NASA Technical Memorandum 101066

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(NASA-TM-101066) R-PARAMETRIZATION AND ITS ROLE IN CLASSIFICATION OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS (NASA) 24 p CSCL 01C

N89-16844

Unclas

G3/08 0191255

December 1988

NASA

National Aeronautics and Space Administration

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R-PARAMETRIZATION AND ITS ROLE IN CLASSIFICATION OF LINEAR MULTIVARIABLE FEEDBACK SYSTEMS

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SUMMARY

A classification of all the compensators that stabilize a given general plant in a linear, time-invariant multi-input, multi-output feedback system is developed. This classification, along with the associated necessary and sufficient conditions for stability of the feedback system, is achieved through the introduction of a new parametrization, referred to as R-Parametrization, which is dual of the familiar Q-Parametrization. The classification is made according to the stability conditions of the compensators and the plant by themselves; and the necessary and sufficient conditions are based on the stability of Q and R themselves.

1. INTRODUCTION

Recently, there is a new formulation of the old problem of designing a compensator for a given time-invariant multi-input, multi-output (MIMO) plant to meet a set of performance specifications. This new formulation (e.g., Callier et al. (1982), Midyasagar (1985), and Boyd et al. (1988)) takes the view of choosing the compensator from among all the compensators that stabilize the given plant and therefore represents a significant departure from the earlier design methods such as Linear-Quadratic-Gaussian (LQG) (Bryson et al. (1969)), Inverse Nyquist Array (Rosenbrock (1969)), and Characteristic Loci (MacFarlane et al. (1977)). Essential to this new formulation is a deeper understanding of the nature and the characteristics of all the compensators that stabilize any given plant. This paper is therefore addressing this specific area.

Consider a multivariable unity feedback system as shown in figure 1. The dimension of the command input u1 (and the plant output y2), and the disturbance input u2 (and the compensator output y1) are n0 and ni respectively. It is assumed that (i) the compensator, C(s), and the plant, P(s), are in general proper, (ii) neither the plant nor the compensator has unstable hidden modes, and (iii) the system is well-posed (i.e., $\det(I + DpDc) \neq 0$). The transfer matrices from u to y and from u to e are given respectively in equations (1) and (2).

*This work was performed during the academic year, 1987-88 while the author was with the Aeronautics/Astronautics Department., Stanford University, as a visiting professor.

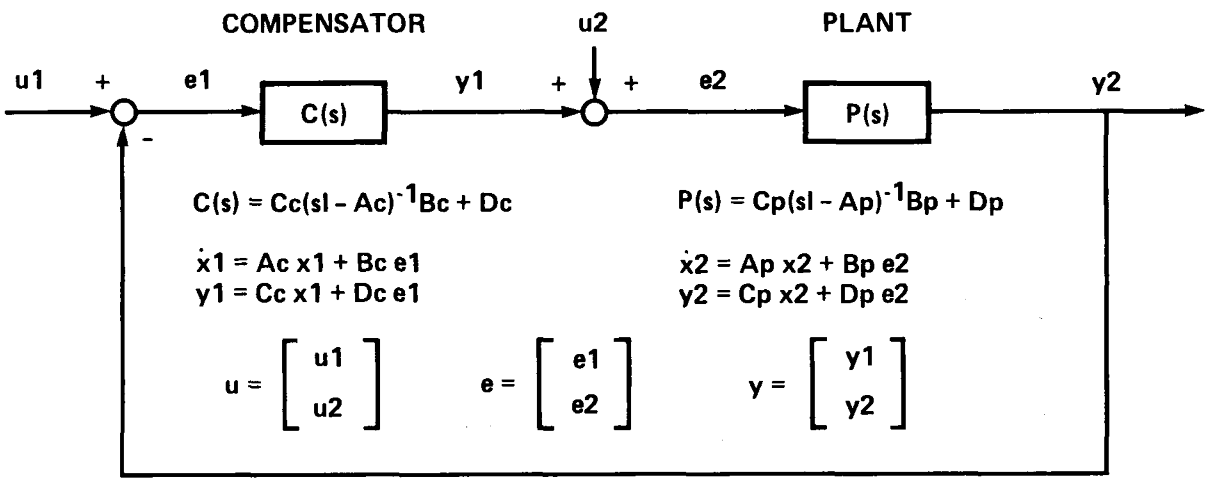

Figure 1.– Multivariable unity feedback system.

$$H_{yu} = \begin{bmatrix} C(I + PC)^{-1} & -CP(I + CP)^{-1} \\ PC(I + PC)^{-1} & P(I + CP)^{-1} \end{bmatrix} \tag{1}$$

$$H_{eu} = \begin{bmatrix} (I + PC)^{-1} & -P(I + CP)^{-1} \\ C(I + PC)^{-1} & (I + CP)^{-1} \end{bmatrix} \tag{2}$$

It is known (e.g., Callier et al. (1982)) that these transfer matrices can be cast into a form which is affine in Q as shown in equations (3) and (4). This "linearizing" process is accomplished using the nonlinear transformation pair, equations (5) and (6), and is commonly referred to as Q-Parametrization.[1]

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & P(I - QP) \end{bmatrix} \tag{3}$$

$$H_{eu} = \begin{bmatrix} I - PQ & -P(I - QP) \\ Q & I - QP \end{bmatrix} \tag{4}$$

$$Q = C(I + PC)^{-1} = (I + CP)^{-1}\, C \tag{5}$$

$$C = Q(I - PQ)^{-1} = (I - QP)^{-1}\, Q \tag{6}$$

It is worth noting, from equations (1)-(4), that Q is the transfer matrix from u1 to y1 (or from u1 to e2), i.e.,

$$Q = H_{y1,u1} = H_{e2,u1} \tag{7}$$

We will defer the discussion of other significant characteristics of the Q-Parametrization to a later section in this paper. For now we will proceed to introduce another parametrization, a new one which will henceforth be referred to as R-Parametrization.

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[1]The idea of the Q-Parametrization for SISO systems was discussed in the literature many years ago (e.g., Newton et al. (1957)).

2. R-PARAMETRIZATION

This linearizing process was motivated by the problem of answering the question: For a given stable $C(s)$ (i.e., $C(s)$ by itself is stable), what are all the plants, P, stable or otherwise, which are stabilizable by $C(s)$? This is "dual" to the problem of answering the question: Given a stable P, what are all the compensators, C, that can stabilize P? (Note that C's may or may not be stable by themselves.) We note that this latter question has been answered by the Q-Parametrization (Zames (1981), Desoer et al. (1981)) described earlier.

To seek a linearizing process directly aiming at answering the first question, it is proposed that the following nonlinear transformation pair, equations (8) and (9), be used:

$$R = P(I + CP)^{-1} = (I + PC)^{-1} P \tag{8}$$

$$P = R(I - CR)^{-1} = (I - RC)^{-1} R \tag{9}$$

It is important to recognize that R is the transfer matrix relating the plant output to disturbance input u2. With this pair of transformations, it can be shown that the transfer matrices (1) and (2) become:

$$H_{yu} = \begin{bmatrix} C(I-RC) & -CR \\ RC & R \end{bmatrix} \tag{10}$$

$$H_{eu} = \begin{bmatrix} I-RC & -R \\ C(I-RC) & I-CR \end{bmatrix} \tag{11}$$

It is seen from equations (10) and (11) that the closed-loop transfer matrices are now affine in R. From equations (3), (4), (10), and (11), the following identities are evident:

$$R = P(I - QP) = (I - PQ)P \tag{12}$$

$$Q = C(I - RC) = (I - CR)C \tag{13}$$

$$PQ = RC \tag{14}$$

$$QP = CR \tag{15}$$

These identities will prove useful in establishing some of the important stability criteria for the feedback system to be discussed next.

3. STABILITY CRITERIA FOR THE FEEDBACK SYSTEM

Since we are concerned with compensators and plants which are proper (not just strictly proper), thereby broadening the engineering applicability of our results, we will be concerned in this paper with bounded-input-bounded-output (BIBO) stability. The following fact is known.

FACT 1: If the plant P is stable by itself, then C stabilizes P (i.e., the feedback system is stable in the sense of BIBO) iff Q is stable (e.g., Callier et al. (1982)).

Fact 1 is clear from equation (3) or (4). An engineering interpretation for that seemingly mysterious necessary and sufficient condition can be made by simply redrawing figure 1 in the form of figure 2. Since the given plant P is stable, d is bounded for bounded u2. We note therefore that the signal d is equivalent to $-u1$. Thus, all the signals in the system are bounded iff Q is stable (recall from equation (7) that Q is the closed-loop transfer matrix from command input u1 to the compensator output y1). The following fact is "dual" of Fact 1.

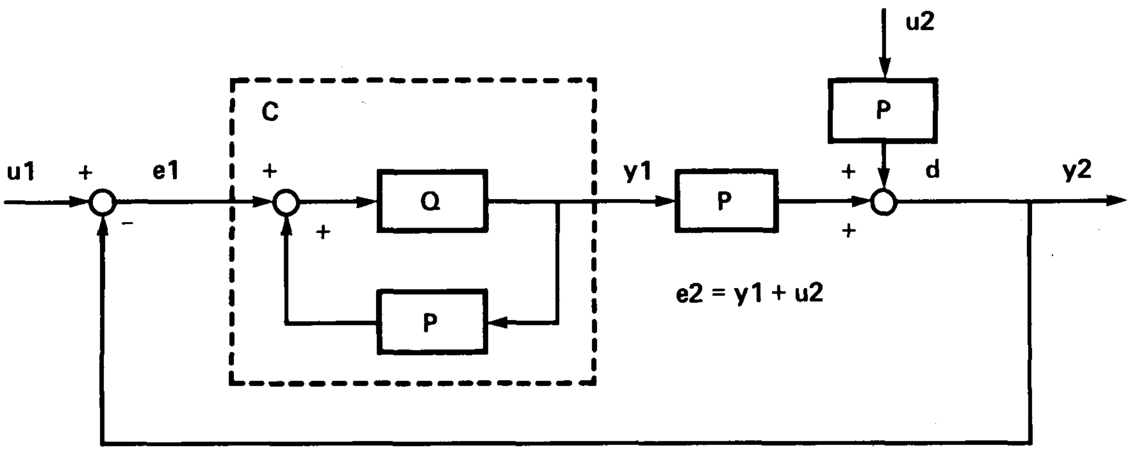

Figure 2.– Multivariable unity feedback system in terms of P and Q.

FACT 2: If the compensator C is stable, then C stabilizes P iff R is stable.

This fact is clear from equations (10) and (11). A somewhat less visualizable proof of this fact is to show algebraically from equation (1) or (2) that the feedback system is stable iff $P(I + CP)^{-1}$ is stable as is usually done in the literature (e.g., Vidyasagar (1985)). Again, a good physical insight into this fact can be made by redrawing figure 1 in the form of figure 3.

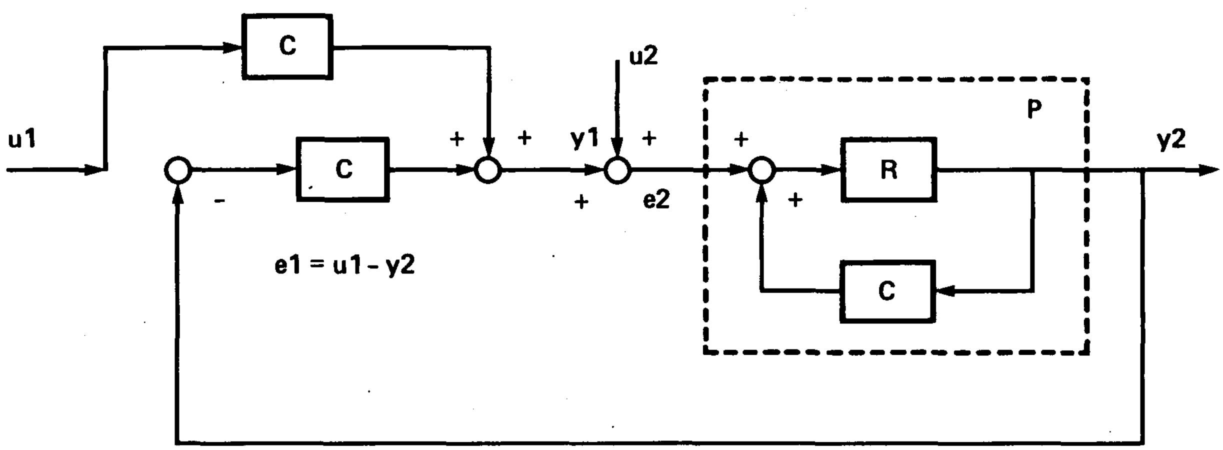

Figure 3.– Multivariable unity feedback system in terms of C and R.

We note that Cu1 is a bounded input similar to u2 since C is stable, and that the closed-loop transfer matrix from u2 to y2 is simply equal to R.

FACT 3: If both C and P are stable by themselves, then C stabilizes P iff $(I + PC)^{-1}$ (or alternatively, $(I + CP)^{-1}$) is stable.

The necessity is clear from equation (2). For sufficiency, we see that (1) can be arranged in the forms of equations (16) and (17) and the fact is established.

$$H_{yu} = \begin{bmatrix} C(I+PC)^{-1} & -C(I+PC)^{-1}P \\ PC(I+PC)^{-1} & (I+PC)^{-1}P \end{bmatrix} \tag{16}$$

$$H_{yu} = \begin{bmatrix} (I+CP)^{-1}C & -CP(I+CP)^{-1} \\ P(I+CP)^{-1}C & P(I+CP)^{-1} \end{bmatrix} \tag{17}$$

From equations (2), (4), and (11) we see that $(I + PC)^{-1} = I - PQ = I - RC$. This, along with equations (5) and (8), indicates that $(I + PC)^{-1}$ is stable iff either Q or R is stable, since both C and P are stable. We have, therefore,

FACT 3A: If C and P are both stable, then C stabilizes P iff either Q or R is stable.

We next establish Fact 4.

FACT 4: C and P are both unstable, but they do not have common unstable poles. C stabilizes P iff both Q and R are stable.

Necessity is clear by combining equations (4) and (11) in the following form

$$H_{eu} = \begin{bmatrix} I-PQ & -R \\ Q & I-QP \end{bmatrix} = \begin{bmatrix} I-RC & -R \\ Q & I-CR \end{bmatrix} \tag{18}$$

or in the form of equation (19),

$$H_{yu} = \begin{bmatrix} Q & -QP \\ PQ & R \end{bmatrix} = \begin{bmatrix} Q & -CR \\ RC & R \end{bmatrix} \tag{19}$$

For sufficiency, we see from the first equality of equation (18) or (19) that if instability occurs in the feedback system it must be caused by the unstable poles of P, since both Q and R are stable. However, the right-hand side of equation (18) or (19) indicates that the instability can only be from the unstable poles of C. Since P and C have no common unstable poles, we conclude that the feedback system cannot be unstable if both Q and R are stable, thus establishing Fact 4.

Fact 4 was first found by Desoer et al. (1975) using complex variable analysis, not in the context of Q- and R-Parametrization. To gain a better insight into this fact, we shall express the terms $I - PQ$ and $I - QP$ in equation (18) in terms of only Q and R. To do this, let $X = I - PQ$ and $Y = I - QP$. Then from equations (9), (14), and (15), we have

$$X = I - X^{-1}RQ \tag{20}$$

$$Y = I - QRY^{-1} \tag{21}$$

Solving these two second-order matrix equations yields

$$X = \frac{1}{2}\left[I \pm (I - 4RQ)^{1/2}\right] \tag{22}$$

$$Y = \frac{1}{2}\left[I \pm (I - 4QR)^{1/2}\right] \tag{23}$$

One of the two solutions in equations (22) and (23) provides the answer for $I - PQ$ and $I - QP$, respectively, as will become clear in the sequel. However, both solutions of X and Y are stable if

$$(I - 4RQ)^{1/2} \text{ and } (I - 4QR)^{1/2}$$

are stable. Note that $(I - 4RQ)$ and $(I - 4QR)$ are stable if both Q and R are stable. For the SISO case, we note that $(1 - 4RQ)^{1/2} = (1 - 4QR)^{1/2}$ and they are stable if $(1 - 4RQ)$ is stable. This fact does not hold for the MIMO case, however. Indeed, from equations (12) and (13), we have

$$I - 4RQ = (I - 2PQ)^2 = (I - 2RC)^2$$

and

$$I - 4QR = (I - 2QP)^2 = (I - 2CR)^2$$

We see that $(I - 4RQ)^{1/2} = I - 2PQ = I - 2RC$ or $(I - 4QR)^{1/2} = I - 2QP = I - 2CR$ can be unstable if P and C have common unstable poles. Examples 1 to 4 in table 1 show the four combinations of the stability characteristics of $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$, given that Q and R are both stable. Example 1 shows that C and P do not have a common unstable pole; consequently, $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$ are both stable. In examples 2 to 4, C and P have common unstable poles, which give rise to three combinations of instability in $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$. It is interesting to observe from these examples that when the instabilities occur in $(I - 4RQ)^{1/2}$ or $(I - 4QR)^{1/2}$, there are transmission zeros coinciding with the unstable common poles of C and P in the associated transfer matrices $I - PQ$, PQ, or $I - QP$, QP, respectively (see table 3). This property of having transmission zeros coinciding with poles, which has no parallel in a SISO system, may exist in a MIMO system, and is the underlying reason why $(I - 4RQ)^{1/2}$ or $(I - 4QR)^{1/2}$ becomes unstable while both Q and R are stable.

It may be appropriate at this point to return to equations (22) and (23), and examine the two solutions of X and Y. Let $X1 = [I + (I - 4RQ)^{1/2}]/2$ and $X2 = [I - (I - 4RQ)^{1/2}]/2$; $Y1 = [I + (I - 4QR)^{1/2}]/2$ and $Y2 = [I - (I - 4QR)^{1/2}]/2$. Then since $(I - 4RQ)^{1/2} = I - 2PQ$ and $(I - 4QR)^{1/2} = I - 2QP$, it follows that $X1 = I - PQ$, $Y1 = I - QP$, and $X2 = PQ$, $Y2 = QP$. This further reinforces the fact that when $(I - 4RQ)^{1/2}$ is stable, both H_{e1u1} and H_{y2u1} are stable, and when $(I - 4QR)^{1/2}$ is stable, H_{e2u2} and H_{y2u1} are stable (see eqs. (18) and (19)).

It is important to point out that, given Q and R being both stable, $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$ can sometimes remain stable (thus the feedback system remains stable) even if there are common unstable poles in C and P. In other words, when Q and R are both stable, having common unstable poles in C and P does not imply that $(I - 4RQ)^{1/2}$ or $(I - 4QR)^{1/2}$ must be unstable. This fact is illustrated by example #5 as listed in tables 1 and 2. Summarizing the above discussion, we can state that, given Q and R being both stable, and C and P being both unstable, (a) C always stabilizes P for the SISO case, and (b) C almost always stabilizes P for the MIMO case with the exception being for some "pathological" cases in which C and P have common unstable poles that happen to cause $(I - 4RQ)^{1/2}$ or $(I - 4QR)^{1/2}$ to become unstable. Fact 5 in the following addresses more precisely those pathological cases.

FACT 5: C and P are unstable and they have common unstable poles. C stabilizes P if (i) Q and R are stable, and (ii) $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$ are stable.

TABLE 1.– FIVE EXAMPLES OF MIMO UNITY FEEDBACK SYSTEMS.

Example	Parameter
#1: $C=\begin{bmatrix}\frac{s+1}{s} & 0\\ 0 & \frac{2(s+1)}{s}\end{bmatrix}$ $P=\begin{bmatrix}\frac{1}{s+1} & \frac{1}{s-1}\\ 0 & \frac{1}{s-1}\end{bmatrix}$	$a=s^2+s+2$ $Q=\begin{bmatrix}1 & -\frac{2(s+1)}{a}\\ 0 & \frac{2(s-1)(s+1)}{a}\end{bmatrix}$ $R=\begin{bmatrix}\frac{s}{(s+1)^2} & \frac{s^2}{a(s+1)}\\ 0 & \frac{s}{a}\end{bmatrix}$ $(I-4RQ)^{1/2}=\begin{bmatrix}\frac{s-1}{s+1} & -\frac{4s}{a}\\ 0 & \frac{s^2-3s-2}{a}\end{bmatrix}$ $(I-4QR)^{1/2}=\begin{bmatrix}\frac{s-1}{s+1} & -\frac{2s}{a}\\ 0 & \frac{s^2-3s-2}{a}\end{bmatrix}$

TABLE 1.– CONTINUED.

Example	Parameter
#2: $C = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix}$ $P = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$	$b = s^2 + s + 1\,; \quad c = s^2 + 3s + 1$ $Q = \begin{bmatrix} \frac{s+1}{c} & \frac{s(s+1)(s+2)}{bc} \\ 0 & \frac{s+1}{b} \end{bmatrix} \; R = \begin{bmatrix} \frac{s(s+1)}{c} & \frac{s(s+1)^2}{bc} \\ 0 & \frac{s+1}{b} \end{bmatrix}$ $(I - 4RQ)^{1/2} = \begin{bmatrix} \frac{b}{c} & -\frac{2(s+1)^2}{bc} \\ 0 & \frac{s^2+s-1}{b} \end{bmatrix}$ $(I - 4QR)^{1/2} = \begin{bmatrix} \frac{b}{c} & -\frac{2(s+1)^2(2s+1)}{sbc} \\ 0 & \frac{s^2+s-1}{b} \end{bmatrix}$

TABLE 1.– CONTINUED.

Example	Parameter
#3: $C = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$ $P = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s+1} \end{bmatrix}$ NOTE: This is the example 9 of Desoer et al. (1975).	$b = s^2 + s + 1 ; \quad c = s^2 + 3s + 1$ $Q = \begin{bmatrix} \frac{s(s+1)}{c} & \frac{s(s+1)^2}{bc} \\ 0 & \frac{s+1}{b} \end{bmatrix} \; R = \begin{bmatrix} \frac{s+1}{c} & \frac{s(s+1)(s+2)}{bc} \\ 0 & \frac{s}{b} \end{bmatrix}$ $(I - 4RQ)^{1/2} = \begin{bmatrix} \frac{b}{c} & -\frac{2(s+1)^2(2s+1)}{sbc} \\ 0 & \frac{s^2+s-1}{b} \end{bmatrix}$ $(I - 4QR)^{1/2} = \begin{bmatrix} \frac{b}{c} & -\frac{2(s+1)^3}{bc} \\ 0 & \frac{s^2+s-1}{b} \end{bmatrix}$

TABLE 1.– CONTINUED.

Example	Parameter
#4: $C = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{s-1} \\ 0 & \frac{2}{s+1} \end{bmatrix}$ $P = C$ Note: This is the example 7 of Desoer et al. (1975).	$d = s^2 + 2s + 5$ $Q = \begin{bmatrix} \frac{2(s+1)}{d} & \frac{(s+1)^2(s+3)}{d^2} \\ 0 & \frac{2(s+1)}{d} \end{bmatrix}; \quad R = Q$ $(I - 4RQ)^{1/2} = \begin{bmatrix} \frac{(s+3)(s-1)}{d} & -\frac{8(s+1)^3}{d^2(s-1)} \\ 0 & \frac{(s+3)(s-1)}{d} \end{bmatrix}$ $(I - 4QR)^{1/2} = (I - 4RQ)^{1/2}$

TABLE 1.– CONCLUDED.

Example	Parameter
#5: $C = \begin{bmatrix} \frac{s+1}{s} & 0 \\ 0 & \frac{2(s+1)}{s} \end{bmatrix}$ $P = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 0 & \frac{1}{s} \end{bmatrix}$ Note: In this example, C and P have common unstable pole at $s = 0$.	$e = s^2 + 2s + 2$ $Q = \begin{bmatrix} 1 & -\frac{2(s+1)}{e} \\ 0 & \frac{2s(s+1)}{e} \end{bmatrix}$; $R = \begin{bmatrix} \frac{s}{(s+1)^2} & \frac{s^2}{e(s+1)} \\ 0 & \frac{s}{e} \end{bmatrix}$ $(I - 4RQ)^{1/2} = \begin{bmatrix} \frac{s-1}{s+1} & -\frac{4s}{e} \\ 0 & \frac{s^2 - 2s - 2}{e} \end{bmatrix}$ $(I - 4QR)^{1/2} = \begin{bmatrix} \frac{s-1}{s+1} & -\frac{2s}{e} \\ 0 & \frac{s^2 - 2s - 2}{e} \end{bmatrix}$

TABLE 2.– STABILITY CHARACTERISTICS OF Q AND R PARAMETERS FOR THE FIVE EXAMPLES IN TABLE 1.

Parameter	Example				
	1	2*	3*	4*	5*
Q	stable	stable	stable	stable	stable
R	stable	stable	stable	stable	stable
$(I - 4RQ)^{1/2}$	stable	stable	unstable	unstable	stable
$(I - 4QR)^{1/2}$	stable	unstable	stable	unstable	stable

*C and P have common unstable poles.

TABLE 3.– POLES AND TRANSMISSION ZEROS OF VARIOUS TRANSFER MATRICES FOR THE FIVE EXAMPLES IN TABLE 1.

Example	Parameter	Poles	Transmission zeros*
#1	$(I-4RQ)^{1/2}$	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{1; (1/2)(3 \pm \sqrt{17})\}$
	$I-PQ$	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{0; 0; 0\}$
	PQ	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{-1\}$
	$(I-4QR)^{1/2}$	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{1; (1/2)(3 \pm \sqrt{17})\}$
	$I-QP$	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{0; 0; 0\}$
	QP	$\{-1; -(1/2)(1 \pm \sqrt{7}j)\}$	$\{-1\}$
#2	$(I-4RQ)^{1/2}$	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{-(1/2)(1 \pm \sqrt{3}j); -(1/2)(1 \pm \sqrt{5})\}$
	$I-PQ$	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; -1; -1; -1\}$
	PQ	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0\}$
	$(I-4QR)^{1/2}$	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; -(1/2)(1 \pm \sqrt{3}j); -(1/2)(1 \pm \sqrt{5})\}$
	$I-QP$	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; 0; -1; -1; -1\}$
	QP	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; 0\}$
#3	$(I-4RQ)^{1/2}$	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; -(1/2)(1 \pm \sqrt{3}j); -(1/2)(1 \pm \sqrt{5})\}$
	$I-PQ$	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; 0; -1;-1;-1\}$
	PQ	$\{0; -(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; 0\}$
	$(I-4QR)^{1/2}$	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{-(1/2)(1 \pm \sqrt{3}j); -(1/2)(1 \pm \sqrt{5})\}$
	$I-QP$	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0; -1; -1; -1\}$
	QP	$\{-(1/2)(3 \pm \sqrt{5}); -(1/2)(1 \pm \sqrt{3}j)\}$	$\{0\}$

*The transmission zeros are defined in terms of the Smith-McMillan form (see, e.g., Sinha (1984)).

TABLE 3.– CONCLUDED.

Example	Parameter	Poles	Transmission zeros
#4	$(I-4RQ)^{1/2}$	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1; 1; 1; -3; -3\}$
	$I-PQ$	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1; -1; -1; -1; -1\}$
	PQ	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1\}$
	$(I-4QR)^{1/2}$	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1; 1; 1; -3; -3\}$
	$I-QP$	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1; -1; -1; -1; -1\}$
	QP	$\{1; -1 \pm 2j; -1 \pm 2j\}$	$\{1\}$
#5	$(I-4RQ)^{1/2}$	$\{-1; -1 \pm j\}$	$\{1; 1 \pm \sqrt{3}\}$
	$I-PQ$	$\{-1; -1 \pm j\}$	$\{0; 0; 0\}$
	PQ	$\{-1; -1 \pm j\}$	$\{-1\}$
	$(I-4QR)^{1/2}$	$\{-1; -1 \pm j\}$	$\{1; 1 \pm \sqrt{3}\}$
	$I-QP$	$\{-1; -1 \pm j\}$	$\{0; 0; 0\}$
	QP	$\{-1; -1 \pm j\}$	$\{-1\}$

Fact 5, along with the preceding four facts, permits a complete classification of the time-invariant linear multivariable unity feedback system examined in this paper. These facts imply that 1) if the given plant P and the compensator C are unstable and they have common unstable poles, then to guarantee the stability of the feedback system it is required to ascertain that (i) both the closed-loop transfer matrix from the command input u1 to the compensator output y1 (which is Q) and the closed-loop transfer matrix from the disturbance input u2 to the plant output y2 (which is R) are stable, and (ii) both $(I-4RQ)^{1/2}$ and $(I-4QR)^{1/2}$ are stable (thus guaranteeing that the closed-loop transfer matrices from the command input to the plant output and from the disturbance input to the compensator output are stable); 2) if the plant and the compensator are unstable but they do not have common unstable poles, then it is required to ascertain only the stability of the associated Q and R; 3) if the plant is unstable but the compensator is stable, then it is required to only ascertain the stability of the associated R (the associated Q will be stable automatically); 4) if the plant is stable but the compensator is unstable, then we only need to make sure that the associated Q is stable (the stability of the associated R will be automatically satisfied); 5) if the given plant and the compensator are both stable, then we only need to ascertain that either the associated R or Q is stable (both will be stable automatically). These facts are summarized in table 4. For the sake of completeness, the SISO case is also included in the table.

TABLE 4.– CLASSIFICATION OF MIMO UNITY FEEDBACK SYSTEMS USING Q- AND R-PARAMETRIZATION.

Stability characteristics of the plant P and the compensator C	Necessary and sufficient conditions for the feedback system to be stable	
	SISO	MIMO
P: unstable C: unstable C and P have common unstable poles	$Q = C(I + PC)^{-1}$ stable and $R = P(I + CP)^{-1}$ stable	Q and R stable, and $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$ stable
P: unstable C: unstable C and P do not have common unstable poles	Q and R stable	Q and R stable
P: unstable C: stable	R stable	R stable
P: stable C: unstable	Q stable	Q stable
P: stable C: stable	Q stable or R stable	Q stable or R stable

4. COMPARISON WITH STABILITY TESTS USING THE CHARACTERISTIC POLYNOMIAL

Using a minimal state-space realization, it has been known for many years (Hsu et al. (1968)) that the characteristic polynomial of the feedback system, $\Delta(s)$ is given by

$$\Delta(s) = \Delta c(s)\ \Delta p(s) \det[I + PC] \tag{24}$$

where $\Delta c(s)$ and $\Delta p(s)$ are respectively the characteristic polynomial of the compensator and the plant. Note that $\det[I + PC] = \det[I + CP]$. Table 5 shows the results of applying equation (24) to the five examples listed in table 1. As expected, the results concerning the stability of the feedback system are identical to those shown in table 2 discussed earlier.

TABLE 5.– CHARACTERISTIC POLYNOMIAL OF THE FIVE EXAMPLES IN TABLE 1.

Examples in table 1	Characteristic polynomial of the feedback system
#1	$(s + 1)^2(s^2 + s + 2)$
#2	$s(s^2 + 3s + 1)(s^2 + s + 1)$
#3	$s(s^2 + 3s + 1)(s^2 + s + 1)$
#4	$(s - 1)^2(s^2 + 2s + 5)^2$
#5	$(s + 1)^2(s^2 + 2s + 2)$

Notice that, to apply equation (24), a minimal state-space realization (or equivalently a co-prime factorization (see Desoer et al. (1975)) of the plant and the compensator are required. By contrast, however, no such procedure is required in table 4 with Q- and R-Parametrization.

5. ILLUSTRATIVE EXAMPLES

The following simple examples serve to illustrate some of the applications that make use of the ideas shown in table 4.

Example 1. To begin with, consider a very simple unstable SISO plant as shown in figure 4.

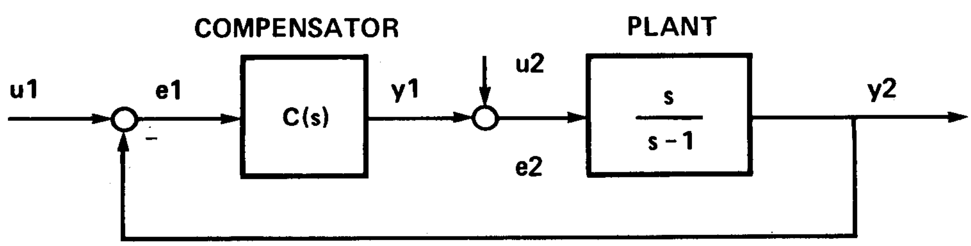

Figure 4.– A simple SISO feedback system.

Can the compensators of the forms, (a) $C(s) = C1(s) = k(s + 1)/s$ and (b) $C(s) = C2(s) = k(s + 1)/(s - 0.5)$, where k being a constant, stabilize the plant? Generate all the compensators that stabilize the plant.

We note that since both the plant and the compensators (a) and (b) are unstable for this SISO feedback system, we need, according to table 4, to ascertain that both the associated Q and R are stable. For C1(s), we found that the associated Q1 and R1, equations (3) and (5), cannot be both stable for any value of k. We conclude therefore that C1(s) cannot stabilize the plant. For C2(s), the associated Q2 and R2 are both stable with values of $k > 1.5$. Thus, the plant can be stabilized by the C2(s) with $k > 1.5$.

To generate all the compensators, stable or otherwise by themselves, which stabilize the unstable SISO plant, we must again ascertain that both the associated Q and R are stable. To meet this requirement, we first choose a general, stable, and proper Q1 such as

$$Qi = \frac{pi(s)}{qi(s)} = \frac{p0s^i + p1s^{i-1} + p2s^{i-2} + \cdots + pi}{s^i + q1s^{i-1} + q2s^{i-2} + \cdots + qi}$$

with qi(s) being Hurwitz. We then enforce the stability requirement of the associated Ri, which from equation (12) is

$$Ri = \frac{n(s)[d(s)qi(s) - n(s)pi(s)]}{[d(s)]^2\, qi(s)}$$

where the plant P(s) is denoted by P(s) = n(s)/d(s). This permits us to express the coefficients of pi(s) in terms of the coefficients of qi(s). Finally the associated compensators, Ci, that stabilize the plant are obtained.

$$Ci = \frac{d(s)pi(s)}{d(s)qi(s) - n(s)pi(s)}$$

The following compensators are generated from the above process with i = 1 and i = 2.

a) i = 1:

$$Q1 = \frac{p0s + p1}{s + q1}; \quad q1 > 0,\ p1 = -(1 + q1),\ p0 = 1 + q1$$

$$R1 = \frac{-q1s}{s + q1}$$

$$C1 = -(1 + 1/q1)$$

b) i = 2:

$$Q2 = \frac{p0s^2 + p1s + p2}{s^2 + q1s + q2}; \quad q1 > 0,\ q2 > 0,\ p0 \text{ arbitrary},$$

$$p1 = (1 + q1 + q2) - 2p0, \text{ and}$$

$$p2 = -(1 + q1 + q2) + p0$$

$$R2 = \frac{(1 - p0)s[s - q2/(1 - p0)]}{s^2 + q1s + q2}$$

$$C2 = \frac{p0s^2 + p1s + p2}{(1 - p0)(s - 1)[s - q2/(1 - p0)]} = \frac{p0}{1 - p0}\,\frac{s - [1 - (1 + q1 + q2)/p0]}{s - q2/(1 - p0)}$$

The above 3-parameter compensators can be reduced to various 2-parameter first-order compensators with a proper selection of p0, p1, and p2.

1) $p0 = 0$:

$$C2 = \frac{1 + q1 + q2}{s - q2}$$

2) $p2 = 0$:

$$C2 = -\frac{1 + q1 + q2}{q1 + q2} \frac{s}{s + q2/(q1 + q2)}$$

3) $p1 = 0$:

$$C2 = \frac{1 + q1 + q2}{1 - q1 - q2} \frac{s + 1}{s + 2q2/(q1 + q2 - 1)}$$

In particular, if it is now restricted to

$$q1 = 1 - 5q2\,, \quad q2 < 0.2$$

then the family of compensators C2(s) of (b) is produced.

$$C2 = k\frac{s + 1}{s - 0.5}\,, \quad k > 1.5$$

Example 2. Consider a simple unstable MIMO plant P(s) and the compensators of the proportional plus integral form:

$$P = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-1} \\ 0 & \frac{1}{s-1} \end{bmatrix} \qquad C = \begin{bmatrix} k1 + \frac{k2}{s} & 0 \\ 0 & k3 + \frac{k4}{s} \end{bmatrix}$$

Can C stabilize P?

Both the C and P are unstable, but they do not have common unstable poles. Thus from table 4, C stabilizes P iff Q and R are stable. We found that both Q and R are stable if

$$k1 > -1\,, \quad k2 > 0 \text{ and } k3 > 1\,, \quad k4 > 0$$

Therefore C can stabilize P if ki (i = 1, 4) are satisfied by the above conditions. In particular, if we set $k1 = k2 = 1$ and $k3 = k4 = 2$, then we reduce the compensators to the special one shown in the example #1 in table 1 discussed previously.

Example 3. Given a two-input, two-output plant P(s),

$$P(s) = \begin{bmatrix} \frac{s-2}{(s-1)(s+2)} & \frac{2(s-2)}{(s-1)(s+2)} \\ \frac{3(s-2)}{(s-1)(s+2)} & \frac{4(s-2)}{(s-1)(s+2)} \end{bmatrix}$$

Can the plant be stabilized with simple proportional compensators of the following form?

$$C(s) = \begin{bmatrix} k1 & 0 \\ 0 & k2 \end{bmatrix}$$

Since the plant is unstable but the compensators are stable, we need, according to table 4, only to ascertain the stability of the associated R. We found that R is

$$R = \frac{s-2}{r(s)} \begin{bmatrix} (s+2)(s-1) - 2(s-2)k2 & 2(s+2)(s-1) \\ 3(s+2)(s-1) & 4(s+2)(s-1) - 2(s-2)k1 \end{bmatrix}$$

$$r(s) = s^4 + (a+b)s^3 + (ab - 2a - 2b - 6k1k2)s^2 + 4(6k1k2 - ab)s - 4(6k1k2 - ab)$$

where $a = 1 + k1$, and $b = 1 + 4k2$. We see from the last two terms of r(s) that r(s) is not Hurwitz and r(s) cannot be the product of a Hurwitz and (s – 2). Thus R is always unstable and hence C(s) cannot stabilize the plant.

Example 4. Generate all the plants which are stabilizable by the following family of compensators,

$$C(s) = \begin{bmatrix} \frac{n1s + n0}{s + d0} & 0 \\ 0 & \frac{n3s + n2}{s + d1} \end{bmatrix}, \quad d0 > 0, d1 > 0$$

We recall that this is the sort of problem which provides a motivation behind our introduction of the R-Parametrization described earlier in the paper. The family of compensators is stable by itself. Therefore, the feedback system is stable iff R is stable. Let us therefore choose a sequence of stable and proper Rs such as

$$Rs = \{R0, R1, R2, \ldots, Ri, \ldots\}$$

where

$$Ri = \frac{1}{ri(s)}\left(K0s^i + K1s^{i-1} + K2s^{i-2} + \cdots + Ki\right)$$

and where K0, K1, K2, . . ., Ki are arbitrary 2 x 2 constant matrices, and ri(s) is a Hurwitz polynomial of order i in s, i.e.,

$$ri(s) = s^i + r1s^{i-1} + r2s^{i-2} + \cdots + ri$$

The sequence of the plants, Ps, associated with the compensator C and the Rs is obtained.

$$Ps = \{P0, P1, P2, \ldots, Pi, \ldots\}$$

where

$$Pi = Ri(I - CRi)^{-1}$$

Since Rs is a general and proper sequence, Ps is a sequence consisting of all the plants stabilizable by the given stable compensator. The following plants are generated from the above process with $i = 0$, and $i = 1$.

1) $i = 0$:

$$R0 = K0 = \begin{bmatrix} k1\,k2 \\ k3\,k4 \end{bmatrix}, \quad \text{ki are arbitrary constants}$$

$$P0 = R0(I - CR0)^{-1}$$

$$= \frac{1}{a0(s)} \begin{bmatrix} t01 & t02 \\ t03 & t04 \end{bmatrix}$$

where

$$a0(s) = [(1 - k4n3)s + (d1 - k4n2)][(1 - k1n1)s + (d0 - k1n0)] - k2k3(n1s + n0)(n3s + n2)$$

$$t01 = (s + d0)\{[k1(1 - k4n3) + k2k3n3]s + k1(d1 - k4n2) + k2k3n2\}$$

$$t02 = (s + d1)\{[k2(1 - k1n1) + k1k2n1]s + k2(d0 - k1n0) + k1k2n0\}$$

$$t03 = (s + d0)\{[k3(1 - k4n3) + k3k4n3]s + k3(d1 - k4n2) + k3k4n2\}$$

and

$$t04 = (s + d1)\{[k4(1 - k1n1) + k2k3n1]s + k4(d0 - k1n0) + k2k3n0\}$$

2) $i = 1$:

$$R1 = \frac{1}{s + r1}(K0s + K1), \quad K1 = \begin{bmatrix} k5 & k6 \\ k7 & k8 \end{bmatrix}, \text{ ki are arbitrary constants}$$

$$P1 = R1(I - CR1)^{-1}$$

$$= \frac{1}{a1(s)} \begin{bmatrix} t11 & t12 \\ t13 & t14 \end{bmatrix}$$

where

$$a1(s) = [(s + r1)(s + d1) - (n3s + n2)(k4s + k8)][(s + r1)(s + d0) - (n1s + n0)(k1s + k5)]$$
$$- (k2s + k6)(k3s + k7)(n1s + n0)(n3s + n2)$$

$$t11 = (s + d0)\{(k1s + k5)[(s + r1)(s + d1) - (n3s + n2)(k4s + k8)]$$
$$+ (k2s + k6)(n3s + n2)(k3s + k7)\}$$

$$t12 = (s + d1)\{(k1s + k5)(n1s + n0)(k2s + k6) + (k2s + k6)[(s + r1)(s + d0)$$
$$- (n1s + n0)(k1s + k5)]\}$$

$$t13 = (s + d0)\{(k3s + k7)[(s + r1)(s + d1) - (n3s + n2)(k4s + k8)]$$
$$+ (k4s + k8)(n3s + n2)(k3s + k7)\}$$

and

$$t14 = (s + d1)\{(k3s + k7)(n1s + n0)(k2s + k6) + (k4s + k8)[(s + r1)(s + d0) - (n1s + n0)(k1s + k5)]\}$$

We note in passing that, for $n0 = d0 = n2 = d1 = 0$, the compensators reduce to the special proportional type of compensators of example 3. As expected, under this condition, the plant in example 3 is not a member of Ps. However, the following plant,

$$P(s) = \frac{(s-4)}{(s-1)(s+4)}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

for instance, is stabilizable by the special proportional type of compensators, and we find that this plant is indeed a member of Ps.

6. CONCLUSIONS

A classification of all the compensators which stabilize a given plant, stable or unstable, in a linear, time-invariant, MIMO unity feedback system is developed. This classification is made possible by introducing a new parametrization referred herein as R-Parametrization, which is dual of the familiar Q-Parametrization. It is shown that (i) if the plant and the compensators are both unstable and they have common unstable poles, then the feedback system is stable iff (a) Q and R are stable, and (b) $(I - 4RQ)^{1/2}$ and $(I - 4QR)^{1/2}$ are stable, (ii) if the plant and the compensators are both unstable but they do not have common unstable poles, then the feedback system is stable iff both Q and R are stable, (iii) if the plant is unstable but the compensators are stable, then the feedback system is stable iff R is stable, (iv) if the plant is stable but the compensators are unstable, then the feedback system is stable iff Q is stable, and (v) if the plant and the compensators are both stable, then the feedback system is stable iff either Q or R is stable.

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Report Documentation Page

NASA
National Aeronautics and Space Administration

Field	Value
1. Report No.	NASA TM 101066
2. Government Accession No.	
3. Recipient's Catalog No.	
4. Title and Subtitle	R-Parametrization and Its Role in Classification of Linear Multivariable Feedback Systems
5. Report Date	December 1988
6. Performing Organization Code	
7. Author(s)	Robert T. N. Chen
8. Performing Organization Report No.	A-89030
9. Performing Organization Name and Address	Ames Research Center Moffett Field, CA 94035
10. Work Unit No.	505-61-51
11. Contract or Grant No.	
12. Sponsoring Agency Name and Address	National Aeronautics and Space Administration Washington, D.C. 20546
13. Type of Report and Period Covered	Technical Memorandum
14. Sponsoring Agency Code	

15. Supplementary Notes

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Moffett Field, CA 94035 (415) 694-5008 or FTS 464-5008

16. Abstract

A classification of all the compensators that stabilize a given general plant in a linear, time-invariant multi-input, multi-output feedback system is developed. This classification, along with the associated necessary and sufficient conditions for stability of the feedback system, is achieved through the introduction of a new parametrization, referred to as R-Parametrization, which is dual of the familiar Q-Parametrization. The classification is made to the stability conditions of the compensators and the plant by themselves; and necessary and sufficient conditions are based on the stability of Q and R themselves.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Multivariable feedback system Stability R-Parametrization Q-Parametrization	Unclassified-Unlimited Subject Category - 08

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	24	A02

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