approximation (4.38) is first-order accurate with (1.1a), i.e., (4.1a) is satisfied with $c_N = Const_0 \cdot \frac{1}{N}$, and hence (4.15) holds.

Appealing to Corollary 4.3, consistency is therefore guaranteed if there exists a positive constant such that the following estimate holds

$$\frac{\Delta x}{2}\left(\frac{\partial}{\partial x}w_N(x), D_N\frac{\partial}{\partial x}w_N(x)\right) + \|w_N(x)\|^2 \geq \frac{1}{2}Const.\Delta x\|\frac{\partial}{\partial x}w_N(x)\|^2\ , \qquad Const. > 0\ ;$$

by the trapezoidal rule this is the same as

$$\frac{1}{2}\sum_{\nu=0}^{2N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}|^2 + \sum_{\nu=0}^{2N}|w_N(x_\nu)|^2\Delta x \geq \frac{1}{2}Const.\sum_{\nu=0}^{2N}|\Delta w_{\nu+\frac{1}{2}}|^2\ , \tag{4.40}$$

and the result follows from our assumption (4.38b), for

$$D_{\nu+\frac{1}{2}} \equiv Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}} \geq Const. > 0\ .$$

Remarks:

1. We observe that the total amount of quadratic entropy dissipated by (4.38a) equals, in view of (4.39),

$$\begin{aligned}\left(\tfrac{\partial}{\partial x}w_N(x), \tilde{E}_N^f w_N(x)\right) &= \tfrac{\Delta x}{2}\left(\tfrac{\partial}{\partial x}w_N(x), D_N\tfrac{\partial}{\partial x}w_N(x)\right) = \\ &= \tfrac{1}{2}\textstyle\sum_{\nu=0}^{2N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}|^2 ,\end{aligned} \tag{4.41}$$

which justifies calling $D_{\nu+\frac{1}{2}}$ the numerical dissipation coefficients of (4.38a); consult [18] [21] for the general case.

2. The finite-difference/element approximations (4.38a), (4.38b), are restricted to first-order accuracy, in agreement with Corollary 7.6. Indeed, the proof of Theorem 5.4 makes no use of the "lower-order" term $\|w_N(x)\|^2$ in (4.40), hence (3.27) holds.

Theorem 4.5 tells us that the conservative approximations (4.27) are consistent with the initial-value problem (1.1a), (1.1b), provided they contain uniformly more viscosity than the entropy conservative ones, (1.24), (1.25). To obtain a sharper localized version of such result, we shall prefer to represent these approximations in terms of local 'hat' basis functions, $\{\Lambda_k(x)\}_{k=-N}^N$, instead of the global trigonometric ones, $\{e^{ikx}\}_{k=-N}^N$, we have used so far. To this end we may proceed as follows.

We first recall that in (4.35), the finite-difference/element methods (4.27) were based on the finite-element approximation with mass lumping (1.24b). Before we turn to consider the viscosity contribution in (4.35), however, we shall prefer to concentrate on a slightly different, yet closely related basic approximation – the finite-element approximation with no mass lumping in (1.24a). Example 1.7 provides us with the piecewise-linear formulation of this basic approximation: it is associated with the piecewise parabolic discretization operator in (1.35), $P_N^f w_N(x) = FE_{\Delta x}f(w_N(x))$, such that

$$\frac{\partial}{\partial x}P_N^f w_N(x) = \sum_{k=-N}^{N}\frac{1}{\Delta x}[C^{-1}\Delta\overline{f}]_k\Lambda_k(x). \tag{4.42}$$

The important ingredient of this approximation, is the orthogonality of its discrete truncation error, $\frac{\partial}{\partial x}E_N^f w_N(x)$, to the ϕ_N-space,

$$\begin{aligned}\left(\tfrac{\partial}{\partial x}E_N^f w_N(x), \Lambda_\nu(x)\right) &= \left(\tfrac{\partial}{\partial x}f(w_N(x)), \Lambda_\nu(x)\right) - \textstyle\sum_{k=-N}^{N}\tfrac{1}{\Delta x}[C^{-1}\Delta\overline{f}_k]_k\left(\Lambda_k(x), \Lambda_\nu(x)\right) = \\ &= [\overline{f}_{\nu+\frac{1}{2}} - \overline{f}_{\nu-\frac{1}{2}}] - \tfrac{1}{\Delta x}[MC^{-1}\Delta\overline{f}]_\nu = 0 .\end{aligned}$$

This brings us to the canonical formulation of the basic finite-element approximation (1.36)

as a Galerkin method, namely, $u_N(x,t)$ is the piecewise-linear approximant such that of <u>all</u> $\varphi_N(x)$ in ϕ_N we have

$$\left(\varphi_N(x), \frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}[f(u_N(x,t))]\right) = 0, \qquad \varphi_N(x) \in \phi_N \ . \tag{4.43}$$

Counterexample 2.4 shows that this method is <u>inconsistent</u> with the entropy condition (1.1b), precisely because it is quadratic entropy conservative: indeed (4.43) implies that (4.14) holds, for

$$\left(\frac{\partial}{\partial x}w_N(x), E_N^f w_N(x)\right) = -\left(w_N(x), \frac{\partial}{\partial x}E_N^f w_N(x)\right) = 0 \ , \qquad w_N(x) \in \phi_N \ . \tag{4.44}$$

We therefore appeal to the vanishing viscosity method. In this context, we seek a piecewise-linear approximant, $u_N(x,t) = \sum_{k=-N}^{N} u_N(x_k,t)\Lambda_k(x)$, such that for <u>all</u> $\varphi_N(x)$ in ϕ_N we have

$$\left(\varphi_N(x), \frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}[f(u_N(x,t))]\right) + \frac{\Delta x}{2}\left(\frac{\partial}{\partial x}\varphi_N(x), \frac{\partial}{\partial x}u_N(x,t)\right)_D = 0 \ . \tag{4.45}$$

Here $D = \{D_{\nu+\frac{1}{2}} > 0\}$ are given numerical dissipation coefficients, so that the vanishing viscosity contribution enters through the <u>weighted</u> inner product on the left

$$(v(x), w(x))_\omega \equiv \sum_{\nu=-N}^{N} \omega_{\nu+\frac{1}{2}} \int_{x_\nu}^{x_{\nu+1}} < v(x), w(x) > dx \ , \qquad \omega = \{\omega_{\nu+\frac{1}{2}}\}_{\nu=-N}^{N} \ .$$

Choosing $\varphi_N(x) = \Lambda_\nu(x)$, then (4.45) reads at the collocation points $x_\nu = \nu\Delta x$,

$$\begin{aligned} &\tfrac{d}{dt}\left[\tfrac{1}{6}u_N(x_{\nu-1},t) + \tfrac{4}{6}u_N(x_\nu,t) + \tfrac{1}{6}u_N(x_{\nu+1},t)\right] + \\ &+\tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t))] = \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)] \ , \end{aligned} \tag{4.46a}$$

where in view of (4.29a), (4.29b) we have, in agreement with (4.35),

$$Q_{\nu+\frac{1}{2}} \equiv Q^*_{\nu+\frac{1}{2}} + D_{\nu+\frac{1}{2}} \ . \tag{4.46b}$$

This approximation corresponds to (the unlumped mass version of) the finite-difference/element approximation (4.27), or equivalently (4.35). It is based on the piecewise-parabolic finite-element approximation in (1.35),

$$P_N^f w(x) = \sum_{k=-N}^{N} \tilde{f}_{k+\frac{1}{2}}\Omega_{k+\frac{1}{2}}(x) \ , \qquad \Delta\tilde{f} = C^{-1}\Delta\overline{f} \ , \tag{4.47a}$$

which is appended by the piecewise parabolic viscosity approximation, $\varepsilon_N D_N \frac{\partial}{\partial x}$, $\varepsilon_N \equiv \frac{\Delta x}{2}$, where

$$D_N w(x) = \sum_{k=-N}^{N} [\widetilde{D\overline{w}}]_{k+\frac{1}{2}} \Omega_{k+\frac{1}{2}}(x) \ , \qquad [D\overline{w}]_{k+\frac{1}{2}} \equiv D_{k+\frac{1}{2}} \overline{w}_{k+\frac{1}{2}} \ . \tag{4.47b}$$

The difficulty of working with the explicit representation of the modified discretization operator, $\tilde{P}_N^f = P_N^f - \frac{\Delta x}{2} D_N \frac{\partial}{\partial x}$, lies in the nonlocal inverse of the mass matrix, C^{-1}, which enters (4.47a), (4.47b). It would be more convenient, therefore, to deal directly with the original weak formulation in (4.45): it asserts that for all C^∞-test functions $\varphi(x)$ and <u>arbitrary</u> ϕ_N-approximants, $\varphi_N(x)$, we have

$$\begin{aligned} \left(\varphi(x), \tfrac{\partial}{\partial t} u_N(x,t) + \tfrac{\partial}{\partial x}[f(u_N(x,t))]\right) &= \left(\varphi(x) - \varphi_N(x), \tfrac{\partial}{\partial t} u_N(x,t)\right) + \\ &+ \left(\varphi(x) - \varphi_N(x), \tfrac{\partial}{\partial x}[f(u_N(x,t))]\right) - \\ &- \tfrac{\Delta x}{2} \left(\tfrac{\partial}{\partial x}\varphi_N(x), \tfrac{\partial}{\partial x} u_N(x,t)\right)_D \ . \end{aligned} \tag{4.48}$$

The novelty of the weak formulation (4.48) is that it allows us to assign to $\varphi(x)$ different ϕ_N-approximants, $\varphi_N(x)$. Different choices of $\varphi_N(x)$ amount to formulation of (4.46) within different setups.

For example, if we take $\varphi_N(x) = \hat{\varphi}_N(x)$ to be the L_2-projection of $\varphi(x)$,

$$\hat{\varphi}_N(x) = \sum_{k=-N}^{N} \hat{\varphi}_k \Lambda_k(x) \ , \qquad \hat{\varphi}_k = C^{-1} \cdot \frac{1}{\Delta x} \int \varphi(x) \Lambda_k(x) dx \ , \tag{4.49}$$

then we recover the piecewise-parabolic discretization operators in (4.47). Indeed, since $\varphi(x) - \hat{\varphi}_N(x)$ is orthogonal to the ϕ_N-space, we are left with the second and third terms on the right of (4.48). The second term is a weak formulation for the local truncation error of the basic approximation

$$\frac{\partial}{\partial x} E_N^f u_N(x,t)[\varphi] = \left(\varphi(x) - \hat{\varphi}(x), \frac{\partial}{\partial x}[f(u_N(x,t))]\right) \ , \tag{4.50a}$$

so that $E_N^f = [I - FE_{\Delta x}] f(\cdot)$, in agreement with (4.47a). The third term represents (in a weak form) the truncation error due to the presence of vanishing viscosity

$$\frac{\Delta x}{2} \frac{\partial}{\partial x} D_N \frac{\partial}{\partial x} u_N(x,t)[\varphi] = -\frac{\Delta x}{2} \left(\frac{\partial}{\partial x}\hat{\varphi}_N(x), \frac{\partial}{\partial x} u_N(x,t)\right)_D \ , \tag{4.50b}$$

in agreement with (4.47b). The difficulty with this representation lies again in the nonlocal inverse, C^{-1}, which enters in the definition of $\hat{\varphi}_N(x)$ in (4.49).

Instead , let us assign $\varphi_N(x)$, as was done in [4], to be the piecewise-linear interpolant of $\varphi(x)$,

$$\varphi_N(x) = \sum_{k=-N}^{N} \varphi(x_k)\Lambda_k(x) \ . \tag{4.51}$$

We now consider the three terms on the right of (4.48). The third term is a weak representation for the truncation error due to the presence of vanishing viscosity

$$\varepsilon_N \frac{\partial}{\partial x} D_N \frac{\partial}{\partial x} w_N(x,t)[\varphi] = -\frac{\Delta x}{2}\left(\frac{\partial}{\partial x}\varphi_N(x), \frac{\partial}{\partial x} w_N(x,t)\right)_D \ , \qquad \varepsilon_N \equiv \frac{\Delta x}{2} \ . \tag{4.52a}$$

This corresponds to the <u>piecewise-constant</u> viscosity operator (here $\chi_{k+\frac{1}{2}}(x)$ denotes the characteristic function of the $[x_k, x_{k+1}]$-cell)

$$D_N w_N(x,t) = \sum_{k=-N}^{N} D_{k+\frac{1}{2}} \overline{w}_{k+\frac{1}{2}}(t)\chi_{k+\frac{1}{2}}(x) \ , \qquad \overline{w}_{k+\frac{1}{2}}(t) \equiv \overline{w}(x_{k+\frac{1}{2}}, t) \ . \tag{4.52b}$$

The second term is related to the discretization of the spatial flux, f_x,

$$\frac{\partial}{\partial x} E_N^f w_N(x,t)[\varphi] = \left(\varphi(x) - \varphi_N(x), \frac{\partial}{\partial x}[f(w_N(x,t))]\right) \ , \tag{4.53a}$$

where, as usual,

$$E_N^f w(x,t) = f(w(x,t)) - P_N^f w(x,t) \ . \tag{4.53b}$$

This uniquely determine a piecewise-constant spatial discretization operator

$$P_N^f w(x,t) = \sum_{k=-N}^{N} \overline{f}_{k+\frac{1}{2}}(t)\chi_{k+\frac{1}{2}}(x) \ , \qquad \overline{f}_{k+\frac{1}{2}}(t) \equiv \overline{f}(w(x_{k+\frac{1}{2}}, t)) \ . \tag{4.53c}$$

We observe that $\frac{\partial}{\partial x} P_N^f w(x,t)$ and $\varepsilon_N \frac{\partial}{\partial x} D_N \frac{\partial}{\partial x} w(x,t)$ do <u>not</u> lie in the ϕ_N-space; they belong to the space of measures $W^{-1,\infty}$. To balance these $W^{-1,\infty}$ terms we have to "discretize" (in space) the temporal flux, u_t, as well. The first term on the right of (4.48) gives us the weak representation of the truncation error due to this "temporal" discretization

$$\frac{\partial}{\partial t} E_N^u w_N(x,t)[\varphi] = \left(\varphi(x) - \varphi_N(x), \frac{\partial}{\partial t} w_N(x,t)\right) \ , \tag{4.54a}$$

or equivalently,

$$E_N^u w(x,t)[\varphi] = (\varphi(x) - \varphi_N(x), w(x,t)) \ . \tag{4.54b}$$

The discrete local error of our approximation consists now of two different sources of errors,

$$E_N w_N(x,t) = E_N^{(t)} w_N(x,t) + E_N^{(x)} w_N(x,t) \ , \tag{4.55a}$$

where $E_N^{(t)} w_N(x,t)$ is associated with discretization of the temporal flux,

$$E_N^{(t)} w(x,t) = E_N^u w(x,t) \ , \tag{4.55b}$$

and $E_N^{(x)} w_N(x,t)$ is associated with the spatial discretization (basic + viscosity approximations)

$$E_N^{(x)} w(x,t) \equiv E_N^f w(x,t) + \frac{\Delta x}{2} D_N \frac{\partial}{\partial x} w(x,t) \ . \tag{4.55c}$$

The (quadratic) entropy consistency estimate corresponds to (3.4) now reads

$$\begin{aligned} &\tfrac{1}{e_N} \| \tfrac{\partial}{\partial t} E_N^{(t)} w_N(x,t) + \tfrac{\partial}{\partial x} E_N^{(x)} w_N(x,t) \|^2_{H^{-1}_{\mathrm{loc}}(x,t)} + \tfrac{1}{\eta_N} U_{\mathrm{prod}}(w_N(x,t)) \leq \\ &\leq \left(\tfrac{\partial}{\partial t} w_N(x,t), E_N^{(t)} w_N(x,t) \right) + \left(\tfrac{\partial}{\partial x} w_N(x,t), E_N^{(x)} w_N(x,t) \right) + \| w_N(x,t) \|^2 \ . \end{aligned} \tag{4.56a}$$

Here, the quadratic entropy production, $U_{\mathrm{prod}}(w_N(x,t))$, takes into account the additional discretization of the temporal flux by modifying (3.2a),

$$\begin{aligned} U_{\mathrm{prod}}(w_N(x,t)) \equiv \ & \tfrac{1}{e_N} \| < w_N(x,t), \tfrac{\partial}{\partial t} E_N^{(t)} w_N(x,t) > \|^2_{H^{-1}_{\mathrm{loc}}(x,t)} + \\ & + \| < \tfrac{\partial}{\partial x} w_N(x,t), E_N^{(x)} w_N(x,t) >^- \|_{L^1_{\mathrm{loc}}(x,t)} \ , \quad e_N \downarrow 0 \ . \end{aligned} \tag{4.56b}$$

We now arrive at the localized version of Theorem 4.5.

Theorem 4.6: (Upwind Differencing). *The finite-difference/element approximation*

$$\begin{aligned} &\tfrac{d}{dt} \left[\tfrac{1}{6} u_N(x_{\nu-1},t) + \tfrac{4}{6} u_N(x_\nu,t) + \tfrac{1}{6} u_N(x_{\nu+1},t) \right] + \\ &+ \tfrac{1}{2\Delta x} [f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t)] = \tfrac{1}{2\Delta x} [Q_{\nu+\frac{1}{2}} \Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}} \Delta u_{\nu-\frac{1}{2}}(t)] \ , \end{aligned} \tag{4.57a}$$

is consistent with scalar initial-value problem (1.1a), (1.1b) w.r.t. the quadratic entropy $\mathcal{U} = \{\frac{1}{2}u^2\}$, *if its viscosity coefficients,* $Q_{\nu+\frac{1}{2}}$, *satisfy*

$$Q_{\nu+\frac{1}{2}}(t) \geq Const. |\overline{a}_{\nu+\frac{1}{2}}(t)| \quad , \quad \overline{a}_{\nu+\frac{1}{2}}(t) \equiv \left[\frac{1}{\Delta x} \int_{x_\nu}^{x_{\nu+1}} |f'(u_N(x,t))|^2 dx \right]^{1/2} \ . \tag{4.57b}$$

Proof: The proof consists of three steps.

(i) Entropy dissipation. The total amount of (quadratic) entropy dissipation minus entropy production of (4.57a) equals – in view of (4.52a), (4.53a), (4.54a) and the piecewise-linearity of $w_N(x,t)$

$$\left(\tfrac{\partial}{\partial t}w_N(x,t), E_N^{(t)}w_N(x,t)\right) + \left(\tfrac{\partial}{\partial x}w_N(x,t), E_N^{(x)}w_N(x,t)\right) \equiv$$

$$\begin{aligned}
&\equiv -\tfrac{\partial}{\partial t}E^{(t)}w_N(x,t)[\varphi = w_N(x,t)] - \tfrac{\partial}{\partial x}E_N^f w_N(x,t)[\varphi = w_N(x,t)] + \\
&+ \tfrac{\Delta x}{2}\tfrac{\partial}{\partial x}D_N\tfrac{\partial}{\partial x}w_N(x,t)[\varphi = w_N(x,t)] = \\
&= \tfrac{\Delta x}{2}\left(\tfrac{\partial}{\partial x}w_N(x,t), \tfrac{\partial}{\partial x}w_N(x,t)\right)_D = \tfrac{1}{2}\textstyle\sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2 ,
\end{aligned} \tag{4.58}$$

in agreement with (4.41).

(ii) Entropy production. The viscosity approximation does not produce entropy, for

$$< \frac{\Delta x}{2}\frac{\partial}{\partial x}w_N(x,t), D_N\frac{\partial}{\partial x}w_N(x,t) >= \frac{1}{2}\sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2\chi_{\nu+\frac{1}{2}}(x) \geq 0 .$$

Consequently, the spatial contribution to the entropy production in (4.56b) is upper bounded by

(4.59a)

$$\begin{aligned}
\| < \tfrac{\partial}{\partial x}w_N(x,t), E_N^{(x)}w_N(x,t) >^- \|_{L^1(x)} &\leq \| < \tfrac{\partial}{\partial x}w_N(x,t), E_N^f w_N(x,t) > \|_{L^1(x)} \leq \\
&\leq \tfrac{e_N}{4}\|\tfrac{\partial}{\partial x}w_N(x,t)\|_D^2 + \tfrac{1}{e_N}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 .
\end{aligned}$$

Using the "super-approximation" estimate [4, Lemma 2.1], we can upper bound the temporal contribution to the entropy production in (4.56b)

$$\begin{aligned}
&\tfrac{1}{e_N}\| < w_N(x,t), \tfrac{\partial}{\partial t}E_N^{(t)}w_N(x,t) > \|_{H^{-1}(x)}^2 \leq \\
&\leq \tfrac{1}{e_N}\sup_{\|\varphi_x\|=1} \left| \left(w_N(x,t)\varphi(x) - (w_N(x,t)\varphi(x))_N, \tfrac{\partial}{\partial t}w_N(x,t)\right) \right|^2 \leq \\
&\leq \eta_\infty \tfrac{(\Delta x)^2}{e_N}\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2, \qquad \eta_\infty = Const.
\end{aligned} \tag{4.59b}$$

(iii) Discrete error. The spatial contribution does not exceed

$$\begin{aligned}
&\tfrac{1}{e_N}\|\tfrac{\partial}{\partial x}E_N^{(x)}w_N(x,t)\|_{H^{-1}(x)}^2 \leq \\
&\leq \tfrac{D_\infty}{e_N}\|\tfrac{\Delta x}{2}D_N\tfrac{\partial}{\partial x}w_N(x,t)\|_{D^{-1}}^2 + \tfrac{D_\infty}{e_N}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 \leq \\
&\leq \tfrac{D_\infty}{4}\cdot\tfrac{(\Delta x)^2}{e_N}\|\tfrac{\partial}{\partial x}w_N(x,t)\|_D^2 + \tfrac{D_\infty}{e_N}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 .
\end{aligned} \tag{4.60a}$$

An upper bound for the temporal contribution is given by

$$
\begin{aligned}
\tfrac{1}{e_N}\|\tfrac{\partial}{\partial t}E_N^{(t)}w_N(x,t)\|_{H^{-1}(x)} &\le \tfrac{1}{e_N}\sup_{\|\varphi_x\|=1}|(\varphi(x)-\varphi_N(x),\tfrac{\partial}{\partial t}w_N(x,t))|^2 \le \\
&\le \tfrac{(\Delta x)^2}{e_N}\|\tfrac{\partial}{\partial x}w_N(x,t)\|^2 \ .
\end{aligned} \tag{4.60b}
$$

Now, let us choose $e_N = Const.\Delta x \downarrow 0$ and $\eta_N \equiv Const.$ with sufficiently large $Const. > 0$. In view of (4.58), (4.59) and (4.60), we conclude that the entropy consistency estimate (4.56) holds, provided there exists a positive constant such that

$$
\Delta x\|\frac{\partial}{\partial t}w_N(x,t)\|^2 + \frac{1}{\Delta x}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 \le Const.\sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2 \ . \tag{4.61}
$$

An energy estimate for the first term on the left is obtained by substituting $\varphi_N(x) = \frac{\partial}{\partial t}w_N(x,t)$ in (4.45):

$$
\|\frac{\partial}{\partial t}w_N(x,t)\|^2 \le 2\|\frac{\partial}{\partial x}f(w_N(x,t))\|^2 + 2D_\infty\|\frac{\partial}{\partial x}w_N(x,t)\|_D^2 \ . \tag{4.62}
$$

To estimate the second term on the left, we take advantage of its local representation in (4.53): since $E_N^f w_N(x,t)$ is a conservative we have

$$
\begin{aligned}
\|E_N^f w_N(x,t)\|_{D^{-1}}^2 &= \sup_{\|\varphi_x\|_D=1}\|\tfrac{\partial}{\partial x}E_N^f w_N(x,t)[\varphi]\|^2 \\
&= \sup_{\|\varphi_x\|_D=1}|\left(\varphi(x)-\varphi_N(x),\tfrac{\partial}{\partial x}[f(w_N(x,t))]\right)|^2 \le \\
&\le (\Delta x)^2\|\tfrac{\partial}{\partial x}[f(w_N(x,t))]\|_{D^{-1}}^2 \ .
\end{aligned} \tag{4.63}
$$

The inequalities (4.61), (4.62) and our assumption (4.57b) imply (4.61), for

$$
\begin{aligned}
&\Delta x\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2 + \tfrac{1}{\Delta x}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 \le \\
&\le Const.\Delta x\left[\|f'(w_N(x,t)\tfrac{\partial}{\partial x}w_N(x,t)\|_{D^{-1}}^2 + \|\tfrac{\partial}{\partial x}w_N(x,t)\|_D^2\right] \le \\
&\le Const\left[\textstyle\sum_{\nu=-N}^{N}\frac{1}{D_{\nu+\frac{1}{2}}}\overline{a}_{\nu+\frac{1}{2}}^2|\Delta w_{\nu+\frac{1}{2}}(t)|^2 + \sum_{\nu=-N}^{N}D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2\right] \le \\
&\le Const.\textstyle\sum_{\nu=-N}^{N}D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)| \ . \qquad \square
\end{aligned}
$$

The proof of Theorem 4.6 hinges on estimate (4.61). The latter was verified with the help of the first-order estimate

$$
\|\varphi(x)-\varphi_N(x)\|_\omega \le \Delta x\|\frac{\partial}{\partial x}\varphi(x)\|_\omega \ .
$$

In fact, $\varphi_N(x)$ is a second-order accurate approximation of $\varphi(x)$, which brings us to the final result of this section.

Theorem 4.7: (Second-Order Differencing). *The finite-difference/element approximation*

$$\frac{d}{dt}\left[\frac{1}{6}u_N(x_{\nu-1},t)+\frac{4}{6}u_N(x_\nu,t)+\frac{1}{6}u_N(x_{\nu+1},t)\right]+ \tag{4.64a}$$
$$+\frac{1}{2\Delta x}[f(u_N(x_{\nu+1},t))-f(u_N(x_{\nu-1}))]=\frac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t)-Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]\ ,$$

is consistent with scalar initial-value problems (1.1a), (1.1b) w.r.t. the quadratic entropy $\mathcal{U}=\{\frac{1}{2}u^2\}$, *if its viscosity coefficients* $Q_{\nu+\frac{1}{2}}$, *satisfy*

$$Q_{\nu+\frac{1}{2}}\geq Const.|\overline{a}'_{\nu+\frac{1}{2}}(t)|\cdot|\Delta u_{\nu+\frac{1}{2}}(t)|\ ,\ \ \overline{a}'_{\nu+\frac{1}{2}}(t)\equiv[\frac{1}{\Delta x}\int_{x_\nu}^{x_{\nu+1}}|f''(u_N(x,t))|^2dx]^{1/2}\ . \tag{4.64b}$$

Proof: We consider a weak representation of (4.64a) of the form

$$\begin{aligned}\left(\varphi_N(x),\tfrac{\partial}{\partial t}u_N(x,t)\right)\ &+\ \left(\overline{\varphi}_N(x),\tfrac{\partial}{\partial x}[f(u_N(x,t))]\right)+\\ &+\ \varepsilon_N\left(\tfrac{\partial\varphi_N}{\partial x},\tfrac{\partial}{\partial x}u_N(x,t)\right)_Q=0\ ,\qquad \varepsilon_N\equiv\tfrac{\Delta x}{2}\ .\end{aligned} \tag{4.65}$$

Here $\overline{\varphi}(x)=\sum_{k=-N}^{N}\frac{\varphi(x_k)+\varphi(x_{k+1})}{2}\chi_{k+\frac{1}{2}}(x)$ is the piecewise-constant projection of an arbitrary ϕ_N-element, $\varphi_N(x)=\sum_{k=-N}^{N}\varphi(x_k)\Lambda_k(x)$.

The local truncation error, $E_N=E_N^{(t)}+E_N^f+\frac{\Delta x}{2}Q_N\frac{\partial}{\partial x}$, consists of temporal contribution

$$\frac{\partial}{\partial t}E_N^{(t)}w_N(x,t)[\varphi]=\left(\varphi(x)-\varphi_N(x),\frac{\partial}{\partial t}w_N(x,t)\right)\ , \tag{4.66a}$$

and spatial contribution $E_N^{(x)}\equiv E_N^f+\frac{\Delta x}{2}Q_N\frac{\partial}{\partial x}$, where

$$\begin{aligned}\tfrac{\partial}{\partial x}E_N^f w_N(x,t)[\varphi]\ &=\left(\varphi(x)-\overline{\varphi}_N(x),\tfrac{\partial}{\partial x}[f(w_N(x,t))]\right)\equiv\\ &\equiv\left(\varphi(x)-\overline{\varphi_N}(x),\tfrac{\partial}{\partial x}[f(w_N(x,t))]-\overline{\tfrac{\partial}{\partial x}[f(w_N(x,t))]}\right)\ ,\end{aligned} \tag{4.66b}$$

and

$$\frac{\Delta x}{2}\frac{\partial}{\partial x}Q_N\frac{\partial}{\partial x}w_N(x,t)[\varphi]=\frac{\Delta x}{2}\left(\frac{\partial}{\partial x}\varphi_N(x),\frac{\partial}{\partial x}w_N(x,t)\right)_Q\ . \tag{4.66c}$$

Once more the entropy dissipation of (4.65) is given by

$$\left(\tfrac{\partial}{\partial t}w_N(x,t), E_N^{(t)}w_N(x,t)\right) + \left(\tfrac{\partial}{\partial x}w_N(x,t), E_N^{(x)}w_N(x,t)\right) = \tag{4.67}$$
$$= \tfrac{1}{2}\textstyle\sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2 \ .$$

The entropy production does not exceed

(4.68)
$$\tfrac{1}{e_N}\| < w_N(x,t), \tfrac{\partial}{\partial t}E_N^{(t)}w_N(x,t) > \|^2_{H^{-1}(x)} + \| < \tfrac{\partial}{\partial x}w_N(x,t), E_N^{(x)}w_N(x,t) >^- \|_{L^1(x)} \leq$$
$$\leq \eta_\infty \tfrac{(\Delta x)^2}{e_N}\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2 + \tfrac{e_N}{4}\|\tfrac{\partial}{\partial x}w_N(x,t)\|^2_D + \tfrac{1}{e_N}\|E_N^f w_N(x,t)\|^2_{D^{-1}} \ ,$$

and the discrete error is upper bounded by

$$\tfrac{1}{e_N}\|\tfrac{\partial}{\partial t}E_N^{(t)}w_N(x,t)\|^2_{H^{-1}(x)} + \tfrac{1}{e_N}\|\tfrac{\partial}{\partial x}E_N^{(x)}w_N(x,t)\|^2_{H^{-1}(x)} \leq \tag{4.69a}$$
$$\leq \tfrac{(\Delta x)^2}{e_N}\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2 + \tfrac{D_\infty}{e_N}\|E_N^f w_N(x,t)\|^2_{D^{-1}} + \tfrac{Q_\infty}{e_N}\|\tfrac{\Delta x}{2}Q_N w_N(x,t)\|^2_{Q^{-1}} \ .$$

Now, we note that $Q_{\nu+\frac{1}{2}}$ and $D_{\nu+\frac{1}{2}} \equiv Q_{\nu+\frac{1}{2}} - Q^*_{\nu+\frac{1}{2}}$ are of the same order of magnitude, for by (4.29b),

$$Q^*_{\nu+\frac{1}{2}} = \int_{\xi=-1/2}^{1/2}\left(\frac{1}{4} - \xi^2\right) f''(w_{\nu+\frac{1}{2}}(\xi))d\xi \cdot \Delta w_{\nu+\frac{1}{2}}(t) \leq \frac{1}{4}|\overline{a}'_{\nu+\frac{1}{2}}(t)| \cdot |\Delta w_{\nu+\frac{1}{2}}(t)| \ . \tag{4.69b}$$

As before, we choose $e_N = Const.\Delta x \downarrow 0$ and $\eta_N = Const.$ with sufficiently large $Const. > 0$, and the entropy consistency (4.56) is reduced to, in view of (4.67), (4.68) and (4.69),

$$\Delta x\|\frac{\partial}{\partial t}w_N(x,t)\|^2 + \frac{1}{\Delta x}\|E_N^f w_N(x,t)\|^2_{D^{-1}} \leq Const. \sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}}|\Delta w_{\nu+\frac{1}{2}}(t)|^2 \ . \tag{4.70}$$

Substituting $\varphi_N(x) = \frac{\partial}{\partial t}w_N(x,t)$ in (4.65), yields an energy-estimate for the first term on the left of (4.70); namely, we have

$$\begin{aligned}
\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2 \ &\leq \ \tfrac{1}{4}\|\overline{\tfrac{\partial}{\partial t}w_N(x,t)}\|^2 + \|\tfrac{\partial}{\partial x}[f(w_N(x,t))] - \overline{\tfrac{\partial}{\partial x}[f(w_N(x,t))]}\|^2 \\
&\quad + \varepsilon_N^2\|\tfrac{\partial^2}{\partial x\partial t}w_N(x,t)\|^2 + \|\tfrac{\partial}{\partial x}w_N(x,t)\|^2_{Q^2} \\
&\leq \ \tfrac{1}{2}\|\tfrac{\partial}{\partial t}w_N(x,t)\|^2 + (\Delta x)^2\|\tfrac{\partial^2}{\partial x^2}[f(w_N(x,t))]\|^2 + Q_\infty\|\tfrac{\partial}{\partial x}w_N(x,t)\|^2_Q \ ,
\end{aligned}$$

which yields, with the help of (4.69b),

$$\|\frac{\partial}{\partial t}w_N(x,t)\|^2 \leq 2(\Delta x)^2\|\frac{\partial^2}{\partial x^2}[f(w_N(x,t)]\|^2 + \|\frac{\partial}{\partial x}w_N(x,t)\|^2_D \ . \tag{4.71}$$

The second term on the left of (4.70) does not exceed

(4.72)

$$\|E_N^f w_N(x,t)\|_{D^{-1}}^2 = \sup_{\|\varphi_x\|_D=1} |\left(\varphi(x) - \overline{\varphi}_N(x), \tfrac{\partial}{\partial x}[f(w_N(x,t))] - \overline{\tfrac{\partial}{\partial x}[f(w_N(x,t))]}\right)|^2$$

$$\leq (\Delta x)^4 \|\tfrac{\partial^2}{\partial x^2}[f(w_N(x,t))]\|_{D^{-1}}^2 .$$

Now, since $w_N(x,t)$ is piecewise-linear, we have at <u>each</u> cell,

$$\frac{\partial^2}{\partial x^2}[f(w_N(x,t))] \equiv f''(w_N(x,t)) \left(\frac{\Delta w_{\nu+\frac{1}{2}}(t)}{\Delta x}\right)^2 .$$

Hence, the inequalities (4.71), (4.72) and our assumption (4.64b) imply (4.70), for

$$\Delta x \|\tfrac{\partial}{\partial t} w_N(x,t)\|^2 + \tfrac{1}{\Delta x}\|E_N^f w_N(x,t)\|_{D^{-1}}^2 \leq$$

$$\leq Const.[\textstyle\sum_{\nu=-N}^{N} \frac{1}{D_{\nu+\frac{1}{2}}}(\overline{a}'_{\nu+\frac{1}{2}})^2 |\Delta w_{\nu+\frac{1}{2}}(t)|^4 + \sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}} |\Delta w_{\nu+\frac{1}{2}}(t)|^2]$$

$$\leq Const. \textstyle\sum_{\nu=-N}^{N} D_{\nu+\frac{1}{2}} |\Delta w_{\nu+\frac{1}{2}}(t)|^2. \qquad \square$$

<u>Remarks</u>:

1. Theorems 4.6 and 4.7 are based on energy-estimating, independently, $\frac{\partial}{\partial t}u_N(x,t)$ and $\frac{\partial}{\partial x}[f(u_N(x,t))]$. In fact we can do with even less, namely, energy-estimating the sum $\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}[f(u_N(x,t))]$. To this end, it will suffice to have a space-time viscosity approximation in the "direction" of the local error, $\frac{\partial}{\partial t}u_N(x,t) + \frac{\partial}{\partial x}[f(u_N(x,t))]$. In this manner one concludes the consistency of finite-element Streamline-Diffusion (SD) method introduced by Hughes and his co-workers and analyzed in [4].

2. The case of <u>no</u> mass lumping, (4.27), can be treated similarly. The temporal "discretization" in such case is represented by the local truncation error,

$$\frac{\partial}{\partial t} E_N^{(t)} w_N(x,t)[\varphi] = (\varphi(x) - \tilde{\varphi}_N(x), \frac{\partial}{\partial t} w_N(x,t)) , \tag{4.73a}$$

where $\tilde{\varphi}_N(x)$ is the ϕ_N-interpolant of $\varphi(x)$ after mass lumping,

$$\tilde{\varphi}_N(x) = \sum_{k=-N}^{N} \tilde{\varphi}_k \Lambda_k(x) , \qquad \tilde{\varphi}_k = C^{-1}\varphi(x_k) . \tag{4.73b}$$

The mass lumping in (4.73) <u>adds</u> entropy dissipation to the unlumped mass version in (4.64).

3. Theorem 4.7 verifies the consistency of modern high-resolution nonlinear finite-difference approximations, e.g., the TVD and UNO methods described in [2], [3], [14], [15]. We observe that the viscosity coefficients in (4.64) need not be "limited" at critical extrema values.

5. L^∞-STABILITY OF THE SCALAR VANISHING VISCOSITY METHOD

In this section we study the L^∞-stability of the scalar vanishing viscosity method. To make our point we shall concentrate on the π_N-framework, where a π_N-approximate solution, $u_N(x,t) = \sum_{k=-N}^{N} \hat{u}_k(t)e^{ikx}$, evolves according to the viscous approximation

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[f(u_N(x,t))] = \frac{\partial}{\partial x}E_N^f u_N(x,t) + \varepsilon_N \frac{\partial}{\partial x}\left[Q_N \frac{\partial}{\partial x}u_N(x,t)\right] . \tag{5.1}$$

In all the examples demonstrated in the previous sections, (5.1) was based on a consistent – in fact (at least) first-order accurate approximation of the conservation law (1.1a),

$$\|E_N^f w_N(x)\| \le Const_0 \cdot \frac{1}{N}\|\frac{\partial}{\partial x}w_N(x)\| . \tag{5.2}$$

In addition, we assume that the viscosity part of (5.1) is parametrized according to the setup of Corollaries 4.2 and 4.3. Namely, we have the inequality (4.10)

$$\varepsilon_N \left(\frac{\partial}{\partial x}w_N(x), Q_N \frac{\partial}{\partial x}w_N(x)\right) + \|w_N(x)\|^2 \ge Const.\varepsilon_N \|\frac{\partial}{\partial x}w_N(x)\|^2 , \tag{5.3}$$

and the viscosity amplitude, ε_N, tends "sufficiently slow" to zero

$$\frac{1}{N} \le \varepsilon_N \downarrow 0 . \tag{5.4}$$

In this case, the vanishing viscosity method (5.1) is a consistent approximation of scalar initial-value problem (1.1a), (1.1b) w.r.t. all strictly convex entropies, $\mathcal{U} = \mathcal{U}_{\text{all}}$. This assertion was verified in Corollaries 4.2 and 4.3 with the help of the inequality

$$\tilde{d}_N \|\frac{\partial}{\partial x}w_N(x)\|^2 \le \left(\frac{\partial}{\partial x}w_N(x), \tilde{E}_N^f w_N(x)\right) + \|w_N(x)\|^2 , \qquad \tilde{d}_N = Const.(1-\theta)\varepsilon_N . \tag{5.5}$$

Here $\theta = \frac{Const_0}{Const.} < 1$ in the general case of Corollary 4.2, see (4.13), and $\theta = 0$ in the entropy conservative case of Corollary 4.3.

Finally we recall that according to a previous remark, see (3.22), $\|u_N(x,t)\|_{L^2(x)}$ is uniformly bounded in time, $\|u_N(x,t)\|_{L^2(x)} \le M_2$. Using this together with the fact that our 'preferred' entropy function is quadratic in the scalar case, $U_*(u) = \frac{1}{2}u^2$, enables us to upper bound the lefthand-side of (3.15) independently of the L^∞-bounds in (3.16),

$$\int_{t=0}^{T} \left(\frac{\partial}{\partial x}u_N(x,t), E_N^f u_N(x,t)\right) dt = \frac{1}{2}\|u_N(x,t)\|^2_{L^2(x)} \Big|_{t=T}^{t=0} \le M_2^2 .$$

Hence, applying (5.5) to $w_N(\cdot) = u_N(\cdot,t)$ we find after temporal integration that

$$\varepsilon_N \|\frac{\partial}{\partial x}u_N(x,t)\|^2_{L^2_{\text{loc}}(x,t)} \le \frac{1}{Const.} \cdot \frac{2M_2^2}{(1-\theta)} . \tag{5.6}$$

The L^∞-stability of (5.1) hinges on L^p-estimates for the discrete error of the basic scheme,

$$\|E_N^f w_N(x)\|_{L^\infty(x)} \leq Const_\infty \cdot \frac{1}{\sqrt{N}} \|\frac{\partial}{\partial x} w_N(x)\| \ , \tag{5.7}$$

and for the discrete error due to the presence of vanishing viscosity,

$$\varepsilon_N \|\frac{\partial}{\partial x}\left[R_N \frac{\partial}{\partial x} w_N(x)\right]\|_{L^\infty(x)} \leq Const_\infty \cdot \|w_N(x)\|_{L^\infty(x)} \ , \qquad R_N + Q_N = Id_N \ . \tag{5.8}$$

The estimates (5.7) and (5.8) are the strengthened L^p-versions of estimates (5.2) and (5.3). Indeed, the L^p-version of (5.7),

$$\|E_N^f w_N(x)\|_{L^p(x)} \leq Const_p \cdot \frac{1}{N^{\frac{1}{2}+\frac{1}{p}}} \|\frac{\partial}{\partial x} w_N(x)\| \ ,$$

with $p = 2$, corresponds to (5.2). The L^p-version of (5.8),

$$\varepsilon_N \|\frac{\partial}{\partial x}\left[R_N \frac{\partial}{\partial x} w_N(x)\right]\|_{L^p(x)} \leq Const_\infty \cdot \|w_N(x)\|_{L^p(x)} \ , \qquad R_N + Q_N = Id_N \ ,$$

with $p = 2$, yields (5.3) for,

$$\begin{aligned} \varepsilon_N \left(\frac{\partial}{\partial x} w_N(x), Q_N \frac{\partial}{\partial x} w_N(x)\right) &= \varepsilon_N \|\frac{\partial}{\partial x} w_N(x)\|^2 + \varepsilon_N \left(w_N(x), \frac{\partial}{\partial x}\left[R_N \frac{\partial}{\partial x} w_N(x)\right]\right) \\ &\geq \varepsilon_N \|\frac{\partial}{\partial x} w_N(x)\|^2 - Const_\infty \cdot \|w_N(x)\|^2 \ . \end{aligned}$$

Equipped with these estimates we can iterate on the $L^p(x)$ norms of $u_N(x,t)$ with the help of

Lemma 5.1: *Consider the vanishing viscosity method* (5.1) *which satisfies the* L^∞-*consistency estimates* (5.7), (5.8). *Then there exists a positive constant (independent of* N *and* p*),* $Const. > 0$*, such that for any even integer* $p \geq 2$ *we have*

$$\|u_N(x,t)\|_{L^p(x)} \leq e^{Const. \cdot t} \left[\|u_N(x, t=0)\|_{L^p(x)} + Const. \frac{\sqrt{p}}{\sqrt{N} \cdot \varepsilon_N}\right] \ . \tag{5.9}$$

Proof: Multiplying (5.1) by $p u_N^{p-1}(x,t)$ and integrating by parts we obtain

$$\begin{aligned} \frac{d}{dt}\|u_N(x,t)\|_{L^p(x)}^p &= -p(p-1) \int_{x=0}^{2\pi} u_N^{p-2}(x,t) \frac{\partial}{\partial x} u_N(x,t) E_N^f u_N(x,t) dx \\ &\quad -p(p-1)\varepsilon_N \int_{x=0}^{2\pi} u_N^{p-2}(x,t) \left[\frac{\partial}{\partial x} u_N(x,t)\right]^2 dx \\ &\quad -p\varepsilon_N \int_{x=0}^{2\pi} u_N^{p-1}(x,t) \frac{\partial}{\partial x}\left[R_N \frac{\partial}{\partial x} u_N(x,t)\right] dx \\ &\equiv I + II + III \ . \end{aligned} \tag{5.10}$$

For even integers $p \geq 2$, the weighted Cauchy-Schwartz inequality is used to upper bound the first term on the right

(5.11)

$$I \leq p(p-1)\varepsilon_N \int_{x=0}^{2\pi} u_N^{p-2}(x,t) \left[\frac{\partial}{\partial x} u_N(x,t)\right]^2 dx + \frac{p(p-1)}{4\varepsilon_N} \int_{x=0}^{2\pi} u_N^{p-2}(x,t) \cdot [E_N^f u_N(x,t)]^2 dx \ ,$$

and together with the second term on the right we have, in view of (5.7),

$$I + II \leq Const_\infty^2 \cdot \frac{p(p-1)}{4N\varepsilon_N} \|u_N(x,t)\|_{L^{p-2}(x)}^{p-2} \cdot \|\frac{\partial}{\partial x} u_N(x,t)\|_{L^2(x)}^2 \ . \tag{5.12}$$

Using (5.8) we can Hölder the third term,

$$\begin{aligned} III &\leq p\varepsilon_N \|u_N^{p-1}(x,t)\|_{L^{\frac{p}{p-1}}(x)} \|\tfrac{\partial}{\partial x}\left[R_N \tfrac{\partial}{\partial x} u_N(x,t)\right]\|_{L^p(x)} \leq \\ &\leq Const_\infty \cdot p \cdot \|u_N(x,t)\|_{L^p(x)}^p \ . \end{aligned} \tag{5.13}$$

Inserting (5.12) and (5.13) into (5.10) we obtain, after division by a common factor of $\frac{p}{2}\|u_N(x,t)\|_{L^p(x)}^{p-2}$,

$$\frac{d}{dt}\|u_N(x,t)\|_{L^p(x)}^2 \leq Const_\infty^2 \cdot \frac{p}{2N\varepsilon_N} \|\frac{\partial}{\partial x} u_N(x,t)\|_{L^2(x)}^2 + 2Const_\infty \|u_N(x,t)\|_{L^p(x)}^2 \ . \tag{5.14}$$

Temporal integration of the last inequality yields

$$\begin{aligned} e^{-2Const_\infty \cdot t}\|u_N(x,t)\|_{L^p(x)}^2 &\leq \|u_N(x,t=0)\|_{L^p(x)}^2 + \\ &+ Const_\infty^2 \cdot \tfrac{p}{2N\varepsilon_N^2} \textstyle\int_{\tau=0}^t e^{-2Const_\infty \cdot \tau} \varepsilon_N \cdot \|\tfrac{\partial}{\partial x} u_N(x,\tau)\|_{L^2(x)}^2 d\tau \ , \end{aligned}$$

and (5.9) follows with the help of (5.6). □

Lemma 5.1 shows that the $L^p(x)$-norms of $u_N(x,t)$ are bounded (w.r.t. N, p and t), at least for "sufficiently high" $L^p(x)$-norms, provided ε_N tends to zero "sufficiently slow". Specifically, if instead of (5.4) we have the stronger

$$\sqrt{\frac{\log N}{N}} \leq \varepsilon_N \downarrow 0 \ , \tag{5.15}$$

then $\|u_N(x,t)\|_{L^p(x)}$ are bounded for $p \leq \log N$,

$$\|u_N(x,t)\|_{L^p(x)} \leq M(t) \ , \qquad p \leq \log N \ . \tag{5.16}$$

Now, we assert that the $L^\infty(x)$-norm of π_N-polynomials does not exceed a constant, say 10, times their "sufficiently high" $L^p(x)$-norm, say $p \sim \log N$, for by one of the Gagliardo-Nirenberg inequalities, e.g., [16, Section 3]

$$\|w_N(x)\|_{L^\infty(x)} \leq \left(\frac{p+2}{4}\right)^{\frac{1}{p+2}} \cdot \|\tfrac{\partial}{\partial x} w_N(x)\|_{L^2(x)}^{\frac{2}{p+2}} \cdot \|w_N(x)\|_{L^p(x)}^{\frac{p}{p+2}} \leq$$

$$\leq 1.1 \cdot N^{\frac{2}{p+2}} \|w_N(x)\|_{L^2(x)}^{\frac{2}{p+2}} \cdot \|w_N(x)\|_{L^p(x)}^{\frac{p}{p+2}} \leq$$

$$\leq 1.1 \cdot e^2 \|w_N(x)\|_{L^{p=\log N}(x)} \ .$$

We conclude

Theorem 5.2: (L^∞-stability).*If the scalar vanishing viscosity method* (5.1) *satisfies* (5.7), (5.8) *and* (5.15), *then it is* L^∞*-stable, i.e., there exists a constant (independent of* N*),* M_∞*, such that*

$$\|u_N(x,t)\|_{L^\infty_{\text{loc}}(x,t)} \leq M_\infty \ .$$

Remarks:

1. The consistency estimate (5.7) together with (5.6) imply

$$\int_{t=0}^{T} \|E_N^f u_N(x,t)\|_{L^\infty(x)}^2 dt \leq Const. \frac{1}{N\varepsilon_N} \ . \tag{5.18}$$

Thus, the presence of vanishing viscosity in (5.4), parametrized according to (5.15), guarantees that the local error of the basic approximation in (5.1) tends to zero.

$$\int_{t=0}^{T} \|E_N^f u_N(x,t)\|_{L^\infty(x)}^2 dt \leq Const. \frac{1}{\sqrt{N \log N}} \longrightarrow 0 \ .$$

2. If instead of (5.15), the viscosity amplitude ε_N is restricted by the weaker (5.4), then (5.18) yields

$$\int_{t=0}^{T} \|E_N^f u_N(x,t)\|_{L^\infty(x)}^2 dt \leq Const. \ ,$$

and we conjecture that the vanishing viscosity method (5.1) remains L^∞-stable in such case.

The first step in implementing Theorem 5.2 requires us to verify the L^∞-consistency estimate (5.7). To this end let us consider basic discretization operators which operate linearly on the flux $f(\cdot)$,

$$P_N^f w_N(x) = P_N f(w_N(x)) \tag{5.19}$$

We recall that the spectral, ψdospectral, finite-difference and element methods discussed in Section 1, are the canonical examples of basic approximations which belong to this "linear" category. The consistency estimate(5.7) then reads

$$\|[I - P_N] f(w_N(x))\|_{L^\infty(x)} \leq Const_\infty \cdot \frac{1}{\sqrt{N}} \|\frac{\partial}{\partial x} w_N(x)\| \ ,$$

with a constant $Const_\infty$ which may depend on the L^∞-bound on $w_N(x)$ but otherwise is dependent on N. Consequently, estimate (5.7) is fulfilled provided for all $w(x)$ in $H^1[0,2\pi]$ we have

$$\|[I-P_N]w(x)\|_{L^\infty(x)} \le Const.\frac{1}{\sqrt{N}}\|\frac{\partial}{\partial x}w(x)\| \ . \tag{5.20}$$

We shall verify the consistency estimate (5.7) – or equivalently (5.20), using the following von Newmann like stability analysis. To this end we define the symbols

$$g_{N,k}(x) = e^{-ikx}P_N e^{ikx} \ , \tag{5.21}$$

and state

Lemma 5.3: *The linear basic approximation $P_N^f = P_N f(\cdot)$, fulfills the L^∞-consistency estimate (5.7), provided its symbols satisfy*

$$\max_x |1-g_{N,k}(x)| \le Const.\min\left(1,\frac{k}{N}\right) \ , \qquad |k| \le \infty \ . \tag{5.22}$$

Proof: For $|k|>N$ we have by (5.22),

$$\begin{aligned} &\|\textstyle\sum_{|k|>N}\hat{w}(k)\cdot[1-g_{N,k}(x)]e^{ikx}\|^2_{L^\infty(x)} \le \\ &\le \textstyle\sum_{|k|>N}k^2|\hat{w}(k)|^2\cdot|1-g_{N,k}(x)|^2\cdot\sum_{|k|>N}\frac{1}{k^2} \le \\ &\le \tfrac{Const.^2}{N}\textstyle\sum_{|k|>N}k^2|\hat{w}(k)|^2 \ . \end{aligned} \tag{5.23a}$$

Moreover, (5.22) implies that for $|k|\le N$ we have

$$\begin{aligned} \|\textstyle\sum_{|k|\le N}\hat{w}(k)\cdot[1-g_{N,k}(x)]e^{ikx}\|^2_{L^\infty(x)} &\le N\cdot\textstyle\sum_{|k|\le N}|\hat{w}(k)|^2\cdot|1-g_{N,k}(x)|^2 \le \\ &\le \tfrac{Const.^2}{N}\textstyle\sum_{|k|\le N}k^2|\hat{w}(k)|^2 \ . \end{aligned} \tag{5.23b}$$

Hence (5.20) follows in view of (5.23a), (5.23b),

$$\begin{aligned} \|(I-P_N)w(x)\|^2_{L^\infty(x)} &= \|\textstyle\sum_{k=-\infty}^{\infty}\hat{w}(k)\cdot[1-g_{N,k}(x)]e^{ikx}\|^2_{L^\infty(x)} \le \\ &\le \tfrac{4Const.^2}{N}\textstyle\sum_{k=-\infty}^{\infty}k^2|\hat{w}(k)|^2 = \left[2Const.\tfrac{1}{\sqrt{N}}\|\tfrac{\partial}{\partial x}w(x)\|\right]^2 \ . \end{aligned}$$

□

We observe that the exponentials $\{e^{ikx}\}_{k=-N}^N$ are eigenfunctions of the linear discretizations operators associated with the spectral method, $P_N = S_N$, ψdospectral method,

$P_N = \psi_N$, finite-difference method, $P_N = FD_{\Delta x}$ and finite-element method (based on quadratic "preferred" entropy $U_*(u) = \frac{1}{2}u^2$), $P_N = FE_{\Delta x}$. We claim that therefore (5.22) is satisfied, which in turn implies that the L^∞-consistency estimate (5.7) is valid in these cases. Indeed, since P_N are bounded

$$\max_x |g_{N,k}(x)| = \max_x |P_N e^{ikx}| \leq Const. \tag{5.24a}$$

Moreover, by the first-order accuracy estimate (5.2) we have

$$\|e^{ikx}[1 - g_{N,k}(x)]\|^2_{L^2(x)} = \|[I - P_N]e^{ikx}\|^2_{L^2(x)} \leq Const_0^2 \cdot \frac{k^2}{N^2} .$$

Now, if $\{e^{ikx}\}_{k=-N}^N$ are eigenfunctions of P_N, then $g_{N,k}(x)$ are constant amplification factors, $g_{N,k}(x) \equiv g_{N,k}$, and the last inequality yields

$$\max_x |1 - g_{N,k}(x)|^2 \leq \frac{1}{2\pi} Const_0^2 \cdot \frac{k^2}{N^2} , \qquad |k| \leq N . \tag{5.24b}$$

This together with (5.24a) imply (5.22) as asserted.

Example 5.4: <u>L^∞-stability of the scalar SV method.</u> We consider the SV method (4.17), (4.23)

$$\frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N f(u_N(x,t))] = \frac{1}{N^\beta}\frac{\partial}{\partial x}[Q_N(x,t) * \frac{\partial}{\partial x}u_N(x,t)] , \tag{5.25a}$$

where $Q_N(x,t) = \sum_{|k|=m_N}^N \hat{Q}_k(t)e^{ikx}$ is a viscosity kernel with monotonically increasing coefficients such that

$$\hat{Q}_k(t) \geq Const. - \frac{N^\beta}{k^2 \cdot \log N} \quad , \quad k^2 \geq m_N^2 = \frac{N^\beta}{Const. \log N} , \qquad 0 < \beta < \frac{1}{2} . \tag{5.25b}$$

The basic spectral or ψdospectral approximation satisfies the L^∞-consistency estimate (5.7) by Lemma 5.3. Next, we verify estimate (5.8) with $R_N + Q_N = Const.Id_N$. In this case, the corresponding kernel $R_N(x,t)$ is given by

$$R_N(x,t) = Const. \sum_{|k|\leq m_N} e^{ikx} + \sum_{|k|>m_N} \hat{R}_k(t)e^{ikx} \equiv R_N^-(x) + R_N^+(x,t) ,$$

where $R_N^-(x)$ is a multiple of the π_{m_N}-Dirichlet kernel,

$$R_N^-(x) = Const.D_{m_N}(x) ,$$

and by (5.25b), $R_N^+(x,t)$ has monotonically decreasing coefficients

$$0 \leq\downarrow \hat{R}_k(t) \leq \frac{N^\beta}{k^2 \cdot \log N} , \qquad |k| \geq m_N .$$

Since $R_N^-(x) * w_N(x)$ is a π_{m_N}-polynomial, we can estimate its derivatives by [23]

$$
\begin{aligned}
\varepsilon\|\tfrac{\partial}{\partial x}[R_N^-(x) * \tfrac{\partial}{\partial x}w_N(x)]\|_{L^\infty(x)} &\le \varepsilon_N m_N^2 \cdot \|R_N^-(x)\|_{L^1(x)} \cdot \|w_N(x)\|_{L^\infty(x)} \le \\
&\le Const.\varepsilon_N m_N^2 \log m_N \|w_N(x)\|_{L^\infty(x)} \le \\
&\le Const_\infty \|w_N(x)\|_{L^\infty(x)} \ .
\end{aligned}
\tag{5.26a}
$$

Since the coefficients of $\log N \cdot N^{-\beta}\frac{\partial^2}{\partial x^2}R_N^+(x,t)$ are monotonically decreasing,

$$
0 \downarrow \log N \cdot N^{-\beta}k^2\hat{R}_k(t) \le 1 \ ,
$$

we can apply [10, Lemma A.1] to obtain

$$
\begin{aligned}
&\varepsilon_N\|\tfrac{\partial}{\partial x}[R_N^+(x,t) * \tfrac{\partial}{\partial x}w_N(x)]\|_{L^\infty(x)} \le \\
&\quad \le \varepsilon_N \tfrac{N^\beta}{\log N} \cdot \|\log N \cdot N^{-\beta}\tfrac{\partial^2}{\partial x^2}R_N^+(x,t)\|_{L^1(x)} \cdot \|w_N(x)\|_{L^\infty(x)} \le \\
&\quad \le \varepsilon_N \tfrac{N^\beta}{\log N} \cdot Const. \log N \cdot \|w_N(x)\|_{L^\infty(x)} \le \\
&\quad \le Const_\infty \cdot \|w_N(x)\|_{L^\infty(x)} \ .
\end{aligned}
\tag{5.26b}
$$

Thus, estimate (5.8) holds in view of (5.26a), (5.26b). Finally, since $\beta < \frac{1}{2}$, the viscosity amplitude $\varepsilon_N = N^{-\beta}$ satisfies (5.15), and Theorem 5.2 guarantees the L^∞-stability of the SV method (5.25a), (5.25b).

6. CONSISTENCY AND L^∞-STABILITY IMPLY CONVERGENCE

We begin with

Theorem 6.1: *Consider the semi-discrete approximation (1.4). We assume that*
(i) The approximation (1.4) *is consistent with the initial-value problem* (1.1a), (1.1b) *w.r.t. a "relevant" class of entropy functions, $\mathcal{U}$,*
and
(ii) The approximation (1.4) is L^∞-stable.
Then for any C^2 entropy pair (U, F) associated with (1.1), such that $U \in \mathcal{U}$,

$$\frac{\partial}{\partial t}[U(u_N(x,t)] + \frac{\partial}{\partial x}[F(u_N(x,t))] \ , \qquad U \in \mathcal{U} \ , \tag{6.1}$$

belongs to a compact subset of $H^{-1}_{\text{loc}}(x,t)$.

Proof: Multiplying (1.4) by $U'(u_N(x,t))$ we obtain (3.1), which we rewrite us

$$\begin{aligned} &\tfrac{\partial}{\partial t}[U(u_N(x,t))] + \tfrac{\partial}{\partial x}[F(u_N(x,t))] = \\ &= \tfrac{\partial}{\partial x} < U'(u_N(x,t)), E^f_N u_N(x,t) > - < \tfrac{\partial}{\partial x} U'(u_N(x,t)), E^f_N u_N(x,t) > \equiv \\ &\equiv I + II. \end{aligned} \tag{6.2}$$

By the first part of Lemma 3.3 we have, see (3.9),

$$\begin{aligned} \|I \ &\equiv \tfrac{\partial}{\partial x} < U'(u_N(x,t)), E^f_N u_N(x,t) > \|_{H^{-1}_{loc}}(x,t) \leq \\ &\leq \|U'(u_N(x,t))\|_{L^\infty_{loc}(x,t)} \cdot Const_I \cdot \sqrt{e_N} \to 0 \end{aligned} \tag{6.3}$$

and hence the term I lies in the compact H^{-1}_{loc}. The third part of Lemma 3.3 gives us

$$\|II \equiv - < \frac{\partial}{\partial x}[U'(u_N(x,t))], E^f_N u_N(x,t) > \|_{L^1_{loc}(x,t)} \leq Const_{III}, \tag{6.4}$$

and this estimate implies, with the help of Murat's Lemma [22], that the term II also lies in the compact of H^{-1}_{loc}, which completes the proof. □

One can use now compensated compactness arguments [22], in order to turn the conclusion of Theorem 6.1 into a convergence proof. The current state of the art of these arguments in this context, includes scalar and 2×2 systems of conservation laws [22], [1].

We start with the scalar case.

Theorem 6.2: *Consider an L^∞-stable approximation (1.4) which is consistent with the scalar initial-value problem (1.1a), (1.1b) w.r.t. all strictly convex entropies, $\mathcal{U} = \mathcal{U}_{\text{all}}$. Then*

(a subsequence of) $u_N(x,t)$ converges weakly to a weak solution, $\overline{u}(x,t)$, of the conservation law (1.1a),

$$\frac{\partial}{\partial t}[\overline{u}(x,t)] + \frac{\partial}{\partial x}[f(\overline{u}(x,t))] = 0. \tag{6.5}$$

Proof: Since the right-hand sides of (1.4),(3.1) were shown to lie in the compact of $H^{-1}_{\text{loc}}(x,t)$, we can apply the div-and lemma [22] to their left-hand sides: abbreviating $\overline{g(u)} = w \lim_{N\to\infty} g(u_N)$, then for any C^2 entropy pair $(U(u), F(u))$ we can extract a subsequence (still denoted by u_N) such that

$$\overline{uF(u)} - \overline{U(u)f(u)} = \overline{u} \cdot \overline{F(u)} - \overline{U(u)} \cdot \overline{f(u)} \tag{6.6}$$

But (6.6) depends continuously on (U, F) in the C^1 topology and therefore it remains valid for piecewise C^1 entropy pairs. Following Krushkov [6] we choose

$$U(u) = |u - c| \quad , \quad F(u) = \operatorname{sgn}(u-c) \cdot (f(u) - f(u)) \qquad c = Const \ ,$$

in which case (6.6) reads

$$\begin{aligned} &\overline{u \cdot \operatorname{sgn}(u-c) \cdot (f(u) - f(c))} - \overline{|u-c| \cdot f(u)} \\ &= \overline{u} \cdot \overline{\operatorname{sgn}(u-c) \cdot (f(u) - f(c))} - \overline{|u-c|} \cdot \overline{f(u)} \ . \end{aligned}$$

Equivalently, we can rewrite this as

$$\overline{(u - \overline{u})\operatorname{sgn}(u-c) \cdot (f(u) - f(c))} = \overline{|u-c| \cdot (f(u) - \overline{f(u)})} \ .$$

Let us examine the last equality by restricting the weak limits to an <u>arbitrary fixed</u> (x,t) location; with $c = \overline{u}(x,t)$ we find after little rearrangement that

$$\overline{|u - \overline{u}|} \cdot (\overline{f(u)} - f(\overline{u})) = 0 \ .$$

This implies that

$$\overline{f(u)} = f(\overline{u}) \ , \tag{6.7}$$

for otherwise $\overline{|u - \overline{u}|}(x,t) = 0$, which in turn leads again to (6.7)[3] Taking the <u>weak</u> limit of (1.4), then (3.9) implies that the righthand-side tends to zero, and by (6.7) the lefthand-side amounts to having (6.5),

$$\frac{\partial}{\partial t}[\overline{u}(x,t)] + \frac{\partial}{\partial x}[f(\overline{u}(x,t))] = \frac{\partial}{\partial t}[\overline{u}(x,t)] + \frac{\partial}{\partial x}[\overline{f(u(x,t))}] = 0 \ . \qquad \square$$

[3] If $\overline{|u - \overline{u}|}(x,t) = 0$ then $\overline{g(u)}(x,t) = g(\overline{u}(x,t))$ for <u>any</u> C^1 function. This follows, for example, by noting that the associated probability measure is concentrated at the single point $\overline{u}$, consult [22].

The above compensated compactness argument for convergence is due to Tartar [22, Theorem 2.6] who made use of (6.6) to further deduce that in case the scalar flux $f(u)$, is nonlinear, i.e., when there is no interval on which $f'(u) \equiv Const.$, then the convergence $u_N(x,t) \to \overline{u}(x,t)$ is in fact strong in $L^p_{loc}(x,t)$, $p < \infty$. Moreover, if the scalar flux $f(u)$ is genuinely nonlinear (GNL), i.e., $f''(u) \neq 0$, then the same conclusion of strong $L^p_{\text{loc}}(x,t)$ convergence holds, by applying the div-curl lemma to a single strictly convex entropy function $U(u)$ in $\mathcal{U}$ [22, Remark 30].

Theorem 6.3: *Consider an L^∞-stable approximation* (1.4) *which is consistent with the nonlinear scalar initial value problem (1.1a), (1.1b) w.r.t. a nonempty "relevant" class of strictly convex entropy functions $\mathcal{U}$. Then we have*
A. *Convergence: If either $\mathcal{U}$ contains all strictly convex entropies, $\mathcal{U} = \mathcal{U}_{\text{all}}$ or $f(u)$ is GNL, then (a subsequence of) $u_N(x,t)$ converges strongly in $L^p_{\text{loc}}(x,t)$, $p < \infty$, to a weak solution $\overline{u}(x,t)$ of the conservation law* (1.1a),

$$\frac{\partial}{\partial t}[\overline{u}(x,t)] + \frac{\partial}{\partial x}[f(\overline{u}(x,t))] = 0 \ . \tag{6.8}$$

B. *Entropy inequality: For each entropy function, $U(u)$ in $\mathcal{U}$, whose entropy production tends weakly to zero*

$$w \lim_{N\to\infty} \left[- < \frac{\partial}{\partial x} U'(u_N(x,t)) \ , \ E^f_N u_N(x,t) >^- \right] = 0 \ , \tag{6.9a}$$

the weak limit solution $\overline{u}(x,t)$ satisfies the entropy inequality

$$\frac{\partial}{\partial t}[U(\overline{u}(x,t))] + \frac{\partial}{\partial x}[F(\overline{u}(x,t))] \leq 0 \ . \tag{6.9b}$$

Proof: The amount of entropy dissipated by (1.4),

$$- < \frac{\partial}{\partial x} U'(u_N(x,t)) \ , \ E^f_N u_N(x,t) >^+$$

tends weakly to a negative measure in view of the $L^1_{\text{loc}}(x,t)$ bound in (3.11). Adding this to (6.9a) we conclude that the second term on the right of (6.2), tends weakly to a negative measure,

$$w \lim_{N\to\infty} [II \equiv - < \frac{\partial}{\partial x} U'(u_N(x,t)) \ , \ E^f_N u_N(x,t) >] \leq 0 \ . \tag{6.10}$$

Thus, in view of (6.3), (6.10) and the strong convergence, $u_N(x,t) \to \overline{u}(x,t)$, the weak limit of (6.2) recovers the entropy inequality (6.9b). □

Example 6.4: Convergence of the SV methods. We consider the SV method (4.17), (4.23)

$$(6.11a)\quad \frac{\partial}{\partial t}[u_N(x,t)] + \frac{\partial}{\partial x}[P_N f(u_N(x,t)] = \frac{1}{N^\beta}\frac{\partial}{\partial x}[Q_N(x,t) * \frac{\partial}{\partial x}u_N(x,t)]\ , \qquad \beta < \frac{1}{2}\ ,$$

where $Q_N(x,t) = \sum_{|k|=m_N}^{N} \hat{Q}_k(t)e^{ikx}$ is a viscosity kernel with monotonically increasing coefficients,

(6.11b)

$$\hat{Q}_k(t) \geq Const. - \frac{N^\beta}{k^2 \cdot \log N}\quad ,\quad k^2 \geq m_N^2 = \frac{N^\beta}{Const. \log N}\quad ,\quad Const. > Const_0\ .$$

The SV method (6.11a), (6.11b) satisfies

(i) It is consistent with the scalar initial-value (1.1a), (1.1b) w.r.t. all strictly convex entropies (by Theorem 4.3),

(ii) It is L^∞-stable (by Theorem 5.2),

and as a consequence of (i),(ii) and the first part of Theorem 6.3,

(iii) The SV solution, $u_N(x,t)$, converges strongly to a weak solution, $\overline{u}(x,t)$, of the nonlinear scalar conservation law (1.1a).

(iv) The entropy produced by the basic approximation of (6.11a) tends strongly to zero (by Theorem 4.3 III).

(v) The entropy produced by the viscosity approximation of (6.11a) tends weakly to zero, consult [10, Section 5].

As a consequence of (iii), (iv), (v) and the second part of Theorem 6.3 we conclude:

The SV method (6.11a), (6.11b) converges strongly to the unique entropy solution of the nonlinear scalar conservation law (1.1).

Remarks:

1. The last conclusion extends the convergence results of [20], [10] which was restricted to the inviscid Burgers' equation where $f(u) = \frac{1}{2}u^2$.

2. The L^∞-stability of the SV method (6.11a), (6.11b) with $\beta < 1$, would imply its convergence along the lines of Example 6.4.

Example 6.5: Convergence of Centered Finite-Difference Methods. We consider the conservative finite-difference/element method (4.27)

$$(6.12a)\quad \begin{aligned} &\tfrac{d}{dt}u_N(x_\nu,t) + \tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t))] = \\ &= \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)]\ , \end{aligned}$$

with numerical viscosity coefficients, $Q_{\nu+\frac{1}{2}} \leq Q_\infty$, such that

$$(6.12b) \qquad Q_{\nu+\frac{1}{2}} \geq Q^*_{\nu+\frac{1}{2}} + Const. \quad , \quad Q^*_{\nu+\frac{1}{2}} = 2\int_{\xi=-1/2}^{1/2} \xi f'(u_{\nu+\frac{1}{2}}(\xi))d\xi \ .$$

This method is consistent with the scalar initial-value problem (1.1a,(1.1b) w.r.t. all strictly convex entropies (by Theorem 4.5). Since (6.12a), (6.12b) is also known to be L^∞-stable, e.g., [14], the first part of Theorem 6.3 implies strong $L^p_{\text{loc}}(x,t)$ convergence to a weak limit solution, $\overline{u(x,t)}$ of (1.1a). Finally, since (6.11a), (6.11b) contains more numerical viscosity than the entropy conservative schemes (1.24b), then $\overline{u}(x,t)$ satisfies (6.9b), (consult [18] 21]). We conclude:

The finite-difference method (6.12a), (6.12b) converges strongly to the unique entropy solution of the nonlinear scalar conservation law (1.1).

Example 6.6: Convergence of Upwind Differencing. We consider the conservative finite-difference/element method

$$(6.13a) \qquad \begin{aligned} &\tfrac{d}{dt}\left[\tfrac{1}{6}u_N(x_{\nu-1},t) + \tfrac{4}{6}u_N(x_\nu,t) + \tfrac{1}{6}u_N(x_{\nu+1},t)\right] + \\ &+\tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t)] = \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)] \ , \end{aligned}$$

with numerical viscosity coefficients, $Q_{\nu+\frac{1}{2}} \leq Q_\infty$, such that

$$(6.13b) \qquad Q_{\nu+\frac{1}{2}} \geq Const.|\overline{a}_{\nu+\frac{1}{2}}| \ , \qquad \overline{a}_{\nu+\frac{1}{2}} \equiv \left[\frac{1}{\Delta x}\int_{x_\nu}^{x_{\nu+1}} |f'(u_N(x,t))|^2 dx\right]^{1/2} \ .$$

This method is consistent with the scalar initial-value problem (1.1a), (1.1b) w.r.t. the quadratic entropy $\mathcal{U} = \{\frac{1}{2}u^2\}$ (by Theorem 4.6). Since (6.13a), (6.13b) is known to be L^∞-stable, e.g., [14], the first part of Theorem 6.3 implies strong $L^p_{\text{loc}}(x,t)$ convergence to a weak limit solution, $\overline{u(x,t)}$, of the nonlinear conservation law (1.1a). Moreover, the quadratic entropy produced by the basic approximation of (6.13a) tends strongly (in $H^{-1}_{\text{loc}}(x,t)$) to zero, while the viscosity approximation is purely dissipative, consult Theorem 4.6. Hence with the help of the second part of Theorem 6.3 we conclude:

The finite-difference/element method (6.13a), (6.13b) converges strongly to the unique entropy solution of the GNL scalar conservation law (1.1).

Example 6.7: Convergence of Second-Order Differencing. We consider the conservative finite-difference/element method

$$(6.14a) \qquad \begin{aligned} &\tfrac{d}{dt}\left[\tfrac{1}{6}u_N(x_{\nu-1},t) + \tfrac{4}{6}u_N(x_\nu,t) + \tfrac{1}{6}u_N(x_{\nu+1},t)\right] + \\ &+\tfrac{1}{2\Delta x}[f(u_N(x_{\nu+1},t)) - f(u_N(x_{\nu-1},t)] = \tfrac{1}{2\Delta x}[Q_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}}(t) - Q_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}(t)] \ , \end{aligned}$$

with numerical viscosity coefficients, $Q_{\nu+\frac{1}{2}} \leq Q_\infty$, such that

$$(6.14b) \quad Q_{\nu+\frac{1}{2}} \geq Const.|\overline{a}'_{\nu+\frac{1}{2}}| \cdot |\Delta u_{\nu+\frac{1}{2}}(t)| \,, \qquad \overline{a}'_{\nu+\frac{1}{2}} \equiv \left[\frac{1}{\Delta x}\int_{x_\nu}^{x_\nu} |f''(u_N(x,t))|^2 dx\right]^{1/2} .$$

This method is consistent with the scalar initial-value problem (1.1a), (1.1b) w.r.t. the quadratic entropy $\mathcal{U} = \{\frac{1}{2}u^2\}$ (by Theorem 4.7). Now, if (6.14a), (6.14b) is L^∞-stable, then the first part of Theorem 6.3 implies strong $L^p_{\text{loc}}(x,t)$ convergence to a weak limit solution, $\overline{u(x,t)}$, of the nonlinear conservation law (1.1a). Moreover, the quadratic entropy produced by the basic approximation of (6.14a) tends strongly (in $H^{-1}_{\text{loc}}(x,t)$) to zero, while the viscosity approximation is purely dissipative, consult Theorem 4.7. Hence with the help of the second part of Theorem 6.3 we conclude:

The finite-difference/element method (6.14a), (6.14b) converges strongly to the unique entropy solution of the GNL scalar conservation law (1.1), provided it is L^∞-stable.

Remarks:

1. The conclusions of Examples 6.6 and 6.7 remain valid with or without mass lumping on the left of (6.13a), (6.14a).

2. The question of L^∞-stability for the second-order accurate approximation (6.14) remains open.

We conclude this section with convergence results for strictly hyperbolic GNL systems of two conservation laws. Making use of Theorem 6.1 and DiPerna's results in [1] we have

Theorem 6.8: *Consider an L^∞-stable approximation* (1.4) *which is consistent with the 2×2 initial-value problem* (1.1*a*), (1.1*b*), *w.r.t. a nonempty "relevant" class of strictly convex entropy function $\mathcal{U}$. Then we have*

I. *If $\|u_N(x,t)\|_{L^\infty_{\text{loc}}(x,t)}$ is sufficiently small then $u_N(x,t)$ converges a.e. to an admissible (i.e., entropy satisfying) solution, $\overline{u}(x,t)$, of* (1.1*a*).

II. *If* (1.1*a*) *is equipped with quasi-convex Riemann invariants and* (5.6) *holds, then $u_N(x,t)$ converges a.e. to an admissible solution $\overline{u}(x,t)$, of* (1.1*a*).

Examples of utilizing Theorem 6.8 in the context of the vanishing viscosity method for 2×2 systems will be given elsewhere; the consistency analysis in such cases can be carried out more conveniently in terms of the entropy variables, v, instead of the conservative variables, u [21].

7. APPENDIX: CONSISTENCY WITH THE CONSERVATION LAW

In this section we study the order of accuracy for various types of discretizations, including spectral and pseudospectral methods, as well as arbitrary recipes of finite-difference and finite-element methods which admit the general conservative form (1.14), (1.15). It is shown that all these methods are at least first-order accurate and hence consistent approximations of the conservation law (1.1a).

We start with

Theorem 7.1: *The spectral method (1.10), (1.11) is infinitely-order accurate, i.e., accurate of any order $s > 0$.*

Proof: With $\hat{f}_k(t)$ denoting the k-th Fourier coefficient of $f(w_N(x))$, we have for <u>any</u> integer $r > 0$,

$$(7.1)\quad \|(I - S_N)f(w_N(x))\|^2 = 2\pi \sum_{|k|>N} |\hat{f}_k|^2 \le \frac{2\pi}{N^{2r}} \sum_{k=-\infty}^{\infty} |k|^{2r}|\hat{f}_k|^2 = \frac{1}{N^{2r}}\|\frac{\partial^r}{\partial x^r} f(w_N(x))\|^2,$$

and (1.6) follows with $C_0 = \|f'(w_N(x))\|_{L^\infty(x)}$. □

<u>Remark:</u> The linear case $f(u) \equiv u$ shows that the above accuracy estimate is indeed sharp. We note that this estimate makes use of the L^∞-bound of $u_N(x,t)$. In the quadratic scalar case, $f(u) = \frac{1}{2}u^2$, a weaker accuracy estimate of order $s = \frac{1}{2}$ was proved in [20, Lemma 3.1], independently of L^∞-bounds.

We turn to the pseudospectral method (1.12),(1.13) where $P_N^f w_N(x) = \psi_N f(w_N(x))$ is the π_N-interpolation of $f(w_N(x))$ at $x_{\nu+\theta} = (\nu+\theta)\Delta x$, $\nu = 0, 1, \cdots, 2N$ with fixed $0 \le \theta < 1$. We recall that

$$(7.2a)\qquad \psi_N = S_N + A_N,$$

where the aliasing projection, A_N, is given by

$$(7.2b)\qquad A_N w(x) = \sum_{|k|\le N} [\sum_{j\neq 0} \hat{w}(k + j(2N+1))e^{ij2\pi\theta}]e^{ikx}.$$

To treat the pseudospectral case, we first prepare

Lemma 7.2: *There exist constants (independent of N), C_r, such that for any $r > \frac{1}{2}$ we have*

$$(7.3)\qquad \|A_N w(x)\| \le C_r \cdot \frac{1}{N^r}\|(I - S_N)\frac{\partial^r}{\partial x^r} w(x)\|, \qquad r > \frac{1}{2}.$$

Proof: Using the aliasing relation (7.2b) we have

$$\|A_N w(x)\|^2 = \sum_{|k|\le N} |\sum_{j\neq 0} \hat{w}(k+j(2N+1)e^{ij2\pi\theta}|^2 \le$$

(7.4)

$$\le \sum_{|k|\le N}[\sum_{j\neq 0} \tfrac{1}{|j|^{2r}N^{2r}}] \cdot \sum_{j\neq 0} |j|^{2r}N^{2r}|\hat{w}(k+j(2N+1)|^2].$$

Since $|j|N \le |k+j(2N+1)|$ for $|k| \le N, j \neq 0$, the right-hand side of (7.4) does not exceed

$$\sum_{j\neq 0} \frac{1}{|j|^{2r}N^{2r}} \sum_{|k|\le N} \sum_{j\neq 0} |k+j(2N+1)|^{2r}|\hat{w}|(k+j(2N+1)|^2 =$$

$$= \frac{1}{N^{2r}} \cdot \sum_{j\neq 0} \frac{1}{|j|^{2r}} \cdot \sum_{|\ell|\ge N} |\ell|^{2r}|\hat{w}(\ell)|^2,$$

and (7.3) follows with $C_r = (\sum_{j\neq 0} \frac{1}{|j|^{2r}})^{\frac{1}{2}}$. □

Equipped with Lemma 7.2, we can show

Theorem 7.3: *The pseudospectral method (1.12), (1.13) is "infinitely-order" accurate, i.e., accurate of any order $s > \frac{1}{2}$.*

Proof: Since A_N and $I - S_N$ are orthogonal projections, we have

(7.5) $$\|(I-\psi_N)f(w_N(x))\|^2 = \|A_N f(w_N(x))\|^2 + \|(I-S_N)f(w_N(x))\|^2;$$

by Lemma 7.2, the first term on the right does not exceed

$$\|A_N f(w_N(x))\|^2 \le C_r^2 \cdot \frac{1}{N^{2r}} \|(I-S_N)\frac{\partial^r}{\partial x^r} f(w_N(x))\|^2 \le$$

$$C_r^2 \frac{1}{N^{2r}} \|\frac{\partial^r}{\partial x^r} f(w_N(x))\|^2,$$

and together with upper bound of the second term from Theorem 7.1, the accuracy estimate (1.6) follows with $C_0 = (1+C_r)\|f'(w_N(x))\|_{L^\infty(x)}$. □

Next, we consider general conservative discretizations of the form (1.14), (1.15). These approximations can be interpreted as evolution schemes in π_N governed by (1.3), with P_N^f which is defined according to (1.17),

(7.6a) $$P_N^f w_N(x) = \sum_{k=-N}^{N} \hat{p}_k e^{ikx}, \qquad \hat{p}_k = \frac{1}{2\pi ik} \sum_{\nu=0}^{2N} [h^f_{\nu+\frac{1}{2}} - h^f_{\nu-\frac{1}{2}}] e^{-ikx_\nu}.$$

Here $h^f_{\nu+\frac{1}{2}} \equiv h^f(w_N(x_{\nu-p+1}), \cdots, w_N(x_{\nu+p}))$ is the Lipschitz continuous numerical flux which is consistent with the differential one

(7.6b) $$h^f(w_N(x_\nu), \cdots, w_N(x_\nu)) \equiv f(w_N(x_\nu)).$$

We claim that such schemes are at least first-order accurate with (1.1a). To see this we first prepare

Lemma 7.4: *Let*

$$\overline{w}(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} w(\xi)d\xi, \qquad \Delta x = \frac{2\pi}{2N+1}, \tag{7.7}$$

denotes the sliding average of a 2π-periodic function $w(x)$. Then we have

$$\|\overline{w}(x)\| \le \|w(x)\|, \tag{7.8a}$$

$$\|w(x) - \overline{w}(x)\| \le Const.\frac{1}{N^r}\|\frac{\partial^r w}{\partial x^r}\|, \qquad r \le 2. \tag{7.8b}$$

Proof: Using the obvious inequality (7.8a) we have

$$\|w(x) - \overline{w}(x)\| \le 2\|w(x) - S_N w(x)\| + \|S_N w(x) - \overline{S_N w}(x)\|. \tag{7.9}$$

By Theorem 7.1, the first term on the right is bounded by $\frac{2}{N^r}\|\frac{\partial^r w}{\partial x^r}\|$. Also, for $w_N(x) \equiv S_N w(x)$ we have

$$\overline{w}_N(x) = \sum_{0\le|k|\le N} \hat{w}(k)\cdot\frac{\sin k\frac{\Delta x}{2}}{k\frac{\Delta x}{2}}\cdot e^{ikx}, \qquad w_N(x) = \sum_{k=-N}^{N}\hat{w}(k)e^{ikx}; \tag{7.10}$$

hence the second term on the right of (7.9) does not exceed

$$\|w_N(x) - \overline{w}_N(x)\| \le \sum_{0<|k|\le N}|\hat{w}(k)|^2|1 - \frac{\sin k\frac{\Delta x}{2}}{k\frac{\Delta x}{2}}|^2 \le \frac{1}{6}(\Delta x)^4\sum_k |k|^4|\hat{w}(k)|^2, \tag{7.11}$$

and (7.8b) follows with $Const = 2 + \frac{\pi^2}{6}$.□

We arrive at

Theorem 7.5: *The conservative approximation (1.14), (1.15) is at least first-order accurate with the conservation law (1.1a).*

Proof: Let us recall (1.16a) where $H_N(x)$ denotes the N-trigonometric polynomial which interpolates $H_N(x_\nu) = h^f_{\nu+\frac{1}{2}}$, i.e.,

$$H_N(x) = \sum_{k=-N}^{N}\hat{H}_k e^{ikx}, \qquad \hat{H}_k = \frac{\Delta x}{2\pi}\sum_{\nu=0}^{2N} h^f_{\nu+\frac{1}{2}}e^{-ikx_{\nu+\frac{1}{2}}}. \tag{7.12}$$

We then have

$$\|f(w_N(x)) - P^f_N w_N(x)\| \le I + II + III + IV. \tag{7.13}$$

By Theorem 7.1 we have

$$(7.14) \qquad I \equiv \|f(w_N(x)) - S_N f(w_N(x))\| \le \frac{1}{N^r}\|\frac{\partial^r}{\partial x^r} f(w_N(x))\| \,, \qquad r \le 2 \,.$$

Lemma 7.4 implies

$$(7.15) \qquad II = \|S_N f(w_N(x)) - \overline{S_N f}(w_N(x))\| \le Const.\frac{1}{N^r}\|\frac{\partial^r}{\partial x^r} f(w_N(x))\| \,, \qquad r \le 2 \,.$$

Let $\psi_N f(w_N(x))$ denotes the π_N-interpolant of $f(w_N(x))$ at $x_{\nu+\frac{1}{2}} = (\nu + \frac{1}{2})\Delta x$, $\nu = 0, 1, \cdots, 2N$. Lemma 7.2 and 7.4 imply

(7.16)

$$III \equiv \|\overline{S_N f}(w_N(x)) - \overline{\psi_N f}(w_N(x))\| \le \|A_N f(w_N(x))\| \le Const.\frac{1}{N^r}\|\frac{\partial^r}{\partial x^r} f(w_N(x))\| \,, \qquad r \le 2$$

Finally, we recall (1.16b) asserting that $P_N^f w_N(x)$ is nothing but the sliding average of $H_N(x)$, i.e.,

$$(7.17) \qquad P_N^f w_N(x) = \overline{H_N}(x).$$

Consequently, for the fourth term on the right of (7.13) we have

$$(7.18) \qquad \begin{aligned} IV \equiv \|\overline{\psi_N f}(w_N(x)) - P_N^f w_N(x)\| \le \|\psi_N f(w_N(x)) - H_N(x)\| = \\ = [\textstyle\sum_{\nu=0}^{2N} |f(w_N(x_{\nu+\frac{1}{2}})) - h^f_{\nu+\frac{1}{2}}|^2 \Delta x]^{\frac{1}{2}}. \end{aligned}$$

Using the Lipshitz continuity of the consistent flux $h^f_{\nu+\frac{1}{2}}$ we can upper bound

$$|f(w_N(x_{\nu+\frac{1}{2}})) - h^f_{\nu+\frac{1}{2}}|^2 \le L^2[\sum_{\ell=-p}^{p-1} |w_N(x_{\nu+\ell}) - w_N(x_\nu)|]^2 \le 2pL^2 \cdot \sum_{\ell=-p}^{p-1} |w_N(x_{\nu+\ell}) - w_N(x_\nu)|^2,$$

and hence the right-hand side of (7.18) does not exceed

$$(7.19) \quad IV \le (2p \cdot L^2 \cdot \sum_{\ell=-p}^{p-1} \sum_{\nu=0}^{2N} |w_N(x_{\nu+\ell}) - w_N(x_\nu)|^2 \Delta x)^{\frac{1}{2}} \le 2pL \cdot Const.\Delta x \|\frac{\partial}{\partial x} w_N(x)\|,$$

and the first-order accuracy estimate follows from (7.13) in view of (7.14), (7.15), (7.16), and (7.19).□

We observe that the first three terms on the right of (7.13) are in fact <u>second</u> order accurate, and hence the second-order accuracy of the conservative approximation (1.14), (1.15) depends on the ℓ_2-distance, (7.18), between the numerical flus $h^f_{\nu+\frac{1}{2}}$, and the midvalues $f(w_N(x_{\nu+\frac{1}{2}}))$. Making use of the detailed structure of $h^f_{\nu+\frac{1}{2}}$ in (4.26) we conclude

Corollary 7.6: *The conservative approximation* (4.27) *is second-order accurate with the conservation law* (1.1*a*), *provided its viscosity coefficients satisfy*

$$\|Q_{\nu+\frac{1}{2}}\| \leq Const.|\Delta w_{\nu+\frac{1}{2}}| \ , \qquad Q_{\nu+\frac{1}{2}} = Q_{\nu+\frac{1}{2}}(w_N(x), x) \ . \tag{7.20}$$

Proof: Inserting (4.26) into (7.18) we obtain

(7.21*a*)

$$IV = [\sum_{\nu=0}^{2N} |f(w_N(x_{\nu+\frac{1}{2}})) - \tfrac{1}{2}(f(w_N(x_\nu)) + f(w_N(x_{\nu+1}))) + \tfrac{1}{2}Q_{\nu+\frac{1}{2}}\Delta w_{\nu+\frac{1}{2}}|^2 \Delta x]^{1/2} \leq$$

$$\leq (\Delta x)^2 \left[Const.\|\tfrac{\partial^2 w_N}{\partial x^2}\| + (\sum_{\nu=0}^{2N} |\tfrac{1}{(\Delta x)^2} Q_{\nu+\frac{1}{2}}\Delta w_{\nu+\frac{1}{2}}|^2 \Delta x)^{1/2}\right] \ .$$

Assumption (7.20) implies that the righthand-side of (7.21a) does not exceed

$$\frac{1}{N^2}\left[C_0 \cdot \|\frac{\partial^2}{\partial x^2} w_N(x)\| + C_1 \cdot \|\frac{\partial}{\partial x} w_N(x)\|\right] \ , \tag{7.21b}$$

and the result follows from (7.13) in view of (7.14),(7.15),(7.16) and (7.21). □

The essentially nonlinear high-resolution approximations surveyed in, e.g., [3],[15], are characterized by viscosity coefficients, $Q_{\nu+\frac{1}{2}}$, which satisfy (7.21) at all but a <u>finite</u> number of gridcells. In this case we have, in agreement with [3],

Corollary 7.7: *The high resolution approximations* (4.27), *whose viscosity coefficients* $Q_{\nu+\frac{1}{2}}$ *satisfy* (7.21) *at all but a finite number of critical gridcells, are accurate of order* $s = 3/2$ *with the conservation law* (1.1a).

Verification of this corollary is immediate noting that the contribution of the <u>finite</u> number of critical gridcells to the summation in (7.21a) is of order $(\Delta x)^{3/2}$.

ACKNOWLEDGMENTS I would like to thank Steve Shochet for stimulating discussions.

REFERENCES

[1] R. J. DiPerna, "Convergence of approximation solutions to conservation laws," Arch. Rational Mech. Anal., v. 82, 1983, pp. 27-70.

[2] A. Harten, "High resolution schemes for hyperbolic conservation laws," J. Comput. Phys., v. 49, 1983, pp. 357-393.

[3] A. Harten, "On the nonlinearity of modern shock-capturing schemes," in Wave Motion: Theory, Modeling and Computation, Mathematical Sciences Research Institute Publications, Vol. 7, (A. Chorin & A. Majda, eds.), Springer-Verlag, 1987, pp. 147-201.

[4] C. Johnson & A. Szepessy, "Convergence of a finite element method for a nonlinear hyperbolic conservation law," Math. Comp., v. 49, 1988, pp. 427-444.

[5] M. Kac & P. van Moerbeke, "On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices," Adv. in Math., v. 16, 1975, pp.160-169.

[6] S. N. Kružhkov, "First-order quasilinear equations in several independent variables," Math. USSR-Sb., v. 10, 1970, pp. 217-243.

[7] P. D. Lax, Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves. SIAM Regional Conference Lectures in Applied Mathematics, No. 11, 1972.

[8] P. D. Lax, "On dispersive difference schemes," Physica 18D, 1986, pp. 250-254.

[9] P. D. Lax & B. Wendroff, "Systems of conservation laws," Comm. Pure Appl. Math., v. III, 1960, pp. 217-237.

[10] Y. Maday & E. Tadmor, "Analysis of the spectral vanishing viscosity method for periodic conservation laws," SIAM J. Numer. Anal., to appear.

[11] M. S. Mock, "Systems of conservation laws of mixed type," J. Differential Equations, v. 37, 1980, pp. 70-88.

[12] J. Moser, "Three integrable Hamiltonian systems connected with isospectral deformations," Advances in Math., v. 16, pp. 197-220.

[13] S. Osher, "Riemann solvers, the entropy condition, and difference approximations," SIAM J. Numer. Anal., v. 21, 1984, pp. 217-235.

[14] S. Osher, "Convergence of generalized MUSCL schemes," SIAM J. Numer. Anal, v. 22, 1984, pp. 947-961.

[15] S. Osher & E. Tadmor, "On the convergence of difference approximations to scalar conservation laws," Math. Comp., v. 50, 1988, pp. 19-51.

[16] S. Schochet, "The instant-response limit in Whitham's nonlinear traffic-flow model: uniform well-posedness and global existence," Asymptotic Analysis, to appear.

[17] E. Tadmor, "Numerical viscosity and the entropy condition for conservative difference schemes," Math. Comp., v. 43, 1984, pp. 369-381.

[18] E. Tadmor, "The numerical viscosity of entropy stable schemes for systems of conservation laws. I," Math. Comp., v. 49, 1987, pp. 91-103.

[19] E. Tadmor, "Stability analysis of finite-difference, Pseudospectral and Fourier-Galerkin approximation for time-dependent problems," SIAM Review, v. 29, 1987, pp. 525-555.

[20] E. Tadmor, "Convergence of spectral methods for nonlinear conservation laws," SIAM J. Numer. Anal., to appear.

[21] E. Tadmor, "The numerical viscosity of entropy stable schemes for systems of conservation laws. II," Math. Comp., to be submitted.

[22] L. Tartar, "Compensated compactness and applications to partial differential equations," in Research Notes in Mathematics 39, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, v. 4 (R.J. Knopps, ed.), Pittman Press (1975), pp. 136-211.

[23] A. Zygmund, Trigonometric Series, Vols. I and II, Cambridge Univ. Press, Cambridge, 1968.

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA CR-181683 ICASE Report No. 88-41		

4. Title and Subtitle	5. Report Date
SEMI-DISCRETE APPROXIMATIONS TO NONLINEAR SYSTEMS OF CONSERVATION LAWS; CONSISTENCY AND L^∞-STABILITY IMPLY CONVERGENCE	July 1988
	6. Performing Organization Code
7. Author(s) Eitan Tadmor	8. Performing Organization Report No. 88-41
	10. Work Unit No. 505-90-21-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665-5225	11. Contract or Grant No. NAS1-18107
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225	13. Type of Report and Period Covered Contractor Report
	14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Richard W. Barnwell

Submitted to Math. Comp.

Final Report

16. Abstract

We develop a convergence theory for semi-discrete approximations to nonlinear systems of conservation laws. We show, by a series of scalar counterexamples, that consistency with the conservation law alone does not guarantee convergence. Instead, we introduce a notion of consistency which takes into account both the conservation law and its augmenting entropy condition. In this context, we conclude that consistency and L^∞-stability guarantee for a "relevant" class of admissible entropy functions, that their entropy production rate belong to a compact subset of $H^{-1}_{\text{loc}}(x,t)$. One can use now compensated compactness arguments in order to turn this conclusion into a convergence proof. The current state of the art for these arguments includes the scalar and a wide class of 2×2 systems of conservation laws.

We study the general framework of the vanishing viscosity method as an effective way to meet our consistency and L^∞-stability requirements. We show how this method is utilized to enforce consistency and L^∞-stability for scalar conservation laws. In this context, we prove under the appropriate assumptions (L^∞-bounds), the convergence of finite-difference approximations (e.g., the high-resolution TVD and UNO methods), finite-element approximations (e.g., the Streamline-Diffusion methods) and spectral and pseudospectral approximations (e.g., the Spectral Viscosity methods).

17. Key Words (Suggested by Author(s))	18. Distribution Statement
conservation laws, entropy condition, finite-difference methods, finite-element methods, spectral methods, consistency, stability, convergence	64 - Numerical Analysis Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	62	A04

NASA FORM 1626 OCT 86

NASA-Langley, 1988

End of Document