PARTITIONING OF REGULAR COMPUTATION ON MULTIPROCESSOR SYSTEMS

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Partitioning of Regular Computation on
Multiprocessor Systems

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Abstract

Problem partitioning of regular computation over two-dimensional meshes on multiprocessor systems is examined. The regular computation model considered involves repetitive evaluation of values at each mesh point with local communication. The computational workload and the communication pattern are the same at each mesh point. The regular computation model arises in numerical solutions of partial differential equations and simulations of cellular automata. Given a communication pattern, a systematic way to generate a family of partitions is presented. The influence of various partitioning schemes on performance is compared on the basis of computation to communication ratio.

Key Words and Phrases: Partition, Partitioning, Parallel Processing, Stencil, Regular Computation.
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1 Introduction

Applying parallel processing in solving computational intensive problems has been of much interest in recent years. There are many scientific and engineering problems in which the major computation structure is regular. This kind of regularity is a great advantage contributing to the good performance of many parallel implementation.

The regular computation model considered involves repetitive evaluation of values at each mesh point with local communication. The computational workload and the communication pattern are the same at each mesh point. This class of computations naturally arises in numerical solutions of partial differential equations and simulations of cellular automata.

The numerical solution of partial differential equations (PDE), by methods such as point Jacobi iteration, involves evaluation of the value at each mesh point at each iteration as the weighted sum of the previous values of its neighbors. The pattern of communicating neighbors is called the stencil. For example, if only the values of the north, south, east and west neighbors of a point is needed, the stencil used is called a 5-point stencil. Interestingly but maybe not surprisingly, a new non-PDE approach for solving physical problems also shares the characteristic of regular computation. Recent research in physics has shown that lattice gas cellular automata [1] have the great potential of simulating fluid flow phenomena. A cellular automaton consists of cells possessing discrete values. At each cycle, the value of a cell is evaluated as a function of the values of itself and its neighbors.

When this kind of regular computation is implemented on a multiprocessor system, it is generally preferable to divide the data space (mesh points) into partitions, and assign each partition to a different processor such that only the values of the boundary points of a partition have to be accessed by other processors [4,2]. Since performance is affected by both the computation and communication costs, the shape of partitions can have important effect on performance.

Historically, rectangular or square partitions have most commonly been assigned to processors, primarily because the resulting data structures can be easily indexed as two-dimensional arrays. Vrsalovic, et al. [4] considered the solution of Poisson’s equation over a square region using a 5-point discretization stencil. They tested triangular, square, and hexagonal partitions. Reed, Adams and Patrick [3] conducted an analytical study on selecting optimal stencil/partition pairs. They considered rectangular, triangular, square and hexagonal partitions. If computation to communication ratio is used as the criterion for comparison, they found that square partitions are best for 9-point star stencils, hexagonal\(^1\) partitions are best for 5-point stencils, 9-point cross stencils and 13-point stencils, and square and hexagonal partitions are equally good for

\(^1\)As explained in Section 2, this kind of hexagons will be referred as R-hex.
2 Generation of Partitions

In solving problems belonging to the class of regular computation, there is usually a choice of stencils. Figure 1 shows several commonly used stencils for two-dimensional meshes. The stencils to be considered in this report are the 5-point stencil, the 7-point stencil, the 9-point star stencil, the 9-point cross stencil and the 13-point stencil.

Let us define a partition to be a set of points in the two dimensional space $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of integers. The neighborhood, $N(p)$, of a point, $p$, is a set which contains the point itself and some points positioned relative to the point, where $N$ is called the neighborhood function. With this notation, we may denote the corresponding neighborhood function of the stencils considered above as $N_5$, $N_7$, $N_{9s}$, $N_{9c}$, and $N_{13}$ respectively. For example, we can express $N_5$ and $N_7$ as follows:

\[
N_5(p) = \{p, p + (1, 0), p + (0, 1), p + (-1, 0), p + (0, -1)\}
\]

\[
N_7(p) = \{p, p + (1, 0), p + (0, 1), p + (-1, 0), p + (0, -1), p + (1, 1), p + (-1, 1)\}
\]
Similarly, the neighborhood functions for other stencils can be written down easily.

The extension of a partition \( P \) under the neighborhood function \( N \) is defined to be

\[
E(P; N) = \{ q : q \in N(p), p \in P \}
\]

(1)

In other words, the extension of a partition is a new partition which contains exactly all the neighboring points of the points in the original partition.

Given any seed (initial) partition \( S \), and neighborhood function \( N \), we can recursively define a family of partitions as follows:

\[
P_k = E(P_{k-1}; N) \quad \text{if } k > 1
\]

(2)

If we denote \( E(E(P; N); N) \) as \( E^2(P; N) \), \( E(E(E(P; N); N); N) \) as \( E^3(P; N) \) and so on, we can rewrite the above partition generation scheme as follows:

\[
P_k = E^{k-1}(S; N)
\]

(3)

where \( E^0(S; N) = S \).

Since it is actually the geometric properties of a partition which are important here, we consider two partitions \( P_1 \) and \( P_2 \) equivalent if \( P_2 \) is a translation of \( P_1 \), that is, if there exists a translation vector \( u = (u_x, u_y) \) such that

\[
P_2 = T(P_1; u) = \{ q : q = p + u, p \in P_1 \}
\]

where \( T \) is the translation function. It is easy to see that the relation defined above is reflexive, symmetric and transitive, hence it is indeed an equivalence relation. This equivalence allows us to freely talk about the shape of the partitions without taking much care about the origin of the coordinate system. For our purposes, rotation equivalence and reflection equivalence are not considered here.

One type of seed we will use very often is a rectangle of size \( m \times n \), denoted as \( S_{m,n} \), where \( S_{m,n} = \{(x, y) : 1 \leq x \leq m, 1 \leq y \leq n \} \). An important special case is the single-point seed, \( S_{1,1} \). Suppose the seed is a single point, what kind of partitions will be generated if \( N \) is one of the corresponding neighborhood functions of the stencils shown in figure 1? Figure 2 shows the cases for \( N_5 \), \( N_7 \) and \( N_{9a} \). We shall call these kinds of partitions diamonds, hexagons and squares respectively. It should be noted that this hexagon is different from the one as discussed in Reed’s paper [3]. Reed’s hexagon, denoted as \( R\text{-hex} \) here, is actually some kind of diamond partition with variable seed size, according to our classification. Suppose we choose \( N_5 \) as the neighboring function. If we set \( S \) to \( S_{2,2} \) for generating \( P_1 \), \( S \) to \( S_{3,2} \) for generating \( P_2 \), \( S \) to \( S_{4,2} \) for generating \( P_3 \), and so on, we will get R-hex (see figure 2).
3. Properties of Partitions

A partition $P$ is said to tessellate $\mathbb{Z}^2$ if and only if for any finite region $R \subseteq \mathbb{Z}^2$, there exists a finite number $n$ of translation vectors $u_i$'s such that

1. $R \subseteq \bigcup_{i=1}^{n} T(P; u_i)$ and
2. $T(P; u_i) \cap T(P; u_j) = \emptyset$ for all $i, j$.

In other words, a partition tessellates if some copies of it cover any given region without overlapping each other. In a given problem, if we use only one kind of partition which tessellates the 2-D plane $\mathbb{Z}^2$, we may reduce the programming effort, because every processor will then see the same data structure and communication patterns (except possibly at boundaries).

In general, only some of the partitions of the form $E^k(S; N)$ tessellate the 2-D plane. The diamond, the hexagon, the square, and the R-hex are some examples. However, the family of partitions derived from the 9-point cross stencil, $E^k(S_{1,1}; N_9)$, does not tessellate, whereas those derived from the 13-point stencil also have the diamond shape. We will only consider those partitions which tessellate.
The grid of a partition under a neighborhood function $N$ is defined to be

$$G(P; N) = E(P; N) - P$$

(4)

The grid points are exactly those external points which have to be accessed by a processor to which the partition is assigned. Figure 3 shows the grids of various partitions under different neighborhood functions (stencil structures).

It is very important to note that the neighborhood function $N$ in equations 3 and 4 can be different. For example, if we start with a single-point seed, and choose $N$ to be $N_{9s}$ in equation 3, and $N$ to be $N_5$ in equation 4, then the number of grid points is equal to $\mid G(E^{k-1}(S_{1,1}; N_{9s}); N_5) \mid$. However, interesting results do occur when the neighborhood function $N$ in equations 3 and 4 are the same.

Since the neighborhood of a point includes itself by definition, it is obvious that $P \subseteq E(P; N)$. Combining this fact with equation 4, we have

$$E(P; N) = P \cup G(P; N)$$

(5)

Since $P \cap G(P; N) = \emptyset$ by definition of $G$ (see equation 4), we also have

$$\mid E(P; N) \mid = \mid P \mid + \mid G(P; N) \mid$$

(6)

Suppose the family of partitions $P_k$ is parametrized by $S$ and $N$, then by applying equation 6 to the definition of $P_k$ (equation 2), we have

$$\mid P_1 \mid = \mid S \mid$$

$$\mid P_k \mid = \mid P_{k-1} \mid + \mid G(P_{k-1}; N) \mid \quad \text{if } k > 1$$

(7)

Solving the recurrence equations, we get the formula for finding the size of a partition:

$$\mid P_k \mid = \mid S \mid + \sum_{i=1}^{k-1} \mid G_i \mid$$

(8)

where $G_i = G(P_i; N)$.

Equation 8 expresses the size of a partition in terms of its successive layers of grids. However, the size of a grid has to be found on a case by case basis. Tables 1 and 2 give the formula for $\mid G_k \mid$ and $\mid P_k \mid$ when the seed is a single point ($S_{1,1}$), and a rectangle ($S_{m,n}$) respectively. They can be readily derived by using mathematical induction.

It is interesting to note that $E^{k-1}(S_{1,1}; N_{9s})$ only generates square with sides of odd length, with $\mid P_k \mid = 1 + 4k(k-1) = (2k-1)^2$, and $E^{k-1}(S_{2,2}; N_{9s})$ only generates square with sides of even length, with $\mid P_k \mid = 4 + (k-1)(4k + 4) = (2k)^2$.

As a special case of diamond partitions with variable seed sizes, R-hex is generated as $P_k$, where $P_k = E^{k-1}(S_{k+1,2}; N_5)$. Substituting $m = k + 1$ and $n = 2$ into the formula for $N_5$ in table 2, we get $\mid G_k \mid = 6k + 2$, and $\mid P_k \mid = 4k^2$. 


Figure 3: Grids of partitions with different stencils. The black circles are partition points, and the white circles are grid points.
Table 1: Size of partitions and grids with single-point seed ($S_{1,1}$)

| name of partition | $N$ | $|G_k|$ | $|P_k|$ |
|-------------------|-----|-------|-------|
| diamond           | $N_5$ | $4k$ | $1 + 2k(k - 1)$ |
| hexagon           | $N_7$ | $6k$ | $1 + 3k(k - 1)$ |
| square            | $N_9$ | $8k$ | $1 + 4k(k - 1)$ |

Table 2: Size of partitions and grids with rectangular seed ($S_{m,n}$)

| $N$ | $|G_k|$ | $|P_k|$ |
|-----|-------|-------|
| $N_5$ | $4k + 2m + 2n - 4$ | $mn + (k - 1)(2k + 2m + 2n - 4)$ |
| $N_7$ | $6k + 2m + 2n - 4$ | $mn + (k - 1)(3k + 2m + 2n - 4)$ |
| $N_9$ | $8k + 2m + 2n - 4$ | $mn + (k - 1)(4k + 2m + 2n - 4)$ |

4 Comparison of Partitions

For a given partition $P$ with $N_c$ as the stencil used in the communication, we assume that the amount of computation workload is equal to the size of the partition, $|P|$, and the amount of communication is equal to the size of the grid, $|G(P; N_c)|$. This assumption was also used in [3]. The computation to communication ratio is thus defined to be

$$CCR = \frac{|P|}{|G(P; N_c)|}$$

For example, if we use the 7-point stencil communication structure, but choose to divide the domain into diamond partitions $P_k$, then the amount of computation is equal to $1 + 2k(k - 1)$, and the amount of communication is equal to $|G(P_k; N_7)| = 6k$ (see figure 3). Table 3 shows the amount of communication for the different combination of stencils and partitions.

Since the partitions have different shapes, it is not always possible to divide a given domain into identical subdomains such that each subdomain matches the right shape and size of a partition one would like to use. We may have to use a bigger partition of the same shape, but this may increase the amount of computation and change the pattern of communication. However, we can still compare the computation to communication ratio (CCR) of the various partitioning schemes in the asymptotic sense (see table 4). It is easy to see that for a given seed, the asymptotic CCR is independent of the seed itself. In table 4, $A$ denotes the number of points contained in a partition. Note that different partitions may have different sets of possible $A$ values.
Table 3: Amount of communication — $|G(P_k; N_c)|$

<table>
<thead>
<tr>
<th>stencil,$N_c$</th>
<th>diamond</th>
<th>hexagon</th>
<th>square</th>
<th>diamond,$[S_{m,n}]$</th>
<th>R-hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_5$</td>
<td>$4k$</td>
<td>$6k - 2$</td>
<td>$8k - 4$</td>
<td>$4k + 2m + 2n - 4$</td>
<td>$6k + 2$</td>
</tr>
<tr>
<td>$N_7$</td>
<td>$6k$</td>
<td>$6k$</td>
<td>$8k - 2$</td>
<td>$6k + 2m + 2n - 4$</td>
<td>$8k + 2$</td>
</tr>
<tr>
<td>$N_9_s$</td>
<td>$8k$</td>
<td>$8k$</td>
<td>$8k$</td>
<td>$8k + 2m + 2n - 4$</td>
<td>$10k + 2$</td>
</tr>
<tr>
<td>$N_9_c$</td>
<td>$8k + 4$</td>
<td>$12k - 4$</td>
<td>$16k - 8$</td>
<td>$8k + 4m + 4n - 4$</td>
<td>$12k + 8$</td>
</tr>
<tr>
<td>$N_{13}$</td>
<td>$8k + 4$</td>
<td>$12k$</td>
<td>$16k - 4$</td>
<td>$8k + 4m + 4n - 4$</td>
<td>$12k + 8$</td>
</tr>
</tbody>
</table>

To calculate the (asymptotic) $CCR$, we let the area of a partition $P_k$ be a constant $A$, and solve for $k$. For example, if we use the diamond partition $P_k$ and the 7-point stencil, then we have

$|P_k| = 1 + 2k(k - 1) = A$

Solving for $k$,

$k = (\sqrt{2A - 1} + 1)/2$

Hence,

$CCR = |P_k|/|G(P_k; N_7)|$

$= A/6k$

$= A/(3(\sqrt{2A - 1} + 1))$

$\times \sqrt{A/18}$

Similarly, we can derive the values in table 4 from tables 1, 2 and 3 according to equation 9.

From tables 4 we have the following observation:

1. In all the cases considered, $CCR$ is proportional to $\sqrt{A}$. This is not surprising, because the size of a partition is a quadratic function of $k$, while the size of the corresponding grid is a linear function of $k$.

2. For each partition, $CCR$ decreases or stays the same as $|N_c|$ increases. It is because for the same area $A$, the number of grid points increases or stays the same as there are more points contained in the communication stencil.

3. Diamond partitions yield the highest $CCR$ ($\sqrt{A/8}$) for $N_5$, hexagons are best ($\sqrt{A/12}$) for $N_7$, squares are best ($\sqrt{A/16}$) for $N_9_s$, and diamond partitions are also best ($\sqrt{A/32}$) for both $N_9_c$ and $N_{13}$ stencils. This pattern suggests that...
Table 4: Asymptotic computation to communication ratio (CCR)

<table>
<thead>
<tr>
<th>stencil, $N_c$</th>
<th>diamond</th>
<th>hexagon</th>
<th>square</th>
<th>R-hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_5$</td>
<td>$\sqrt{A}/8$</td>
<td>$\sqrt{A}/12$</td>
<td>$\sqrt{A}/16$</td>
<td>$\sqrt{A}/9$</td>
</tr>
<tr>
<td>$N_7$</td>
<td>$\sqrt{A}/18$</td>
<td>$\sqrt{A}/12$</td>
<td>$\sqrt{A}/16$</td>
<td>$\sqrt{A}/16$</td>
</tr>
<tr>
<td>$N_{9a}$</td>
<td>$\sqrt{A}/32$</td>
<td>$\sqrt{3A}/64$</td>
<td>$\sqrt{A}/16$</td>
<td>$\sqrt{A}/25$</td>
</tr>
<tr>
<td>$N_{9c}$</td>
<td>$\sqrt{A}/32$</td>
<td>$\sqrt{A}/48$</td>
<td>$\sqrt{A}/64$</td>
<td>$\sqrt{A}/36$</td>
</tr>
<tr>
<td>$N_{13}$</td>
<td>$\sqrt{A}/32$</td>
<td>$\sqrt{A}/48$</td>
<td>$\sqrt{A}/64$</td>
<td>$\sqrt{A}/36$</td>
</tr>
</tbody>
</table>

Table 5: Normalized asymptotic CCR

<table>
<thead>
<tr>
<th>stencil, $N_c$</th>
<th>diamond</th>
<th>hexagon</th>
<th>square</th>
<th>R-hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_5$</td>
<td>1.41</td>
<td>1.15</td>
<td>1</td>
<td>1.33</td>
</tr>
<tr>
<td>$N_7$</td>
<td>0.94</td>
<td>1.15</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$N_{9a}$</td>
<td>0.71</td>
<td>0.87</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>$N_{9c}$</td>
<td>1.41</td>
<td>1.15</td>
<td>1</td>
<td>1.33</td>
</tr>
<tr>
<td>$N_{13}$</td>
<td>1.41</td>
<td>1.15</td>
<td>1</td>
<td>1.33</td>
</tr>
</tbody>
</table>
Table 6: Number of neighboring partitions (for $P_k$ when $k \geq 3$)

<table>
<thead>
<tr>
<th>stencil, $N_c$</th>
<th>diamond</th>
<th>hexagon</th>
<th>square</th>
<th>R-hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_5$</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$N_7$</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>$N_{9g}$</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>$N_{9c}$</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>$N_{13}$</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

there is a formal relationship between the optimal partition and the chosen stencil: The partition derived from a stencil $N_c$ is the optimal partition in terms of computation to communication ratio when $N_c$ is also the communication stencil, for $N_c = N_5, N_7, N_{9g}$.

4. The results on selecting the optimal stencil/partition pairs reported in [3] (see section 1) correspond to the last two columns of table 4.

5. R-hex is the second best whenever diamond is the best (when $N_c = N_5, N_{9g}, N_{13}$). It is the second worst whenever diamond is the worst (when $N_c = N_7, N_{9g}$). It is never the optimal partition in any of the cases considered.

Since square partitions probably result in most regular data structures, we are especially interested in knowing how well square partitions compare with other partitions. Hence, the normalized asymptotic CCR with respect to square partitions are calculated and displayed in table 5. It shows that square partition is never more than 41% worse than any other partitions under all the cases considered.

For our purpose of finding the optimal partitions under different cases, rectangular stripes, rectangular partitions and triangular partitions are not considered. They have been previously shown to be inferior to squares or R-hex’s [3].

Good performance involves many factors. Communication cost not only depends on the total amount of communication, but also depends on the actual patterns of communication, such as the number of communicating neighbors (see table 6) and the underlying machine architectures. This report intends to give the asymptotic bound on one of the issue — optimal partitioning with respect to the computation to communication ratio. Maximizing the computation to communication ratio does not necessarily guarantee minimum execution time of a parallel program, but it is still an important indicator of the potential performance of the program. It is interesting to see how much this ratio varies under different combination of stencils and partitions.
5 Conclusion

This report has presented an analysis for selecting optimal partitions for regular computation over two-dimensional meshes given the communication stencil. The criterion used is the computation to communication ratio, which is defined to be the ratio of the size of a partition to that of its gird. It is shown that diamond partitions are best for 5-point stencils, 9-point cross stencils and 13-point stencils, hexagonal partitions are best for 7-point stencils, and square partitions are best for 9-point star stencils.
References


