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ON THE GÖRTLER VORTEX INSTABILITY MECHANISM AT HYPERSONIC SPEEDS.

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Abstract

The linear instability of the hypersonic boundary layer on a curved wall is considered. As a starting point the viscosity of the fluid is taken to be a linear function of temperature and real-gas effects are ignored. It is shown that the flow is susceptible to Görtler vortices and that they are trapped in the logarithmically thin adjustment layer in which the temperature of the basic flow changes rapidly to its free stream value. The vortices decay exponentially in both directions away from this layer and are most unstable when their wavelength is comparable with the depth of the adjustment layer. The non-uniqueness of the neutral stability curve associated with incompressible Görtler vortices is shown to disappear at high Mach numbers if the appropriate 'fast' streamwise dependence of the instability is built into the disturbance flow structure. It is shown that in the hypersonic limit wall-cooling has a negligible effect on the stability of a fluid with a given value of the Chapman constant.

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1. Introduction

Our concern is with the linear development of Görtler vortices in hypersonic boundary layers. Interest in the stability properties of hypersonic boundary layers has been stimulated by recent research and development into hypersonic aircraft. In some situations it is envisaged that Mach numbers as high as 20-25 will be achieved; in that situation real gas effects will certainly be important. Here, as a starting point, we ignore such effects and assume that the fluid under investigation behaves as a perfect gas. The other major simplification which we make is that the viscosity of the fluid is given by the Chapman's viscosity law. This simplification can of course be justified but at any rate it is probable that our analysis could be extended to deal with more realistic viscosity-temperature laws.

We shall take into account the effect of boundary layer growth on the vortices using asymptotic methods related to those used for the incompressible case by Hall (1982a,b, 1983, 1985, 1988) and Hall and Lakin (1988). The main feature of the incompressible case is that the growth of vortices with wavelengths comparable with the boundary layer thickness is dominated by non-parallel effects and can only be described self-consistently by integrating numerically the linear partial differential equations obtained by perturbing appropriately the Navier-Stokes equations. These equations were solved by Hall (1983) who showed that the position of neutral stability of a Görtler vortex is a function of its upstream origin; this means that in this wavelength regime the concept of a unique neutral curve is not tenable. Thus the results of, for example, Görtler (1940), Hammerlin (1956) which were obtained by ignoring the streamwise dependence of the vortices are not valid.

However, at smaller vortex wavelengths the vortices only feel the local structure of the boundary layer and localize themselves so as to maximize their downstream growth. In the neutral case this position corresponds to the location where Rayleigh's criterion is most violated. Moreover the vortices then develop downstream in a quasi-parallel manner and Hall (1982a) showed how the Görtler number could then be simply expressed as an asymptotic expansion in terms of the wavenumber. An asymptotic investigation of the equation used by the parallel flow theories shows that they give the first few terms in this expansion but that they are in error when non-parallel effects become important. Thus, in the only regime where the parallel flow theories are valid, a relatively trivial asymptotic expansion of the Görtler number is available and is indeed more accurate than that obtained by a numerical solution of the parallel flow eigenvalue problem.

In a recent paper Hall and Malik (1987) applied the approach of Hall (1982a) to compressible boundary layers at $O(1)$ Mach numbers. A parallel flow theory of Görtler vortices in compressible boundary layers at $O(1)$ Mach numbers had previously been given by El-Hady and Verma (1981) who followed the approach of Floryan and Saric (1979). Here, as a first step towards an understanding of the structure of Görtler vortices at hypersonic speeds, we look at the limiting form of the Hall-Malik calculation at high Mach numbers.

It turns out that the "most dangerous" location in a hypersonic boundary layer is the logarithmically small region where the basic state temperature field adjusts to its free stream value. It is in this layer that asymptotically small wavelength vortices described by the structure found for the incompressible case by Hall (1982a) become trapped.

The main result of the present calculation is that the logarithmically small layer at the edge of a hypersonic boundary layer can support a range of Görtler vortex wavelengths other than those appropriate to the high wavenumber limit of Hall (1982a). Moreover the most dangerous wavelengths of Görtler vortices at hypersonic speeds are comparable with the thickness of the logarithmic layer. Furthermore, the streamwise development of the vortices can be taken care of by using a similarity variable and a multiple scale or WKB approach so that the structure of the vortices is obtained by solving ordinary differential equations. However, it should be stressed that the replacement of the streamwise derivative of the disturbance by a constant in the work of El-Hady and Verma means that their results are in error. Thus at hypersonic speeds Görtler vortices have a quasi-parallel behaviour and a structure considerably simpler than that relevant to the $O(1)$ Mach number regime. We shall see that the neutral curve at hypersonic speeds is well-defined with left and right hand branches and a unique minimum Görtler number above which linear perturbations will grow.

The procedure adopted in the rest of the paper is as follows: in §2 we formulate the linear stability problem for a compressible boundary layer. We then write down the eigenrelation found by Hall and Malik (1987) and by taking the further limit of large Mach number show the importance of the logarithmically thin layer at the edge of the boundary layer. In §3 we show that this layer can in fact support a whole range of vortex wavenumbers and show that there exists a unique critical Görtler number at hypersonic speeds. In §4 We present results for a variety of flow properties and draw some conclusions.

2. Formulation of the stability equations and the small wavelength hypersonic limit.

Our formulation follows closely the derivation of El-Hady and Verma (1981) who extended the usual incompressible approach in a straightforward manner. Suppose that L is a typical streamwise length scale while U_∞, ρ_∞, T_∞, μ_∞ are the free stream values for velocity, density, temperature and viscosity. If we take (x^*, y^*, z^*) to be a coordinate system with x^* measuring distance along a rigid wall of variable curvature $\frac{1}{A}\kappa(\frac{x^*}{L})$ then we define a curvature parameter δ by

$$\delta = \frac{L}{A}, \tag{2.1}$$

and consider the limit $\delta \to 0$ with the Reynolds number R defined by

$$R = \frac{U_\infty L \rho_\infty}{\mu_\infty}, \tag{2.2}$$

taken to be so large that the Görtler number

$$G = 2R^{\frac{1}{2}}\delta \tag{2.3}$$

is $O(1)$. We define a Mach number M by

$$M^2 = \frac{U_\infty^2}{\gamma R^* T_\infty}. \tag{2.4}$$

Here R^* is the gas constant whilst γ is the ratio of the specific heats of the gas. The coordinate y^* measures distance normal to the body whilst z^* measures distance in the spanwise direction .The reader is referred to the book by Stewartson(1964) for a more detailed discussion of compressible boundary layers. We assume that the basic flow corresponds to a zero pressure gradient and that the viscosity μ^* and temperature are related by Chapman's Law:

$$\frac{\mu^*}{\mu_\infty} = C\frac{T^*}{T_\infty}. \tag{2.5}$$

Here C is a constant and T^* is the temperature. We denote the Prandtl number by Γ, and define dimensionless variables x, y, z by

$$(x, y, z) = (x^* L^{-1}, y^* R^{\frac{1}{2}} L^{-1}, z^* R^{\frac{1}{2}} L^{-1}),$$

whilst (u, v, w), scaled on U_∞, $R^{-\frac{1}{2}} U_\infty$, and $R^{-\frac{1}{2}} U_\infty$, denote the velocity corresponding to (x, y, z). Finally we let ρ, T and μ denote density, temperature and viscosity scaled on their free-stream values. The basic state is then given by

$$(u, v, w) = (\bar{u}(x, y), \bar{v}(x, y), 0), \quad T = \bar{T}(x, y), \quad \rho = \bar{\rho}(x, y), \quad \mu = \bar{\mu}(x, y). \tag{2.6}$$

If we define the Howarth-Dorodnitsyn variable $\bar{y}$ by

$$\bar{y} = \int_0^y \frac{dy}{\bar{T}},$$

then $\bar{u}$ and $\bar{v}$ can be written as

$$\bar{u} = f'(\eta), \bar{v} = \frac{1}{\sqrt{2x}}(\bar{T}(\eta)(\eta f' - f) - f' \int_0^\eta \eta \bar{T}'(\eta) d\eta), \tag{2.7}$$

where

$$\eta = \frac{\bar{y}}{\sqrt{2x}} \tag{2.8a}$$

and

$$f(\eta) = \sqrt{C} F(\frac{\eta}{\sqrt{C}}), \tag{2.8b}$$

where F is the Blasius function. The temperature field $\bar{T}(\eta)$ is then written in the form

$$\bar{T} = 1+B\int_{\eta}^{\infty}\{f''(\eta_1)\}^{\Gamma}d\eta_1+\Gamma(\gamma-1)M^2\int_{\eta}^{\infty}\{f''(\eta_1)\}^{\Gamma}d\eta_1\int_0^{\eta_1}\{f''(\eta_2)\}^{2-\Gamma}d\eta_2. \quad (2.9)$$

Here B is determined by the appropriate boundary condition to be imposed on $\bar{T}$ at the wall. In the aerodynamic situation it is expected that the wall will be cooled since otherwise the wall temperature will become intolerably large at high Mach numbers. However, in order to indicate the appropriate vortex structure at high Mach numbers we shall, in the first instance, assume that the Prandtl number is 1 and that the wall is adiabatic. In that case $\bar{T}$ reduces to

$$\bar{T} = 1+\frac{\gamma-1}{2}M^2(1-f'^2). \quad (2.10)$$

Next suppose ,as in Hall and Malik(1987), that the flow is perturbed to a spanwise periodic stationary vortex structure with wavenumber a. The linearized stability equations for Görtler vortices are then found by linearizing the Navier-Stokes equations about (2.7), (2.9) and retaining the leading order terms in the high Reynolds number limit. We obtain

$$\frac{1}{\bar{T}}(\bar{u}U)_x+\bar{\mu}a^2U+\frac{1}{\bar{T}}\bar{v}U_y-(\bar{\mu}U_y)_y+\frac{1}{\bar{T}}\bar{u}_yV-\left[\frac{1}{\bar{T}^2}(\bar{u}\bar{u}_x+\bar{v}\bar{u}_y)+(\tilde{\mu}\bar{u}_y)_y\right]T-\tilde{\mu}\bar{u}_yT_y = 0, (2.11a)$$

$$\frac{1}{\bar{T}}(\bar{v}_x+\kappa\bar{u}G)U-c\bar{\mu}_yU_x-(c+1)\bar{\mu}U_{xy}-\bar{\mu}_xU_y+\frac{1}{\bar{T}}(\bar{v}V)_y+\frac{\bar{u}}{\bar{T}}V_x+\bar{\mu}a^2V-(c+2)(\bar{\mu}V_y)_y+P_y$$

$$-\left[\frac{1}{\bar{T}^2}(\bar{u}\bar{v}_x+\bar{v}\bar{v}_y+\tfrac{1}{2}\kappa G\bar{u}^2)+(c+1)\tilde{\mu}\bar{u}_{xy}+c\tilde{\mu}_y\bar{u}_x+(c+2)(\tilde{\mu}\bar{v}_y)_y+\tilde{\mu}_x\bar{u}_y\right]T-\tilde{\mu}\bar{u}_yT_x$$

$$-\left[c\tilde{\mu}\bar{u}_x+(c+2)\tilde{\mu}\bar{v}_y\right]T_y-c\bar{\mu}_yiaW-(c+1)ia\bar{\mu}W_y = 0, (2.11b)$$

$$\bar{\mu}_xiaU+(c+1)\bar{\mu}iaU_x+\bar{\mu}_yiaV+(c+1)\bar{\mu}iaV_y-iaP+c\tilde{\mu}(\bar{u}_x+\bar{v}_y)$$

$$iaT-\frac{\bar{u}}{\bar{T}}W_x-(c+2)\mu a^2W-\frac{\bar{v}}{\bar{T}}W_y+(\bar{\mu}W_y)_y = 0, (2.11c)$$

$$(\frac{U}{\bar{T}})_x + (\frac{V}{\bar{T}})_y + ia(\frac{W}{\bar{T}}) - (\frac{T\bar{u}}{\bar{T}^2})_x - (\frac{T\bar{v}}{\bar{T}^2})_y = 0, (2.11d)$$

$$\frac{1}{\bar{T}}\bar{T}_x U - 2(\gamma-1)M^2\bar{\mu}\bar{u}_y U_y + \frac{1}{\bar{T}}\bar{T}_y V - [\frac{1}{\bar{T}^2}(\bar{u}\bar{T}_x + \bar{v}\bar{T}_y) + (\gamma-1)M^2\tilde{\mu}\bar{u}_y^2 + \frac{1}{\Gamma}$$

$$(\tilde{\mu}\bar{T}_y)_y]T + \frac{1}{\bar{T}}\bar{u}T_x + \frac{\bar{\mu}}{\Gamma}a^2 T + (\frac{1}{\bar{T}}\bar{v} - \frac{1}{\Gamma}\tilde{\mu}\bar{T}_y)T_y - \frac{1}{\Gamma}(\bar{\mu}T_y)_y = 0.(2.11e)$$

Here $\tilde{\mu} = d\bar{\mu}/d\bar{T}$ and $c = (\bar{\lambda}/\bar{\mu}) - 2/3$ where $\bar{\lambda}$ is the bulk viscosity, whilst (U, V, W) , P, and T denote the vortex velocity field ,pressure and temperature respectively. In the calculations reported later in this paper c has the numerical value $-.666666$

It was shown by Hall (1982a) that in the incompressible case the neutral curve for small wavelength vortices has $G \sim a^4$ and the vortices are confined to a layer of depth $a^{-\frac{1}{2}}$ where the flow is locally most unstable. Hall and Malik (1987) extended this approach to the above system for $M = O(1)$ and wrote

$$G = g_0 a^4 + g_1 a^3 + \cdots. \tag{2.12}$$

In the neutral case the vortices are trapped at the location (in y) where

$$\delta^* = \frac{\bar{\mu}^2}{\Gamma} + \frac{g_0\bar{u}^2}{2\bar{T}^3}\bar{T}_y - \frac{\bar{u}\bar{u}_y g_0}{\bar{T}^2\Gamma} = 0, \tag{2.13}$$

and

$$\frac{\partial\delta^*}{\partial y} = 0, \quad \frac{\partial^2\delta}{\partial y^2} < 0. \tag{2.14}$$

In fact (2.13) determines the constant g_0 for a given value of the streamwise variable x such that the vortices are locally neutrally stable there. Wadey (1989) has discussed the strongly nonlinear compressible Görtler vortex regime using the approach of Hall and Lakin (1988) and shows that (2.13) then determines $\bar{u}$ such that a strongly nonlinear vortex can exist. For simplicity we take $\Gamma = 1$, in which case (2.13) reduces, after some simplification, to

$$\frac{g_0}{\sqrt{2x}} = \frac{C^2\{1 + \frac{(\gamma-1)M^2}{2}(1 - f'^2)\}^6}{f'f''\{1 + \frac{(\gamma-1)M^2}{2}\}}, \tag{2.15}$$

where (2.5) has been used and a prime denotes differentiation with respect to η. In Figure 1 we have shown $g_0/\sqrt{2x}$ calculated from (2.15), subject to $g_0' = 0$, $g_0'' > 0$, as a function of Mach number with $C = 1$, $\gamma = 1.4$. We see that g_0 initially increases from its incompressible value, reaches a maximum at $M_0 \sim 5$, and then decays. The corresponding value of η where g_0 satisfies (2.15) is denoted by η_c and is shown as a function of M in Figure 2. We see that η_c moves into the free stream as M increases. The precise large M asymptotic structure of g_0 and η_c is found by first noting that for $\eta >> 1$

$$f = \sqrt{C}\{\frac{\eta}{\sqrt{C}} - \beta + \frac{\tilde{c}e^{-\frac{1}{2}(\eta/\sqrt{C}-\beta)^2}}{(\eta/\sqrt{C}-\beta)^2} + \cdots\},$$

where $\beta = 1.2168$ and $\tilde{c}$ is a constant. It then follows that the minimum value of g_0 for $M >> 1$ occurs for

$$1 - f'^2 = \frac{2\tilde{c}e^{-(\eta/\sqrt{C}-\beta)^2/2}}{(\eta/\sqrt{C}-\beta)} \sim M^{-2}.$$

More precisely if we define $\tilde{y}$ by

$$\frac{\eta}{\sqrt{C}} - \beta = \sqrt{2\log M^2} - \frac{\{\tilde{y} + \log\sqrt{2C\log M^2}\}}{\sqrt{2\log M^2}},$$

we find that (2.15) reduces to

$$\frac{g_0}{\sqrt{2x}} = \frac{2C^{5/2}\{1 + Ne^{\tilde{y}}\}^6}{Ne^{\tilde{y}}\sqrt{2\log M^2}} \tag{2.16}$$

with $N = (\gamma - 1)\tilde{c}\sqrt{C}$, so that g_0 attains its minimum value when $Ne^{\tilde{y}} = \frac{1}{5}$ and then

$$\frac{g_0}{\sqrt{2x}} = \frac{10C^{\frac{5}{2}}(\frac{6}{5})^6}{\sqrt{2\log M^2}}, \tag{2.17}$$

and this asymptotic prediction is shown in Figure 1by the dotted curve. In fact in the layer where $\tilde{y} = O(1)$ the temperature is given by

$$\bar{T} = 1 + Ne^{\tilde{y}}, \tag{2.18}$$

so that the vortices are trapped in the logarithmically thin layer where $\bar{T}$ adjusts from its free stream value of 1 to an $O(M^2)$ temperature typical of a hypersonic boundary layer. Thus in the double limit $a \to \infty$, $M \to \infty$ the vortices are trapped in a thin layer embedded within the logarithmic layer. We now show in the next section that other more unstable vortices persisting throughout the adjustment layer can exist.

3. Hypersonic Görtler vortices with wavelength $0\{(\sqrt{2\log M^2})^{-1}\}$

We now investigate the possibility that the logarithmic adjustment layer at the edge of the boundary layer can support vortices with larger wavelengths than those described by Hall and Malik. If the nondimensional wall temperature of the basic state is T_w then B in (2.9) is determined by

$$B = \frac{T_w - 1 - \Gamma(\gamma-1)M^2\int_0^\infty (f''(\eta_1))^\Gamma d\eta_1 \int_0^{\eta_1}(f''(\eta_2))^{2-\Gamma}d\eta_2}{\int_0^\infty (f''(\eta_1)^\Gamma d\eta_1}. \tag{3.1}$$

If we impose an adiabatic wall temperature condition then (3.1) determines the wall recovery temperature T_w and $B = 0$. Since the Prandtl number is not necessarily unity now we now define a stretched variable $\tilde{y}$ in the adjustment zone by

$$\eta - \sqrt{C}\beta = \sqrt{\frac{C}{\Gamma}}\sqrt{2\log M^2} - \frac{\sqrt{\frac{C}{\Gamma}}(\tilde{y} + \log\sqrt{\frac{2C}{\Gamma}\log M^2})}{\sqrt{2\log M^2}}. \tag{3.2}$$

After some manipulation we can then show from (2.9) that for $\tilde{y} = O(1)$ the temperature $\bar{T}$ may be expressed as

$$\bar{T} = 1 + Ne^{\tilde{y}} \tag{3.3}$$

where

$$N = \frac{C}{\Gamma}\left(\frac{\tilde{c}}{\sqrt{C}}\right)^{\Gamma}\{(\gamma - 1)\,\Gamma \int_0^{\infty} (f''(\eta))^{2-\Gamma} d\eta + \lim_{M\to\infty} (BM^{-2})\}. \tag{3.4}$$

Thus the wall condition on $\bar{T}$ determines the constant N appearing in (3.3) which gives the temperature in the logarithmic adjustment zone. In fact we shall see that the critical Görtler number at which instability first occurs is independent of N, hence in a hypersonic boundary layer wall cooling has no effect on the onset of the Görtler instability.

For convenience we now define M_1 and M_2 by

$$M_1 = M^2 \quad , \quad M_2 = \sqrt{2\log M^2}, \tag{3.5a, b}$$

and if we are to seek a vortex structure with wavelength comparable with the depth of the adjustment layer we must expand the wavenumber a in the form

$$a = \frac{M_2 \tilde{k}}{\sqrt{2x}} + \cdots. \tag{3.6}$$

Here the factor $\sqrt{2x}$ has been inverted in order to scale x out of the eigenvalue problem $G = G(x, a)$; we are thus determining G such that the flow is locally neutral at the position x. The growth or decay rates of a vortex trapped in the adjustment layer must be $O(M_2)$ in order to enter the zeroth order stability problem so that in the hypersonic limit the parallel flow approximation becomes valid. Here we concentrate on finding the neutral curve and therefore determine $G = G(x, a)$ such that the vortex with wavenumber a is neutrally stable at x. The previous discussion of the double limit $a \to \infty$, $M \to \infty$ suggests that the Görtler number G should be expanded as

$$G = \frac{M_2^3 \tilde{G}}{(2x)^{\frac{3}{2}}} + \cdots .$$

However an examination of (2.11b) shows that the term $\frac{1}{T^2}\frac{1}{2}kG\bar{u}^2T$ which balances with the term $\bar{u}a^2V$ in the limit $G \to \infty$ will be comparable with the term $\frac{1}{T^2}\bar{u}\bar{v}_xT$ when $G = 0(M^2)$. Thus , as pointed out by F.T.Smith (private communication) , the curvature of the basic state produces an effective negative Görtler number of order M^2 in the absence of curvature so that instability cannot occur for $G << M^2$. Hence we must expand G in such a way that the dominant contributions from the terms $\bar{u}\bar{v}_x$ and $\frac{1}{2}GK\bar{u}^2$ cancel in the adjustment layer. To achieve this we note that in this layer $\bar{u}\bar{v}_x = \frac{-1}{(2x)^3/2}QM^2 + \cdots$ where $Q = \underset{M\to\infty}{\mathcal{L}t}\frac{1}{M^2}\int_0^\infty \eta\bar{T}'dy$. Thus G must be expanded in the form

$$G = \frac{1}{(2x)^{3/2}}\{QM^2 + M_2^3\tilde{G} + \cdots\}, \tag{3.7}$$

and the disturbance quantities expand as

$$U = \left[\tilde{U}(x,\tilde{y}) + \cdots\right]E, V = \left[M_1M_2\tilde{V}(x,\tilde{y}) + \cdots\right]E, W = \left[M_1M_2\tilde{W}(x,\tilde{y}) + \cdots\right]E, (3.8a-e)$$

$$P = \left[M_1M_2^2\frac{\tilde{P}}{\sqrt{2x}} + \cdots\right]E., T = \left[M_1\sqrt{2x}\tilde{\theta} + \cdots\right]E$$

where $E = e^{\int^x M_2\sigma(x)dx}$. Thus the local growth rate of the vortex is $M_2\sigma(x)$ but here we restrict our attention to the situation $\sigma(x) = 0$. Again the factors of $\sqrt{2x}$ inserted in (3.8) have been introduced in order to scale x out of the eigenvalue problem at the neutral location. If the above expansions are inserted into (2.11) we find after some manipulation that

$$\sqrt{\frac{\Gamma}{C}}\frac{d\tilde{V}}{d\tilde{y}} + \frac{d\tilde{\theta}}{d\tilde{y}} = \frac{Ne^{\tilde{y}}}{\mathcal{F}}(\sqrt{\frac{\Gamma}{C}}\tilde{V} + \tilde{\theta}) + i\tilde{k}\mathcal{F}\tilde{W},$$

$$\sqrt{\frac{\Gamma}{C}}\frac{d\tilde{P}}{d\tilde{y}} = (c+2)\sqrt{\Gamma C}\frac{d^2\tilde{\theta}}{d\tilde{y}^2} + \{(c+2)\frac{\Gamma N e^{\tilde{y}}}{\mathcal{F}} - 1\}\sqrt{\frac{C}{\Gamma}}\frac{d\tilde{\theta}}{d\tilde{y}}$$

$$-\frac{\tilde{\theta}}{\mathcal{F}}\{\frac{1}{2}\tilde{G} - 2\sqrt{\frac{C}{\Gamma}}Ne^{\tilde{y}} - \sqrt{\frac{\Gamma}{c}}Ne^{y} + (c+2)\frac{\sqrt{\Gamma C}N^2e^{2\tilde{y}}}{\mathcal{F}}\} - i\tilde{k}\mathcal{F}\sqrt{C\Gamma}\frac{d\tilde{W}}{d\tilde{y}} + \{\sqrt{\frac{C}{\Gamma}}\mathcal{F} - (c+4)$$

$$\sqrt{\Gamma C}Ne^{\tilde{y}}\}i\tilde{k}\tilde{W} + \{C\mathcal{F}^2\tilde{k}^2 + \frac{2Ne^{\tilde{y}}}{\mathcal{F}} - (c+2)\Gamma\frac{Ne^{\tilde{y}}}{\mathcal{F}}\}\tilde{V}, \Gamma\frac{d^2\tilde{W}}{d\tilde{y}^2} = \frac{d\tilde{W}}{d\tilde{y}} + C\mathcal{F}^2\tilde{k}^2\tilde{W} \quad (3.9a,b,c,d,e)$$

$$+(c+2)\sqrt{C\Gamma}Ne^{\tilde{y}}i\tilde{k}\tilde{V} + i\tilde{k}\mathcal{F}\tilde{P} - (c+1)C\mathcal{F}i\tilde{k}\frac{d\tilde{\theta}}{d\tilde{y}} + CNe^{\tilde{y}}i\tilde{k}\tilde{\theta}, \frac{d^2\tilde{\theta}}{d\tilde{y}^2} = \frac{d\tilde{\theta}}{d\tilde{y}}\mathcal{F}^{-1} + \{\frac{C\tilde{k}^2\mathcal{F}^2}{\Gamma} - N$$

$$e^{\tilde{y}}(1+\mathcal{F}^{-1})\mathcal{F}^{-1}\}\tilde{\theta} - \frac{Ne^{\tilde{y}}}{\mathcal{F}}\sqrt{\frac{\Gamma}{C}}\tilde{V}, \tilde{\theta} = \tilde{V} = \tilde{W} = 0 \quad , \quad \tilde{y} = \pm\infty.$$

Here the function $\mathcal{F}(\tilde{y}) = 1 + Ne^{\tilde{y}}$.

The zeroth order approximation to the x momentum equation has not been written down here because it decouples from the other equations and thus plays no role in the eigenvalue problem. A result of some importance follows immediately from (3.9) if we note that by a simple change of origin in $\tilde{y}$ we can, without any loss of generality, set $N = 1$ since whenever it appears it is multiplied by $e^{\tilde{y}}$. Again this means that the wall conditions on the basic temperature field have no effect on the Görtler number for the onset of Görtler vortices in the hypersonic limit.

The eigenvalue problem specified by (3.9) was solved using the compact finite difference scheme of Malik, Chuang and Hussaini (1982). In order to apply the scheme the boundary conditions (3.9e) must be applied at finite values of $\tilde{y}$ so it is necessary to investigate the required decay of the disturbance quantities at $\tilde{y} = \pm\infty$ in order to derive asymptotic boundary conditions. After some manipulation it is found from (3.9) that when $\tilde{y} \to -\infty$

$$\tilde{\theta} = \bar{P}e^{\mu_1 \tilde{y}}, \tilde{V} = \bar{Q}e^{\mu_2 \tilde{y}} + \bar{R}e^{\mu_3 \tilde{y}} + \frac{\{\frac{1}{2}\tilde{k}^2\tilde{G} - \mu_1^3(\frac{\Gamma}{C})^{\frac{1}{2}}(\Gamma - 1)\}}{\frac{\Gamma^2}{C}(\mu_1^2 - \frac{\mu_1}{\Gamma} - \frac{\tilde{k}^2 C}{\Gamma})(\mu_1^2 - \frac{\tilde{k}^2 C}{\Gamma})}\bar{P}e^{\mu_1 \tilde{y}}, \quad (3.10)$$

where

$$\mu_1 = \frac{1}{2}\{1 + \sqrt{1 + \frac{4C\tilde{k}^2}{\Gamma}}\}, \mu_2 = \frac{1}{2}\{\frac{1}{\Gamma} + \sqrt{\frac{1}{\Gamma^2} + \frac{4C\tilde{k}^2}{\Gamma}}\}, \mu_3 = \tilde{k}\sqrt{\frac{C}{\Gamma}}. \quad (3.11a,b,c)$$

The constants $\bar{P}, \bar{Q}$ and $\bar{R}$ are arbitrary so that as $\tilde{y} \to \infty$ we have three independent solutions of (3.9). The quantities $\tilde{W}, \tilde{P}, \frac{d\tilde{W}}{d\tilde{y}}, \frac{d\tilde{\theta}}{d\tilde{y}}$ can then be expressed in terms of $\bar{P}, \bar{Q}$ and $\bar{R}$ and when the three latter constants are eliminated we obtain the asymptotic boundary conditions

$$\frac{d\tilde{\theta}}{d\tilde{y}} - \mu_1\tilde{\theta} = 0, i\tilde{k}\tilde{W} + F\tilde{\theta} = \sqrt{\frac{\Gamma}{C}}\mu_2\tilde{V} + \{-S\mu_2\sqrt{\frac{\Gamma}{C}} + \frac{H(\mu_2 - \mu_3)}{\mu_3^2}\}\tilde{\theta} + \frac{\tilde{k}^2}{\mu_3^2}(\mu_3 - \mu_2)\tilde{P},$$

$$i\tilde{k}\frac{d\tilde{W}}{d\tilde{y}} + F\frac{d\tilde{\theta}}{d\tilde{y}} = \sqrt{\frac{\Gamma}{C}}\mu_2^2\tilde{V} + \{-S\mu_2^2\sqrt{\frac{\Gamma}{C}} + H(-1 + \frac{\mu_2^2}{u_3^2})\}\tilde{\theta} + \tilde{k}^2\tilde{P}(1 - \frac{\mu_2^2}{\mu_3^2}), \quad (3.12a,bc)$$

where

$$S = \frac{\frac{1}{2}\tilde{k}^2\tilde{G} - \mu_1^3(\frac{\Gamma}{C})^{\frac{1}{2}}(\Gamma - 1)}{\frac{\Gamma^2}{C}\{\mu_1^2 - \frac{\mu_1}{\Gamma} - \frac{\tilde{k}^2 C}{\Gamma}\}\{\mu_1^2 - \frac{\tilde{k}^2 C}{\Gamma}\}}, F = -\mu_1 - \sqrt{\frac{\Gamma}{C}}S\mu_1, H = C(c+1)\tilde{k}^2\mu_1 + F\mu_1(\Gamma - 1).$$

When $\tilde{y} \to \infty$ a related set of boundary conditions can be derived using the WKB method. We do not give the full details here because it is a routine but tedious calculation to derive these boundary conditions. However in order to point out the rather abrupt decay of the disturbance which occurs for $\tilde{y} \to \infty$ we note that in this limit (3.9d) reduces to

$$\frac{d^2\tilde{\theta}}{d\tilde{y}^2} - \tilde{k}^2 e^{2\tilde{y}}\frac{C}{\Gamma}\tilde{\theta} = -\sqrt{\frac{\Gamma}{C}}\tilde{V},$$

and if we set $\tilde{V} = 0$ the required decaying solution of this equation has

$$\tilde{\theta} \sim e^{-\sqrt{C/\Gamma}\tilde{k}e^{\tilde{y}}}.$$

In fact, if we do not set $\tilde{V} = 0$ and consider the limit of the other disturbance equations we find two further decaying solutions with a similar structure to that above. The appropriate boundary conditions, as $\tilde{y} \to +\infty$ are then found by eliminating the constants associated with the three decaying solutions. However the structure of the decaying solutions for $\tilde{y} \to \infty$ means that the lower boundary of the region of vortex activity will be much sharper than the upper one. This difference will clearly be most pronounced in the small wavenumber limit.

We shall present the results of our numerical investigation in the next section. Here we note that in order to find the neutral Görtler number correct to the graphical accuracy of the figures shown in that section it was sufficient to use 200 equally spaced grid points and approximate $(-\infty, +\infty)$ by $(-10., 15.0)$. Without the asymptotic boundary conditions at $\pm\infty$ the magnitude of the required approximation to $-\infty$ must, particularly for $\tilde{k} << 1$, be significantly increased.

4. Results and Discussion

We shall in the first place discuss the results of our numerical solution of the eigenvalue problem $\tilde{G} = \tilde{G}(\tilde{k})$. This was carried out using the scheme discussed in the previous section and Figure 3 shows the neutral curves for three different values of the Chapman constant C and $\Gamma = 0.72$. We see that the curves have well defined left and right hand branches typical of Taylor vortex problems. The critical Görtler number is seen to increase monotonically with the Chapman constant C. We note that for a given value of C the basic hypersonic boundary layer is unstable to Görtler vortex disturbances if $(\tilde{k}, \tilde{G})$ is above the appropriate neutral curve of Figure 3.

Thus we see that in the hypersonic limit a fluid with viscosity given by Chapman's Law supports Görtler vortices whose growth is governed by a parallel flow theory. The parallel flow approach is valid in this situation because of the development of the thin logarithmic adjustment layer in a hypersonic boundary layer. At $O(1)$ Mach numbers there is no such adjustment region and the growth of the Görtler vortex mechanism is controlled by non-parallel effects.

The eigenfunctions associated with the $C = 0.5$ calculations are shown in Figures 4 and 5 for two values of the vortex wavenumber. We note that the rate of decay of the vortex when $\tilde{y} \to \infty$ is much greater than when $\tilde{y} \to -\infty$. This is consistent with the discussion of the previous section.

Now let us turn to a discussion of the relationship of our results with those found in earlier investigations of Görtler vortices in compressible boundary layers. Firstly, we note that the scalings used in the present paper are those inferred by the small wavelength limit of Hall and Malik (1987) if the further limit $M \to \infty$ is taken. However, the fact that the logarithmic adjustment layer is relatively thin means that both the left and right hand branches of the neutral curve can be described by a quasi-parallel theory. In actual fact it is perhaps strictly not correct to describe the analysis of §3 as a parallel flow theory because the whole basic flow and disturbance structure in the adjustment layer depends on x. In a parallel flow theory, e.g. El Hady and Verma (1981), the operator $\bar{u}\partial_x + \bar{v}\partial_y$, which appears in (2.11), would be replaced by $\beta\bar{u} + \bar{v}\partial_y$ where β is a constant growth rate. In our analysis this operator is rewritten in terms of ∂_x and $\partial_{\tilde{y}}$ with the result that the x derivative in the resulting operator becomes formally $O\{(\sqrt{2\log M^2})^{-1}\}$ smaller than the $\tilde{y}$ derivative and therefore negligible. In other words the disturbance has a 'fast' x dependence built in when we change variables; this structure is lost when a parallel flow theory of the type used by El Hady and Verma arbitrarily replaces $\frac{\partial}{\partial x}$ acting on a disturbance quantity in (2.11) by a constant.

El Hady and Verma gave neutral curves for Mach numbers in the range $0 < M < 5$ and found that the critical Görtler number increased monotonically with M. (See Figure 1 of El Hady and Verma (1981)). This is entirely consistent with our prediction that the Görtler number scales on $(\log M^2)^{\frac{3}{2}}$ for $M >> 1$. However in our calculations we find that the critical wavenumber increases like $(\log M^2)^{\frac{1}{2}}$ whilst they found that , at least in the range $0 < M < 5$, it decreased with M.

The first possible explanation for this difference is that El Hady and Verna did not perform computations at large enough values of M in order to see the correct large M dependence of the wavenumber. Alternatively, it could be that the parallel flow approximation in their calculation necessarily leads to the incorrect wavenumber behaviour for large M. At $O(1)$ values of M their results are clearly incorrect and the disturbance growth must be described using the approach of Hall (1983). In any case it is clear that in the only Mach number regions where a parallel flow theory is valid the approach of El Hady and Verna effectively ignores the 'fast' x dependence of the disturbance in the adjustment layer.The authors acknowledge the comments made by the referees on the original form of this paper.

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Legends

Figure 1. The dependence of $g_0/\sqrt{2x}$ on M calculated from (2.15)

Figure 2. The dependence of η_c, the value of η at which g_0 is calculated, on M

Figure 3. The neutral curves in the $\tilde{G} - \tilde{k}$ plane for three different values of the Chapman constant C.

Figure 4. The eigenfunctions for the case $C = .5, \tilde{k} = .4$

Figure 5. The eigenfunctions for the case $C = .5, \tilde{k} = .2$

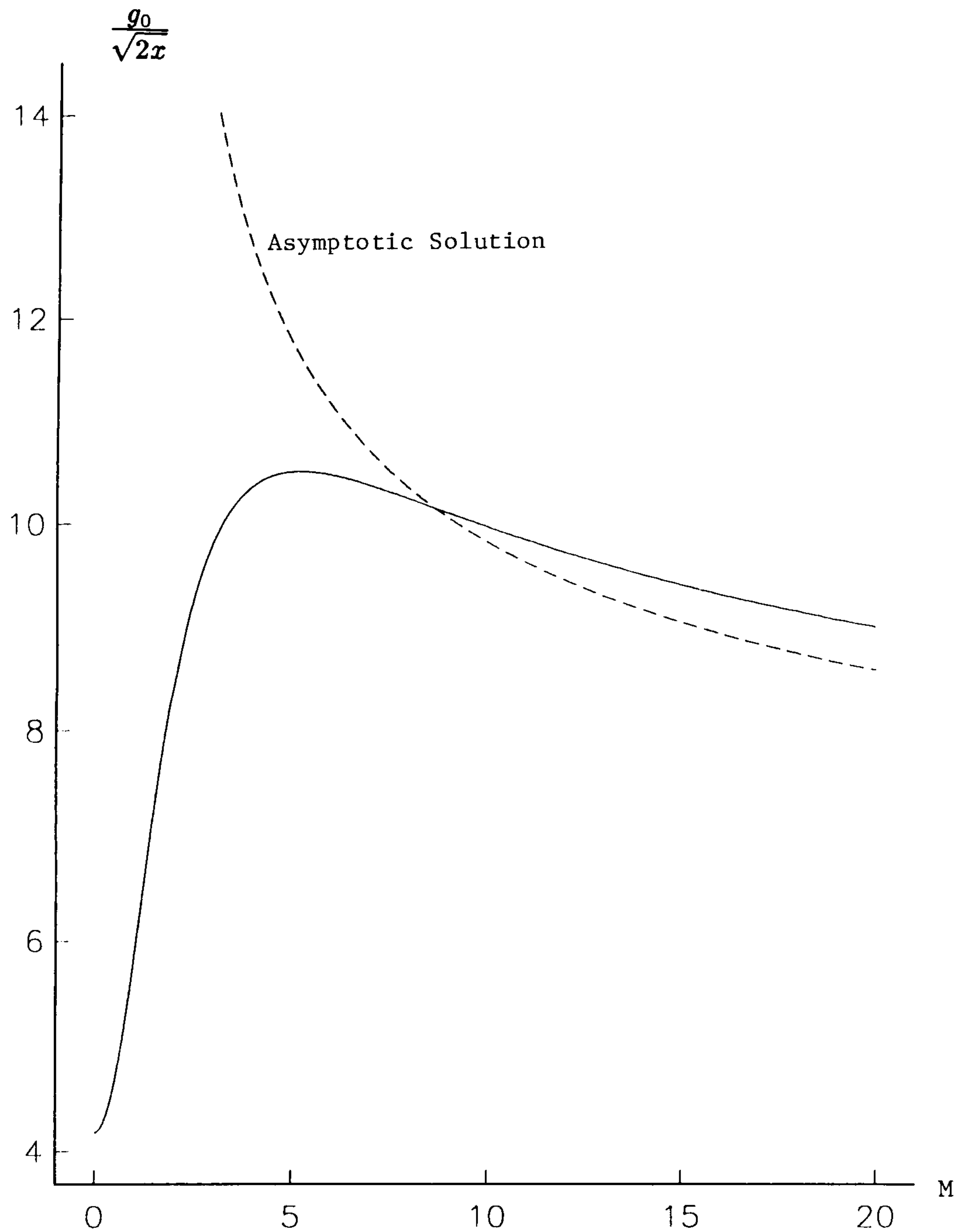


Figure 1

Figure 2

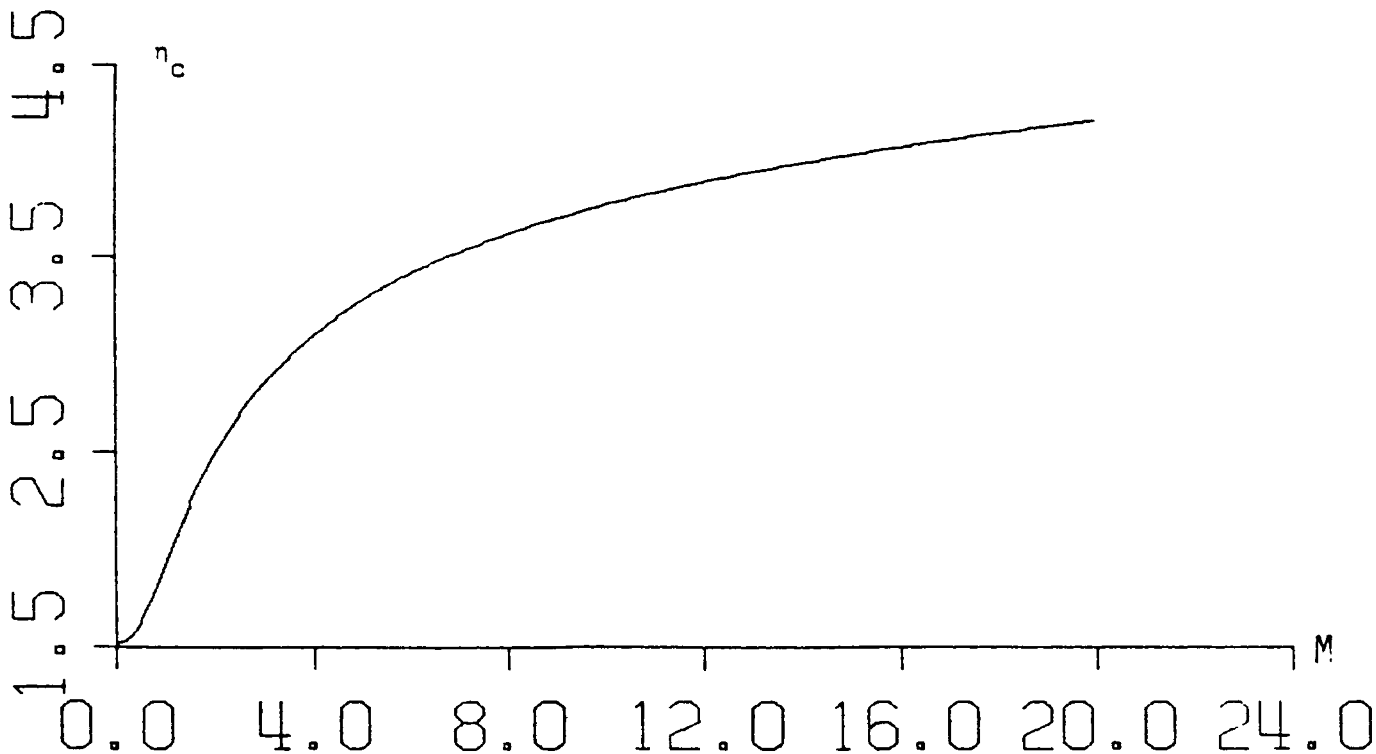


Figure 3

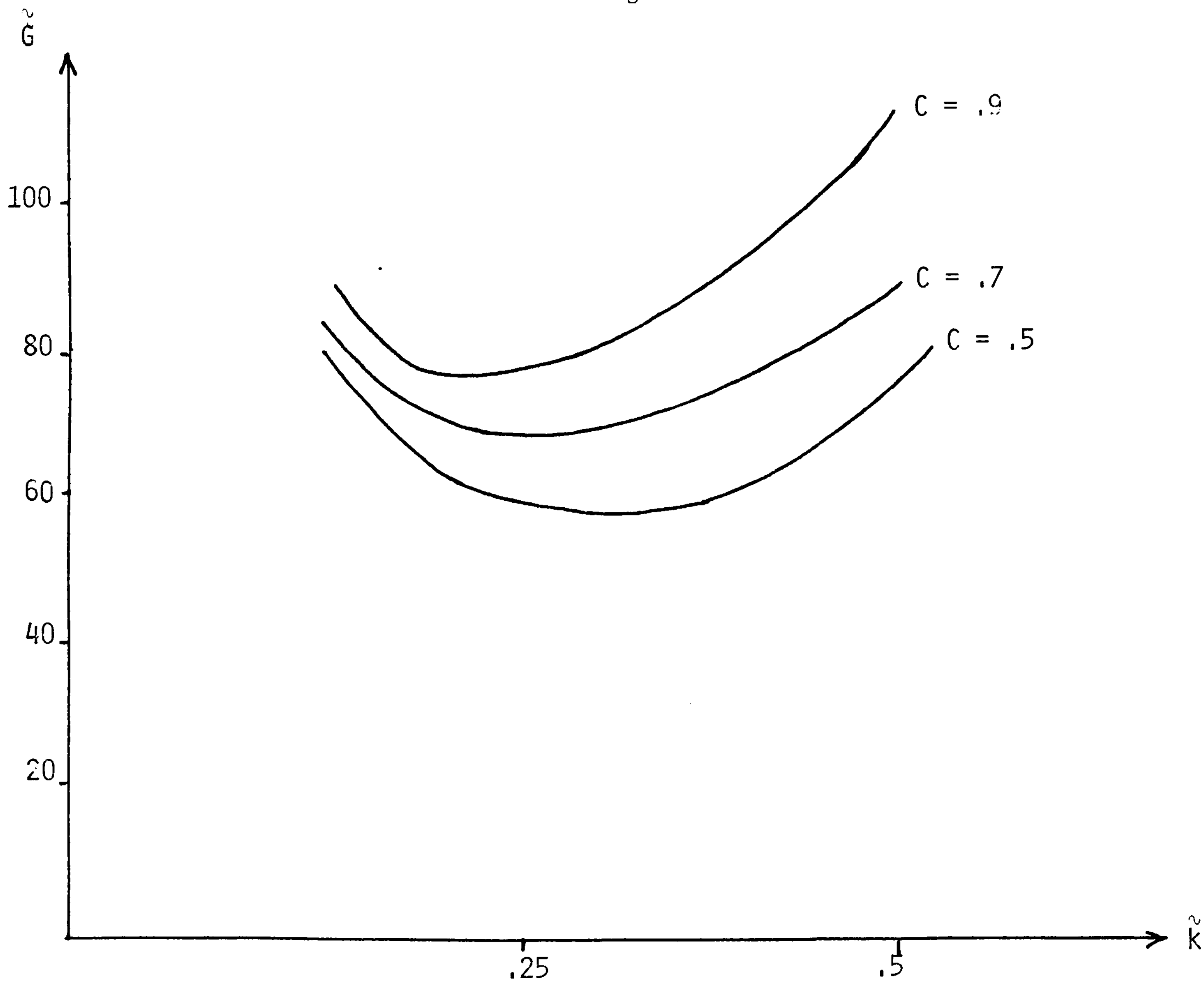


Figure 4

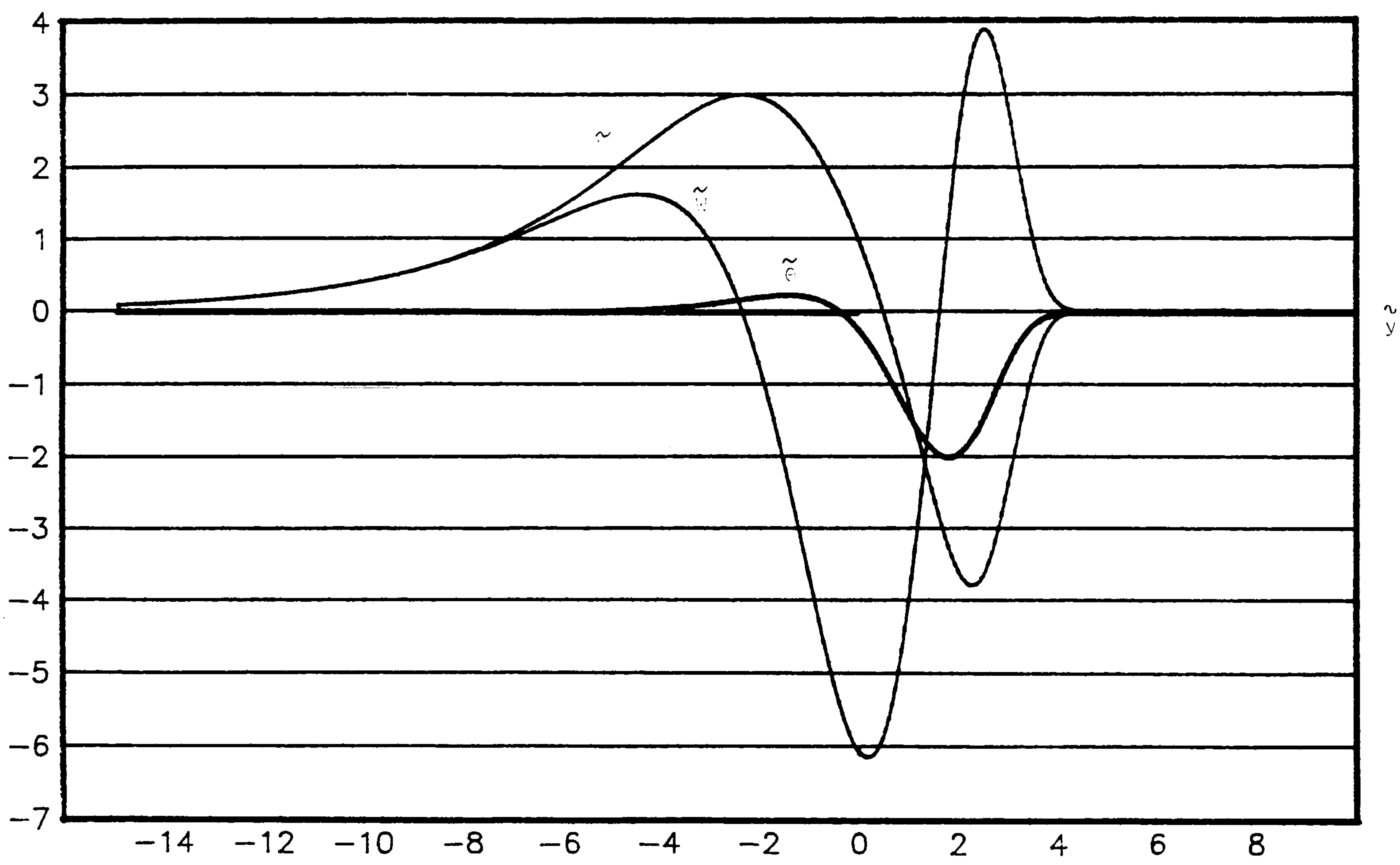
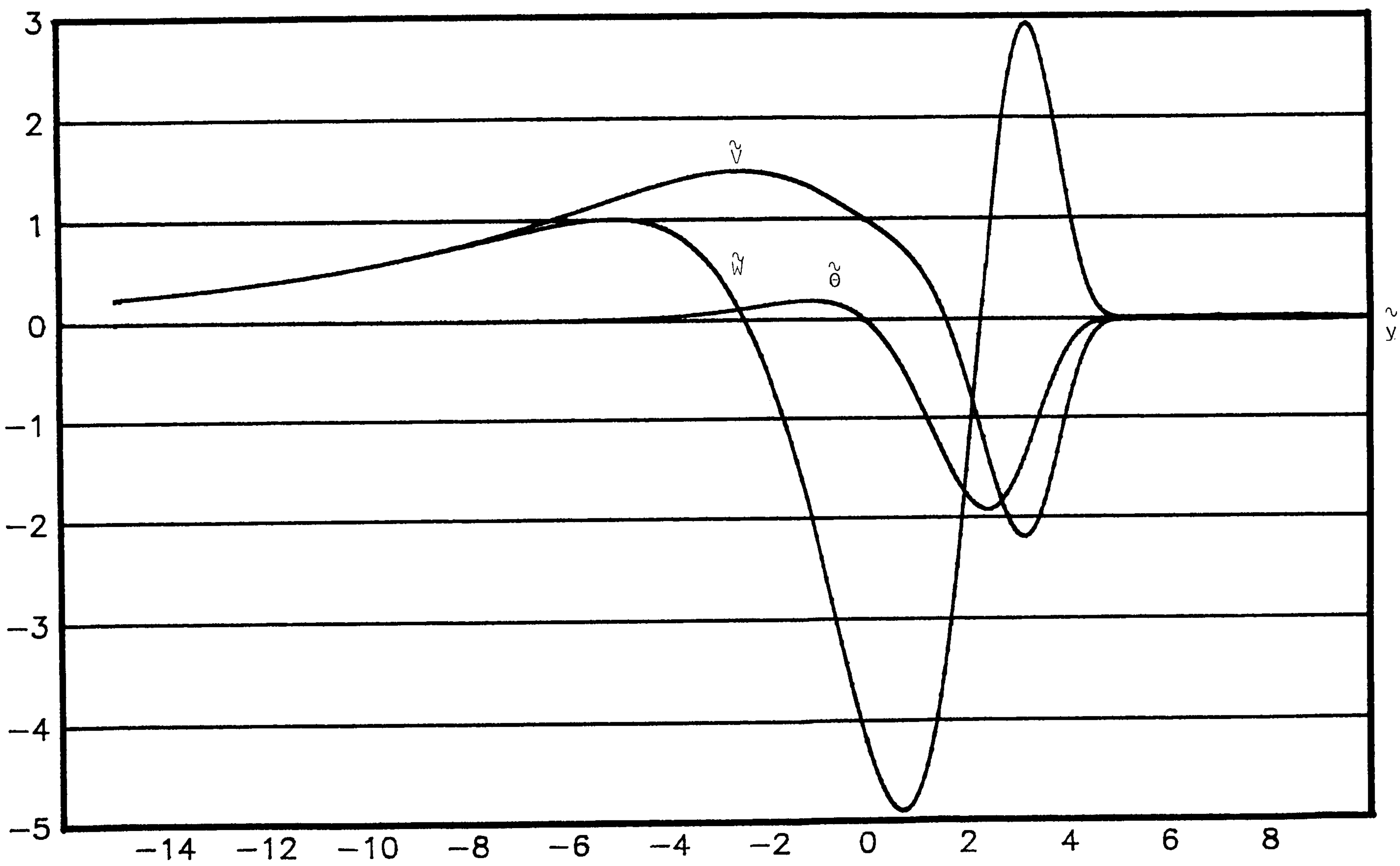


Figure 5



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16. Abstract

The linear instability of the hypersonic boundary layer on a curved wall is considered. As a starting point the viscosity of the fluid is taken to be a linear function of temperature and real-gas effects are ignored. It is shown that the flow is susceptible to Gortler vortices and that they are trapped in the logarithmically thin adjustment layer in which the temperature of the basic flow changes rapidly to its free stream value. The vortices decay exponentially in both directions away from this layer and are most unstable when their wavelength is comparable with the depth of the adjustment layer. The non-uniqueness of the neutral stability curve associated with incompressible Gortler vortices is shown to disappear at high Mach numbers if the appropriate "fast" streamwise dependence of the instability is built into the disturbance flow structure. It is shown that in the hypersonic limit wall-cooling has a negligible effect on the stability of a fluid with a given value of the Chapman constant.

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