Maximum Dynamic Responses Using Matched Filter Theory and Random Process Theory

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SEPTEMBER 1988
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and Random Process Theory  

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Introduction  

A primary requirement of aircraft structures is that they withstand all the static and dynamic loads the structure is expected to encounter during its life. Such loads include landing and taxi loads, maneuver loads, and gust loads. There are several methods used by airframe manufacturers to compute gust loads to satisfy certification requirements and include both stochastic methods and deterministic methods. The U.S. Federal Aviation Administration (FAA) recently asked the National Aeronautics and Space Administration (NASA) for assistance in evaluating the Statistical Discrete Gust (SDG) Method (ref. 1) as a candidate gust-loads analysis method for complying with FAA certification requirements. The SDG Method is a time-domain approach and yields time-correlated gust loads by employing a computationally expensive search procedure.

During the course of the NASA evaluation of the SDG Method, the authors recognized that Matched Filter Theory (MFT) (ref. 2) could be applied to the gust problem to compute time-correlated gust loads. Computing time-correlated gust loads in this manner has the twin advantages of being computationally fast and of solving for the answers directly. Historically, Matched Filter Theory was first utilized in the optimal detection of returning radar signals. Papoulis (ref. 3) showed that Matched Filter Theory can be used to obtain maximized responses in fields other than signal detection. The first purpose of this paper is to demonstrate that Matched Filter Theory is also applicable to the general fields of structural dynamics and aeroelasticity and is specifically applicable to the computation of time-correlated gust loads.

During the course of the MFT investigation the authors also recognized that time-correlated gust loads, theoretically identical to those computed by MFT, could also be obtained using Random Process Theory (RPT) (ref. 4) with the same twin advantages. To the knowledge of the authors, Random Process Theory has, until now, not been applied in the computation of time-correlated gust loads. The second purpose of this paper is to demonstrate this applicability.

Both the MFT and the RPT ways of computing time-correlated gust loads involve novel applications of the theories and unconventional interpretations of the intermediate and final results. This paper outlines the mathematical developments, recognizes a duality between MFT and RPT, and presents example calculations using both MFT and RPT for computing time-correlated gust loads.

This paper was originally presented orally at the TTCP HAG-6 Workshop on Active Controls and Structural Integrity at the Royal Aerospace Establishment, Farnborough, England, September 28 - 29, 1988.
Time-Correlated Loads

This paper deals specifically with time-correlated gust loads, and this figure illustrates two types of such loads. Time-correlated loads are time histories of two or more different load responses to the same disturbance quantity. As illustrated in the figure, the disturbance quantity is the vertical component of one-dimensional atmospheric turbulence and the time-correlated loads (the output quantities) are the resulting bending moments and torsion moments at several locations on the airplane wing.

The first type of time-correlated load is illustrated on the right wing: loads (two bending moments in this illustration) at two different locations on the airplane. The second type is illustrated on the left wing: two different loads (bending moment and torsion moment in this illustration) at the same location on the airplane.

As indicated in the time histories in the figure, time correlation provides knowledge of the value (magnitude and sign) of one load when another is maximum (positive or negative), and vice versa. Such information may be used directly during analyses and testing of aircraft structures (ref. 5).
Time-Correlated Loads

Figure 1
Some Features of the Statistical Discrete Gust Method

Some background information on the features of the Statistical Discrete Gust Method (ref. 1) is offered so that the contributions of the present paper may be put in context.

The SDG Method determines the response time histories of "worst case" gust loads (such as shear forces, bending moments, and torsion moments) and the corresponding "critical gust profiles" which produce them. These loads are time correlated and this feature is a major advantage of the SDG Method over some other gust loads analysis methods.

Another advantage of the SDG Method is its applicability to nonlinear, as well as to linear, systems. This feature allows one to obtain time-correlated gust loads for a nonlinear system. As indicated at the bottom of the figure, the SDG Method employs a search procedure to obtain its answers. For a linear system, by taking advantage of superposition, the search procedure may be simplified and reduced to a one-dimensional search. For a nonlinear system, however, this is not the case and the resulting search procedure remains multi-dimensional and can become exhaustive (ref. 6).
Some Features of the Statistical Discrete Gust Method

* Determines maximum time-correlated loads due to gust
* Applicable to linear and nonlinear systems
* Employs search procedure to find critical gust profiles and loads time histories
  - Linear - - - - Superposition simplifies search
  - Nonlinear - - Search can become exhaustive

Figure 2
Scope - Problem Definition - Proposed Solution

With the information from figure 2 in mind, this figure states the conditions under which and the means by which this paper makes a technical contribution to the area of time-correlated gust-load calculations.

Whereas the SDG Method is capable of performing both linear and nonlinear analyses, the present methods are restricted to linear systems only.

Whereas the SDG Method obtained time-correlated gust loads and the corresponding critical gust profiles using a search procedure, the goal in the present paper is to obtain the same quantities directly and to achieve a significant reduction in computation time.

Novel applications of Matched Filter Theory and Random Process Theory and unconventional interpretations of the intermediate and final results from these theories will be used to achieve the goal.
Scope

Restricting attention to linear systems...

Problem Definition

... directly obtain time-correlated loads and the corresponding critical gust profile ...

Proposed Solution

... by applying Matched Filter Theory (and, later, Random Process Theory).

Figure 3
Original and New Applications of Matched Filter Theory

The objective of Matched Filter Theory, as originally developed, is the design of an electronic filter such that its response to a known input signal is maximum (ref. 7). It found early application to radar considering the "filter" to be a correlation detector that, in response to a known input signal, produces an output signal for further processing (ref. 2). In this case the correlation detector design is the optimum design for maximizing the output signal-to-noise ratio.

In the present application the "filter" is considered to be a system whose dynamics are known. Specifically, the system is characterized by the combination, in series, of the dynamics of atmospheric turbulence and the dynamics of aircraft load response. The simple result of Matched Filter Theory allows direct determination of the input signal, or excitation, that produces a maximum response of the system. The result, as will be shown later, is the maximum load response and the critical gust profile that produces the response.
Original and New Applications of Matched Filter Theory

Original Application: RADAR

New Application: Time-Correlated Gust Loads

Figure 4
"Matched" Excitation Waveform

The theoretical result of Matched Filter Theory is that the excitation that produces the maximum response is proportional to the system's unit impulse response, lagged and reversed in time (refs. 3 and 7). The excitation is said to be "matched" to the particular output in question. This figure outlines the analytical steps necessary to obtain the matched excitation waveform.

The constant of proportionality, \( K \), and the lag time, \( t_0 \), (shown in the figure) are arbitrary in principle and may be selected for convenience or by other requirements of the problem at hand. In practice, a unit impulse is applied to the system (as shown in the figure) and the lag time is chosen at a point at which the impulse response has attenuated to a small fraction of the maximum response. This is so that when that waveform is reversed its amplitude builds up smoothly from zero (or near zero).

Taking the Fourier transform of the impulse response gives the frequency response function, \( H_F \), of the system in terms of the Fourier transform, \( X \), of the excitation waveform, \( x(t) \), the constant of proportionality, \( K \), and the lag time, \( t_0 \). The root-mean-square (r.m.s.), \( \sigma_h \), of the impulse response is evaluated by integrating \( H_F^*H_F \) (where * denotes the complex conjugate) with respect to frequency from minus to plus infinity. If the constant of proportionality is chosen as the r.m.s. of the impulse response, then the excitation, \( x(t) \), has an r.m.s. of unity. Alternate normalizations have been suggested that bring statistical properties of atmospheric turbulence into the problem solution (ref. 8). In the present application the normalization is a convenient device for comparing the effects of different excitations. Thus, the end result is that the excitation waveform appears as a mirror image of the impulse response normalized by \( \sigma_h \).
"Matched" Excitation Waveform

**Matched Filter Theory**: Impulse response and maximizing signal (reversed and lagged) are proportional

**Impulse Response**: $\delta(t) \xrightarrow{F} h(t)$

where $h(t) = Kh\left(-t - t_0\right)$ with r.m.s. $\sigma_h$ and $x(t)$ is the desired signal

**Normalization of Waveform**:

$$H_F = KX^* e^{-i\omega t} \text{ (Freq. Resp. Function)}$$

$$\sigma_h^2 = K^2 \frac{1}{2\pi} \int X^*X \, d\omega$$

If $K = \sigma_h$, then $x(t)$ must satisfy

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*X \, d\omega = 1$$

"Matched" Excitation Waveform: $x(t)$

**Figure 5**
Response to "Matched" Excitation Waveform

If the excitation x(t), described in figure 5, is now applied to the system just as the unit impulse was, a response y(t) results. The Fourier transform, Y, of the response may be written directly as the product of the system frequency response function, HF, and the transform, X, of the excitation. As derived in figure 5, HF may be written in terms of X.

Taking the inverse transform it is noted that the product X*X is a positive, even function of frequency allowing the exponential to be written as a cosine function without changing the result of the integration. Thus, the total integral is maximum when t = t₀. So it can be seen that the maximum value of the response y(t) occurs at t = t₀ and is equal to σ₁ times the r.m.s. of the excitation waveform, which has been normalized to be unity (figure 5). Thus the maximum response of the system produced by the matched excitation waveform is equal to the r.m.s. of the impulse response. Of course, if the normalized excitation waveform were multiplied by a scalar, then the response would be multiplied by the same scalar.
Response to "Matched" Excitation Waveform

Fourier Transform of Response: \[ Y = H_F X = \sigma_h X^*X e^{-i\omega t_0} \]

Inverse Fourier Transform:
\[
y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_h X^*X e^{i\omega(t-t_0)} d\omega \]
\[ |y|_{\text{max}} = |y(t_0)| = \sigma_h \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*X d\omega \]

\[ |y|_{\text{max}} = \sigma_h \]

Figure 6
Response to Arbitrary Waveform

If any other arbitrary waveform, \( x'(t) \), subject to the same normalizing constraint (that its r.m.s. be unity) were applied to the system, some response, \( y'(t) \), results. This response can be found, as in figure 6, by writing the transform of the response as the product of the frequency response function, \( H_F \), and the transform of the excitation, \( x'(t) \), and taking the inverse transform. Applying Schwarz's Inequality, it is seen that \( y'(t) \) can never exceed the maximum value of \( y(t) \), which is the response resulting from applying the matched waveform. Thus, the response of an output to any waveform (appropriately normalized) will never exceed the maximum response of that output to its own matched waveform.
Response to Arbitrary Waveform

Arbitrary waveform, $x'$, satisfying

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} X^* X' \, d\omega = 1$$

Response to $x'$:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_h X^* X' e^{i \omega (t - t_o)} \, d\omega$$

Schwarz's Inequality:

$$\left| \int f \, g \, dt \right|^2 \leq \left( \int f^* f \, dt \right) \left( \int g^* g \, dt \right)$$

Letting: $f = \sigma_h X^*$ and $g = X' e^{i \omega (t - t_o)}$

$$|y'(t)| \leq \frac{1}{2\pi} \int \sigma_h^2 X^* X \, d\omega \cdot \frac{1}{2\pi} \int X'^* X' \, d\omega$$

Thus, response to any $x'$ never exceeds $|y|_{\text{max}}$

Figure 7
Matched Filter Theory Applied to Time-Correlated Gust Loads

This figure contains a signal flow diagram of the steps necessary to generate a maximum dynamic response at some point in the aircraft structure. It expands on the information presented in figures 5 and 6. The signal flow diagram is presented as two paths; the top path illustrates the generation of the system impulse response; the bottom path illustrates the generation of the maximum response of the system.

In the top path the gust spectrum is excited by an impulse of unit strength to generate an intermediate gust impulse response which, in turn, is the excitation to the aircraft. Computationally, the time history of the response is carried out until the magnitude of the response dies out to a small fraction of the largest amplitude of the response. For the example shown in the figure, this occurs at about 10 seconds and corresponds to the lag time \( t_0 \) referred to in figures 5, 6, and 7. The response is normalized as described in connection with figure 5.

The bottom path illustrates how the maximum response of the system and the critical gust profile are obtained. For this part of the analysis, it is usually necessary to carry out the time history of the response to time \( 2t_0 \). The response builds to a maximum at time \( t_0 \) at which point the excitation ends. The response then decays to near zero. The maximum response, \( Y_{\text{max}} \), is equal to the r.m.s. of the impulse response, as was shown in figure 6. It should be mentioned that both the critical gust profile and maximum response of the system are unique to only one load output and for other maximum load responses a separate but similar analysis needs to be performed.

An important detail illustrated in this figure is the introduction of a pre-filter. The effect of the pre-filter is to provide dynamics of the input disturbance which itself contributes to the shape and magnitude of "matched" excitation waveform. In this example the pre-filter is an s-plane approximation of the von Karman spectrum, but in other applications it could be landing or taxiing disturbance dynamics for obtaining landing or taxi loads or possibly "pilot" dynamics for obtaining maneuver loads.
Matched Filter Theory
Applied to
Time-Correlated Gust Loads

Figure 8
Configuration Used For MFT and RPT Applications

In the application of MFT and RPT existing structural and aerodynamic models of the NASA DAST ARW-2 were used. This configuration, a Firebee II target drone fitted with an Aeroelastic Research Wing (ARW), was especially suited for the study since it has structural flexibility, a stable and dominant short period, and several load outputs. The load outputs are comprised of shear forces, torsion moments, and bending moments at several points along the span of the wing (ref. 9).

The figure presents relevant information about the vehicle itself and about the analytical representation. The structural part of the model was derived from a finite element code and the unsteady aerodynamics (at a Mach number of 0.7) from a doublet lattice code. Two rigid-body modes and eight symmetric flexible modes were retained for this study. The final dynamics equations (the quadruple equations), constructed with a matrix analysis code, consisted of 97 first order equations, 9 output equations and 1 input. These final equations contained the dynamics of the structure, unsteady aerodynamics, loads, and the von Karman spectrum.
Configuration Used For MFT and RPT Applications
NASA DAST ARW-2

Analysis Conditions
Altitude = 15,000 ft
Mach No. = 0.7

Vehicle Description
Weight = 2500 lb
Wing Span = 227.84 in
Reference Length = 23.47 in
Aspect Ratio = 10.3

Analytical Representation of Vehicle
2 Longitudinal Rigid-Body Modes
8 Symmetric Flexible Modes

Analytical Representation of Atmosphere
von Karman Spectrum
L = 2500 ft
σwg = 1 in/sec

Figure 9
Time-Correlated Gust Loads Using Matched Filter Theory

The two adjacent figures present time-correlated loads at the wing root of the DAST ARW-2 vehicle. Both figures contain time histories of bending moment and torsion moment at the same point in the structure. The top figure contains the bending moment and the corresponding torsion moment responses resulting from the excitation matched to root bending moment. The bottom figure contains the torsion moment and the corresponding bending moment responses resulting from the excitation matched to root torsion moment. In each figure the solid arrow indicates the response of the output to which the excitation is matched while the shaded arrow indicates the other response to the same excitation.

The critical gust profiles for both torsion moment and bending moment are generally of the same shape and magnitude but are of opposite sign. A careful examination of the plots reveals that the maximum bending moment in figure 10a is greater in magnitude than in figure 10b. Similarly, the maximum torsion moment in figure 10b is greater than in figure 10a. These results are in accord with theory which states that the excitation that is matched to a particular output is guaranteed to produce the maximum response of that output.
Time-Correlated Gust Loads
Using Matched Filter Theory

Figure 10(a)

Critical Gust Profile for Excitation Waveform
Matched to Root Bending Moment

Figure 10(b)

Critical Gust Profile for Excitation Waveform
Matched to Root Torsion Moment
Overlap of Matched Filter Theory and Random Process Theory

It has been observed in the results obtained thus far with Matched Filter Theory that the time responses to matched and nonmatched excitation waveforms resemble the auto- and cross-correlation functions encountered in Random Process Theory (ref. 10). It can be shown that this is indeed the case by writing again the response, \( y(t) \), resulting from the matched excitation, \( x(t) \). As indicated in the figure, if the transform, \( X \), of the excitation is written in terms of \( HF \) and substituted, the product \( HFHF^* \) appears in the integral. This product is the power spectral density (PSD) function of the impulse response for the output \( y \). The inverse transform is \( R(t - t_0) \), the auto-correlation function (with time argument \( t - t_0 \)) for \( y \), and the time response, \( y(t) \) is equal to \( R(t - t_0) \) divided by the r.m.s. of the impulse response.

This figure presents only the derivation showing that the response of \( y \) to the excitation matched to \( y \) is the auto-correlation function for \( y \). For example, the time response of bending moment to the excitation waveform matched to bending moment would be the auto-correlation function for bending moment divided by the r.m.s. of the bending moment impulse response.

If the response of \( y \) to an excitation matched to some other output, say \( z \), were being considered, then it would be the cross-correlation function between \( y \) and \( z \) divided by the r.m.s. of the \( z \) impulse response. Thus, the response of bending moment to the excitation matched to torsion moment would be the cross-correlation function between bending and torsion moments divided by the r.m.s. of the torsion moment impulse response. The mathematical derivation for this result is not included here but proceeds in a fashion similar to that presented here.
Overlap of Matched Filter Theory and Random Process Theory

Recall Matched Filter Result: \[ y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma_h X^* X e^{i\omega(t-t_0)} d\omega \]

From definition of \( H_F \) (Figure 5): \[ X = \frac{H^*_F}{\sigma_h} e^{-i\omega t_0} \]

Then, \[ y(t) = \frac{1}{\sigma_h} \frac{1}{2\pi} \int_{-\infty}^{\infty} H_F H_F e^{i\omega(t-t_0)} d\omega \]

or, \[ y(t) = \frac{1}{\sigma_h} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_F e^{i\omega(t-t_0)} d\omega \]

So, \[ y(t) = \frac{1}{\sigma_h} R(t-t_0) \]

Auto-Correlation Function

Figure 11
Random Process Theory Applied to Time-Correlated Gust Loads

This figure contains a signal flow diagram with two paths and is analogous to the diagram in figure 8. This figure illustrates the steps necessary to generate time-correlated gust loads using Random Process Theory. Whereas the signals in figure 8 were all in the time domain, all but one of the signals in this figure are in the frequency domain.

From the top path, the "Known Dynamics" box is the same as that in figure 8. The input to this box is Gaussian white noise, and the output is an auto-power spectral density function of some aircraft response with an intermediate output being the von Karman power spectral density function of atmospheric turbulence. Not shown in the figure, but also generated at the same time, are cross-power spectral density functions of other aircraft responses.

Time-correlated gust loads are obtained in the bottom path of the figure by taking the inverse Fourier transforms of the auto- and cross-power spectral density functions obtained in the top path. It should be mentioned that to obtain precise representations of the time correlated loads it was necessary to deal numerically with two-sided spectra (that is, with both the positive and negative frequency components present). In this figure the time axis, $\tau$, of the auto-correlation function is equivalent to the time argument, $t - t_0$, in figure 11.
Random Process Theory
Applied to
Time-Correlated Gust Loads

Gaussian White Noise

Known Dynamics

von Karman Gust (Pre-Filter)

Aircraft Load

Gust Response

Power Spectral Density

Inverse Fourier Transform

von Karman Gust Spectrum

Gust Response

Power Spectral Density Function

Auto-Correlation Function

Figure 12
Comparison of Time-Correlated Gust Loads
Using Matched Filter and Random Process Theories

This figure shows a comparison of wing-root bending-moment time responses and wing-root torsion-moment time responses calculated with the Matched Filter Theory and Random Process Theory approaches. The Matched Filter Theory results are the same as those shown in figure 10a. For purposes of comparison, the s-plane approximation of the von Karman power spectral density function is used for both the matched filter and random process calculations. Except for some slight differences in the peaks and troughs, depicted in the insets, results from the two approaches are practically indistinguishable. This is in accord with the theory presented in figure 11.
Comparison of Time-Correlated Gust Loads Using Matched Filter and Random Process Theories

Responses to Excitation Waveform Matched to Root Bending Moment

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Root Bending Moment</th>
<th>Root Torsion Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td>100</td>
<td>2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>40</td>
<td>-2.0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-4.0</td>
</tr>
</tbody>
</table>

Figure 13
Contributions

The technical contributions made by this paper are listed in this figure.

This paper has presented a new use of Matched Filter Theory and a novel interpretation of intermediate and final results. Compared to the original application of the theory (to radar), the "knowns" and "unknowns" have been reversed. In addition, the "known dynamics" have been expanded to include not only the dynamics of aircraft loads but also, through the introduction of a pre-filter, the dynamics of atmospheric turbulence. An intermediate result from the application of Matched Filter Theory to time-correlated gust loads is the critical gust profile.

The new use of Random Process Theory is another contribution of this paper. It has been shown that the time-correlated gust loads predicted by Matched Filter Theory are theoretically identical to the auto- and cross-correlation functions predicted by Random Process Theory and that there is thus a duality between the two approaches. That is, these correlation functions are now interpreted as time histories. Time-correlated gust loads may be obtained by taking the inverse Fourier transform of the auto- and cross-power spectral density functions obtained in a conventional power spectral density analysis.
Contributions

1. Unconventional Use of Matched Filter Theory
   - Switching "knowns" and "unknowns"
   - "Pre-filter" included in "known dynamics" to incorporate dynamics of atmospheric turbulence

2. Unconventional Use of Random Process Theory
   - Interpretation of auto- and cross-correlation functions as time histories

3. Demonstrating Duality Between MFT and RPT
   - Time responses obtained from MFT are theoretically identical to the auto- and cross-correlation functions of RPT

Figure 14
Concluding Remarks

This paper has described and illustrated two approaches for computing time-correlated gust loads. The first is based on Matched Filter Theory and is a time-domain approach; the second is based on Random Process Theory and is a frequency-domain approach. These approaches involve new applications of the theories and novel interpretations of the intermediate and final results.

The two approaches yield theoretically identical results and the choice of which to use depends on the intended application. Both approaches are computationally fast and are general enough to be applied a variety of dynamic-response problems, such as taxi and landing loads, maneuver loads, and gust loads.

As indicated by the bottom bullet, applying Matched Filter Theory to the calculation of time-correlated gust loads has the advantage of yielding, as an intermediate result, the critical gust profile. An additional advantage of Matched Filter Theory over Random Process Theory is that it may be applied to problems in which no input power spectral density functions are available.

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August 19, 1988
Concluding Remarks

* Two approaches, based on new interpretations of old theories, have been presented for computing time-correlated gust loads:

1. Matched Filter Theory
2. Random Process Theory

* Matched Filter and Random Process Theories

- Yield theoretically identical results
- Are computationally fast
- Are general enough that they may be applicable to a wider variety of dynamic response problems

* Matched Filter Theory has the additional advantages of

- Yielding the critical gust profile
- Being applicable if no power spectra are available

Figure 15
References


This paper describes and illustrates two ways of performing time-correlated gust-load calculations. The first is based on Matched Filter Theory; the second on Random Process Theory. The two yield theoretically identical results and both employ novel applications of the theories and unconventional interpretations of the intermediate and final results. Both approaches are computationally fast and are general enough to be applied to dynamic-response problems other than gust loads. A brief mathematical development and example calculations using both Matched Filter Theory and Random Process Theory are presented.