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Finite-Dimensional Approximation for Optimal Fixed-Order Compensation of Distributed Parameter Systems

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Abstract

In controlling distributed parameter systems it is often desirable to obtain low-order, finite-dimensional controllers in order to minimize real-time computational requirements. Standard approaches to this problem employ model/controller reduction techniques in conjunction with LQG theory. In this paper we consider the finite-dimensional approximation of the infinite-dimensional Bernstein/Hyland optimal projection theory. Our approach yields fixed-finite-order controllers which are optimal with respect to high-order, approximating, finite-dimensional plant models. We illustrate the technique by computing a sequence of first-order controllers for one-dimensional, single-input/single-output, parabolic (heat/diffusion) and hereditary systems using spline-based, Ritz- Galerkin, finite element approximation. Our numerical studies indicate convergence of the feedback gains with less than 2% performance degradation over full-order LQG controllers for the parabolic system and 10% degradation for the hereditary system.

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1. Introduction

Approximation methods for the optimal control of distributed parameter systems have been widely studied. In particular, the approach taken in [1–12] involves approximating the original distributed parameter system by a sequence of finite-dimensional systems and then using finite-dimensional control-design techniques to obtain a sequence of approximating, sub-optimal control laws, observers, or compensators. Furthermore, in these treatments it was demonstrated that if the open-loop system is approximated appropriately, then it is possible to guarantee convergence of the sequence of sub-optimal controllers, observers, or compensators, respectively, to the optimal controller, observer, or compensator for the original infinite-dimensional system. In addition, it can also be shown that when the approximating sub-optimal control laws or estimators are applied to the original system, near-optimal performance can frequently be obtained. These ideas were pursued in the context of both open- and closed-loop control, in both continuous and discrete-time, and for both full-state-feedback control and LQG (i.e., Kalman-filter-based) state estimation and compensation.

In practical situations, however, it is often of interest to obtain the simplest (i.e., the lowest order) controller which provides a given, desired feedback performance. This is usually achieved in one of two ways. Either the plant approximation order is reduced prior to controller design, or, alternatively, reduction techniques are applied to a given high-order control law. Unfortunately, the former approach may result in undesirable spillover effects while the latter may yield low-order controllers of low authority which perform unacceptably. In fact, with the second approach, this may occur even when a suitable controller is known to exist. For example, as is shown in [13], controller reduction techniques may even destabilize the closed-loop system.

A third, more direct approach involves fixing the controller order a priori, and then optimizing a performance criterion over the class of fixed-order controllers. In a finite-dimensional setting, a set of necessary conditions in the form of four coupled matrix equations (as a direct extension of the pair of the separated Riccati equations of LQG theory) which characterize the optimal fixed-order compensator was derived in [14]. These four equations are coupled via an oblique projection (idempotent) matrix. In the full-order case, this projection becomes the identity thus effectively eliminating the additional two equations, and the necessary conditions reduce to the standard LQG Riccati equations.

The notion that this direct (i.e., fixed-finite-order) approach can be applied to distributed parameter systems was first suggested by Johnson in [15] and further developed in [16] and [17]. To realize such an approach, however, would require a suitable generalization of the optimality conditions for the finite-dimensional fixed-order theory. This result was subsequently obtained in [18] where the matrix optimal projection equations obtained in [14] for finite-dimensional systems were extended to a set of four coupled *operator* Riccati and Lyapunov equations characterizing optimal fixed-finite-order controllers for infinite-dimensional systems.

In developing numerical schemes to actually compute fixed-finite-order compensators for infinite-dimensional systems, one might consider an approach wherein LQG reduction procedures are applied to a sequence of controllers obtained by using finite-dimensional full-order design techniques in conjunction with high-order finite-dimensional plant approximations. However, such an approach is unappealing for two reasons. First, since such methods are not predicated on the minimization of a performance index, prospects for convergence are slim. And, second, controller-reduction methods have not proven to be reliable in producing stabilizing compensators (see, for example, [13]).

Hence, on the other hand, we develop an abstract approximation framework (and ultimately computational schemes) which combine the infinite-dimensional optimal projection theory of [18] with the approximation ideas developed in [9–12] for infinite-dimensional LQG problems. More precisely, our approach involves constructing a sequence of approximating finite-dimensional subspaces of the original, underlying, infinite-dimensional Hilbert state space along with corresponding sequences of bounded linear operators which approximate the given input, output, and system operators. Then, by choosing bases for these approximating subspaces and applying the finite-dimensional optimal projection theory, a sequence of matrix equations characterizing a sequence of approximating optimal, fixed-finite-order compensators for the distributed system is obtained. Finally, numerical techniques for solving the matrix optimal projection equations (for example, the homotopic continuation algorithm described in [19] and [20]) can be used to compute the sequence of approximating gains.

Our primary aim in this paper is to describe the general approach we are proposing, to discuss its implementation, and to demonstrate its feasibility and practicality. We offer no convergence arguments here, but rather reserve them for a more theoretical paper to follow. Instead, we consider the application of our technique to two examples. One involves the control of a one-dimensional,

single-input, single-output parabolic (heat/diffusion) system while the other involves a single-input single-output one-dimensional hereditary control system. These relatively simple examples have been used throughout the distributed parameter control literature to illustrate the application of new theories and techniques. A detailed discussion of the application of our ideas to more complex control systems, for instance, the vibration control of flexible structures, will also appear elsewhere. We use spline- based Ritz-Galerkin finite element schemes to approximate the open-loop systems (one for which convergence can be demonstrated in the LQG case) and present and discuss some of the numerical results which we have obtained using our general approximation framework.

We now outline the remainder of the paper. In Section 2 we briefly review the infinite-dimensional optimal projection theory from [18], describe the approximation framework, and derive the corresponding equivalent matrix equations and feedback gains which characterize the approximating fixed-finite-order compensator. In Section 3 we consider the examples, construct the approximation schemes, and discuss our numerical findings. Section 4 contains a summary and some concluding remarks.

2. Optimal Projection Theory and Finite-Dimensional Approximation

We consider the following fixed-finite-order dynamic-compensation problem. Given the infinite-dimensional control system

$$\dot{x}(t) = Ax(t) + Bu(t) + H_1 w(t) \tag{2.1}$$

with measurements

$$y(t) = Cx(t) + H_2 w(t), \tag{2.2}$$

where $t \in [0, \infty)$, design a finite-dimensional, n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \tag{2.3}$$

$$u(t) = C_c x_c(t) \tag{2.4}$$

which minimizes the steady-state performance criterion

$$J(A_c, B_c, C_c) \triangleq \lim_{t\to\infty} \mathbb{E}\big[\langle R_1 x(t), x(t)\rangle + u(t)^{\mathrm{T}} R_2 u(t)\big]. \tag{2.5}$$

For convenience we denote the infinite-dimensional plant by Π; that is,

$$\Pi \triangleq \{A, B, C, R_1, R_2, V_1, V_2\}.$$

Here $x(t)$ lies in a real, separable Hilbert space $\mathcal{X}$ with inner product $\langle\cdot,\cdot\rangle, A : \mathrm{Dom}\,(A) \subset \mathcal{X} \to \mathcal{X}$ is a closed, densely defined operator which generates a C_0 semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators on $\mathcal{X}, B \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$, and $C \in \mathcal{L}(\mathcal{X}, \mathbb{R}^\ell)$. We assume that the state and measurement are corrupted by a white noise signal $w(t)$ in a real, separable Hilbert space $\hat{\mathcal{X}}$ (see [21] or [22]), that $H_1 \in \mathcal{L}(\hat{\mathcal{X}}, \mathcal{X}), H_2 \in \mathcal{L}(\hat{\mathcal{X}}, \mathbb{R}^\ell), R_1 \in \mathcal{L}(\mathcal{X})$ is (self-adjoint) nonnegative definite, and that R_2 is an $m \times m$ (symmetric) positive-definite matrix. We define $V_1 = H_1 H_1^*$ and $V_2 = H_2 H_2^*$, where $(\ )^*$ denotes adjoint, and assume for convenience that $H_1 H_2^* = 0$ and that V_2 is positive definite. The compensator is assumed to be of fixed, finite order n_c (i.e., $x_c(t) \in \mathbb{R}^{n_c}$) and that A_c, B_c, and C_c are matrices of appropriate dimension. For further details and discussion on the problem statement and the above assumptions, see [18].

We summarize here the primary result from [18] characterizing optimal fixed-finite-order controllers. For convenience define $\Sigma \triangleq BR_2^{-1}B^*$ and $\bar{\Sigma} \triangleq C^*V_2^{-1}C$. Also let I_{n_c} and $I_{\mathcal{X}}$ denote respectively the $n_c \times n_c$ identity matrix and the identity operator on $\mathcal{X}$.

Theorem 2.1. Let n_c be given and suppose there exists a controllable and observable n_cth-order dynamic compensator (A_c, B_c, C_c) which minimizes J given by (2.5) and for which the closed-loop semigroup generated by

$$\mathcal{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$$

is uniformly exponentially stable. Then there exist nonnegative-definite operators $Q, P, \hat{Q}, \hat{P}$ on $\mathcal{X}$ such that A_c, B_c, and C_c are given by

$$A_c = \Gamma(A - Q\bar{\Sigma} - \Sigma P)G^*, \tag{2.6}$$

$$B_c = \Gamma Q C^* V_2^{-1}, \tag{2.7}$$

$$C_c = -R_2^{-1}B^*PG^*, \tag{2.8}$$

where $Q : \mathrm{Dom}\,(A^*) \to \mathrm{Dom}\,(A), P : \mathrm{Dom}\,(A) \to \mathrm{Dom}\,(A^*), \hat{Q} : \mathcal{X} \to \mathrm{Dom}\,(A), \hat{P} : \mathcal{X} \to \mathrm{Dom}\,(A^*)$, and $G, \Gamma \in \mathcal{L}(\mathcal{X}, \mathbb{R}^{n_c})$, and such that the following conditions are satisfied:

$$\mathrm{rank}\ \hat{Q} = \mathrm{rank}\ \hat{P} = \mathrm{rank}\ \hat{Q}\hat{P} = n_c, \tag{2.9}$$

$$\hat{Q}\hat{P} = G^*M\Gamma, \qquad \Gamma G^* = I_{n_c}, \tag{2.10}$$

for some $M \in \mathbb{R}^{n_c \times n_c}$,

$$0 = AQ + QA^* + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^*, \tag{2.11}$$

$$0 = A^*P + PA + R_1 - P\Sigma P + \tau_\perp^* P\Sigma P\tau_\perp, \tag{2.12}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^* + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^*, \tag{2.13}$$

$$0 = (A - Q\bar{\Sigma})^*\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^* P\Sigma P\tau_\perp, \tag{2.14}$$

where

$$\tau \triangleq G^*\Gamma, \qquad \tau_\perp \triangleq I_{\mathcal{X}} - \tau.$$

It is shown in [18] that the factorization (2.10) for the nonnegative-definite operators $\hat{Q}$ and $\hat{P}$ satisfying rank $\hat{Q}\hat{P} = n_c$ always exists and is unique except for a change of basis in $\mathbb{R}^{n_c}$. It is also shown in [18] that $\Gamma^* : \mathbb{R}^{n_c} \to \mathrm{Dom}\,(A^*)$ so that the expression (2.6) is well defined.

Equations (2.11)–(2.14) are, in general, infinite-dimensional operator equations. To actually use them to compute the optimal fixed-order compensator, a finite-dimensional plant approximation is required. For each $N = 1, 2, \ldots$, let $\mathcal{X}^N$ denote a finite-dimensional subspace of $\mathcal{X}$ and let $\mathcal{P}^N : \mathcal{X} \to \mathcal{X}^N$ denote the corresponding orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}^N$. Let $A^N \in \mathcal{L}(\mathcal{X}^N), B^N \in \mathcal{L}(\mathbb{R}^m, \mathcal{X}^N), C^N \in \mathcal{L}(\mathcal{X}^N, \mathbb{R}^\ell), R_1^N \in \mathcal{L}(\mathcal{X}^N)$, and $V_1^N \in \mathcal{L}(\mathcal{X}^N)$. We consider the system (2.6)–(2.14) with the plant Π replaced by the plant Π^N given by

$$\Pi^N \triangleq \{A^N, B^N, C^N, R_1^N, R_2, V_1^N, V_2\}.$$

Typically, the operators B^N, C^N, R_1^N and V_1^N are chosen as $B^N = \mathcal{P}^N B, C^N = C\mathcal{P}^N, R_1^N = \mathcal{P}^N R_1$ and $V_1^N = \mathcal{P}^N V_1$ with the requirement that $\mathcal{P}^N$ converge strongly to the identity $I_{\mathcal{X}}$ as $N \to \infty$. The operator A^N is chosen so that it and its adjoint satisfy the hypotheses of the Trotter-Kato semigroup approximation theorem (i.e., stability and consistency, see, for example, [23]). That is, A^N is chosen so that $\lim_{N\to\infty} T^N(t)\mathcal{P}^N\phi = T(t)\phi$, and $\lim_{N\to\infty} T^N(t)^*\mathcal{P}^N\phi = T(t)^*\phi$, uniformly in t for t in bounded intervals, for each $\phi \in \mathcal{X}$, where $T^N(t) = \exp(tA^N)$, $t \geq 0$. We shall say more about these choices for A^N, B^N, C^N, R_1^N, and V_1^N when we address convergence questions below.

Although with the plant Π^N equations (2.11)–(2.14) are finite dimensional, they are still operator equations. It is their *matrix* equivalents which are used in computations. Unless orthonormal bases are chosen for the subspaces $\mathcal{X}^N$ (which is typically not the case in practice) some care must be taken to obtain the appropriate matrix system.

For each $N = 1, 2, \ldots$, let $\{\phi_j^N\}_{j=1}^{k^N}$ be a basis for $\mathcal{X}^N$ and choose the standard bases for all Euclidean spaces. For a linear operator L with domain and range $\mathcal{X}^N$ or any Euclidean space, let $[L]$ denote its matrix representation with respect to the bases chosen above. Also, let Φ^N denote the k^N-square Gram matrix corresponding to the basis $\{\phi_j^N\}_{j=1}^{k^N}$; that is, $\Phi_{ij}^N = \langle \phi_i^N, \phi_j^N \rangle$, $i, j = 1, 2, \ldots, k^N$. Noting that

$$[(A^N)^*] = (\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N, \quad [(B^N)^*] = [B^N]^{\mathrm{T}}\Phi^N, \quad [(C^N)] = (\Phi^N)^{-1}[C^N]^{\mathrm{T}},$$
$$[(\tau_\perp^N)^*] = (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N, \quad [\Sigma^N] = [B^N]R_2^{-1}[B^N]^{\mathrm{T}}\Phi^N, \quad [\bar{\Sigma}^N] = (\Phi^N)^{-1}[C^N]^{\mathrm{T}}V_2^{-1}[C^N],$$

the matrix equivalents of the operator equations (2.11)–(2.14) become

$$\begin{aligned} 0 &= [A^N][Q^N] + [Q^N](\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N + [V_1^N] - [Q^N][\bar{\Sigma}^N][Q^N] \\ &\quad + [\tau_\perp^N][Q^N][\bar{\Sigma}^N][Q^N] - (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N, \end{aligned} \tag{2.15}$$

$$\begin{aligned} 0 &= (\Phi^N)^{-1}[A^N]^{\mathrm{T}}\Phi^N[P^N] + [P^N] - [A^N] + [R_1^N] - [P^N][\Sigma^N][P^N] \\ &\quad + (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N[P^N] - [\Sigma^N][P^N][\tau_\perp^N], \end{aligned} \tag{2.16}$$

$$\begin{aligned} 0 &= \big([A^N] - [\Sigma^N][P^N]\big)[\hat{Q}^N] + [\hat{Q}^N](\Phi^N)^{-1}\big([A^N] - [\Sigma^N][P^N]\big)^{\mathrm{T}}\Phi^N \\ &\quad + [Q^N][\bar{\Sigma}^N][Q^N] - [\tau_\perp^N][Q^N][\bar{\Sigma}^N][Q^N](\Phi^N)^{-1} - [\tau_\perp^N]^{\mathrm{T}}\Phi^N, \end{aligned} \tag{2.17}$$

$$\begin{aligned} 0 &= (\Phi^N)^{-1}\big([A^N] - [Q^N][\bar{\Sigma}^N]\big)^{\mathrm{T}}\Phi^N[\hat{P}^N] + [\hat{P}^N]\big([A^N] - [Q^N][\bar{\Sigma}^N]\big) \\ &\quad + [P^N][\Sigma^N][P^N] - (\Phi^N)^{-1}[\tau_\perp^N]^{\mathrm{T}}\Phi^N[P^N][\Sigma^N][P^N][\tau_\perp^N]. \end{aligned} \tag{2.18}$$

Therefore, if we define the $k^N \times k^N$ nonnegative-definite matrices

$$\begin{aligned} Q_0^N &\triangleq [Q^N](\Phi^N)^{-1}, & P_0^N &\triangleq \Phi^N[P^N], \\ \hat{Q}_0^N &\triangleq [\hat{Q}^N](\Phi^N)^{-1}, & \hat{P}_0^N &\triangleq \Phi^N[\hat{P}^N], \\ V_0^N &\triangleq [V_1^N](\Phi^N)^{-1}, & R_0^N &\triangleq \Phi^N[R_1^N], \\ \Sigma_0^N &\triangleq [B^N]R_2^{-1}[B^N]^{\mathrm{T}}, & \bar{\Sigma}_0^N &\triangleq [C^N]^{\mathrm{T}}V_2^{-1}[C^N], \end{aligned}$$

we can solve the *matrix* optimal projection equations given in [14] corresponding to the matrix plant model

$$\Pi_0^N \triangleq \{[A^N], [B^N], [C^N], R_0^N, R_2, V_0^N, V_2\},$$

to obtain the matrices $Q_0^N, P_0^N, \hat{Q}_0^N$ and $\hat{P}_0^N$. The approximating optimal n_cth-order dynamic compensator $\{A_c^N, B_c^N, C_c^N\}$ is then given by

$$\begin{aligned} A_c^N &= \Gamma_0^N\big([A^N] - Q_0^N\bar{\Sigma}_0^N - \Sigma_0^N P_0^N\big)(G_0^N)^{\mathrm{T}}, \\ B_c^N &= \Gamma_0^N Q_0^N[C^N]^{\mathrm{T}}V_2^{-1}, \\ C_c^N &= -R_2^{-1}[B^N]^{\mathrm{T}}P_0^N(G_0^N)^{\mathrm{T}}, \end{aligned}$$

where $\Gamma_0^N, G_0^N \in \mathbb{R}^{n_c \times k^N}$, and $M^N \in \mathbb{R}^{n_c \times n_c}$ satisfy

$$\hat{Q}_0^N \hat{P}_0^N = G_0^N M^N \Gamma_0^N, \qquad \Gamma_0^N (G_0^N)^{\mathrm{T}} = I_{n_c},$$

$$[\tau^N] = (G_0^N)^{\mathrm{T}} \Gamma_0^N, \qquad [\tau_\perp^N] = I_{k^N} - [\tau^N].$$

When an infinite-dimensional controller will suffice, $C_c = -R_2^{-1} B^* P \in \mathcal{L}(\mathcal{X}, \mathbb{R}^m)$ and $B_c = QC^* V_2^{-1} \in \mathcal{L}(R^\ell, \mathcal{X})$ are the usual infinite-dimensional LQG controller and observer gains (see [9]). The operators $P, Q \in \mathcal{L}(\mathcal{X})$ are the nonnegative-definite solutions to the two decoupled operator algebraic Riccati equations (2.11) and (2.12) with τ and $\tau_\perp$ formally set to $I_{\mathcal{X}}$ and 0, respectively. Since C_c has range in $\mathbb{R}^m$ and B_c has domain $\mathbb{R}^\ell$, there exist vectors $c_c = (c_c^1, \ldots, c_c^m)^{\mathrm{T}} \in \times_{j=1}^m \mathcal{X}$ and $b_c = (b_c^1, \ldots, b_c^\ell)^{\mathrm{T}} \in \times_{j=1}^\ell \mathcal{X}$ such that

$$[C_c x]_i = \langle c_c^i,\ x \rangle, \qquad i = 1, 2, \ldots, m, \quad x \in \mathcal{X},$$

and

$$B_c y = b_c^{\mathrm{T}} y = \sum_{i=1}^{\ell} y_i b_c^i, \qquad y = (y_1, \ldots, y_\ell) \in \mathbb{R}^\ell.$$

The vectors c_c and b_c are referred to as the optimal LQG controller and observer functional gains respectively.

With regard to approximation for the full-order LQG problem, for each $N = 1, 2, \ldots$ we take $n_c = k^N$. Then it is not difficult to show that

$$C_c^N [P^N x] = \langle c_c^N,\ x \rangle, \qquad x \in \mathcal{X},$$

and

$$B_c^N y = (b_c^N)^{\mathrm{T}} y, \qquad y \in \mathbb{R}^\ell,$$

where $c_c^N \in \times_{j=1}^m \mathcal{X}^N$ and $b_c^N \in \times_{j=1}^\ell \mathcal{X}^N$ are given by $c_c^N = C_c^N (\Phi^N)^{-1} \phi^N$ and $b_c^N = (B_c^N)^{\mathrm{T}} \phi^N$ respectively with $\phi^N = (\phi_1^N, \ldots, \phi_{k^N}^N) \in \times_{j=1}^{k^N} \mathcal{X}^N$. The vectors c_c^N and b_c^N are referred to as the approximating optimal LQG controller and observer functional gains. To compute them we need only solve two standard decoupled matrix algebraic Riccati equations for the $k^N \times k^N$ nonnegative-definite matrices Q_0^N and P_0^N.

A rather complete convergence theory for LQG approximation can be found in [9]. Essentially, it is shown there that if the approximating subspaces $\mathcal{X}^N$ are chosen so that the projections $\mathcal{P}^N$

converge strongly to the identity as $N \to \infty$, the operators A^N, B^N, C^N, R_1^N, and V_1^N are chosen as was described above, and the operators Q^N and P^N are uniformly bounded in N, then Q^N and P^N converge weakly to Q and P, respectively as $N \to \infty$. This in turn implies that $C_c^N \to C_c$, strongly, $B_c^N \to B_c$, weakly, $c_c^N \to c_c$ and $b_c^N \to b_c$, weakly, and the closed-loop semigroup for the approximating optimal LQG compensator converges weakly to the closed-loop semigroup for the optimal infinite-dimensional LQG compensator, as $N \to \infty$. If, in addition, the operators $S^N(t) = T^N(t) + B^N C_c^N$ and $\hat{S}^N(t) = T^N(t) - B_c^N C^N$ are uniformly exponentially stable, uniformly in N, then $Q^N \to Q$ and $P^N \to P$, strongly, $C_c^N \to C_c$ and $B_c^N \to B_c$, in norm, $c_c^N \to c_c$ and $b_c^N \to b_c$, strongly, and the closed-loop semigroups converge strongly, as $N \to \infty$. If R_1^N and V_1^N are coercive, uniformly in N, then $S^N(t)$ and $\hat{S}^N(t)$ will be uniformly exponentially stable. If it is also true that R_1 and V_1 are trace class and $R_1^N P^N \to R_1$ and $V_1^N P^N \to V_1$ in trace norm then Q and P are trace class and $Q^N P^N \to Q$ and $P^N P^N \to P$ in trace norm as $N \to \infty$.

Returning to the fixed-finite-order case, we note that in general the approximating optimal projection equations may not possess a unique solution. However, in [19] it is shown for the finite-dimensional case that it is possible to obtain an upper bound for the number of stabilizing solutions. Using topological degree theory, the following result was obtained in [19].

Theorem 2.2. Consider the equations (2.11)–(2.14) with the infinite-dimensional plant Π replaced by the finite-dimensional plant Π^N. Let n_u denote the dimension of the unstable subspace of A^N and assume that $n_c \geq n_u$. Then in the class of nonnegative-definite operators $Q^N, P^N, \hat{Q}^N, \hat{P}^N$ on $\mathcal{X}^N$ satisfying rank $\hat{Q}^N$ = rank $\hat{P}^N$ = rank $\hat{Q}^N \hat{P}^N = n_c$, there exist at most

$$\begin{pmatrix} \min(k^N, m, \ell) - n_u \\ n_c - n_u \end{pmatrix}, \quad n_c \leq \min(k^N, m, \ell),$$
$$1, \qquad \text{otherwise},$$

solutions of (2.11)–(2.14), each of which is stabilizing. If, in addition, the plant (A^N, B^N, C^N) is stabilizable by an n_cth-order controller, then there exists at least one stabilizing solution of (2.9)–(2.14).

Theorem 2.2 shows that while there may exist multiple solutions to the finite-dimensional optimal projection equations, in practice this number can be quite small. For example, if $n_c \geq n_u$ and the system is either single input ($m = 1$) or single output ($\ell = 1$) then there exists at most one solution to (2.9)–(2.14) for the plant Π^N. The existence of at least one stabilizing solution of course depends upon whether or not the plant is stabilizable by an n_cth-order controller

(for relevant results, see [24]). Finally, while it may be possible to stabilize a plant with $n_c < n_u$, this case lies outside the scope of the analysis given in [19].

3. Examples and Numerical Results

We first consider the one-dimensional, single-input/ single-output, parabolic (heat/diffusion) control system with Dirichlet boundary conditions given by

$$\frac{\partial v}{\partial t}(t,\eta) = a\frac{\partial^2 v}{\partial \eta^2}(t,\eta) + b(\eta)u(t) + h_1(\eta)w_1(t,\eta), \quad 0 < \eta < 1, \quad t > 0, \tag{3.1}$$

$$v(t,0) = 0, \qquad v(t,1) = 0, \quad t > 0, \tag{3.2}$$

$$y(t) = \int_0^1 c(\eta)v(t,\eta)d\eta + h_2 w_2(t), \qquad t > 0, \tag{3.3}$$

where $a > 0$, and $b(\cdot)$ and $c(\cdot)$ are given by

$$b(\eta) = \begin{cases} \frac{1}{\beta_2-\beta_1}, & \beta_1 \le \eta \le \beta_2, \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$c(\eta) = \begin{cases} \frac{1}{\gamma_2-\gamma_1}, & \gamma_1 \le \eta \le \gamma_2, \\ 0, & \text{elsewhere}, \end{cases}$$

with $0 \le \beta_1 < \beta_2 \le 1$ and $0 \le \gamma_1 < \gamma_2 \le 1$. In (3.1) and (3.3), $h(\cdot) \in L_\infty(0,1), w_1(t,\cdot) \in L_2(0,1), a.a.\ t \in [0,\infty)$, (see [22], p. 314), h_2 is a nonzero constant and $w_2(\cdot)$ is unit-intensity white noise.

To rewrite (3.1)–(3.3) in the form (2.1), (2.2), in the usual way we take $\mathcal{X} = L_2(0,1)$ endowed with the standard L_2 inner product, let $x(t) = v(t,\cdot), t \ge 0$, define $A : \mathrm{Dom}\,(A) \subset \mathcal{X} \to \mathcal{X}$ by $A\phi = aD^2\phi$ for $\phi \in \mathrm{Dom}\,A \triangleq H^2(0,1) \cap H_0^1(0,1)$, and define $B \in \mathcal{L}(\mathbb{R}^1, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathbb{R}^1)$ by $Bu = b(\cdot)u$ for $u \in \mathbb{R}^1$, and $C\phi = \int_0^1 c(\eta)\phi(\eta)d\eta$, for $\phi \in L_2(0,1)$, respectively. Furthermore, let $\hat{\mathcal{X}} \triangleq L_2(0,1) \times \mathbb{R}$, set $w(t) \triangleq (w_1(t,\cdot), w_2(t)) \in \hat{\mathcal{X}}$, and define $H_1 \in \mathcal{L}(\hat{\mathcal{X}}, \mathcal{X})$ and $H_2 \in \mathcal{L}(\hat{\mathcal{X}}, \mathbb{R}^1)$ by $H_1 z = h_1(\cdot)z_1$ and $H_2 z = h_2 z_2$ for $z = (z_1, z_2) \in \hat{\mathcal{X}}$.

It is well known (see, for example, [23]) that A is closed, densely defined, and negative definite. Furthermore, A is the infinitesimal generator of a uniformly exponentially stable, analytic (abstract parabolic) semigroup $\{T(t) : t \ge 0\}$ of bounded, self-adjoint linear operators on $\mathcal{X}$.

We consider linear spline-based Ritz-Galerkin approximation for the open-loop system. For each $N = 2, 3, \ldots$, let $\{\phi_j^N\}_{j=1}^{N-1}$ be the linear spline ("hat") functions defined on the interval [0,1]

with respect to the uniform partition $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$, i.e.,

$$\phi_j^N(\eta) = \begin{cases} N\eta - j + 1, & \eta \in [\frac{j-1}{N}, \frac{j}{N}), \\ j + 1 - N\eta, & \eta \in [\frac{j}{N}, \frac{j+1}{N}), \\ 0, & \text{elsewhere on } [0,1], \end{cases}$$

$j = 1, 2, \ldots, N-1$. Set $\mathcal{X}^N = \operatorname{span}\{\phi_j^N\}_{j=1}^{N-1}$ and note that $k^N = \dim \mathcal{X}^N = N - 1$, and $\mathcal{X}^N \subset H_0^1(0,1)$ for all N. If $\mathcal{P}^N : \mathcal{X} \to \mathcal{X}^N$ denotes the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}^N$, then standard convergence estimates for interpolatory splines (see [25]) can be used to show that $\lim_{N\to\infty} \mathcal{P}^N\phi = \phi$ in $L_2(0,1)$ for $\phi \in L_2(0,1)$.

There are two equivalent ways to obtain an operator representation for the usual Ritz-Galerkin approximation to A. First, A can be extended to a bounded linear operator from $H_0^1(0,1)$ onto its dual, $H^{-1}(0,1)$, via

$$(A\phi)(\psi) = -a\langle D\phi, D\psi\rangle, \qquad \phi, \psi \in H_0^1(0,1). \tag{3.4}$$

Since $\mathcal{X}^N \subset H_0^1(0,1)$ for all $N = 2, 3, \ldots$, we define $A^N \in \mathcal{L}(\mathcal{X}^N)$ by $A^N\phi^N = A\phi^N, \phi^N \in \mathcal{X}^N$, with $A\phi^N \in H^{-1}(0,1)$ considered to be a linear functional on $\mathcal{X}^N$. From the Riesz Representation theorem we obtain $A^N\phi^N = \psi^N$ where ψ^N is that element in $\mathcal{X}^N$ which satisfies $(A^N\phi^N)(\chi^N) = -a\langle D\phi^N, D\chi^N\rangle = \langle\psi^N, \chi^N\rangle$.

Alternatively and equivalently, by using the fact that A is self-adjoint, we can define A^N as follows. Let $\mathcal{P}_1^N : H_0^1(0,1) \to \mathcal{X}^N$ denote the orthogonal projection of the Hilbert space $H_0^1(0,1)$ onto $\mathcal{X}^N$. Using the definition (3.4), it is not difficult to show that $-A \in \mathcal{L}\big(H_0^1(0,1), H^{-1}(0,1)\big)$ is coercive and, therefore, that $A^{-1} : H^{-1}(0,1) \to H_0^1(0,1)$ exists and is bounded. We then define $A^N \in \mathcal{L}(\mathcal{X}^N)$ to be the inverse of the operator given by $(A^N)^{-1} = \mathcal{P}_1^N A^{-1}\,|_{\mathcal{X}^N}$.

Using either definition, it is easily argued that A^N is well defined, self-adjoint, and is the infinitesimal generator of a uniformly exponentially stable (uniformly in N) semigroup, $T^N(t) = \exp(tA^N), t \geq 0$, of bounded linear operators on $\mathcal{X}^N$. Also, using the approximation properties of splines, it is not difficult to show that $\lim_{N\to\infty}(A^N)^{-1}\mathcal{P}^N\phi = A^{-1}\phi, \phi \in \mathcal{X}$. Consequently, the hypotheses of the Trotter-Kato theorem (see [23]) are satisfied and we have $\lim_{N\to\infty} T^N(t)\mathcal{P}^N\phi = T(t)\phi$ and $\lim_{N\to\infty} T^N(t)^*\mathcal{P}^N\phi = T(t)^*\phi, \phi \in \mathcal{X}$, uniformly in t for t in bounded intervals. A detailed discussion of the results just outlined can be found in [8].

We define $B^N = \mathcal{P}^N B$ and $C^N = C\mathcal{P}^N$, from which it immediately follows that $\lim_{N\to\infty} B^N = B$ and $\lim_{N\to\infty} C^N = C$ in norm and similarly for their adjoints. For the example we shall consider

here, we have chosen $R_1 = r_1 I_\chi, R_2 = r_2 I_m$, with $r_1, r_2 > 0$. Setting $h_1(\eta) = v_1^{\frac{1}{2}}, 0 < \eta < 1$, and $h_2 = v_2^{\frac{1}{2}}$ with $v_1, v_2 > 0$, we obtain $V_1 = v_1 I_\chi$ and $V_2 = v_2$. We then take $R_1^N = \mathcal{P}^N R_1$ and $V_1^N = \mathcal{P}^N V_1$. For the LQG problem, the open-loop uniform exponential stability of both the infinite-dimensional system and the approximating systems is sufficient to conclude the strong convergence of the approximating Riccati operators to the solutions of the infinite-dimensional Riccati equations, the uniform norm convergence of the approximating controller and observer gains, and the strong convergence of the functional gains, as $N \to \infty$.

Since the basis elements $\{\phi_j^N\}_{j=1}^{N-1}$ are piecewise linear with respect to the uniform mesh $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$ on [0,1], the equivalent matrix representations for the operators defined above can be computed directly and in closed form. The Gram matrix $\Phi_{ij}^N = \langle \phi_i^N, \phi_j^N \rangle, i, j = 1, 2, \ldots, N-1$ is given by $\Phi^N = \frac{1}{N}\text{Tridiag}\{\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\}$, and if we define the generalized stiffness matrix Ψ^N by $\Psi_{ij}^N = -a\langle D\phi_i^N, D\phi_j^N \rangle, i, j = 1, 2, \ldots, N-1$, then $\Psi^N = aN\ \text{Tridiag}\{1, -2, 1\}$. It follows that $[A^N] = (\Phi^N)^{-1}\Psi^N, [B^N] = (\Phi^N)^{-1}b^N, [C^N] = c^N$, with $b_i^N = \langle b, \phi_i^N \rangle = \frac{1}{\beta_2 - \beta_1}\int_{\beta_1}^{\beta_2} \phi_i^N(\eta)d\eta$, and $c_i^N = \langle c, \phi_i^N \rangle = \frac{1}{\gamma_2 - \gamma_1}\int_{\gamma_1}^{\gamma_2} \phi_i^N(\eta)d\eta, i = 1, 2, \ldots, N-1$, and that $R_0^N = r_1\Phi^N$ and $V_0^N = v_1(\Phi^N)^{-1}$.

For our numerical study we set $a = 1, \beta_1 = .75 - .03\sqrt{2}, \beta_2 = 75 + .04\sqrt{2}, \gamma_1 = .25 - .04\sqrt{2}, \gamma_2 = .25 + .03\sqrt{2}, r_1 = v_1 = 1, r_2 = v_2 = 10^{-4}, h_1(\eta) \equiv 1$, and used our technique to compute approximating optimal LQG (i.e., $n_c = N-1$) and 1*st* order (i.e., $n_c = 1$) compensators for various values of N. The open-loop stability of system (3.1)–(3.3) and the approximating systems imply that the finite-dimensional approximating optimal projection equations have a solution. Theorem 2.2 on the other hand, with $n_u = 0$ and $n_c = 1$ or $n_c = N-1$, implies that they have at most one solution. Consequently, the system of equations (2.11)–(2.14) with the plants Π^N admits a unique solution.

The optimal projection equations (2.11)–(2.14) were solved using the homotopic continuation algorithm described in [19]. It is shown in [19] that the operation count for the algorithm is proportional to $p\big(2n^3 + (m+\ell)n^2 + (m+\ell)^3 n_c^3\big)$ where p is the number of integration steps and n is the dimension of the finite-dimensional plant. This is competitive with the operation count for the Hamiltonian solution of the standard Riccati equations which is $O(16n^3)$ for LQG. Also, note that the computational burden for the solution of the optimal projection equations decreases with n_c.

Since $m = \ell = 1$ in the LQG case, the optimal functional observer and feedback control gains b_c and c_c and the approximating gains b_c^N and c_c^N, are all simply L_2 functions with b_c^N and c_c^N elements in $\mathcal{X}^N$. We plot the functions b_c^N and c_c^N we obtained for various values of N respectively

in Figures 3.1 and 3.2 below. That convergence is indeed achieved can immediately be observed in the figures.

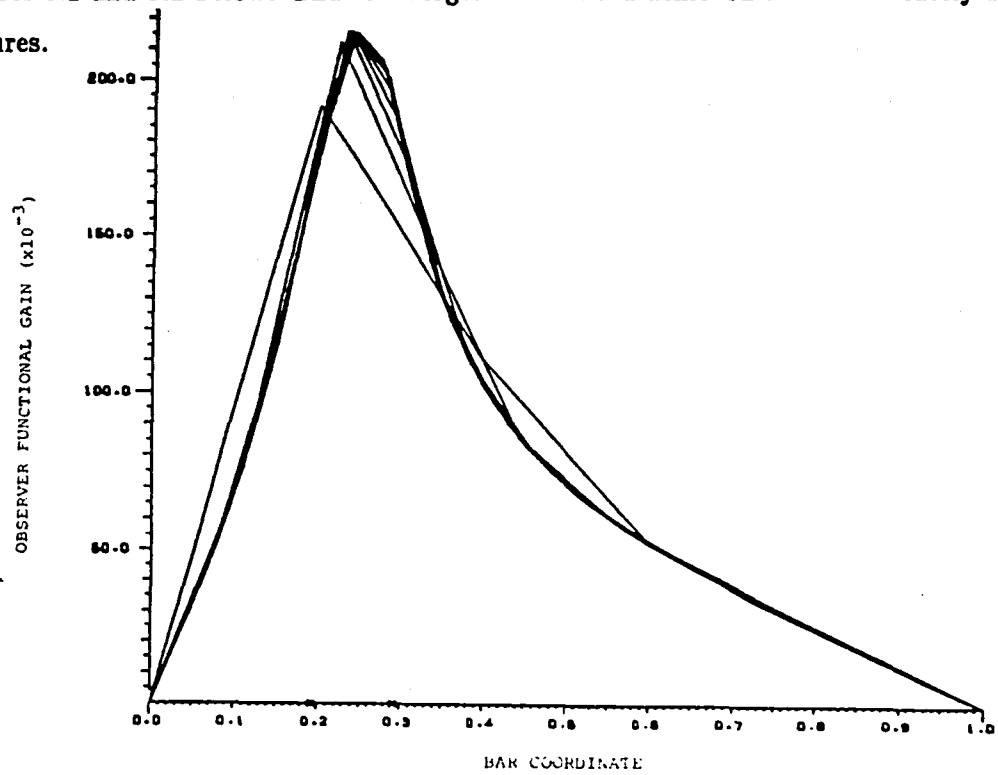

Figure 3.1

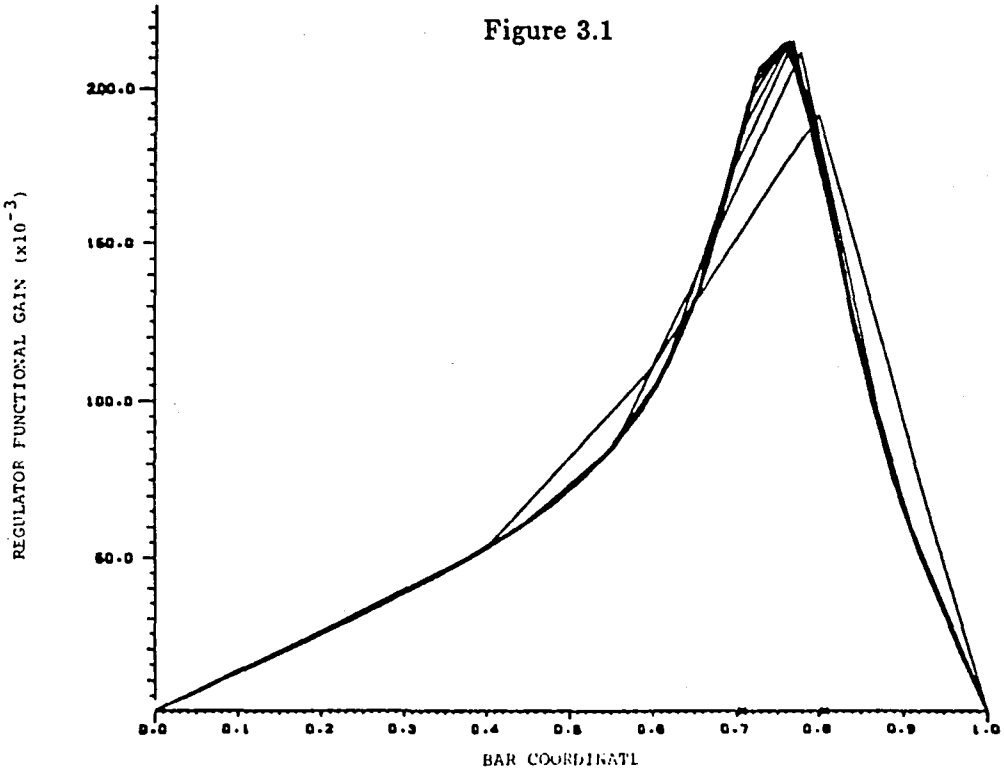

Figure 3.2

In the fixed-order case with $n_c = 1$, the compensator gains A_c, B_c, and C_c are all scalars. Also, for a first-order controller there are only two independent parameters, A_c and B_cC_c. In Table 3.1 below we give the values we obtained for A_c^N and $B_c^NC_c^N$ for various values of N. Once again, it is clear that the gains are converging as N increases. In addition, in Table 3.1 we provide the closed-loop costs J_{LQG}^N and J_{FO}^N for the LQG and first-order controllers. These closed-loop costs were evaluated using a 64th-order modal approximation to the infinite-dimensional system. For all values of N the performance of the fixed-order compensator was within 2% of the corresponding LQG controller. Thus, for example, the replacement of a 32nd-order approximating optimal LQG controller by an approximating optimal first-order controller will yield considerable implementation simplification with only minor performance degradation. Note that for the example we consider here, it is possible to compute the open-loop cost for the infinite-dimensional system in closed form. We have

$$\begin{aligned} J_{OL} &= \mathrm{tr}\int_0^\infty V_1T^*(t)R_1T(t)dt = v_1r_1\mathrm{tr}\int_0^\infty T(t)^2dt \\ &= v_1r_1\sum_{n=1}^\infty\int_0^\infty e^{-2n^2\pi^2at}dt = \frac{v_1r_1}{2\pi^2a}\sum_{n=1}^\infty\frac{1}{n^2} \\ &= \frac{v_1r_1}{12a} = \frac{1}{12} \approx .08333. \end{aligned}$$

Finally, for comparison purposes, we tried applying balancing techniques to the LQG controllers to reduce their order. However, with $n_c = 1$, such controllers were found to be destabilizing. Based upon the results in [13], this was not unexpected.

N	A_c^N	$B_c^NC_c^N$	J_{LQG}^N	J_{FO}^N
4	-687.6	5470	.06999	.07014
8	-720.9	5231	.06870	.06993
12	-730.9	5182	.06872	.06991
16	-734.3	5145	.06874	.06990
20	-738.0	5127	.06875	.06990
24	-737.6	5108	.06876	.06990
28	-739.8	5109	.06876	.06990
32	-738.7	5099	.06877	.06990

Table 3.1

As a second example we consider the one-dimensional, single-input, single-output hereditary control system given by

$$\dot{v}(t) = a_0 v(t) + a_1 v(t-\rho) + b_0 u(t) + h_1 w(t), \quad t > 0, \tag{3.5}$$

$$y(t) = c_0 v(t) + h_2 w(t), \quad t > 0, \tag{3.6}$$

where $a_0, a_1, b_0, c_0, h_1, h_2, \rho \in \mathbb{R}^1$ with $h_2 \neq 0$, and w is a unit-intensity white noise process. To rewrite (3.5), (3.6) in the form (2.1), (2.2), we take $\mathcal{X} = \mathbb{R}^1 \times L_2(-\rho, 0)$ endowed with the usual product space inner product, $\langle(\eta,\phi),(\xi,\psi)\rangle = \eta\xi + \int_{-\rho}^{0} \phi\psi$, and let $x(t) = (v(t), v_t)$, $t \geq 0$, where for $t \geq 0$, $v_t \in L_2(-\rho, 0)$ is given by $v_t(\theta) = v(t+\theta)$, $-\rho \leq \theta \leq 0$. Define A: $\mathrm{Dom}(A) \subset \mathcal{X} \to \mathcal{X}$ by $A\hat{\phi} = (a_0\phi(0) + a_1\phi(-\rho), D\phi)$ for $\hat{\phi} = (\phi(0), \phi) \in \mathrm{Dom}(A) \triangleq \{(\xi,\psi) \in \mathcal{X} : \psi \in H^1(-\rho, 0), \psi(0) = \xi\}$, and let $B \in \mathcal{L}(\mathbb{R}^1, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathbb{R}^1)$ be given by $Bu = (b_0 u, 0)$ and $C(\eta,\phi) = c_0\eta$, respectively. Let $\hat{\mathcal{X}} = \mathbb{R}^1$ and define $H_1 \in \mathcal{L}(\hat{\mathcal{X}}, \mathcal{X})$ and $H_2 \in \mathcal{L}(\hat{\mathcal{X}}, \mathbb{R}^1)$ by $H_1 z = (h_1 z, 0)$ and $H_2 z = h_2 z$, for $z \in \mathbb{R}^1$.

The operator A is densely defined and is the infinitesimal generator of a C_0 semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators on $\mathcal{X}$ with $T(t)(\eta,\phi) = \big(v(t;\eta,\phi),\ v_t(\eta,\phi)\big)$, $t \geq 0$, where $v(\cdot;\eta,\phi)$ is the unique solution to (3.5) with $b_0 = h_1 = 0$, and initial conditions $v(0) = \eta, v_0 = \phi$. We take $R_1 \in \mathcal{L}(\mathcal{X})$ and $R_2 \in \mathcal{L}(\mathbb{R}^1)$ to be $R_1(\eta,\phi) = (r_1\eta, 0)$ and $R_2 u = r_2 u$, respectively, with $r_1, r_2 > 0$. The definitions of H_1 and H_2 given above imply that $V_1 \in \mathcal{L}(\mathcal{X})$ and $V_2 \in \mathcal{L}(\mathbb{R}^1)$ are given by $V_1(\eta,\phi) = (h_1^2\eta, 0)$ and $V_2 z = h_2^2 z$, for $(\eta,\phi) \in \mathcal{X}$ and $z \in \mathbb{R}^1$.

We employ an approximation scheme recently proposed by Ito and Kappel in [26]. We briefly outline it here; a more detailed discussion can be found in [26]. For each $N = 1, 2, \ldots$ let $\chi_j^N \in L_2(-\rho, 0)$ denote the characteristic function for the interval $[-j\rho/N, -(j-1)\rho/N), j = 1, 2, \ldots, N$, and let $\mathcal{X}^N$ be the $(N+1)$-dimensional subspace of $\mathcal{X}$ defined by

$$\mathcal{X}^N = \mathrm{span}\{(1,0), (0,\chi_1^N), \ldots, (0,\chi_N^N)\}.$$

Let $\mathcal{P}^N : \mathcal{X} \to \mathcal{X}^n$ denote the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}^N$. Let $\{\phi_j^N\}_{j=0}^N$ denote the linear B-spline functions defined on the interval $[-\rho, 0]$ with respect to the uniform mesh $[-\rho, \ldots, -\rho/N, 0\}$, and set $\mathcal{X}_1^N = \mathrm{span}\{(\phi_j^N(0), \phi_j^N)\}_{j=0}^N$. Then $\mathcal{X}_1^N$ is an $(N+1)$-dimensional subspace of $\mathrm{Dom}(A)$ and it is not difficult to demonstrate that the restriction of $\mathcal{P}^N$ to $\mathcal{X}_1^N$ is a bijection onto $\mathcal{X}^N$. Using the fact that A restricted to $\mathcal{X}_1^N$ has range in $\mathcal{X}^N$, we define $A^N \in \mathcal{L}(\mathcal{X}^N)$ by $A^N = A(\mathcal{P}^N)^{-1}$, and set $T^N(t) = \exp(A^N t)$, $t \geq 0$. Noting that $R(B) \subset \mathcal{X}^N$, we take $B^N \in \mathcal{L}(\mathbb{R}^1, \mathcal{X}^N)$ to be given by $B^N = B$. Similarly, we take $R_1^N = R_1$ and $V_1^N = V_1$. We set $C^N = C$.

It is shown in [26] that $\mathcal{P}^N(\eta,\phi) \to (\eta,\phi)$, $T^N(t)\mathcal{P}^N(\eta,\phi) \to T(t)(\eta,\phi)$, and $T^N(t)^*\mathcal{P}^N(\eta,\phi) \to T(t)^*(\eta,\phi)$ for $(\eta,\phi) \in \mathcal{X}$ as $N \to \infty$, uniformly in t, for t in bounded subsets of $[0,\infty)$. It then follows that $\lim_{N\to\infty} B^N = B$ and $\lim_{N\to\infty} C^N\mathcal{P}^N = C$, in norm.

For the LQG (full-order) problem, the optimal functional observer and feedback control gains b_c and c_c are of the form $b_c = (\beta_0,\beta_1)$ and $c_c = (\gamma_0,\gamma_1)$ with $\beta_0,\gamma_0 \in \mathbb{R}^1$, and $\beta_1,\gamma_1 \in L_2(-\rho,0)$. The approximating gains are of the form $b_c^N = (\beta_0^N,\beta_1^N)$ and $c_c^N = (\gamma_0^N,\gamma_1^N)$ with $\beta_0^N,\gamma_0^N \in \mathbb{R}^1$ and $\beta_1^N,\gamma_1^N \in$ span $\{\chi_j^N\}_{j=1}^N$. Since we are treating a one-dimensional example, if $b_0 \neq 0$, the theory in [26] implies that $\beta_0^N \to \beta_0$ and $\gamma_0^N \to \gamma_0$ in $\mathbb{R}^1$, and $\beta_1^N \to \beta_1$, and $\gamma_1^N \to \gamma_1$ in $L_2(-\rho,0)$, as $N \to \infty$.

Once again, as in the first example, matrix representations for the operators A^N, B^N, C^N, R_1^N, and V_1^N are not difficult to compute in closed form. Indeed, the $(N+1)\times(N+1)$ matrix representation for the bijection $\mathcal{P}^N\colon \mathcal{X}_1^N \to \mathcal{X}_1$ is given by

$$[\mathcal{P}^N] = \begin{bmatrix} 1 & 0 & & & & & 0 & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & & & & & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ & 0 & & & & & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Then $[A^N] = [K^N][\mathcal{P}^N]^{-1}$, where

$$[K^N] = \begin{bmatrix} a_0 & 0 & & & & & 0 & & a_1 \\ \frac{N}{\rho} & -\frac{N}{\rho} & 0 & & & & & & \\ 0 & \frac{N}{\rho} & -\frac{N}{\rho} & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & \frac{N}{\rho} & -\frac{N}{\rho} & 0 \\ & 0 & & & & & 0 & \frac{N}{\rho} & -\frac{N}{\rho} \end{bmatrix}.$$

We have the $(N+1)\times 1$ matrix $[B^N] = [b_0\ 0 \ldots 0]^{\mathrm{T}}$ and the $1\times(N+1)$ matrix $[C^N] = [c_0\ 0 \ldots 0]$, while $[R_1^N] = r_1[\mathcal{M}^N]$ and $[V_1^N] = h_1^2[\mathcal{M}^N]$ where the $(N+1)\times(N+1)$ matrix $[\mathcal{M}^N]$ is given by

$$[\mathcal{M}^N] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}.$$

We set $a_0 = a_1 = b_0 = c_0 = r_1 = h_1 = \rho = 1$, $r_2 = .1$, and $h_2 = \sqrt{.1}$ and computed approximating optimal LQG (i.e., $n_c = N+1$) and first-order (i.e., $n_c = 1$) compensators for

$N = 8, 16, 24$, and 32. The optimal LQG observer gains are given in Table 3.3 and Figure 3.3; the control gains are given in Table 3.4 and Figure 3.4. The first 23 open-loop poles of the system (see [27]) are given in Table 3.2. The approximating first-order compensator gains along with the corresponding and LQG closed-loop costs are given in Table 3.5 below. These costs were computed using an evaluation model obtained by setting $N = 64$.. Note that the performance of the first-order controllers is within 10% of the performance of the LQG controllers. Once again it is clear that convergence is achieved.

1.278465
-1.588317 ± 4.155305i
-2.417631 ± 10.68603i
-2.861502 ± 17.05611i
-3.167754 ± 23.38558i
-3.401945 ± 29.69798i
-3.591627 ± 36.00146i
-3.751047 ± 42.29965i
-3.888543 ± 48.59442i
-4.009422 ± 54.88686i
-4.117267 ± 61.17761i
-4.214618 ± 67.46710i

Table 3.2

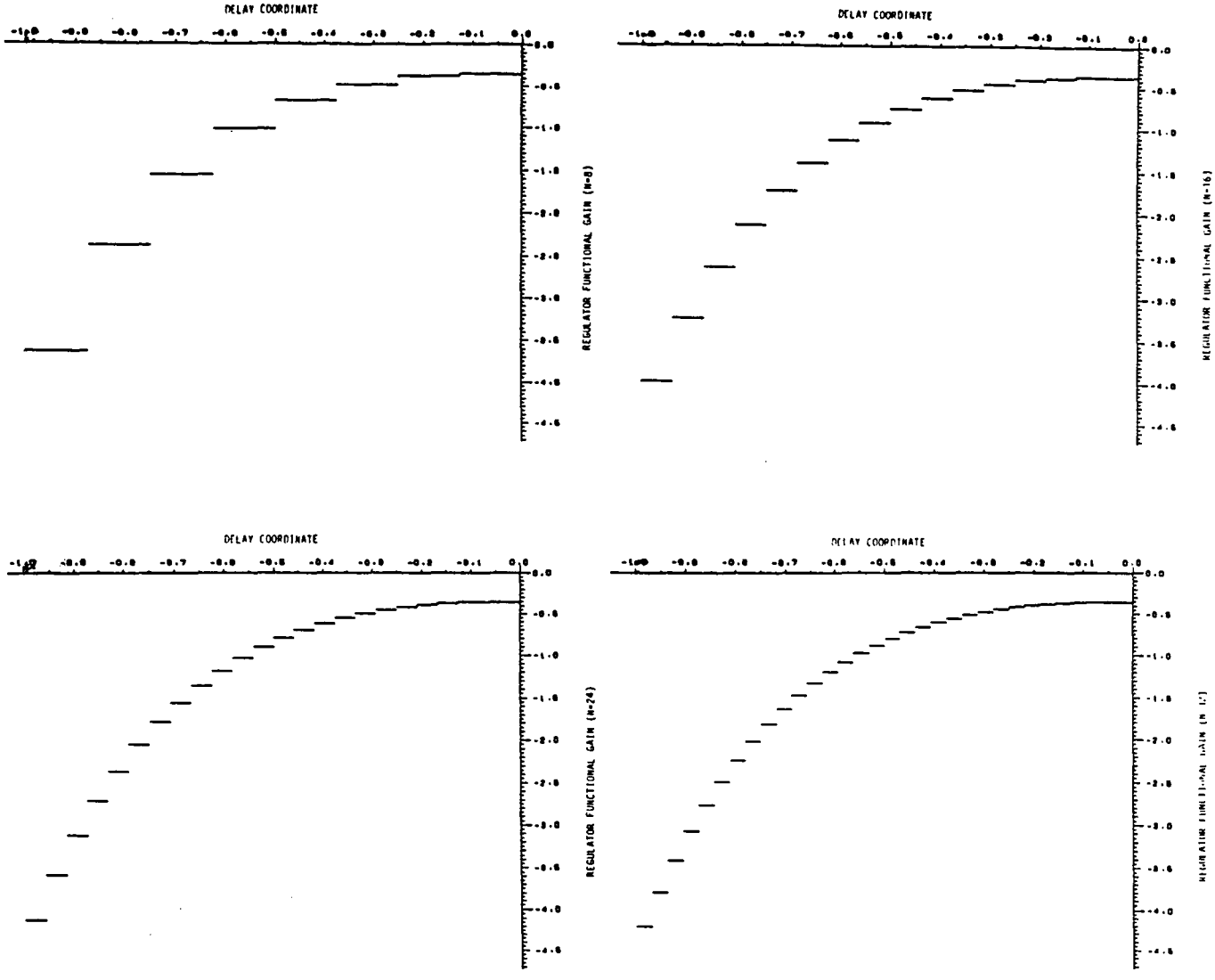

Figure 3.4

N	8	16	24	32
γ_0^N	-4.4213	-4.4229	-4.4233	-4.4234

Table 3.4

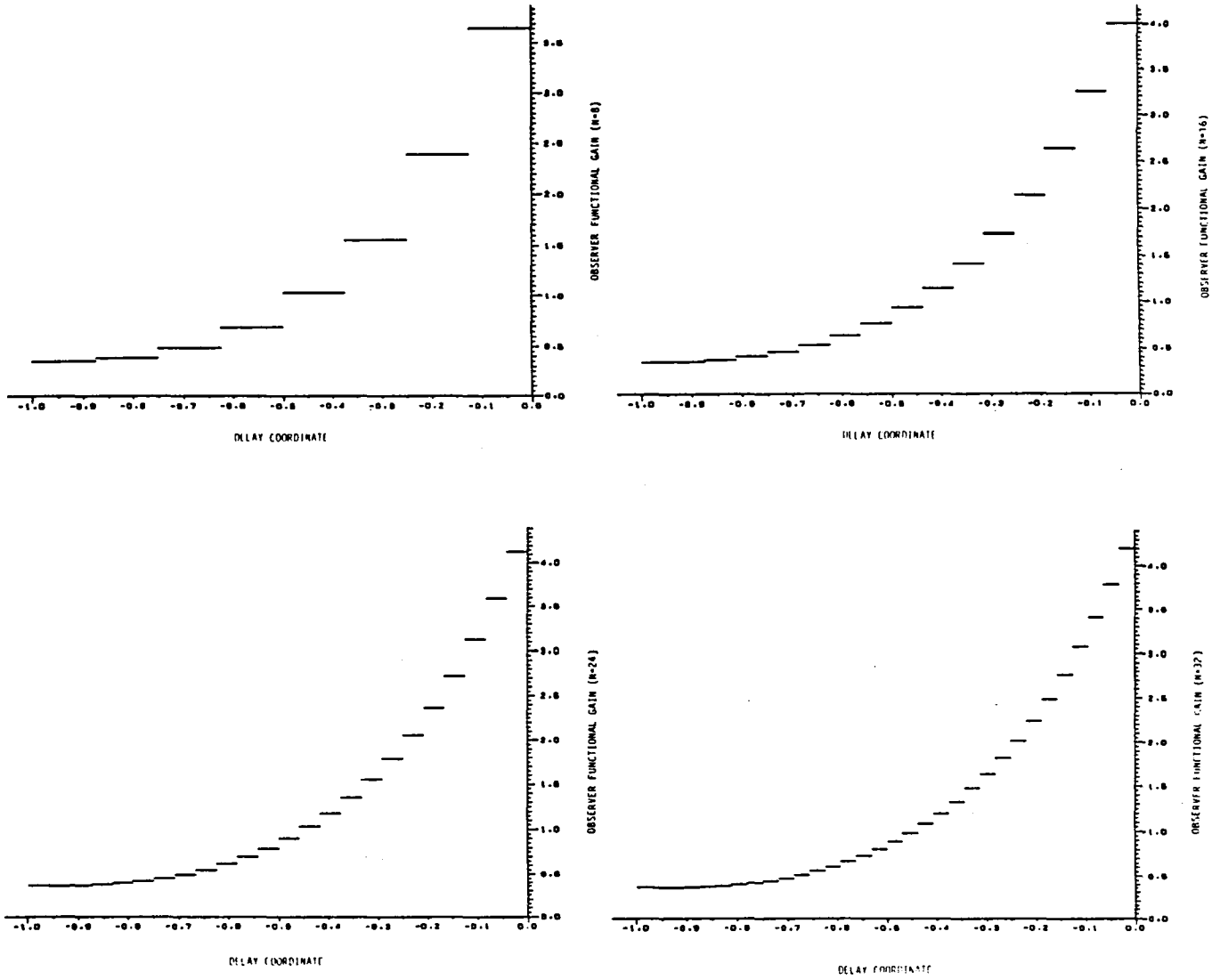

Figure 3.3

N	8	16	24	32
β_0^N	4.4213	4.4229	4.4233	4.4234

Table 3.3

N	A_c^N	$B_c^N C_c^N$	J_{LQG}^N	J_{FO}^N
8	-4.835	-16.057	1.4042	1.5221
16	-4.936	-16.343	1.403877	1.5298
24	-4.959	-16.378	1.403856	1.5309
32	-4.962	-16.404	1.403852	1.5317

Table 3.5

4. Summary and Concluding Remarks

We have proposed an approximation technique for computing optimal fixed-order compensators for distributed parameter systems. Our approach involves using the optimal projection theory for infinite-dimensional systems (which characterizes the optimal fixed-order compensator) developed in [18] in conjunction with finite-dimensional approximation of the infinite-dimensional plant. We demonstrated the feasibility of our approach with two examples wherein we used spline-based Ritz-Galerkin finite element schemes to compute approximating optimal first-order controllers for one-dimensional, singe-input/output, parabolic (heat/diffusion) and hereditary control systems. Our numerical studies indicate that convergence of the compensator gains is achieved and that using the first-order controller would lead to only minimal performance degradation over a standard LQG compensator while yielding significant implementation simplification.

At this point one is led naturally to ask the question of whether or not a satisfactory convergence theory could be developed. We are working on this at present and expect that such a theory would conform closely in form and spirit to the convergence results for LQG approximation found in [9] and [10] and outlined in Section 2 above. We also intend to consider our approximation ideas in the context of discrete-time or sampled-data systems, and for continuous-time systems involving unbounded input and/or output (for example, boundary control systems), and systems with control or measurement delays, see [11],[12]). Finally, we intend to investigate the application of our approximation framework to other infinite-dimensional control systems, in particular the vibration control of flexible structures (i.e., second-order systems such as wave, beam, or plate equations).

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16. Abstract

In controlling distributed parameter systems it is often desirable to obtain low-order, finite-dimensional controllers in order to minimize real-time computational requirements. Standard approaches to this problem employ model/controller reduction techniques in conjunction with LQG theory. In this paper we consider the finite-dimensional approximation of the inifinite-dimensional Bernstein/Hyland optimal projection theory. Our approach yields fixed-finite-order controllers which are optimal with respect to high-order, approximating, finite-dimensional plant models. We illustrate the technique by cmoputing a sequence of first-order controllers for one-dimensional, single-input/single-output, parabolic (heat/diffusion) and hereditary systems using spline-based, Ritz-Galerkin, finite element approximation. Our numerical studies indicate convergence of the feedback gains with less than 2% performance degradation over full-order LQG controllers for the parabolic system and 10% degradation for the hereditary system.

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