Algorithms for Adaptive Control of Two-Arm Flexible Manipulators Under Uncertainty

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1. Abstract

The paper presents a nonlinear extension of model reference adaptive control (MRAC) technique to guide a double arm nonlinearizable robot manipulator with flexible links, driven by actuators collocated with joints subject to uncertain payload and inertia. The objective is to track a given simple linear and rigid but computationally simple model in real, possibly stipulated time and within stipulated degree of accuracy of convergence while avoiding collision of the arms. The objective is attained by a specified signal adaptive feedback controller and by adaptive laws, both given in closed form. A case of 4 DOF manipulator illustrates the technique.

2. Introduction

The MRAC technique becomes popular proposition for guidance of recent robot manipulators, with demand for precision pointing in difficult conditions, under the action of full scale dynamic forces, and subject to uncertainty in parameters. Such manipulators, particularly these used on spacecraft are highly nonlinear and nonlinearizable structures (geometric nonlinearity of elastic links, large angle articulation, nonlinear coupling of DOF's, nonignorable gyro and Coriolis forces, several equilibria), while classical MRAC is linear and applicable to rigid bodies only. Thus the extension is needed for handling nonlinearity, see [1], and flexible links, see [2]. On the other hand many robotic objectives, again particularly these in difficult space conditions require at least two arm systems. Thus the tracking has to be a double MRAC (mutual reference adaptive control) which secures tracking the same model by two arms while avoiding mutual collision - cf. [3], [4]. If adaptive (self-organizing) control is intended, the tracking relates not to a given path but to a given dynamic target-model with prescribed target-parameters. We take the model simple thus rigid and linear, but locally compatible with the nonlinear arms regarding equilibria. Each arm is represented as an open chain with \( n \) DOF, nonlinear characteristics and coupling, elastic links, driven by actuators collocated with joints, under uncertain inertia parameters and uncertain payload. The tracking is done in real possibly stipulated time by a designed signal adaptive feedback controller and integrable adaptive laws in the state space, while avoiding collision between arms of all the joints (and elastic nodes) in Cartesian configuration space.

3. Motion Equations

Lagrangian motion equations give the rigid dynamics of the arms in the general format

\[
\dot{\lambda}^j(q^j,\dot{q}^j) + \Gamma^j(q^j,\dot{q}^j,\ddot{q}^j) + \Sigma^j(q^j,\dot{q}^j,\ddot{q}^j,\lambda^j) = \beta^j(q^j,\dot{q}^j)u_j, \quad j = 1, 2, \tag{1}
\]

where \( q^j(t) \in \mathbb{R}^n \), \( t > t_0 = 0 \), is the configuration vector of the joint variables \( q_{ij} \) of the \( j \)-th arm varying in the known bounded work region \( \Lambda_q \) of the configuration space \( \mathbb{R}^n \); \( \dot{q}^j(t) \) is the corresponding vector of joint velocities in the specified bounded subset \( \Lambda_{\dot{q}} \) of the space tangent to \( \mathbb{R}^n \); \( u^j(t) \in \mathbb{R}^n \) are the control vectors in given compact set of constraints \( \Lambda_u \); \( \lambda^j(t) \in \mathbb{R}^l \), \( l \leq 2n \), are the vectors of adjustable system parameters in bounded bands of values \( \Lambda_{\lambda} \), and \( \Sigma^j(t) \in \mathbb{R}^m \) is an uncertainty parameter within the known band \( S \). Moreover \( \lambda^j(q^j,\dot{q}^j) \) are the inertia \( n \times n \) matrices obtained in the known way from the quadratic form of kinetic energy. The vectors \( \beta^j = (\beta^j_1, \ldots, \beta^j_n)^T \) represent potential forces (gravity, spring) while \( \Gamma^j = (\Gamma^j_1, \ldots, \Gamma^j_n)^T \) represent the internal nonpotential acting forces (Coriolis, gyro, centrifugal, damping structural or viscous, etc.) and \( \beta^j \) is the actuator transmission (gear) nonsingular \( n \times m \) matrix. The control vectors \( u^j(t) \) are selected for the objectives of tracking and avoidance by adaptive feedback control programs \( \dot{u}^j = p^j(q^j,\dot{q}^j,\ddot{q}^j,\lambda^j,\dot{\lambda}^j,\ddot{\lambda}^j) \) on corresponding products of \( \Lambda_q \times \Lambda_{\dot{q}} \times \Lambda_{\lambda} \). For convenience the superscripts "\( ^j \)" will be dropped until they are needed to avoid ambiguity.

Considering the links elastic we introduce the deformation coordinates for the \( i \)-th link as shown in Fig. 1, while using the Ritz-Kantorovitch series expansion.

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where $r_i(y_i,t) = \sum_{i=1}^{m} r_i^+(y_i(t)) r_i^-(t) = r_i(y_i(t)) r_i(t)$.

and for $v_i(y_i,t)$, $w_i(y_i,t)$ analogously, with the exact solution expected for $m \to \infty$. We take $m$ large enough so that the Kantorovitch linearization is physically justified. The technical way about it is to stepwise subdividing the links between grid as long as the difference of results for successive $m$'s becomes small.

Having specified (2) we form the vector $n(t) \Delta (n_1(t), \ldots, n_N(t))^T$, where $n_i(t) \Delta (r_i^+(t), v_i^+(t), w_i^+(t))^T$ and following [5] write the hybrid system as

$$
A(q,n,s) - D(q,\dot{q},n,\dot{n},s) + P(q,\dot{q},n,\dot{n}) = B(q,s)u
$$

where $A_n(n,s)$, $\Gamma_n(n,s)$, $\Pi_n(n,s)$ are the elastic correspondents of $A$, $\Gamma$, $\Pi$ while $A_c(q,n)$, $D_c(q,\dot{q},n,\dot{n})$, $P_c(q,n)$ and the internal damping $D(q,\dot{q},n,\dot{n})$ as well as the hybrid restoring coefficients $P(q,n)$ are matrices coupling the elastic and joint coordinates. These matrices are formed by integrals over the shape functions, see [5]. Letting

$$
A(q,n,s) = \begin{bmatrix} A & A_c \\ A_T & A_n \end{bmatrix}
$$

we obtain the hybrid inertia matrix which is nonsingular positive definite, we inertially decouple (4):

$$(\ddot{q},\ddot{\dot{n}})^T = D(q,\dot{q},n,\dot{n},s) + P(q,\dot{q},n,\dot{n})u
$$

where $D = A^{-1}(D_c [n_1, \ldots, n_N])^T$ and $P = A^{-1}(P_c[n_1, \ldots, n_N])^T$ are successively vectors of nonpotential and potential forces and the meaning of the matrix $B$ is obvious. The vectors $q, \dot{q}, \dot{\dot{n}}$ form the state vector $x(t) = (x_1(t), \ldots, x_N(t))^T \Delta (q(t), \dot{q}(t), \ddot{q}(t), n(t), \dot{n}(t), \ddot{n}(t))^T$. For convenience (4) may be then written in the general state form

$$
\dot{x} = f(x,u,\lambda,s)
$$

with $f = (f_1, \ldots, f_N)$ of the shape specified by (4) in an obvious way. Formally (5) may be written in the contingent form:

$$
\dot{x} - f(x,u,\lambda,s) | s \in S
$$

which for suitable $f(\cdot)$, $p(\cdot)$ has solutions $x(t) = k(x^0,t)$, $t \geq 0$, absolutely continuous curves through each $x^0 = x(0)$ in $S$. We shall consider the class of such solutions $K(x^0)$ by exhausting all values of $s(t)$ in (5) at each $t$.

4. The Reference Model

We let the given Cartesian "world" coordinates representation of the reference model in general terms

$$
\dot{\xi}_m = F(\lambda_m)\dot{\xi}_m
$$

with $2n$ DOF, $\xi(t) \in \mathbb{R}^{1,2n}$, and $F(\lambda_m)$ suitable matrix, be off-line recalculated to the joint coordinate format of the rigid linear system

$$
\dot{q}_m + D_m(q_m)\dot{q}_m + P_m(q_m)q_m = 0
$$

with the $2n$-vectors $q_m$, $\dot{q}_m$ of joint coordinates and velocities, state $x_m(t) = (q_m(t), \dot{q}_m(t))^T \in \mathbb{R}^N$, and
suitable matrices, while \( \lambda = (\lambda_1, \ldots, \lambda_L) = \text{const} \in \mathbb{R}^L \), \( t = n \). Moreover
\[
P_m(\lambda) (q^m, \dot{q}^m) = 0
\]
(8) with \((q^m, \dot{q}^m)\) denoting the equilibria of (3) on the surface \( \dot{q} = 0, \dot{\dot{q}} = 0 \). The total energy of the model will be denoted by \( E_m(\xi_m, \dot{\xi}_m) \) in the world coordinates and \( E_m(\xi_m, \dot{\xi}_m) \) in the joint coordinates, obviously equal to one another. Then
\[
E_m(q_m, \dot{q}_m) = \frac{1}{4} T_m + \int_{q_0}^{q_m} P_m(\lambda) \, dq
\]
(9) and substituting (7),
\[
E_m(q_m, \dot{q}_m) = -D_m(\lambda)(\dot{q}_m)^2
\]
(10) The model is selected such as to allow achieving of a stipulated target behavior in the state space. To focus attention on something specific and yet general enough, let it be stability of the origin, guaranteed by the nonaccumulation of the total energy i.e. non-negative damping
\[
E_m(q_m, \dot{q}_m) \leq 0, \quad \forall q_m \neq 0
\]
(11) while
\[
\forall E_m(q_m, \dot{q}_m) > 0
\]
(12) in-the-large i.e. on same \( \mathcal{C}_L = \Delta - \Delta_L \), where \( \Delta_L \) is the set in \( \mathbb{R}^N \) enclosing all the equilibria.

5. Objectives

Now we consider both arms \( j = 1, 2 \) and the model together. The block scheme of the system is shown in Fig. 2.

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**Figure 2.** Block scheme of the system

Define two product 2N-vectors \( X^j(t) = (x^j(t), x_m(t))^T \), \( \Delta = \Delta \), generating the product trajectories \( \lambda^j(t) = \lambda^j(0) \), \( x^j(t) = x^j(0) \). Then we define the "diagonal" sets
\[
W^j = \{(x^j, x^j) \mid \Delta x^j \leq 1, x^j = x_m, x^j = 0 \}, \quad j = 1, 2
\]
and given stipulated \( \mu^j > 0 \), their neighbourhoods
\[
W^j = \{(x^j, x^j) \mid \Delta x^j \leq 1, x^j < x_m, x^j < 0 \}, \quad j = 2
\]
Moreover we let \( \Delta_U \) be a desired subset of \( \Delta \) where we want the tracking to occur, and let \( t_c \) be the
stipulated time after which the tracking is attained with accuracy $\mu$.  

**First Objective:** The manipulator arms (i) are mutually u-tracking the target (7) on $A_0$. If there is a pair of controllers $p^j(\cdot)$, $j = 1, 2$ such that for each solution $k^j(x^j_0, t)$, $t \geq 0$ of (4) in $K(x^j_0)$, the set $\Delta^2 \times A$ is positively invariant: $(x^j_0, a^j_0) \in \Delta^2 \times A = (x^j(t), a^j(t)) \in \Delta^2 \times A$ and given $t_c$, for each $k^j(\cdot) \in K(x^j_0)$ the product trajectories satisfy

$$\begin{align*}
(x^j(t), a^j(t)) &\in M^j, \forall t \leq t_c.
\end{align*}$$

The convergence is illustrated in Fig. 3.

![Figure 3. Convergence of product trajectories](image)

Suppose the transformation from joint to world coordinates (forward kinematics) is given by

$$r_j(x) = r_j(q_j, n_j), c = 1, \ldots, 3 \cdot 2n$$
and denote $z(t) = (x^1(t), x^2(t))$. Then we let the set

$$A \triangleq [Z \times \Delta^2 | |z(t) - q(t)| < d, 0 \leq t \leq t_c] \times \Delta^2$$
be the collision set between arms to be avoided. We define $CA \triangleq A_0 = A$, specified by $|\lambda_0 - \epsilon_0| > d$, and let

$$A \triangleq [Z \times \Delta^2 | |z(t) - q(t)| < d + \epsilon]$$
be the "slow down" safety zone, with $\epsilon > 0$ suitable constant.

**Second Objective:** The tracking arms (1) avoid collision iff there is $\Delta^0$ such that for any $z^0 \in CA$, and any pair $k^j(\cdot) \in K(x^j_0)$ the corresponding product trajectory

$$z(z^0, t) \in CA, \forall t \geq 0.$$  

6. Sufficient Conditions

We return now to the first objective and specify by $N[\mu(\Delta^2 \times A)]$ a neighborhood of the boundary $|b(\Delta^2 \times A)|$ of the region $\Delta^2 \times A$. Then let $N, \Delta^2, [\lambda(\Delta^2 \times A)] \n \Delta^2 \times A^2$, and introduce open $D^j = \mathbb{R}^j$ such that $D^j \cap M = \emptyset$. Further we consider four $C^1$-functions $V_j(\cdot)$: $V_j(\cdot) : \mathbb{R}^j \rightarrow \mathbb{R}^j$, with the positive constants

$$\begin{align*}
V_j^j &= V_j^j(x^j, x^j) \in (x^j, x^j) \cap \Delta^2, \\
V_j^i &= \inf_{z_j^j} V_j(x^j, x^j) \in (x^j, x^j) \cap \Delta^2, \\
V_j^j &= \sup_{z_j^j} V_j(x^j, x^j) \in (x^j, x^j) \cap \Delta^2.
\end{align*}$$

The first relation obviously requires forming $V_j(\cdot)$ from suitable $z_j^j (x^j, x^j)$ taken at each level, or conversely, forming $\lambda_0, \lambda$ from levels of suitable $V_j^j(\cdot)$, in the latter case $\lambda_0, \lambda$ smaller than these desired will be the secure choice.

**THEOREM 1:** Objective 1 is attained if, given $\lambda_0, \lambda, \mu$ there are programs $p^j(\cdot)$ and functions $V_j^j(\cdot)$, $V_j^i(\cdot)$ such that for all $(x^j, x^j), x^j, x^j \in \Delta^2 \times \Delta^2$.  

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\[
\begin{align*}
(1) & \quad \psi(x^j, a^j) \leq \psi_j, \quad V(x^j, a^j) \in U \subset J, \quad j = 1, 2, \\
(2) & \quad \text{for each } u^j \in \mathcal{P}^j(X^j, X^j), \\
& \quad \psi(x^j(t), a^j(t)) < 0, \quad \forall s \in S \\
& \quad \text{along the product trajectories } (x^j(x^0, t), a^j(x^0, t)), \quad t \geq 0, \quad j = 1, 2; \\
(3) & \quad 0 \leq \psi(x^j, a^j) \leq \psi_j, \quad (x^j, a^j) \in C \mathcal{W}^j, \quad j = 1, 2; \\
(4) & \quad \psi(x^j, a^j) \leq \psi_j, \quad \forall x^j, a^j \in D \cap N \mathcal{W}_j, \quad j = 1, 2; \\
(5) & \quad \text{for each } u^j = \mathcal{P}^j(x^j, x^j), \quad \text{there is a constant } c_j > 0 \text{ such that} \\
& \quad \psi(x^j(t), a^j(t)) \leq c_j, \quad \forall s \in S \\
& \quad \text{along the product trajectories } (x^j(x^0, t), a^j(x^0, t)), \quad t \geq 0, \quad j = 1, 2.
\end{align*}
\]

Remark 1: The Objective 1 holds after a stipulated \( t_c \) if Theorem 1 is satisfied with \( c_j \) selected by
\[
c_j \leq \frac{\psi^*_j}{c^*_j}, \quad j = 1, 2.
\]

THEOREM 2: Objective 2 is attained if Theorem 1 holds and given \( d \) there is a \( C^1 \)-function \( V_A(z) : \Delta A \to \mathbb{R} \)
\text{ such that for the tracking pair } \( p_t^j(z) \), \text{ for all } z \in \Delta A,
\[
\begin{align*}
(6) & \quad V_A(z) = V_A(z), \quad \forall z \in \Delta A; \\
(7) & \quad \text{for each } u^j \in p_t^j(z), \\
& \quad \hat{V}_A(z(t_0, t)) < 0, \quad z(t_0, t) \in \Delta A, \quad \forall s \in S \\
& \quad \text{along product trajectories } z(t_0, t), \quad t \geq 0.
\end{align*}
\]

PROOF. Suppose some \( z(t_0, t), \quad t \geq 0, \quad z(t_0, t) \in \Delta A \) crosses \( \Delta A \) at \( t_1 > 0 \). Then by (6), \( V_A(z(t_1)) < V_A(z(t_0)) \)
which contradicts (7).

7. Controllers and Adaptive Laws

Let us set up
\[
\begin{align*}
\psi(x^j, a^j) & = \psi_j, \quad \forall x^j, a^j \in D, \\
\psi(x^j, a^j) & = \psi_j, \quad (x^j, a^j) \in C \mathcal{W}^j, \quad j = 1, 2, \\
\psi(x^j, a^j) & = \psi_j, \quad (x^j, a^j) \in D \cap N \mathcal{W}_j, \quad j = 1, 2, \\
\psi(x^j, a^j) & = \psi_j, \quad (x^j, a^j) \in D \cap N \mathcal{W}_j, \quad j = 1, 2.
\end{align*}
\]

where \( a^j = (\text{sign } v^j, \ldots, \text{sign } v^j), \quad j = 1, 2, \) and \( E(x^j) \) is \( E_x \) with \( x \) exchanged for \( x^j \). Choosing
\( N_x \) in \( \Delta A \), the character of \( E_x \) specified additionally by (12), satisfies (ii), (iii) and (iv).

To see that (v) holds, observe that \( E(x^j) \) is \( E_x \) of (6) and that increasing the distance
\( (E_x^j)^{1/2} \) for at least one \( j \) from its \( \Delta A \) value increases the value of \( V_A(z(t_1)) \).

To check upon conditions (ii), (v), (vi) we differentiate (21) - (23) with respect to time
\[
\begin{align*}
\dot{\psi}_j(t) & = \dot{\psi}_j(x^j) + \dot{\psi}_j(x^j); \\
\dot{\psi}_j(t) & = \begin{cases} 
\dot{E}(x^j) - E(x^j) + x^{j1}, & (x^j, x^j) \in C \mathcal{W}^j, \\
\dot{x}^{j1}, & (x^j, x^j) \in D \cap N \mathcal{W}_j, \\
\dot{x}^{j1}, & (x^j, x^j) \in D \cap N \mathcal{W}_j,
\end{cases} \\
\dot{\psi}_j(t) & = [E(x^j) - E(x^j)] \cdot [E(x^j) - E(x^j)],
\end{align*}
\]

where
\[
\begin{align*}
\dot{E}(x^j) & = E_x^j \cdot x^{j1}, \quad (x^j, x^j) \in (E_u - P_x P_x x^j, \dot{q}, \dot{q}).
\end{align*}
\]
The brackets of the functions $B$, $D$, $P$ dropped for clarity. Moreover $C^0_{\mu} J$ are subsets of $C^0_{\mu}$ defined by

$$
C^0_{\mu}: E_{\mu}(x^J) \geq E_{\mu}(x) \\
C^+_{\mu}: E_{\mu}(x^J) < E_{\mu}(x) .
$$

With a suitable choice of initial states the following set of conditions implies (ii), (v) and (vii):

(a) $\min \max \hat{E}_{\mu}(x^J) \leq \hat{E}_{\mu}(x), \forall(x^J, u^J), C^0_{\mu}$,

$$
\max \min \hat{E}_{\mu}(x^J) \geq \hat{E}_{\mu}(x), \forall(x^J, u^J) \in C^+_{\mu} ;
$$

(b) $\max \min \hat{E}_{\mu}(x^J) > \min \max \hat{E}_{\mu}(x^J), \forall(x^J, u^J) \in C^0_{\mu}$,

$$
\min \max \hat{E}_{\mu}(x^J) < \min \max \hat{E}_{\mu}(x^J), \forall(x^J, u^J) \in C^+_{\mu} ;
$$

for $i \neq 0$, $\mu \neq 0$, $j = 1,2$. In the above $C^\mu_{\mu}$ are subsets of $C^\mu_{\mu}$ defined by:

$$
C^0_{\mu}: E_{\mu}(x^J) \geq E_{\mu}(x) ,
$$

$$
C^+_{\mu}: E_{\mu}(x^J) < E_{\mu}(x) .
$$

Observe that for $\gamma^J = 0$ there is no need for adaptation and that the system (4) crosses the surface $\tilde{\gamma} = 0$, $\tilde{\gamma} = 0$ time instantaneously (vertically) so there is no need for control in view of the smoothness of trajectories. Conditions (a), (b) are called control conditions helping to design $p^*(\cdot)$, condition (c) is called adaptive, helping to design adaptive laws. Let us check that (a), (b), (c) indeed imply (ii), (v), (vii). Consider first the case $E_{\mu}(x^J) \neq E_{\mu}(x)$. Substituting (c) into (25) in view of (ii) we obtain $\gamma^J = \text{negative terms} \cdot \hat{E}_{\mu}(x^J)$. Boundedness of the work space necessitates the power; $\hat{E}_{\mu}(x^J) \leq 0$ thus (11). Substituting (a), (c), and (11) into (24) with (18) we satisfy (v) in stipulated time $t_c$. Note that this holds for any initial states. The case $E_{\mu}(x^J) = E_{\mu}(x)$ is trivial as then $\gamma^J = \text{negative terms} \cdot \hat{E}_{\mu}(x^J) - \gamma^J = \hat{E}_{\mu}(x^J) - c_J = -c_J$. Finally we check (vii). Again first let $E_{\mu}(x^J) \neq E_{\mu}(x^J)$ and observe that (b) substituted to (26) implies (vii). The case $\hat{E}_{\mu}(x^J) = \hat{E}_{\mu}(x^J)$ is obviously trivial.

Observe that, with (10), (c) is implied by the following adaptive laws

$$
\gamma^J_t = \text{sign} \left[ \left( \frac{\gamma^J_{\mu}}{\gamma^J_{\mu} \gamma^J_{\mu} - \gamma^J_{\mu}} \right) \right].
$$

for $i^J \neq 0$, $i = 1, \ldots, n$. Physically the solutions $\gamma^J_j(0, t)$ represent the model energy flux which become positive or negative depending upon where $\gamma_j^J$ is located (below or above the surface $\gamma^J = 0$) thus regulating the increment of $\gamma^J$ to zero from anywhere outside the surface $\gamma^J = 0$.

8. Modular Double RP-Manipulator

Our technique is illustrated below on the case study of the four DOF manipulator with two arms shown in Fig. 1.
The Lagrange equations of motion for each arm result in the following motion equations
\[
\begin{align*}
(m_1 + m_2 + m_3)\ddot{q}_1 + 2m_2 q_1 \dot{q}_2 + \lambda_1 q_1 |\dot{q}_1| + g(m_1 + m_2) \cos q_1 - m_2 \ddot{q}_2 + \lambda_2 q_2 |\dot{q}_2| + 2a_1^2 = u_1 \\
m_2 \ddot{q}_2 - m_2 q_2 \dot{q}_2 + \lambda_4 q_2 |\dot{q}_2| + m_2 \dot{q}_2 \sin q_1 = u_2
\end{align*}
\]  
(29)

Here \(\lambda_3, \lambda_4\) are damping coefficients, \(\lambda_1, \lambda_2\) spring coefficients, \(g\) gravity acceleration, the remainder of notations shown in Fig. 4. The superscripts \(i^T\), \(j = 1, 2\), are ignored for the time being. We take the possible payload on the grippers as unknown but within known bounds which makes \(m_2\) specified by

\[
m \leq m_2 \leq \bar{m},
\]

where \(m, \bar{m}\) positive constants. Allowing \(\sin q_1 = q_1 - \frac{1}{2}q_1^3\), \(\cos q_1 = 1 - \frac{1}{2}q_1^2\), and subdividing the equations (29) by corresponding inertia coefficients we obtain:

\[
\ddot{q}_1 + \Gamma_1 + \Pi_1 = u_1^1, \quad i = 1, 2
\]

where

\[
\begin{align*}
\Gamma_1 &= \frac{2m_2 q_1 \dot{q}_2 + \lambda_1 |\dot{q}_1| q_1}{m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2} \\
\Gamma_2 &= -q_2 \dot{q}_2^2 / m_2 + \lambda_2 |\dot{q}_2| q_2 \\
\Pi_1 &= \frac{\lambda_2 q_2 |\dot{q}_2| + \frac{1}{2}m_2 q_2^2 - \frac{1}{2}m_1 q_1^2}{m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2} \\
\Pi_2 &= g q_1 - \frac{1}{2} \dot{q}_1^2 \\
\end{align*}
\]  
(31)

The reference model is taken as

\[
\begin{align*}
\dot{q}_m &= \lambda_3 q_m + \lambda_4 q_m + g q_m = 0 \\
\dot{q}_m &= \lambda_4 q_m + \lambda_3 q_m = 0
\end{align*}
\]  
(32)

The total energy of the model is

\[
E_m(q_m, \dot{q}_m) = \frac{1}{2}(\dot{q}_m^2 + \dot{q}_m^2) + \lambda_3 q_m^2 + \lambda_4 q_m^2 + 2g q_m q_m
\]  
(33)

Differentiating it with respect to time and substituting (32),

\[
\dot{E}_m(q_m, \dot{q}_m) = -\lambda_3 q_m^2 - \lambda_4 q_m^2
\]

Accordingly,

\[
\dot{E}_m(q, \dot{q}) = (B_1 \dot{u}_1 + B_2 \dot{u}_2) q_1 + (B_2 \dot{u}_1 + B_1 \dot{u}_2) q_2
\]

Choose \((X^0, \tau^0) \cdot C M, j = 1, 2\) and \(X^0 \cdot C A\). Then the control conditions (a), (b) hold if successively

\[
\begin{align*}
\min \max \left\{ \left| \frac{B_1 \dot{u}_1 + B_2 \dot{u}_2}{m_1} q_1 \right| : j = 1, 2 \right\} \\
\min \max \left\{ \left| \frac{B_2 \dot{u}_1 + B_1 \dot{u}_2}{m_2} q_2 \right| : j = 1, 2 \right\}
\end{align*}
\]  
(34)

and

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Thus we choose $u^*_1$ such that for $\dot{q}_1^* \neq 0$,
$$\min \max(\mathbf{8}^T u^{*1} q^1) < -\lambda_3 (\dot{q}_m)$$
and for such $u^*_1$, we choose $u^*_2$ satisfying
$$\min \max(\mathbf{8}^T u^{*2} q^2) < \min \max(\mathbf{8}^T u^{*1} q^1)$$

The procedure for $u^*_2$ is identical utilizing the second inequalities of (34), (35). Assuming symmetry of arms: $m^*_1 = m^*_2 = m_1^*$, $m^*_2 = m^*_2 = m_2$, $r = r = r_0$, and substituting the expressions for $\mathbf{1}^T$, $\mathbf{e}^T$, $\mathbf{e}^T$, $i = 1$,

$$u^*_1(t) = \begin{cases} 
\frac{m^*(q^*_m)}{q^*_1} [m^*_1 r^2 + \overline{m}(q^*_2)^2] + \frac{2m^*_1}{q^*_1} \dot{q}^*_1 \dot{q}^*_1, \quad \forall t \neq 0 \\
\text{suitable constant, } Vq_1^* = 0
\end{cases}$$

$$u^*_2(t) = \begin{cases} 
\frac{m^*(q^*_m)}{q^*_2} [m^*_2 r^2 + \overline{m}(q^*_2)^2] + \frac{2m^*_2}{q^*_2} \dot{q}^*_2 \dot{q}^*_2, \quad \forall t \neq 0 \\
\text{suitable constant, } Vq_2^* = 0
\end{cases}$$

and the collision avoidance controllers

$$u^2(t) = \begin{cases} 
\frac{m^*(q^*_m)}{q^*_1} [m^*_1 r^2 + \overline{m}(q^*_2)^2] - \frac{2m^*_1}{q^*_1} \dot{q}^*_1 \dot{q}^*_1, \quad \forall t \neq 0 \\
\text{suitable constant, } Vq_1^* = 0
\end{cases}$$

$$u^2(t) = \begin{cases} 
\frac{m^*(q^*_m)}{q^*_2} [m^*_2 r^2 + \overline{m}(q^*_2)^2] - \frac{2m^*_2}{q^*_2} \dot{q}^*_2 \dot{q}^*_2, \quad \forall t \neq 0 \\
\text{suitable constant, } Vq_2^* = 0
\end{cases}$$

which imply the control conditions (a), (b) for our example. The adaptive laws (24) are

$$\dot{\dot{q}}_1 = 0, \quad \dot{\dot{q}}_2 = 0$$

$$\dot{\dot{q}}_3 = -\text{sign}(\dot{q}_3) \lambda_3 \dot{q}_m^2 - \dot{c}_3$$

$$\dot{\dot{q}}_4 = -\text{sign}(\dot{q}_4) \lambda_4 \dot{q}_m^2 - \dot{c}_4$$

for $j = 1, 2$. The first two laws vanish identically, since by design $\dot{\dot{q}}_1 = \dot{\dot{q}}_2$. Numerical simulation of our modular case, with the data $m_1 = 70$, $m_2 = 30$, $m_3 = 40$, $r = 0.06$, $\dot{\lambda}_1 = 20$, $\dot{\lambda}_2 = 20$, $\dot{\lambda}_3 = 5$, $\dot{\lambda}_4 = 2$, is shown in Fig. 5, and confirms the convergence-avoidance required.

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**Figure 5. Numerical simulation**
9. References


