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ABSTRACT

In General Relativity the Principle of General Covariance allows one to describe phenomena by means of any convenient choice of coordinate system. In this paper it is shown that the geodetic precession of a gyroscope orbiting a spherically symmetric, nonrotating mass can be recast as a Lense-Thirring frame-dragging effect, in an appropriately chosen coordinate frame whose origin falls freely along with the gyroscope and whose spatial coordinate axes point in fixed directions.

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The theory of general relativity[1,2,3,4] has had remarkable success in the last few decades in describing gravitational interactions in cosmological and intergalactic as well as solar system domains. Among the many relativistic phenomena which have been observed, there are still two effects which await confirmation which we address in this paper.

According to general relativity, observers fixed with respect to distant stars will note that a spinning gyroscope, falling freely in the gravitational field of a rotating source, will undergo two kinds of effects known respectively as geodetic precession, and the often-called motional or "Lense-Thirring" precession[5]. Geodetic precession is usually associated with motion of the gyroscope through the static gravitational field of the source. Conventionally one derives this precession by parallel transport of the spin vector in the curved spacetime near the mass[6]; this effect is present even if the mass is not rotating. The motional precession, or "hyperfine precession"[7] is due to the interaction of the spin angular momentum $\vec{J}$ of the source, and the spin $\vec{S}$ of the gyroscope. This effect resembles the interaction of the spin of the electron with the magnetic field of the nucleus of an atom. Thorne[8] has taken this analogy further to describe the two effects respectively in terms of interaction of the gyroscope with the "gravitoelectric" and the "gravitomagnetic" fields derived from the various components of the metric including spatial curvature contributions. Schwinger[9] also derived the motional precession in terms of source theory where space-time components of the metric yield a gravitomagnetic field tensor which interacts with the spin of the gyroscope. Schiff[10] suggested that motional precession could be seen as a "dragging" of inertial frames, analagous to the effect predicted by Thirring[11] who showed that inertial frames inside a rotating hollow shell undergo precession with respect to observers whose orientation is fixed with respect to distant stars. Outside a rotating spherical source, the precession of a nearby gyroscope could be conveniently pictured by considering a spinning sphere submerged in a viscous fluid. Small toothpicks placed in the fluid near the poles, rotate in the

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same direction as the sphere rotates, while those placed at the equator rotate in the opposite direction.

It is hoped that in the next decade Gravity Probe B[12], a drag-free satellite carrying a gyroscope around the earth, will be launched. For an orbit of altitude 480 km, the gyroscope's geodetic precession should be 6.9 arc-sec/year, and the Lense-Thirring precession should be .044 arcsec/year. These precessions are to be measured when gyroscope orientation is checked against distant fixed stars. The Lense-Thirring drag due to the earth may be observed using the orbit of a satellite such as the recently proposed LAGEOS III[13].

While efforts are already being put into detecting these two effects, we suggest that the two effects can be considered to be based on the same fundamental physical phenomenon–gravitomagnetism. Our purpose here is to recast the entire geodetic precession as a "Lense-Thirring" drag. This would further extend the framework introduced by Thorne to show that geodetic precession in its entirety is related to a gravitomagnetic effect. This is via a boost to a reference frame with origin falling freely with the gyroscope, having axes pointing in fixed directions, with spatial and time units rescaled slightly due to Lorentz contraction and other small relativistic effects which we shall discuss. In this reference frame the massive source is revolving around the gyroscope, giving rise to a gravitomagnetic drag which is precisely of the magnitude necessary to explain that precession of the gyroscope which was interpreted in the original frame as geodetic precession. By the principle of general covariance, a phenomenon may be described in any convenient coordinate system provided that the experimental observations are interpreted properly, in terms of invariant quantities.

This suggests that experimental observation of either the geodetic precession or the Lense-Thirring frame dragging effect would confirm them both, otherwise general relativity would not be self-consistent. Both effects can be considered as manifestations of gravitomagnetism.

In this paper we shall use the simplest Parameterized Post-Newtonian (PPN) formulation[14] and shall neglect preferred frame and energy-momentum nonconser-

vation effects. Therefore the only relevant PPN parameters would be γ and β; however we shall not find it necessary to include nonlinear effects due to β as these are of higher order.

We consider a model of a non-rotating, spherically symmetric mass M placed at the origin of the PPN frame. Given the metric to post-Newtonian order, we construct an orthonormal tetrad of basis vectors falling along a geodesic curve around the source, but with orientation fixed with respect to points at infinity. The origin, at position $\vec{R}_0$, is thus in motion through the PPN grid. The coordinate reference frame is erected using this tetrad as a basis. Observers in this frame do not feel radial accelerations towards the source as they are falling along geodesics; however they "see" a mass revolving around their origin. We call this a quasi-inertial frame. The goal is to calculate the metric tensor in this frame by coordinate transformation. Upon expanding the metric tensor to linear order in local coordinates near the origin, the equations of motion of a spinning gyroscope at the origin may be obtained, and it will be seen that in this coordinate system the geodetic precession is entirely of gravitomagnetic origin.

We use the formalism derived previously by us[15] to calculate the local metric $g_{\mu\nu}$ in the quasi-inertial frame. We will show that there exist terms in the components g_{0i} linear in coordinates leading to the expected geodetic precession of the gyroscope. The magnitude of the precession, and its dependence on γ is as expected, and the direction of the precessional angular velocity is the same as the direction of the orbital angular momentum of the source as seen by observers in the quasi-inertial frame. Thus the local inertial frame is truly dragged by the source as it traverses its orbit.

CALCULATION OF THE METRIC IN THE QUASI-INERTIAL FRAME

The metric in the PPN frame with the static source at the origin up to the desired order is given by:

$$G_{00} = -1 + 2U \tag{1a}$$

$$G_{0i} = 0 \tag{1b}$$

$$G_{ij} = \delta_{ij}(1 + 2\gamma U), \tag{1c}$$

where $-c^2U = -GM/R$ is the gravitational potential due to the central mass. One can construct an orthonormal tetrad falling along the geodesic in the gravitational field given by Eqs. (1), but with spatial axes directionally fixed with respect to the PPN frame. The construction is straightforward[16] and for a circular orbit the components of the spatial members of the tetrad are:

$$\Lambda^0_{(j)} = V^j(1 + 2(1+\gamma)U_0 + \frac{1}{2}V^2), \tag{2a}$$

$$\Lambda^k_{(j)} = \delta^k_j(1 - \gamma U_0) + \frac{1}{2}V^kV^j, \tag{2b}$$

where V^k are the components of velocity of the gyroscope. Following the procedure developed in Ref. (15), one can now construct a set of coordinate transformations, relating the PPN coordinates X^μ to the local coordinates x^μ:

$$X^\mu = X^\mu|_{orbit} + \Lambda^\mu_{(j)}x^j - \frac{1}{2}\Gamma^\mu_{\alpha\beta}\Lambda^\alpha_{(k)}\Lambda^\beta_{(j)}x^kx^j + \ldots \tag{3}$$

where we have included terms to quadratic order in the local coordinates. This is required in order to obtain terms linear in coordinates in the local metric. Evaluating the coordinate transformations gives:

$$X^0 = Kx^0 + \vec{r}\cdot\vec{V}[1 + (1+\gamma)U_0] + (U_{,j}x^j)(\vec{r}\cdot\vec{V}), \tag{4a}$$

$$X^j = X^j\big|_{orbit} + x^j(1 - \gamma U_0) + \frac{1}{2}V^j(\vec{r}\cdot\vec{V}) - \gamma x^j(U_{,k}x^k) + \frac{1}{2}\gamma U_{,j}(\vec{r}\cdot\vec{r}), \tag{4b}$$

where

$$K = 1 + U_0 + \frac{1}{2}V^2, \tag{4c}$$

and $\vec{r}$ is the local coordinate position vector. $U_{,j}$ is the gradient of the potential of the mass evaluated at the position of the gyroscope. Having found the coordinate transformations one may simply regard them as exact transformations from PPN coordinates to another reference frame. The transformations (4) include resynchronization of clocks, Lorentz contraction, and rescaling of lengths due to the mass, as well as several quadratic terms needed to make the metric at the position of the gyroscope reduce to the Minkowski values. The metric tensor in the new frame is obtained by tensor transformation:

$$g_{\mu\nu} = G_{\alpha\beta}\frac{\partial X^\alpha}{\partial x^\mu}\frac{\partial X^\beta}{\partial x^\nu} \ . \tag{5}$$

For a circular orbit, after expanding to quadratic order in local coordinates one finds the following expressions for the metric tensor components g_{00} and g_{ij}:

$$g_{00} = -1 + O(x^i x^j) \ , \tag{6a}$$

$$g_{ij} = 1 + O(x^i x^j) \ . \tag{6b}$$

Before expansion of the potential $U(R)$ for small values of the local coordinates, the expression for the "gravitomagnetic" metric tensor components g_{0i} is

$$\begin{aligned} g_{0i} = 2(\gamma+1)\frac{GM}{c^2|\vec{R}_0 + \vec{r}|}V^i - 2U_0(\gamma+1)V^i - (\gamma+\frac{3}{2})U_{,j}x^jV^i \\ -(\gamma+\frac{1}{2})U_{,i}(\vec{V}\cdot\vec{r}) \ . \end{aligned} \tag{6c}$$

The first term in the above equation is what would be expected for a mass moving with velocity $-V^i$, as is observed in the quasi-inertial frame. After expansion of the above expression for g_{0i} to linear order in quasi-inertial coordinates, one finds after cancellation that

$$g_{0i} = (\gamma + \frac{1}{2})x^j(U_{,j}V^i - U_{,i}V_j) + O(x^i x^j) \ . \tag{6d}$$

The equations of motion of a gyroscope with spin $\vec{S}$, placed at the origin of the local frame can now be found. The only Christoffel symbols which contribute are:

$$\Gamma^k_{0j} = \frac{1}{2}(g_{0k,j} - g_{0j,k}). \tag{7}$$

For a circular orbit, one arrives at:

$$\frac{d\vec{S}}{dt} = \frac{1}{2}\vec{S} \times (\vec{\nabla} \times \vec{g}), \tag{8}$$

where $\vec{g} = (g_{01}, g_{02}, g_{03})$ is the gravitomagnetic field vector. Evaluating Eq. (8) in terms of the potential U gives:

$$\frac{d\vec{S}}{dt} = (\gamma + \frac{1}{2})\vec{S} \times (\vec{\nabla U}\Big|_0 \times \vec{V}). \tag{9}$$

This represents the precession of a gyroscope due to gravitomagnetic components only of the metric, in quasi-inertial coordinates. In a more illuminating form, Eq. (9) can be written as:

$$\frac{d\vec{S}}{dt} = (\gamma + \frac{1}{2})\frac{G}{c^2R^3}\vec{L} \times \vec{S}, \tag{10}$$

where $\vec{L}$ is the angular momentum per unit mass of the gyroscope. The spin axis of the gyroscope is dragging behind the revolving source with angular velocity of magnitude proportional to the angular momentum of the source measured by quasi-inertial observers.

If the source also possesses spin angular momentum per unit mass $\vec{J}$, there will be additional contributions to g_{0i}, as in the standard literature, and the spin precession rate will then be given by:

$$\frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S} \ , \tag{11}$$

with[6]

$$\vec{\Omega} = \frac{G}{c^2 R^3}\Big\{\frac{1}{2}(\gamma+1)[-\vec{J} + \frac{3(\vec{J}\cdot\vec{R})\vec{R}}{R^2}] + (\gamma+\frac{1}{2})\vec{L}\Big\}. \tag{12}$$

CONCLUSIONS

We have shown that the seemingly different geodetic precession and Lense-Thirring drag can be recast, in the spirit of Mach's principle, in terms of a single gravitomagnetic effect consisting of two contributions. One arises from orbital angular momentum of the source mass as it revolves around the origin of quasi-inertial coordinates; the other arises from spin angular momentum of the source. In this coordinate frame, the net effect arises from g_{0i} components of the metric alone, and may be considered entirely gravitomagnetic in origin. There are no contributions leading to either precession from spatial curvature.

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In General Relativity, the Principle of General Covariance allows one to describe phenomena by means of any convenient choice of coordinate system. In this paper it is shown that the geodetic precession of a gyroscope orbiting a spherically symmetric, nonrotating mass can by recast as a Lense-Thirring frame-dragging effect, in an appropriately chosen coordinate frame whose origin falls freely along with the gyroscope and whose spatial coordinate axes point in fixed directions.

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