

INTERPRETATION AND MODELING OF THE AVERAGED EQUATIONS
FOR A FLUID-SOLID FLOW

Hayley H. Shen and Guann-Jiun Hwang

Department of Civil and Environmental Engineering

Clarkson University, Potsdam, NY 13676

ABSTRACT

In this study, a self-consistent derivation of the conservation laws is given for flows of a fluid-solid mixture. A unified analytical framework for obtaining constitutive relations is provided. This analysis uses a control volume/control surface approach that is widely used in fluid mechanics. All terms in the governing equations and the constitutive relations are written in terms of the mass-weighted averages except solid concentration. It is believed strongly that the mass-weighted average is the natural bridge between micromechanics and constitutive relations. The derived momentum equations contain terms that differ from all existing models except that of Prosperetti and Jones (1984). However, their assumptions are not needed here. Special attention is given to the solid phase pressure. The physical basis of previously assumed form for this pressure (Givler 1987) becomes clear. A number of related phenomena are also discussed. These include the anti-diffusion and anisotropic normal stresses. The energy equations are also different from existing models. But detail discussion on the energy equations is left to future work.

I. INTRODUCTION

Modeling a flowing fluid-solid mixture starts from writing down a set of governing equations. These equations describe the conservation of mass, momentum and energy. In the early stage, the popular approach was to view the mixture as a single phase material. Consequently, the following type of equations were used for mass and momentum conservations (see Zuber 1964, Ishii 1975, 1977).

$$\frac{\partial \langle \rho_m \rangle}{\partial t} + \nabla \cdot \langle \rho_m \rangle \mathbf{u}_m = 0 \tag{1}$$

$$\frac{\partial}{\partial t} \langle \rho_m \rangle \mathbf{u}_m + \nabla \cdot \langle \rho_m \rangle \mathbf{u}_m \mathbf{u}_m = \nabla \cdot \mathbf{T}^m + \langle \rho_m \rangle \mathbf{g} \tag{2}$$

In which, $\langle \rangle$ represents the ensemble average, ρ_m is the mixture density, \mathbf{u}_m is the mixture velocity, \mathbf{T}^m is the mixture stress and \mathbf{g} is the body force per unit mass.

Recently, the two-phase flow approach becomes more popular. Using this approach, the conservation equations are formulated for each individual phase separately. This approach allows for direct modeling of a fluid-solid mixture when the two phases have distinctly different dynamics. For mass and momentum, these conservation equations are generically written in the following way (see Ishii 1975, Drew 1976, 1983, McTigue et al. 1986).

$$\partial\langle\rho_s c\rangle/\partial t + \nabla \cdot \langle\rho_s c\mathbf{u}\rangle = 0 \quad (3)$$

$$\partial\langle\rho_f(1-c)\rangle/\partial t + \nabla \cdot \langle\rho_f(1-c)\mathbf{v}\rangle = 0 \quad (4)$$

$$\partial\langle\rho_s c\mathbf{u}\rangle/\partial t + \nabla \cdot \langle\rho_s c\mathbf{u}\mathbf{u}\rangle = \langle\rho_s c\mathbf{g}\rangle + \langle\mathbf{m}\rangle + \nabla \cdot \langle c\mathbf{T}_s\rangle \quad (5)$$

$$\partial\langle\rho_f(1-c)\mathbf{v}\rangle/\partial t + \nabla \cdot \langle\rho_f(1-c)\mathbf{v}\mathbf{v}\rangle = \langle\rho_f(1-c)\mathbf{g}\rangle - \langle\mathbf{m}\rangle + \nabla \cdot \langle(1-c)\mathbf{T}_f\rangle \quad (6)$$

where ρ_s and ρ_f are densities of the solid and fluid phases, \mathbf{u} and \mathbf{v} are the velocities of the solid and fluid phases, \mathbf{m} is the interaction force per unit volume of the mixture, c is the local solid concentration (equals to 1 at a solid point and 0 at a fluid point), \mathbf{T}_s and \mathbf{T}_f are the solid and fluid stress respectively. The above equations do not consider the phase changes at the interface. The energy conservation equations have not been studied as extensively as the mass and momentum conservation equations.

Because the two phases are separated, available information on a single particle's motion and the particle-particle interactions in a fluid environment are incorporated directly. In a mixture model, these informations will first be utilized to obtain the drift flux term in \mathbf{T}^m of Eq. (2). Preference of the two approaches apparently depends on the application. However, since the mixture model can be derived from adding together the two-phase equations, the two-phase approach is considered more fundamental.

Many mathematical models have been derived based on Eqs. (3)-(6). There are two issues in modeling terms appeared in those conservation equations. First, what kind of averaging is used in defining the macro quantities, such as concentration, velocity, velocity divergence, strain-rate, etc. Second, how to obtain the required constitutive relations for

averaged terms like stress, pressure, phase interaction, etc. Neither of these two issues is settled at the moment.

We will refer to the mechanics that governs particle's motion as the "micromechanics". Such micromechanics includes particle-particle and particle-fluid interactions. In order to explicitly formulate constitutive relations, knowledge of micromechanics is essential.

It is understood that the formulation of constitutive relation is extremely complicated, because the micromechanics of fluid-solid interaction is itself not well understood. Another reason for this difficulty is less obvious but more significant. That is, up to the moment, there has not been a set of governing equations in which all terms are interpreted with a consistent averaging method, without such governing equations, and a consistent bookkeeping to account for the micromechanics. It is impossible to correctly formulate the constitutive relations. The authors believe that this is the source of the recent argument about the "solid phase pressure" and related phenomena. A survey of the recent literature shows that different interpretations have been given to the solid phase pressure. It has been suggested to be equal to the (i) fluid phase pressure (Drew 1976), (ii) averaged fluid pressure around the surface of a particle (Givler 1987), or (iii) a more sophisticated version of (ii) with additional consideration of Brownian forces and bulk viscosity (McTigue et al. 1986). All of which are intended to apply to an arbitrary flow of a fluid-solid mixture. Similarly, the phase interaction $\langle \mathbf{m} \rangle$ in Eqs. (5) and (6) has also been modeled in many different ways. Essentially, in all the more recent works this term has been modeled as

$$\langle \mathbf{m} \rangle = n \mathbf{h} + \bar{p} \nabla \langle c \rangle \quad (7)$$

where n is the averaged number of particles per unit volume of the mixture, \mathbf{h} is the averaged hydrodynamic force per particle. The term \bar{p} is a source of confusion again. It has been equated to the (i) averaged fluid pressure over the fluid-solid interface (Drew 1983), (ii) fluid phase pressure (McTigue et al. 1986) or (iii) hydrostatic fluid pressure (Ahmadi 1987). Again, all the above are suggested for general flow condition.

Although the existing two-phase flow models appear to be inconsistent, development of the above models has provided a great deal of insights. These insights are essential to the work presented here.

In the following, we will give a step by step derivation of the two-phase flow governing equations based on the mass-weighted average defined as

$$\{\psi\} = \frac{\langle c_k \psi \rangle}{\langle c_k \rangle} \quad (8)$$

where c_k equals to c for the solid phase and $1 - c$ for the fluid phase. This averaging method has been exploited in the theory of compressible turbulence (Van Driest 1951), and applied to granular flow (Ahmadi and Shahinpoor 1983).

The mass-weighted average of any given quantity is the quantity averaged within that phase only. For instance, mass-weighted solid velocity at a point is the average velocity of all observed particles passing that point. This average is the easiest one to measure in most real flows. Moreover, through using this average, a direct bridge between micromechanics and the constitutive relations may be established. Applying this average to the solid phase stress, terms such as the solid phase pressure will have a clear meaning. Thus a unique definition for these quantities in terms of micromechanics is possible. In addition, a number of interesting phenomena that seem to defy the well accepted property of fluids are observed when a fluid-solid mixture is viewed as a composition of two separate continuums.

In the present study, we concentrate on the derivation of governing equations, interpretation of the averaged terms and the development of constitutive relations from micromechanics. Derivation of the actual constitutive relations for various flow conditions is beyond our scope and present ability. A few exceptionally simple cases will however be studied. In order to complete the mathematical modeling, boundary conditions must be derived. Due to the existence of the discrete solid phase, derivation of boundary conditions is equally difficult as constitutive relations. This is also left to future development.

II. GOVERNING EQUATIONS

Consider an arbitrary control volume V as shown in Fig. 1. Its surface S is the control surface. For visual reason this control volume is drawn large compared with the solid particle's size. The derivation is not restricted to this size. The particles are not necessarily spherical or uniform either. However, in order to simplify the notations, we will discuss the case of uniform spherical particles. The essence of the analysis is captured in this simplified case. There has been previous work deriving conservation equations for this type of control volume for a fluid-solid mixture (Soo 1981). Nevertheless, such equations have not been derived in terms of the mass-weighted average nor the interpretation in terms of micromechanics given.

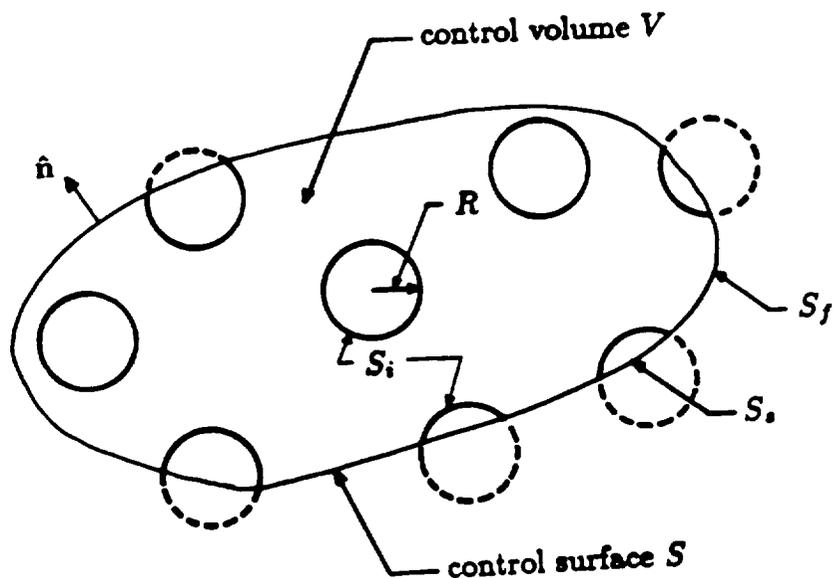


Fig. 1 A control volume V with control surface $S = S_o + S_f$ and internal interface S_i .

In general, the flow situation is such that the condition in any given control volume

from different observations is a random phenomenon. Only an average behavior may be described by a deterministic mathematical model. To allow for cases where flow conditions may vary over a length scale comparable with the particle size, the ensemble average over many realizations of a control volume smaller than the “representative volume” commonly used in fluid mechanics may be necessary.

For one realization, the rate of increase of solid mass in the control volume is

$$\frac{\partial}{\partial t} \int_V \rho_s c dV + \int_S \rho_s c \mathbf{u} \cdot \hat{\mathbf{n}} dS = 0 \quad (9)$$

where $\hat{\mathbf{n}}$ is the unit outward normal. After ensemble averaging, the resulting integrands are smooth functions. Applying Green’s theorem to the second term and removing the integrals, the mass conservation of the solid phase in this control volume is obtained as

$$\frac{\partial \langle \rho^s \rangle}{\partial t} + \nabla \cdot \langle \rho^s \rangle \{ \mathbf{u} \} = 0 \quad (10)$$

In the above, $\langle \rho_s c \rangle$ is replaced by $\langle \rho^s \rangle$ and $\langle \rho_s c \mathbf{u} \rangle$ is replaced by $\langle \rho^s \rangle \{ \mathbf{u} \}$ where $\{ \}$ is the mass-weighted average defined in Eq. (8). Similarly, for the fluid phase,

$$\frac{\partial \langle \rho^f \rangle}{\partial t} + \nabla \cdot \langle \rho^f \rangle \{ \mathbf{v} \} = 0. \quad (11)$$

The momentum conservation equations require representation of the forcing terms which include the surface and body forces. This conservation for the solid and fluid phases in one realization are, respectively,

$$\frac{\partial}{\partial t} \int_V \rho_s c \mathbf{u} dV + \int_S \rho_s c \mathbf{u} \mathbf{u} \cdot \hat{\mathbf{n}} dS = \int_S c \mathbf{T}_s \cdot \hat{\mathbf{n}} dS + \int_V \rho_s c \mathbf{g} dV + \int_V \mathbf{m} dV \quad (12)$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho_f (1 - c) \mathbf{v} dV + \int_S \rho_f (1 - c) \mathbf{v} \mathbf{v} \cdot \hat{\mathbf{n}} dS &= \int_S (1 - c) \mathbf{T}_f \cdot \hat{\mathbf{n}} dS \\ &+ \int_V \rho_f (1 - c) \mathbf{g} dV - \int_V \mathbf{m} dV \end{aligned} \quad (13)$$

Again the ensemble average is first applied to smooth the above integrands. The Green's theorem may then be used to change the surface integral to volume integral. After removing the integral sign, the above equations become

$$\frac{\partial}{\partial t} \langle \rho_s c \mathbf{u} \rangle + \nabla \cdot \langle \rho_s c \mathbf{u} \mathbf{u} \rangle = \nabla \cdot \langle c \mathbf{T}_s \rangle + \langle \rho_s c \mathbf{g} \rangle + \langle \mathbf{m} \rangle \quad (14)$$

$$\frac{\partial}{\partial t} \langle \rho_f (1 - c) \mathbf{v} \rangle + \nabla \cdot \langle \rho_f (1 - c) \mathbf{v} \mathbf{v} \rangle = \nabla \cdot \langle (1 - c) \mathbf{T}_f \rangle + \langle \rho_f (1 - c) \mathbf{g} \rangle - \langle \mathbf{m} \rangle \quad (15)$$

Substituting \mathbf{u} by $\{\mathbf{u}\} + \mathbf{u}''$ and \mathbf{v} by $\{\mathbf{v}\} + \mathbf{v}''$ and making use of the mass conservation equations, we obtain the following equations with the Reynolds stresses for both phases. In these equations mass-weighted velocities and stresses appear.

$$\langle \rho^s \rangle (\partial \{\mathbf{u}\} / \partial t + \{\mathbf{u}\} \cdot \nabla \{\mathbf{u}\}) = \nabla \cdot \langle c \rangle \{\mathbf{T}_s\} - \nabla \cdot \langle \rho^s \rangle \{\mathbf{u}'' \mathbf{u}''\} + \langle \rho^s \rangle \mathbf{g} + \langle \mathbf{m} \rangle \quad (16)$$

$$\langle \rho^f \rangle (\partial \{\mathbf{v}\} / \partial t + \{\mathbf{v}\} \cdot \nabla \{\mathbf{v}\}) = \nabla \cdot \langle (1 - c) \rangle \{\mathbf{T}_f\} - \nabla \cdot \langle \rho^f \rangle \{\mathbf{v}'' \mathbf{v}''\} + \langle \rho^f \rangle \mathbf{g} - \langle \mathbf{m} \rangle \quad (17)$$

The Reynolds stress in the solid phase is also called the kinetic stress in the granular flow terminology.

In a realization over a period of time, particles cut by the control surface may interact with neighboring particles through collisions. The rate of momentum transfer to the interior of the control volume resulted from these collisions is part of the surface force \mathbf{T}_s . Moreover, the hydrodynamic forces acting on particles cut by the control surface also produce surface forces that contribute to \mathbf{T}_s . We denote these two stresses as \mathbf{T}^c and \mathbf{T}^p respectively.

Concept for modeling the mass-weighted average of \mathbf{T}^c has been described in the granular flow literature (e.g. Bagnold 1954, Jenkins and Savage 1983, an excellent survey to appear by Campbell 1990). Although most of the work deals with negligible fluid effect, the route to extend to fluid-solid mixture is, though complicated, quite clear (e.g. Ackermann and Shen 1982, Shen et al. 1988). On the other hand, the explanation of the hydrodynamic stress on the solid phase is not readily available. Intuitively, one would

quite comfortably accept that the fluid only acts on each particle in the control volume through the drag, added mass, etc. Hence, it should contribute to the body force only. Consequently the effect of fluid-solid interaction is the hydrodynamic force per particle multiply the number of particles per unit mixture volume. This concept is proved doubtful as demonstrated by the existing various models discussed in Eq. (7). In the following we will rigorously formulate the fluid effect in a two-phase flow.

Consider a surface particle P in Fig. 2. Part of this particle, P^i , is inside the control volume, part of it, P^o is outside of it. The hydrodynamic force acting on P produces a pair of internal forces, $\pm \mathbf{t}$, on the intersection of the particle and the control surface S_s . The total hydrodynamic force acting on this particle, \mathbf{h} , can similarly be decomposed into that on the outside of P , \mathbf{d}^o , and that on the inside of P , \mathbf{d}^i .

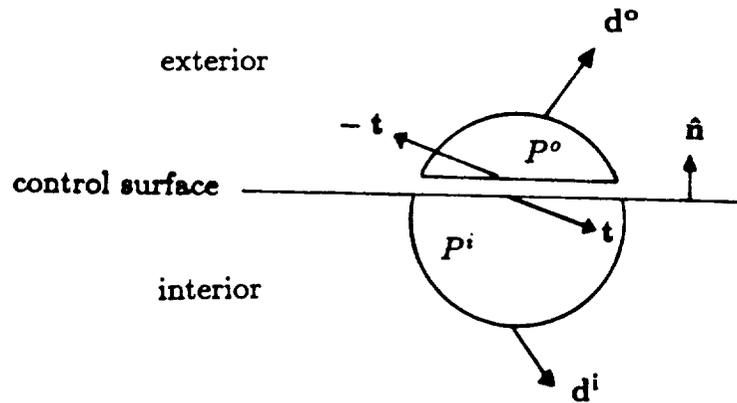


Fig. 2 Decomposing the hydrodynamic force on a surface particle P .

$$\text{The total hydrodynamic force is } \mathbf{h} = \mathbf{d}^i + \mathbf{d}^o.$$

In budgeting the total force acting on the solid portion of V , the interface force \mathbf{t} acting on S_s most naturally belongs to the solid phase stress. In fact it may equally well be classified as part of the body force since after all it acts on P^i which is inside of V . As long as the budgeting of all forces is done in a consistent way, the resulting equation of motion should not depend on the detail of the bookkeeping. Most of the existing

two-phase models appear to fail in using a consistent bookkeeping. That is, force acting on the control surface and that acting on the internal interface are not always carefully distinguished. We choose to call \mathbf{t} the surface force for V and \mathbf{d}^i the body force for V due to fluid-solid interaction.

The internal force \mathbf{t} is the difference between \mathbf{d}^o and the total surface force acting on P^o . The first may be determined if the hydrodynamic force distribution on a particle's surface and the particle's location relative to the control surface are both given.

The second is determined using Newton's second law. That is, the total surface force acting on the partial particle P^o equals to its inertia subtract the body force acting on it. Ensemble average of the inertia and the body force on P^o may also be obtained if the particle's relative position with respect to the control surface is given. For a control volume reasonably away from the boundary of the flow field, the position of particles on the control surface S_o may be assumed to uniformly distribute inside or outside the control volume. Using these arguments it has been shown in Hwang and Shen (1989a) that the ensemble average of the sum of \mathbf{t} in a unit area produces the following stress,

$$\{\mathbf{T}^P\} = \frac{1}{V_o} \left(\int_{A_o} \{\boldsymbol{\Sigma}\} \cdot \hat{\mathbf{n}} \mathbf{r} dA - \int_{V_o} (\nabla \cdot \{\boldsymbol{\Sigma}\}) \mathbf{r} dV \right), \quad (18)$$

where A_o and V_o are the surface area and volume of particle P respectively, $\boldsymbol{\Sigma}$ is the local fluid stress on the particle's surface, and \mathbf{r} is the position vector. The very same equation has been obtained by Batchelor (1970) with a quite different derivation. This term has been named the "particle-presence stress" in Hwang and Shen (1989a), and it contains the "interaction stress" in McTigue et al. (1986). The second term inside the parenthesis in Eq. (18) is related to the particle's rotational inertia (Hwang and Shen 1989a).

Next we discuss the interaction term \mathbf{m} in the solid momentum equation. The most natural way consistent with the above bookkeeping is to define it as the total force acting on the interface of the two phases inside the control volume V . This interface is denoted by S_i in Fig. 1, which consists of the surfaces of whole particles if they are entirely in V ,

otherwise only the part of particle's surface that is inside V belongs to S_i . Total force acting on S_i is the sum of all \mathbf{h} , the hydrodynamic force on a whole particle, and all \mathbf{d}^i , the portion of the hydrodynamic force on P inside of V . As shown in Hwang and Shen (1988), the ensemble average of the total interaction force \mathbf{m} is

$$\langle \mathbf{m} \rangle = \frac{\langle c \rangle}{V_o} \langle \mathbf{h} \rangle + \langle c \rangle \nabla \cdot \langle \mathbf{T}^f \rangle - \nabla \cdot c \langle \mathbf{T}^p \rangle \quad (19)$$

In the above, $\langle \mathbf{T}^f \rangle = \langle \mathbf{T}_f \rangle - \rho_f \langle \mathbf{v}'' \mathbf{v}'' \rangle$ is used to indicate the total fluid stress. Substituting Eq. (19) and $\mathbf{T}_s = \mathbf{T}^c + \mathbf{T}^p$ into Eq. (16), the solid momentum equation becomes

$$\langle \rho^s \rangle (\partial \langle \mathbf{u} \rangle / \partial t + \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle) = \nabla \cdot (\langle c \rangle \langle \mathbf{T}^c \rangle) - \nabla \cdot \langle \rho^s \rangle \langle \mathbf{u}'' \mathbf{u}'' \rangle + \langle \rho^s \rangle \mathbf{g} + \frac{\langle c \rangle}{V_o} \langle \mathbf{h} \rangle + \langle c \rangle \nabla \cdot \langle \mathbf{T}^f \rangle \quad (20)$$

This equation is identical to that derived in Prosperetti and Jones (1984). In that work, a couple of assumptions were made to arrive at Eq. (20). It is shown here that those assumptions are not necessary. Similarly, the fluid momentum equation may be derived as

$$\langle \rho^f \rangle (\partial \langle \mathbf{v} \rangle / \partial t + \langle \mathbf{v} \rangle \cdot \nabla \langle \mathbf{v} \rangle) = (1 - 2\langle c \rangle) \nabla \cdot \langle \mathbf{T}^f \rangle - \langle \mathbf{T}^f \rangle \cdot \nabla \langle c \rangle + \langle \rho^f \rangle \mathbf{g} - \frac{\langle c \rangle}{V_o} \langle \mathbf{h} \rangle + \nabla \cdot (\langle c \rangle \langle \mathbf{T}^p \rangle) \quad (21)$$

As it shows in above equations, the effect of particle-presence stress $\langle \mathbf{T}^p \rangle$ appears only in the fluid momentum equation.

Derivation of the energy equations follows the same spirit as the above. Hwang and Shen (1989c) gave the following equations in indicial form for the turbulence energy in the solid and fluid phases respectively,

$$\begin{aligned} \langle \rho^s \rangle \left(\frac{\partial}{\partial t} \left\{ \frac{\mathbf{u}'' \cdot \mathbf{u}''}{2} \right\} + \langle \mathbf{u} \rangle \cdot \nabla \left\{ \frac{\mathbf{u}'' \cdot \mathbf{u}''}{2} \right\} \right) &= \langle c \rangle \langle \mathbf{T}^c \rangle : \nabla \langle \mathbf{u} \rangle + \langle c \rangle \langle \mathbf{T}^p \rangle : \nabla \langle \mathbf{u} \rangle \\ &- \langle \rho^s \rangle \langle \mathbf{u}'' \mathbf{u}'' \rangle : \nabla \langle \mathbf{u} \rangle + \nabla \cdot (\langle c \rangle \langle \mathbf{T}^c \cdot \mathbf{u}'' \rangle) + \nabla \cdot (\langle c \rangle \langle \mathbf{T}^p \cdot \mathbf{u}'' \rangle) \\ &- \nabla \cdot (\langle \rho^s \rangle \left\{ \frac{\mathbf{u}'' \mathbf{u}''}{2} \cdot \mathbf{u}'' \right\}) + \langle \mathbf{m} \cdot \mathbf{u}'' \rangle - \langle c \rangle \langle \gamma^s \rangle \end{aligned} \quad (22)$$

$$\begin{aligned} \langle \rho^f \rangle \left(\frac{\partial}{\partial t} \left\{ \frac{\mathbf{v}'' \cdot \mathbf{v}''}{2} \right\} + \langle \mathbf{v} \rangle \cdot \nabla \left\{ \frac{\mathbf{v}'' \cdot \mathbf{v}''}{2} \right\} \right) &= -(1 - c) \langle p^f \rangle \nabla \cdot \langle \mathbf{v} \rangle - \langle \rho^f \rangle \langle \mathbf{v}'' \mathbf{v}'' \rangle : \nabla \langle \mathbf{v} \rangle \\ &- \nabla \cdot (\langle \rho^f \rangle \left\{ \left(\frac{\mathbf{v}'' \cdot \mathbf{v}''}{2} + \frac{p^{f''}}{\rho_f} \right) \mathbf{v}'' \right\}) + \mu \nabla^2 (\langle (1 - c) \left\{ \frac{\mathbf{v}'' \cdot \mathbf{v}''}{2} \right\} \rangle) \\ &- \mu (1 - c) \langle \nabla \mathbf{v}'' : \nabla \mathbf{v}'' \rangle - \langle \mathbf{m} \cdot \mathbf{v}'' \rangle \end{aligned} \quad (23)$$

where γ^s is the sink of solid turbulence energy from particle-particle interaction (e.g. collisional dissipation), p^f is the fluid pressure, $p^{f''} = p^f - \{p^f\}$, and μ is the dynamic viscosity of the fluid.

The interaction force \mathbf{m} does unequal amount of work to the turbulence energy in the fluid and solid phases. This difference, as given in Hwang and Shen (1989c), comes from the slip velocity between the two phases.

The combination of mass, momentum and energy conservation equations given by the six equations: Eqs. (10), (11), (20), (21), (22) and (23) governs the six unknowns c , \mathbf{u} , \mathbf{v} , $\{\mathbf{u}'' \cdot \mathbf{u}''/2\}$, $\{\mathbf{v}'' \cdot \mathbf{v}''/2\}$ and $\{p^f\}$. To form a closed system, a large number of constitutive relations must be obtained. Most of these constitutive relations require knowledge beyond the current understanding of fluid mechanics around a particle. One example is the term $\{\mathbf{m} \cdot \mathbf{u}''\}$ appeared in Eq. (22). As shown in Eq. (19), \mathbf{m} contains the hydrodynamic force \mathbf{h} acting on a particle. If one considers the drag force part of \mathbf{h} , which is in general in terms of $\mathbf{u}'' - \mathbf{v}''$, the term $\{\mathbf{m} \cdot \mathbf{u}''\}$ will produce $\{\mathbf{v}'' \cdot \mathbf{u}''\}$. Namely, the correlation of the fluid and solid turbulence must be formulated as part of the constitutive relations. A number of analyses have indicated that this correlation depends greatly on the particle's size and density (Chao 1964, Xie 1987, Abou-Arab and Roco 1988). Quantitative study of such correlation is far from complete.

Despite the complexity and the lack of necessary knowledge in fluid mechanics, a number of results may be obtained for the constitutive relations appeared in the governing equations. In the next section, we will derive a term associated with the fluid-solid interactions. Namely, the particle-presence stress \mathbf{T}^P including a detailed discussion on the solid phase pressure. A number of interesting observations will be given at the end of the derivation.

III. CONSTITUTIVE MODELING

To simplify the notation, we remove all the average symbols ($\langle \rangle$ and $\{ \}$) and adopt indicial notation from this section on.

In this section, we will discuss the particle-presence stress T_{ij}^p . Detail of this term is given in Eq. (18). The term in Eq. (18) related to the particle's rotational inertia cancels with the rotational contribution in the particle's Reynolds stress in Eq. (16) (Babič 1989). Hence the net particle-presence stress is only the first term on the right side of Eq. (18). This term may be quantified if the distribution of the hydrodynamic stress Σ is known. Before quantifying this stress for special cases, we will provide an interpretation of the solid phase pressure first.

As defined in Continuum Mechanics, pressure means the negative average of the normal stresses. Therefore, in addition to the contributions from the collisional and Reynolds stress, the particle-presence stress will also produce the following solid phase pressure,

$$\begin{aligned}
 p^p &= -\frac{R^3}{3V_0} \int_0^{2\pi} \int_0^\pi \Sigma_{ik} n_k n_i \sin \phi \, d\phi \, d\theta \\
 &= -\frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi (-p^f \delta_{ik} - \frac{2}{3} \mu e_{ll} \delta_{ik} + 2\mu e_{ik}) n_k n_i \sin \phi \, d\phi \, d\theta \\
 &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi p^f \sin \phi \, d\phi \, d\theta
 \end{aligned} \tag{24}$$

where ϕ and θ are the polar and azimuthal angles respectively. In the above, a Newtonian fluid is assumed such that

$$\Sigma_{ik} = -p^f \delta_{ik} - \frac{2}{3} \mu e_{ll} \delta_{ik} + 2\mu e_{ik} \tag{25}$$

where e_{ik} is the component of local strain-rate. Moreover, for Eq. (24) to be valid, the fluid flow near the particle must be incompressible and the strain-rate must possess certain symmetry property. Thus the viscous contribution vanishes from Eq. (24). Under these special conditions, it is seen that the solid phase pressure is the numerical average of the

fluid pressure around the particles. This very result has been postulated by Givler (1987) and derived by Prosperetti and Jones (1984) using a couple of assumptions.

We now derive the particle-presence stress in two extreme cases for a very dilute mixture. The two extreme cases are: Stokes regime and inviscid regime. In both cases we assume the fluid is Newtonian and the flow is incompressible.

In case of a shear flow in the Stokes regime, the local strain-rate around a spherical particle in an infinite flow field is (Batchelor 1967)

$$e_{ij}(r_i) = E_{ij} \left(1 - \frac{R^5}{r^5}\right) + E_{kl} \left(\frac{r_i r_k \delta_{jl} + r_j r_k \delta_{il}}{r^2} - \frac{2}{3} \frac{r_k r_l}{r^2} \delta_{ij}\right) \left(-\frac{5R^3}{2r^3} + \frac{5R^5}{r^5}\right) + E_{kl} \frac{r_k r_l}{r^2} \left(\frac{r_i r_j}{r^2} - \frac{1}{3} \delta_{ij}\right) \left(\frac{25R^3}{2r^3} - \frac{35R^5}{2r^5}\right) \quad (26)$$

where r_i is the i th component of a position vector \mathbf{r} , $r = |\mathbf{r}|$, E_{ij} is the component the undisturbed strain-rate and R is the particle's radius.

The fluid pressure around a spherical particle in a uniform incoming flow U_∞ is (Chester and Breach 1969)

$$p = p^f + \frac{\mu U_\infty}{r} \left[-\frac{3}{2} \left(1 + \frac{3}{8} R_e\right) \cos \phi + \frac{27}{32} R_e \cos^2 \phi + O(R_e^2 \log R_e)\right], \quad (27)$$

where p^f is the undisturbed fluid pressure; U_∞ is equivalent to the relative velocity of the particle to the fluid flow, or $U_\infty = v_r$ where $v_r = |\mathbf{v} - \mathbf{u}|$; R_e is the particle Reynolds number defined as $2\rho_f R v_r / \mu$; and ϕ is the polar angle of a point on the surface of the particle measured from the direction of U_∞ .

Substituting Eqs. (25) and (26) into Eq. (18), the viscous contribution of the particle-presence stress is obtained as

$$T_{ij}^{p\mu} = \frac{5}{2} (2\mu E_{ij}) \quad (28)$$

As discussed in Batchelor (1970), this stress together with the viscous stress in the fluid phase reproduces Einstein's formula for the effective viscosity (1956). The solid phase pressure is shown below after substituting Eq. (27) into Eq. (24),

$$p^p = p^f + \frac{9}{32} \rho_f v_r^2. \quad (29)$$

In the above, we used p^p to indicate the fact that only particle-presence stress \mathbf{T}^p is included in the total solid stress. Neither collisional nor Reynolds pressure is considered.

The above is not the whole story about the particle-presence stress. If one by-passes the solid phase pressure and investigate the normal stresses directly, the following is found.

$$\begin{aligned} T_{11}^{pp} &= -p^f - \frac{81}{160} \rho_f v_r^2 \\ T_{22}^{pp} &= T_{33}^{pp} \\ &= -p^f - \frac{27}{160} \rho_f v_r^2 \end{aligned} \quad (30)$$

The total particle-presence stress is the sum of Eqs. (28) and (30). It is apparent that Eq. (29) can be reproduced by taking the negative average of T_{11}^{pp} , T_{22}^{pp} and T_{33}^{pp} .

Eq. (30) shows that on top of the viscous effect the isotropic fluid pressure induces anisotropic normal stress in the solid phase stress. This anisotropy of the normal stress is the product of the distinctly different dynamics of the two phases involved. If both phases move with exactly same velocity, this phenomenon will disappear.

In case of an inviscid flow, only the fluid pressure contributes to the particle-presence stress. This pressure around the spherical particle is (Lamb 1932),

$$p = p_\infty + \frac{1}{2} \rho_f v_r^2 \left(-\frac{1}{8} + \frac{9}{8} \cos 2\phi \right). \quad (31)$$

Substituting Eq. (31) into Eq. (25) and neglecting the viscous part, Eq. (18) becomes

$$\begin{aligned} T_{11}^p &= -p^f - \frac{1}{20} \rho_f v_r^2 \\ T_{22}^p &= T_{33}^p \\ &= -p^f + \frac{2}{5} \rho_f v_r^2 \end{aligned} \quad (32)$$

and the solid phase pressure from the hydrodynamic effect alone is the negative average of the above,

$$p^p = p^f - \frac{1}{4} \rho_f v_r^2 \quad (33)$$

The anisotropy of the normal stresses is again observed. Theoretical speaking, without considering Reynolds stress, one may observe the shear force in a two-phase mixture even though the fluid is inviscid.

Comparing the coefficient of the solid phase pressure in the above two extreme cases, one would expect that as the particle Reynolds number increases from nearly zero to nearly infinity, the coefficient in front of the solid phase pressure should vary gradually from $\frac{9}{32}$ to $-\frac{1}{4}$. Explicit determination of this coefficient depends on the knowledge of the hydrodynamic stress distribution around the particle. Unfortunately this information is not available for the entire range of the particle Reynolds number. However, for particle Reynolds number greater than 3000, experimental data shows that the viscous contribution is negligible. In this case, sufficient information exists to empirically determine the pressure distribution, and accordingly the solid phase pressure (Hwang and Shen 1989b).

IV. SOME INTRIGUING POINTS

In deriving the governing equations, a few points have struck the authors as being quite non-trivial. Some of those may have significant implications that is unclear at the moment.

First, the governing equations are for the mass-weighted quantities. All constitutive relations must eventually be written in these quantities to produce a closed system. The kinematic quantities appeared in these governing equations are the mass-weighted velocities. On the other hand, in constitutive relations, we look for mathematical description of the stresses. The average stresses, at least for the viscous component of the fluid stress, is however a function of the average local fluid strain-rate instead of the gradient of the mass-weighted velocity. Namely, this stress is in terms of the mass-weighted strain-rate instead of the gradient of the mass-weighted velocity. For a fluid-solid mixture with rigid

particles, this relation has been derived as (Hwang and Shen 1989c)

$$\left\{ \frac{\partial v_j}{\partial x_i} \right\} = \frac{\partial \{v_j\}}{\partial x_i} + \frac{\partial \langle 1-c \rangle}{\partial x_i} \frac{\{v_j\}}{\langle 1-c \rangle} + \frac{1}{\langle 1-c \rangle} \frac{\partial}{\partial x_i} (\langle c \rangle \{u_j\}) - \frac{\langle c \rangle}{\langle 1-c \rangle} \{\Omega_{ij}^s\} \quad (34)$$

where $\{\Omega_{ij}^s\}$ is the spin tensor (or particle rotation) of the solid phase. In addition to the above two possible definitions for the phase strain-rate, a third strain-rate is also important. This is called the bulk (or mixture) strain-rate defined by

$$\begin{aligned} E_{ij} &= \langle 1-c \rangle \{e^f_{ij}\} + \langle c \rangle \{e^s_{ij}\} \\ &= \langle 1-c \rangle \{e^f_{ij}\} \end{aligned} \quad (35)$$

where e^f_{ij} and e^s_{ij} are the local strain-rate for the fluid and solid phase respectively. In case of rigid particles, $e^s_{ij}=0$. In representing the effective viscous stress in a fluid-solid mixture, the above bulk strain-rate must be used.

In a pure fluid flow, such distinction shown in Eq. (34) or Eq. (35) does not exist. In a two-phase flow, this point becomes essential in many places. In fact, if one interprets the bulk viscous stress using $\partial\{v_j\}/\partial x_i$ instead of $\{\partial v_j/\partial x_i\}$, the coefficient of effective viscosity in Eq. (28), i.e. $\frac{5}{2}$, will become $\frac{3}{2}$ or $\frac{7}{2}$ depends on different interpretations of the mixture strain-rate.

Second, in Givler (1987), a phenomenon called anti-diffusion in the solid phase was mentioned. This term is believed to come from an inconsistent bookkeeping system of the surface and interaction forces in the control volume V . The equation given there for the solid phase was

$$\rho^s \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho^s g_i + \frac{c}{V_0} h_i - c \frac{\partial p^p}{\partial x_i} - (p^p - p^f) \frac{\partial c}{\partial x_i} \quad (36)$$

When the potential flow is considered, the $\partial c/\partial x_i$ term has a positive coefficient. This presents a force that moves the solid phase from a low concentration towards high concentration, which is a puzzling phenomenon. However, in the present derivation, it is shown that in the solid momentum equation such $\partial c/\partial x_i$ term does not exist. The

associated diffusive or anti-diffusive forces are thus absent. But, investigating the fluid momentum equation, we do observe such $\partial c/\partial x_i$ term. Through the mass conservation, any diffusion in the fluid phase results an effective anti-diffusion in the solid phase and vice versa. At this point, we have observed a totally opposite trend between Givler (1987) and the present model. Namely, Givler's model would predict a diffusion phenomenon for solid particles in the Stokes regime and anti-diffusion phenomenon in inviscid flow regime. The present model however predicts exactly the opposite. Further verification with detailed experimental data is desirable.

On top of the above, the anisotropy of the solid phase normal stress produced by fluid pressure is a new observation. This may produce interesting thermodynamic properties that are peculiar to a two-phase flow only.

It is natural to ask whether such a great care in deriving governing equations shown here is of importance. In order to see this, two models have been applied to a vertical pipe flow of a fluid-solid mixture (Hwang and Shen 1988). The two models are identical except the phase interaction term m_i . For case (A), m_i is modeled as shown in Eq. (7) with $\tilde{p} = p^p$ and case (B) is the present model given by Eq. (19). The resulting non-dimensional slip velocity $u^* - v^*$ verses the non-dimensional fluid pressure gradient $(\frac{dp}{dx})^*$ is reproduced in Fig. 3 for three different density ratios $\rho_s^* = \frac{\rho_s}{\rho_f}$. Qualitatively different results are obtained from these two models. From the behavior of the neutrally buoyant particles, it is believed that the present model is more reasonable.

V. CONCLUSION

For a two-phase flow, the interpretation of terms in the governing equations is the first step towards deriving constitutive relations using micromechanics. Therefore it has been surprising to us that the work presented here has not been available in the vast amount of two-phase flow literature. On the other hand, this may be explained by the fact that only

recently researchers have started to derive the constitutive relations from micromechanics. Hence the need for such derivation of governing equations is also very recent.

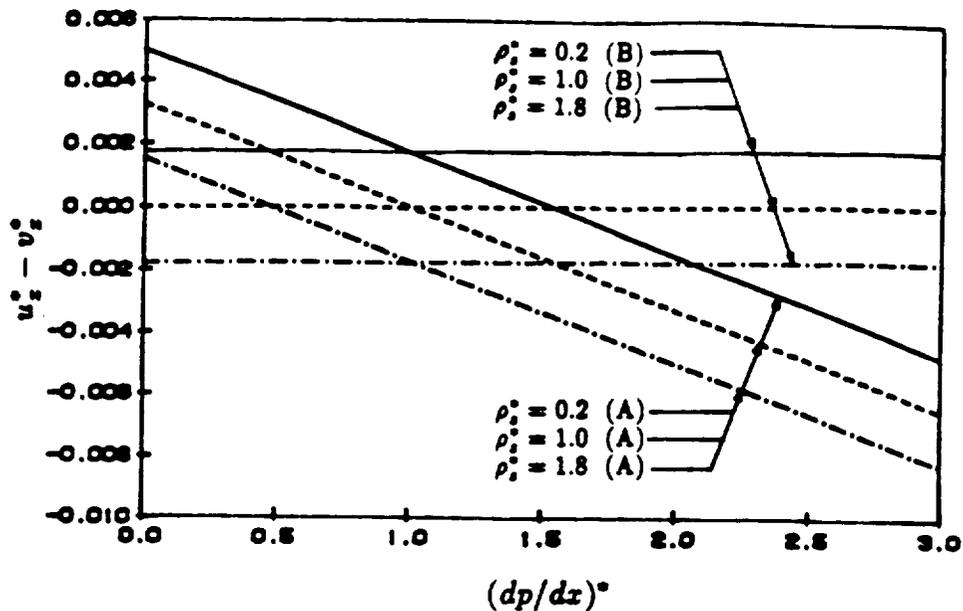


Fig. 3 Effects of ρ_s^* and $(dp/dx)^*$ on slip velocity from model A and model B.

It is shown in this work that the derivation of governing equations for flows of a fluid-solid mixture is not as straightforward as any single phase continuum. This derivation is done with a careful and almost tedious method to account for all transfer quantities between each phase. The resulting equations in terms of mass-weighted average quantities differ from the existing models.

These governing equations provide a foundation to incorporate micromechanics in deriving the constitutive relations. A number of previously well accepted facts about fluid-solid flows emerge naturally from this analysis. In addition, interesting phenomena such as: non-equality of average strain-rate and gradient of the average velocity, anti-diffusion, and anisotropic normal stress have been observed in these equations. Interpretation and

implications of these phenomena is of great interest.

REFERENCES

- Abou-Arab, T. W. and Roco, M. C., "Solid Phase Contribution in the Two-Phase Turbulence Kinetic Energy Equation", *Proc. 3rd Int. Symp. Liquid-Solid Flow*, Chicago, Nov. 28-Dec. 2, 1988, pp. 13-28.
- Ackermann, N. L. and Shen, H. H., "Stresses in Rapidly Sheared Fluid-Solid Mixtures", *ASCE, J. Eng. Mech. Div.*, **108**, No. EM1, 1982, pp. 95-113.
- Ahmadi, G. and Shahinpoor, M., "Towards a Turbulent Modeling of Rapid Flow of Granular Materials", *Powder Technology*, **35**, 1983, pp. 241-248.
- Ahmadi, G., "On the Mechanics of Incompressible Multiphase Suspensions", *Adv. Water Resources*, **10**, 1987, pp. 32-43.*
- Babič, M., "Contact Stress Tensor in Granular Materials", Department of CEE, Report No. 89-1, Clarkson University, Potsdam, N.Y., 1989.
- Bagnold, R. A., "Experiments of Gravity Free Dispersion of Large Solid Spheres in a Newtonian Fluid under Shear", *Proceedings of the Royal Society of London, Series A*, **225**, 1954, pp. 49-63.
- Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.
- Batchelor, G. K., "The Stress System in a Suspension of Force-Free Particles", *J. Fluid Mech.*, **41**, Part 3, 1970, pp. 545-570.
- Campbell, C. S., "Rapid Granular Flow", *Annual Review of Fluid Mechanics*, **22**, 1990 (to be published).
- Chao, B. T., "Turbulent Transport Behavior of Small Particles in Dilute Suspensions", *Ostereichisches Ingenieur-Archiv.*, **18**(122), 1964, pp. 7-21.
- Chester, W. and Breach, D. R., "On the Flow Past a Sphere at Low Reynolds Number",

- J. Fluid Mech.*, **37**, 1969, pp. 751–760.
- Drew, D. A., “Two-Phase Flows: Constitutive Equations for Lift and Brownian Motion and Some Basic Flows”, *Arch. Rat. Mech. Anal.*, **62**, 1976, pp. 149–163.
- Drew, D. A., “Mathematical Modeling of Two-Phase Flow”, *Ann. Rev. Fluid Mech.*, **15**, 1983, pp. 261–291.
- Einstein, A., *Investigations on the Theory of the Brownian Movement*, Dover Publications, Inc., New York, N.Y., 1956.
- Givler, R. C., “An Interpretation for the Solid Phase Pressure in Slow, Fluid-Particle Flows”, *Int. J. Multiphase Flow*, **13**(5), 1987, pp. 717–722.
- Hwang, G. J. and Shen, H. H., “A Consistent Formulation for the Phase Interaction in Flows of a Fluid-Solid Mixture”, *Proc. 3rd Int. Symp. Liquid-Solid Flow*, Chicago, Nov. 28-Dec. 2, 1988, pp. 65–73.
- Hwang, G. J. and Shen, H. H., “Modeling the Solid Phase Stress in a Fluid-Solid Mixture”, *Int. J. Multiphase Flow*, 1989a, in print.
- Hwang, G. J. and Shen, H. H., “Modeling Two-Phase Flows of a Fluid and Solid Mixture : Part (III) – Solid-Phase Pressure in a Rapid Flow”, Department of CEE, Report in print, Clarkson University, Potsdam, N.Y., 1989b.
- Hwang, G. J. and Shen, H. H., “Modeling Two-Phase Flows of a Fluid and Solid Mixture : Part (IV) – Energy Equations and Closure of the System”, Department of CEE, Report in print, Clarkson University, Potsdam, N.Y., 1989c.
- Ishii, M., “Thermo-Fluid Dynamic Theory of Two-Phase Flow”, Eyrolles, Paris, Scientific and Medical Publication of France, N.Y., 1975.
- Ishii, M., “One-Dimensional Drift-Flux Model and Constitutive Equations for Relative Motion between Phases in Various Two-Phase Flow Regimes”, Argonne National Laboratory Report, ANL-77-47, Argonne, Illinois, 1977.
- Jenkins, J. T. and Savage, S. B., “A Theory for the Rapid Flow of Identical, Smooth,

- Nearly Elastic, Spherical Particles”, *J. Fluid Mech.*, **130**, 1983, pp. 187–202.
- Lamb, H., *Hydrodynamics*, 6th ed., Cambridge University Press, 1932.
- McTigue, D. F., Givler, R. C., and Nunziato, J. W., “Rheological Effects of Nonuniform Particle Distributions in Dilute Suspensions”, *J. of Rheology*, **30**, Issue 5, 1986, pp. 1053–1076.
- Prosperetti, A. and Jones, A. V., “Pressure Forces in Disperse Two-phase Flows”, *Int. J. Multiphase Flow*, **10**, 1984, pp. 426–440.
- Shen, H. H., Hopkins, M. A., and Ackermann, N. L., “Modeling Collisional Stresses in A dense Fluid-Solid Mixture”, *Journal of Fluids Engineering*, **110**, March 1988, pp. 85–90.
- Soo, S. L., “Effects of Configuration of Phases on Dynamic Relations”, *AIChE Symposium Series*, Heat Transfer - Milwaukee, No. 208, **77**, 1981, pp. 152–160.
- Van Driest, E. R., “Turbulent Boundary Layer in Compressible Fluids”, *J. Aeronaut. Sci.*, **18**, 1951, pp. 145–160.
- Xie, D., “Turbulent Kinetic Energy of Particle Phase in Solid-Fluid Suspension Turbulence”, *Proc. Int. Symp. Multiphase Flow-China*, 1987, pp. 396–401.
- Zuber, N., “On the Dispersed Two-Phase Flow in the Laminar Flow Regime”, *Chemical Engineering Science*, **19**, 1964, pp. 897–917.

