TURBULENCE KINETIC ENERGY EQUATION FOR DILUTE SUSPENSIONS

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ABSTRACT

This paper presents a multiphase turbulence closure employing one transport equation, namely the turbulence kinetic energy equation. The proposed form of this equation is different from the earlier formulations in some aspects. The power spectrum of the carrier fluid is divided into two regions, which interact in different ways and at different rates with the suspended particles as a function of the particle-eddy size ratio and density ratio. The length scale is described algebraically. A mass/time averaging procedure for the momentum and kinetic energy equations is adopted. The resulting turbulence correlations are modeled under less restrictive assumptions comparative to the previous work. The closures for the momentum and kinetic energy equations are given. Comparisons of the predictions with experimental results on liquid-solid jet and gas-solid pipe flow show satisfactory agreement.

NOMENCLATURE

a amplitude ratio
b body force
C1, C2, C3, C4, C5, C6 = constants
d particle diameter
D turbulent diffusion coefficient for solid phase
E(k) energy spectrum
f flow variable
(fK)_K interaction term
J flux vector for a variable f
k kinetic energy of turbulence
l length scale
NS Stokes number
K(K, t) phase distribution function, Eq. (1)
p pressure
x position vector
Re Reynolds number based on the most energetic eddy size
t average over area

Superscripts

f mass average of f
fT turbulence fluctuation of f
f1 fluctuation of f at low wave number (production range)
fn fluctuation of f at intermediate wave number (transfer range)
P production
T transfer
K volume/time and mass/time averages of f over K

Subscripts

i, j, n denote Cartesian coordinates (= 1, 2, 3)
e eddy
e laminar
k turbulence kinetic energy
K flow component K
KP phase K in the production range
KT phase K in the transfer range
p production range
T transfer range
Total total
average over area

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On sabbatical leave at Caltech 104-44,
1. INTRODUCTION

Multiphase flows are widely applied in engineering processes from chemical, petroleum, mining and other industries. Various theoretical and experimental techniques for the investigation of these flows are available. Some of them are a straightforward extension from the single phase flow models by introducing some ad hoc modifications. Other investigations originate from the gas-solid flow (Soo, 1983) or fluidized bed models (Wang et al., 1986).

Increasing concern for the prediction of turbulent multiphase flows have been noticed during the last twenty years (Danon et al., 1974), Al-Taweel and Landau (1977), Genchev and Karpuzov (1980), Melville and Bray (1979), Crowe and Shams (1978), Michaelides and Farmer (1983), and Shuen et al. (1983)). Two equation turbulence models have been proposed for dilute particulate flows by Elghobashi et al. (1982, 1983, 1984) and Crowder et al. (1984). Algebraic and one equation turbulence models have been suggested also for dense liquid-solid flows (Roco et al., 1983, 1985, 1986) in which the particle-particle interactions play an important role besides the fluid-fluid and fluid-solid interactions. Most of these studies as well as other earlier investigations have some limitations. In the above mentioned studies the response of solids to the turbulent fluctuations of the carrier fluid is obtained under restrictions similar to those referred by Hinze (1975, p. 460), which limit their use. In addition to that, empirical constants and empirical functions are usually introduced in these models.

The purpose of the present paper is

1) To propose a specific mass/time averaging approach for multiphase turbulent flows. Even if the approach is developed for incompressible flow, its application for other multiphase flows is foreseen: From the liquid-solid interaction forces only the drag force is considered in this paper.
2) To improve the one-equation turbulence model reported in [28] by including the modulation of turbulence by particles as a function of particle size and density.
3) To test the proposed model with other models and experimental data for various two-phase flows, without adopting any adjusting empirical coefficients.

2. MIXED AVERAGING APPROACH

The continuum transport equations for multiphase flows can be obtained by assuming a continuum medium with averaged field quantities by using either time, local volume, local mass or spectral averaging (see Buyevich (1971), Soo (1967), Vernier and Delhaye (1968), Hetners (1982)). The averaging for multiphase flow systems may be performed in various ways. Mass averaging technique was applied by Abou-Arab (1985) for turbulent incompressible and compressible flows. To express the spatial nonuniformities and interactions between the flow components Roco and Shook (1985) have developed a specific volume/time averaging technique for turbulent multicomponent systems, in which the size of the averaging volume Av is related to the turbulence scale. Since the Eulerian description of the flow is more convenient than the Lagrangian description and there are more comprehensive mathematical schemes for such formulation, they transformed the volume/time averaging into double time averaging. For any flow function f and any component K in the mixture, at a position r and time t

\[ f^K(t, t) = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} < f^K > dt \]  

(1)

where

\[ \Delta t = \text{time averaging interval corresponding to}\ ]

the turbulence production range (\( \Delta t \rightarrow -1 \)).

\[ < f^K > = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} f(r, t) K(r, t) R(t-t) \ dr \]

(2a)

= intrinsic averaging of f over \( \Delta t \) and the flow component K.

\[ \Delta t = \text{Eulerian time scale for the most energetic}\ ]

eddies \[ = \int_0 R(t-t) \ dr \] (corresponding to the

Taylor's length scale). Note that \( \Delta t < \Delta T \),

but much larger than particle residence time at t.

\[ f^K(t, t) = \begin{cases} 1 & \text{if the considered flow component}\ 
\text{resides at point r and time t} \\
0 & \text{otherwise} \end{cases} \]

(2b)

\[ R(t-t) = \frac{V''(t)V''(t-t)}{V^2(t)} \]

(2c)

= auto-correlation coefficient of the velocity fluctuations of the most energetic eddies (Taylor's scale)

The time averaging over \( \Delta t \) corresponds to the averaging over local volume \( \text{Av} \). The dimension of \( \text{Av} \) is given by the turbulence mixing length, and \( \Delta t = (\text{length scale of Av})/(\text{mean velocity}) [29]. \)

By integrating over a flow component K its interfaces with other flow components become boundary of integration, and the interaction terms are derived in a straightforward manner in the differential formulation. According to (1), any instantaneous value differs from the mean value by a turbulence fluctuation with two components \( f^K \) and \( f''^K \);

\[ f^K = f^K + f''^K = f^K + f_K + f''^K \]  

(3)

where

\[ f''^K = f_K - < f^K > \]

(4)

= spatial nonuniformities within \( \text{Av} \) (or

temporal nonuniformities within \( \Delta t \))

\[ f_K = < f^K > - \overline{f^K} \]

(5)
Since the averaging domain (Δv or Δt) has the dimensions of the mixing length or Eulerian time scale, respectively, the turbulence fluctuation f' corresponds to the turbulence transfer range. The temporal nonuniformities f_k reflect the turbulence fluctuations in the production range.

By averaging with formula (1) the point instantaneous conservation equations, one obtains the double time averaged equations. The formulation is equivalent to the volume/time averaging. The momentum and kinetic energy equations are given in Appendix A.

The local mass average of f over a flow component k is denoted $\bar{f}_k$. It is obtained by applying (1), in which the phase distribution function $\bar{f}(\Sigma, \tau)$ is weighted by the specific mass $\rho_k$.

In this paper we model the phase interaction by using spectral analysis and suggest a closure of the mass averaged equations for linear momentum and kinetic energy. The averaged equations are initially written with all the terms, and then simplified formulations for various flow conditions are suggested.

### 3. ENERGY SPECTRUM AND SOLIDS - EDDY INTERACTION

It is well accepted that turbulence is characterized by fluctuating motions defined by an energy spectrum (Tennekes and Lumley (1972)). Single time scale models, which are normally used for the prediction of turbulent flows, seems simplistic because different turbulent interactions are associated with different parts of the energy spectrum (Mumjic and et al. (1979)).

A typical energy spectrum can be divided into three regions. The first region is the production region of large eddies and low wave number. The second region is the dissipation region with small eddies and high wave number, in which the total kinetic energy produced at the lower wave number is dissipated. The intermediate range of wave numbers represents the Taylor's transfer range. The total kinetic energy k of turbulence may be divided into production range ($k_p$) and transfer range ($k_t$) because there is negligible kinetic energy in the dissipation range:

$$ k = k_p + k_t $$

where

$$ k_p = \frac{1}{2} u_i^2 $$

$$ k_t = \frac{1}{2} u_i^2 $$

$u_i$ = fluctuating velocities in the production and transfer range, respectively.

This partitioning of the energy spectrum was shown to be important for swirling flows (Chen (1966)). and heterogeneous mixture flows such as two-phase jet (Al-Taweel and Landau (1977)).

By using spectral analysis in conjunction with mass/time averaging some additional turbulent correlations will result in the mixture flow equations comparative to homogenous flows. These correlations can be classified into five categories:

1. Eddy-eddy interaction
2. Eddy-mean flow interaction
3. Eddy-particle interaction
4. Particle-mean flow interaction
5. Particle-particle interaction (for dense suspension flow).

These correlations have to be modeled. Since the suspended particles may be of different sizes and different materials, their response to the carrier fluid fluctuations will vary as a function of the mean and fluctuating properties of the flow. The present work will consider a two-way interaction mechanism between solid particles and fluid vortices in dilute suspensions. This interaction mechanism depends on the ratio between the particle size $d_p$ and the turbulent vortex (eddy) size $l_e$. These length scales are compared with the turbulence dissipation micro-scale ($\eta$).

To determine the particle-eddy interaction the energy spectrum for multiphase flow system is divided into three typical zones (Figure 1):

1. "Large vortex zone" (H), where the turbulence energy is extracted from the mean flow by low frequency eddies. Here, the eddy length scale $\ell_e$ is larger than the particle size $d_p$.

![Figure 1. Schematic showing the relative particle size to different eddy sizes in the energy spectrum.](image)
2. "Medium vortex zone" (\#2), where the solid particles are about the same size with the vortex size, i.e.,

$$1_{e2} \approx d_S > \eta$$  \hspace{1cm} (6b)

3. "Small vortex zone" (\#3), which would correspond to the Kolmogorov’s length scale, i.e.,

$$d_S > 1_{e3} \approx \eta$$  \hspace{1cm} (6c)

In zone \#1 the solid particles generally follow the motion within a vortex, and have an energy dissipation effect. The particle response to the turbulent fluctuations (turbulence modulation) is fully determined (see Hinze, 1975). In the small vortex zone \#3 the solid particles can not significantly affect the turbulence microstructure. For the intermediate zone \#2 a linear variation of the particle response is considered. This partitioning allows for the particles-nonuniform size eddies interaction to be efficiently modeled.

The present closure formulation originates from the idea of subgrid scale modeling. If this idea is to be accepted, any flow quantity \( u, v, \alpha, k, \ldots \) etc. may be separated into three parts according to (3), where \( f^K \) and \( a^K \) define the fluctuations in the production and transfer ranges of the energy spectrum, respectively. By starting from the particle equation of motion in its general form, the relation between the particle motion and different fluid eddies can be determined, and from here the fluctuation components \( f^K \) and \( a^K \) (see section 6).

4. COMPOSED AVERAGED EQUATIONS

4.1 Mean Flow Governing Equations

The mass/time averaged momentum equation (Appendix A, Eq. A2) with \( r^f_K = f^K + a^K \) yields:

$$\rho_K \frac{\partial}{\partial t} (\frac{\bar{a}^{u_{KI}}_K}{K} + \frac{\bar{a}^{u_{Ki}}}{K} + \frac{a^{u_{KI}}}{K} + a^{u_{Ki}})$$

Time rate change of the mean flow convection

$$+ \rho_K \frac{\partial}{\partial x^j} (\frac{\bar{a}^{u_{KI}}_K}{K} + \frac{\bar{a}^{u_{Ki}}}{K} + \frac{a^{u_{KL}}}{K} + a^{u_{Ki}})$$

Mean flow convection

$$+ \rho_K \frac{\partial}{\partial x^j} (\frac{\bar{a}^{u_{KI}}_K}{K} + \frac{\bar{a}^{u_{Ki}}}{K} + \frac{a^{u_{KL}}}{K} + a^{u_{Ki}})$$

Inertial effect

$$+ \rho_K \frac{\partial}{\partial x^j} (\frac{\bar{a}^{u_{KI}}_K}{K} + \frac{\bar{a}^{u_{Ki}}}{K} + \frac{a^{u_{KL}}}{K} + a^{u_{Ki}})$$

Collisional/Inertial Effects

$$= u_{Kj} \frac{\partial}{\partial x^j} (\frac{\bar{a}^{u_{KI}}_K}{K} + \frac{\bar{a}^{u_{Ki}}}{K} + \frac{a^{u_{KL}}}{K} + a^{u_{Ki}})$$

Pressure effect

$$+ \rho_K \frac{\partial}{\partial x^j} (\frac{\bar{a}^{x_{KI}}_K}{K} + \frac{\bar{a}^{x_{Ki}}}{K} + \frac{a^{x_{KL}}}{K} + a^{x_{Ki}})$$

Frictional effect

$$+ \frac{\partial}{\partial x^j} (\frac{\bar{a}^{x_{KI}}_K}{K} + \frac{\bar{a}^{x_{Ki}}}{K} + \frac{a^{x_{KL}}}{K} + a^{x_{Ki}})$$

Phase interaction

where \( K \) is a flow component, \( b_{Ki} \) is the body force in the \( i \)-th direction, and \( (1_{KL})_{-K} \) is the projection in the \( i \)-th direction of the interaction vector \( (1_{KL})_{-K} \)

Equation (9) contains correlations which are related to the production and transfer wave number ranges of the turbulence spectrum. It contains also mixed correlations.

4.2 The Kinetic Energy Equation

Similar to the mean flow governing equation the mass/time averaging form of the turbulent kinetic energy equation can be derived for a flow component \( K \). The exact form of this equation for steady state turbulent flow is given in Appendix A (Eq. A5). It contains more than one hundred correlations which are related to the eddies in the production and transfer ranges as well as some mixed correlations. However, only some of these are predominant in a given flow situation as a function of relative particle-vortex size and density ratio.

4.3 Some Modeling Principles and Assumptions

By analysing the derived mean momentum and kinetic energy conservation equations, it can be easily recognized that some modeling assumptions must be made, based on the physical interpretation and the nature of each term. Previous experimental and theoretical findings can help in modeling "collectively" similar terms with minimum number of empirical constants. Secondly, carrying out an order of magnitude analysis for different correlations which appear in the governing equations some terms may be neglected. Thirdly, the micromechanics which control the ability of the flow variables to correlate with each other and the factors affecting the magnitude of these correlations should be considered. With the previous remarks in mind, one can assume for the sake of simplicity that:
1) The correlations between the large eddies from the production range and the small eddies from the transfer range (mixed correlations e.g. $u'u''$) can be neglected as they originate differently and they are related to different ranges of the power spectrum. Similar assumptions are accepted in the classical single fluid turbulence theory.

2) The void fraction fluctuations occur mainly at low frequencies i.e.

$$\alpha = \alpha_0 + \alpha' + \alpha''$$

with $\alpha' \gg \alpha''$.

This is a simplifying assumption which is acceptable for such complicated problems. If the particles are of small diameter their concentration is relatively uniform distributed in the Taylor length scale. High void fractions are mostly associated with large size and high density particulate flows. These large size particles are mainly fluctuating at low frequencies due to its high inertia, hence they in turn correlate weakly at high frequencies.

3) The correlations of higher order than three, for instance $\overline{u'u'u'}\overline{\partial p}/\overline{\partial x_1}$, $\overline{u'u'u'}/\overline{\partial x_j}$, etc., are neglected. These are at least an order magnitude smaller than those of the third order (see Hanjalic' and Launder (1972)).

4) Pressure diffusion contribution to the total turbulent diffusion in the kinetic energy equation will be neglected because of its relatively small magnitude (Hanjalic' and Launder (1972)).

5) The Boussinesq gradient type approximation is adopted for modeling of different fluxes and triple correlations, with assumptions similar to Elghobashi and Abou-Arab (1963) and Roco and Kabadavan (1966).

6) The following constitutive relations are employed for the shear stress of carrier fluid:

$$- \rho \overline{u_i' u_j'} = \mu_{LP} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} k_{LP} \overline{u_i u_j} - \frac{2}{3} \mu_{LP} \overline{u_i u_j} \overline{u_i u_j}$$

$$- \rho \overline{u_i' u_j'} = \mu_{LT} \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} k_{LT} \overline{u_i u_j} - \frac{2}{3} \mu_{LT} \overline{u_i u_j} \overline{u_i u_j}$$

(12) (13)

while the total shear stress $-\rho \overline{u_i' u_j'}$ can be given by

$$\rho \overline{u_i' u_j'} = \rho \overline{u_i u_j'} + \rho \overline{u_i' u_j}$$

(14)

where

$$\mu_{LP} = C_{\mu P} \rho_{SP} \frac{1}{2} l_p$$

(15)

$$\mu_{LT} = C_{\mu T} \rho_{ST} \frac{1}{2} l_T$$

(16)

$$\mu_{LT} = \mu_{LP} + \mu_{LT} = C_{\mu L} \rho_{SL} \frac{1}{2} l_L$$

(17)

Similar relations can be written for the viscosity of the dispersed phase $\mu_{ST}$ (see Roco and Balakrishnan (1965)). However, in the present work we choose to define the eddy viscosity of solids as follows:

$$\nu_{ST} = \frac{\nu_{LT}}{\sigma}$$

(18)

where

$$\sigma = \sigma_{dS} / \sigma_S$$

(19)

$$\sigma_{dS} = \nu_{LT} / D_S$$

(20)

$$\sigma_S = \nu_{ST} / D_S$$

(21)

Appropriate expressions for $\sigma_{dS}$ are cited in many articles such as Peskin (1971), Picart et al. (1960), and Hetsroni (1982). $\sigma_S$ is a Schmidt number and its value is about 1.5 (Abou-Ellail and Abou-Arab (1965)).

7) In the present approach for dilute particulate flows the turbulence kinetic energy equation is written only for the carrier flow. The solid phase turbulence kinetic energy and turbulence correlations are evaluated from the available solution of the linearized equation of motion of a solid particle after its transformation from the real time to the frequency domain.

8) Terms which are of similar nature i.e. convection, diffusion, dissipation, etc. can be modeled collectively. The length-, velocity- and time-scale which are appropriate for the description of their rate of change must be identified from the physical interpretation of these terms.

9) The response function which shows the ability of solid particles to follow the eddies is obtained from the equation of motion of particles for different local dimensionless parameter, $d_S / l_0$ and $\rho_S / \rho_L$.

10) To establish the degree of generality of the proposed model validation tests were carried out for air-laden and water-laden jet, and air-solid pipe flow.

5. CLOSURE FOR THE MEAN FLOW EQUATIONS

With the modeling assumptions given in the previous section, the steady state mean flow momentum equations for any flow component, reads

$$\rho_k \frac{\partial}{\partial x_j} \left( \overline{a_k u_k u_j} + \overline{a_k u_k u_j} \right) + \frac{\partial}{\partial x_k} \rho_k \left( \overline{a_k u_k u_j} + \overline{a_k u_k u_j} \right) +$$

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\[
\begin{align*}
&+ \frac{\partial}{\partial x_j}(\rho_k(u_k u_i \partial u_j + u_i u_j \partial u_k) + \partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k) \\
&+ \frac{\partial}{\partial x_j}(\partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k) + (\partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k)) \\
&= \rho_k(\partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k + \partial u_k \partial u_i u_j) - \frac{\partial}{\partial x_i}(\partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k + \partial u_k \partial u_i u_j) \\
&+ \left(\frac{\partial}{\partial x_j}(\partial u_i \partial u_j u_k) + (\partial \partial u_i \partial u_j u_k)\right) \\
&+ \frac{\partial}{\partial x_i}(\partial u_i \partial u_j u_k + \partial u_j \partial u_i u_k + \partial u_k \partial u_i u_j) \\
&+ \frac{\partial}{\partial x_j}(\partial u_i \partial u_j u_k) + (\partial \partial u_i \partial u_j u_k)
\end{align*}
\] (22)

In Eq. (22) there are 16 correlations, half of them in the production "large eddy" range of the spectrum. The terms are modeled following the criteria: i) Physically correct behavior, ii) Minimum number of empirical functions and constants, and iii) Comparisons against experimental data over a wide range of conditions are required to check the validity of the model.

The turbulent stresses caused by the large and small size energetic eddies, \( -\rho_L u_i u_j \) and \( -\rho_L u_i u_j \), are defined by Eqs. (12) and (13). The correlation between \( u_i \) and \( u_j \) is weak when \( i \neq j \). This finding will be explained in the next section.

The collisional effect correlations are modeled after Launder (1976):

\[
\begin{align*}
\frac{\partial}{\partial x_1} u_k u_i u_j &= - C_{\Phi \Sigma} \left( \frac{k_{P\Sigma}}{\epsilon_{P\Sigma}} \right) \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
&= \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
\end{align*}
\] (23)

where \( C_{\Phi \Sigma} \) is a constant of a value of about 0.1. Similarly

\[
\begin{align*}
\frac{\partial}{\partial x_1} u_k u_i u_j &= - C_{\Phi \Sigma} \left( \frac{k_{P\Sigma}}{\epsilon_{P\Sigma}} \right) \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
&= \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
\end{align*}
\] (24)

where \( C_{\Phi \Sigma} \) must be optimized by comparison with experimental data. The fluxes in (23) and (24) can also be collectively modeled as:

\[
\begin{align*}
\frac{\partial}{\partial x_1} u_k u_i u_j &= - C_{\Phi \Sigma} \left( \frac{k_{P\Sigma}}{\epsilon_{P\Sigma}} \right) \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
&= \left( u_k u_i u_j \right) \frac{\partial}{\partial x_1} u_i \partial u_j \\
\end{align*}
\] (25)

The diffusion fluxes namely \( \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} \), etc. are modeled using Boussinesq approximation as:

\[
- \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} = \frac{u_k \partial \tilde{\alpha}}{\partial x_i} \frac{\partial \tilde{\alpha}}{\partial x_k} \quad (26)
\]

while

\[
- \frac{\partial u_i}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \frac{u_k \partial \tilde{\alpha}}{\partial x_i} \frac{\partial \tilde{\alpha}}{\partial x_k} \quad (27)
\]

Similar expression can be written for \( \frac{\partial u_i}{\partial x_k} \) and \( \frac{\partial \tilde{\alpha}}{\partial x_k} \), etc. However, according to our assumptions, especially those concerning the void fraction fluctuation at high frequency \( \alpha^* \), the fluxes in Eqs. (26) and (27) can be modeled collectively as:

\[
- \frac{\partial u_i}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \frac{u_k \partial \tilde{\alpha}}{\partial x_i} \frac{\partial \tilde{\alpha}}{\partial x_k} \quad (28)
\]

The solution of the two transport equations (for \( k_p \) and \( k_T \)) would require also a description for the length scales. This point will be explained in more details in the next section.

The fourth group of terms with the void fraction-shear stress turbulent correlations contains the laminar viscosity as a multiplier, and will be neglected due to its smaller order of magnitude.

The pressure effect contribution to the mean flow equation consists of two groups. Each contains three terms. The first of these terms is the mean pressure-void fraction. The second and the third terms are the pressure-void fraction correlations which can be modeled after Elghobashi and Abou-Arab (1983) by

\[
- \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial \tilde{\alpha}}{\partial x_k} \right) = \psi_1 + \psi_2 \quad (29)
\]

where

\[
\psi_1 = - C_{\Phi \Sigma} k_{P\Sigma} \frac{k_{P\Sigma}}{\epsilon_{P\Sigma}} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \quad \text{and}
\]

\[
\psi_2 = - C_{\Phi \Sigma} k_{P\Sigma} \frac{k_{P\Sigma}}{\epsilon_{P\Sigma}} \frac{u_k \partial \tilde{\alpha}}{\partial x_k}
\]

The values of the constants \( C_{\Phi \Sigma} \) and \( C_{\Phi \Sigma} \) are about unity.

The second correlation in the second group of terms can be also modeled following Launder (1976). The final form is

\[
\frac{\partial \tilde{\alpha}}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \rho_k (\tilde{\alpha} \frac{\partial \tilde{\alpha}}{\partial x_k}) \left[ C_{\Phi \Sigma} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \right]
\]

\[
= \frac{\partial \tilde{\alpha}}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \rho_k (\tilde{\alpha} \frac{\partial \tilde{\alpha}}{\partial x_k}) \left[ C_{\Phi \Sigma} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \right]
\]

\[
= \frac{\partial \tilde{\alpha}}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \rho_k (\tilde{\alpha} \frac{\partial \tilde{\alpha}}{\partial x_k}) \left[ C_{\Phi \Sigma} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \right]
\]

\[
= \frac{\partial \tilde{\alpha}}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \rho_k (\tilde{\alpha} \frac{\partial \tilde{\alpha}}{\partial x_k}) \left[ C_{\Phi \Sigma} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \right]
\]

\[
= \frac{\partial \tilde{\alpha}}{\partial x_k} \frac{\partial \tilde{\alpha}}{\partial x_k} = \rho_k (\tilde{\alpha} \frac{\partial \tilde{\alpha}}{\partial x_k}) \left[ C_{\Phi \Sigma} \frac{u_k \partial \tilde{\alpha}}{\partial x_k} \right]
\]
The gradient transport model with the exchange minimum number of empirical engineering energy equation A5 scale "I" is described equation (Ex. AS A5).

The values of the constants $C_1$ and $C_2$ are 4.3 and -3.2, respectively. The modeling of these correlations suffers from the embodied assumptions concerning the velocity and length scale description. It would require a large number of transport equations to model accurately each of the above correlations.

The interaction term $(\overline{I}_{K_1})_K$ for $K = L$ (liquid) and $-K = S$ (solid) is modeled for dilute suspensions with particle Reynolds numbers less than unity:

$$\overline{(I_{K_1})_S} = - (18\mu_k/\sigma) \left[ (\overline{u_{Li}^2} - \overline{u_{Si}^2}) \sigma_S + \frac{\nu_{LP}}{\sigma} + \frac{\nu_{LT}}{\sigma} \right] \overline{u_{S}} \overline{u_{L}}$$

where the first term is the drag interaction for particles in the Stokes range. The second and third terms are the turbulent fluxes due to the relative motion between the particles and fluid. The gradient transport model with the exchange coefficients $u_{LP}$ and $u_{LT}$ corresponding to the production and transfer ranges is adopted for these fluxes ($\overline{u_{LP}}$, $\overline{u_{LT}}$, $\overline{u_{Si}^2}$, $\overline{u_{S} u_{L}^2}$, and $\overline{u_{S} u_{L} u_{Li}}$).

If only single velocity scale is chosen for the whole energy spectrum, $k_L$, there will be only one momentum exchange coefficient $u_{LT}$ instead of $u_{LP}$ and $u_{LT}$

$$v_{LT} = \frac{\nu_{LP}}{\sigma_{aS}} + \frac{\nu_{LT}}{\sigma_{aS}}$$

6. CLOSURE FOR THE KINETIC ENERGY EQUATION

In the present work, the turbulence kinetic energy for the liquid phase "$k_L$" (turbulence velocity scale) is obtained from an exact transport equation (Eq. A5 in Appendix A). and the length scale "I" is described algebraically. The kinetic energy equation A5 contains a large number of turbulence correlations. In order to obtain an engineering turbulence model, it is sufficient to consider the principles and the assumptions given in Section 4.3. By engineering turbulence model, it is meant a physically correct model with minimum number of empirical coefficients.

The first group of terms in the $k$-equation (Group #1) is the convection of the total specific kinetic energy, where

$$\frac{\partial k_L}{\partial t} + \frac{\partial}{\partial x_j} \left( \frac{k_L u_{Li}}{\sigma_x} \right)$$

Diffusion transport of $k$ is composed of two main parts. The first part (Group #2) is the velocity diffusion and it contains the 3rd order velocity correlations, while the second part (Group #3) is the pressure diffusion with the pressure-velocity correlations. The modeling of the velocity diffusion part is obtained as follows:

$$\rho_L \frac{\partial}{\partial x_j} \left( \frac{k_L u_{Li} u_{Lj}}{\sigma_x} \right) = \rho_L \frac{\partial}{\partial x_j} \left( \frac{k_L u_{Li} u_{Lj}}{\sigma_x} \right)$$

where $\sigma_{k}$ and $\sigma_{T}$ are the turbulent Prandtl/Schmidt numbers for the kinetic energy in the production and transfer ranges. To reduce the number of empirical constants and the number of governing equations the above two correlations are modeled collectively as follows

$$\rho_L \frac{\partial}{\partial x_j} \left( \frac{k_L u_{Li} u_{Lj}}{\sigma_x} \right)$$

Mixed and higher order correlations (Groups #3 and #5) can be neglected according to the modeling assumptions stated and discussed in section 4.3 of the present paper.

The pressure diffusion term is negligible relative to the velocity diffusion (Hanjalic and Launder (1972)).

The production terms are divided into two groups. The first group (Group #6) is common for single and multiphase incompressible flows.

$$\rho_L \frac{\partial}{\partial x_j} \left( \frac{k_L u_{Li} u_{Lj}}{\sigma_x} \right) + \rho_L \frac{\partial}{\partial x_j} \left( \frac{\overline{u_{Si}^2}}{\sigma_x} \right) \frac{\partial u_{Li}}{\partial x_j}$$

The physics of turbulence and the consideration of the spectral energy transfer and the production is only due to the interaction between the mean flow and the large eddy. Since the $\overline{u_{Li} u_{Lj}}$ correlation is for medium size eddies which have almost no direct interaction with the mean flow, it results that the
The contribution of these small eddies to the turbulence production via the mean field is smaller than that of the large eddies. This means also that the correlation is weak if \( i \neq j \), i.e., it is only of significant value if the turbulent normal stress components \( (u_i''u_j) \) are considered. According to Hanjalic' et al. (1979) multiple scale model the turbulent viscosity is defined as follows:

\[
\nu_L/\rho_L = C_\mu k_L (k_L/\varepsilon_L)^{0.5}
\]

where \( k_L = k_{LP} + k_{LT} \)

and \( C_\mu = 0.09 \)

This equation can be rewritten as

\[
\nu_L/\rho_L = C_\mu (k_{LP}^{1.5}/\varepsilon_L^{1.5})
\]

where \( C_{LP} \) and \( C_{LT} \) are two additional constants and \( k_{LP} \) and \( k_{LT} \) are also two additional length scales for the large and medium size eddies. The length scale can be related with the following relations (from Eqs. (38) and (40))

\[
l_{LT} = C_1 l_{LP} (k_{LP}/k_{LP})
\]

Since the ratio \( k_{LT}/k_{LP} \) is of the order of unity and \( l_{LP} > l_{LT} \), the constant \( C_1 \) should be smaller than unity. Thus if the multiple time scale model is not recommended (due to its large number of additional constants) an alternative approach is to consider a multiple velocity scale model. In this model only two differential equations for \( k_{LP} \) and \( k_{LT} \) have to be solved. The length scales can be obtained by using the previous relation Eq. (41) and any expression for the length scale of the large eddies \( l_{LP} \), for example that used by Roco and Shook (1963) for cylindrical pipes.

The modeling of the additional production terms (Group #7) is achieved as follows:

\[
- \frac{a_{LT}u_i''u_j''}{\sigma_A} \frac{\partial p_i}{\partial x_1} = \frac{u_{LT}}{\sigma_A} \frac{\partial p_i'}{\partial x_1} \frac{\partial p_j}{\partial x_1}
\]

\[
- \frac{a_{LT}u_i''u_j''}{\sigma_A} \frac{\partial p_i'}{\partial x_1} = \frac{u_{LT}}{\sigma_A} \frac{\partial p_j}{\partial x_1} \frac{\partial p_i'}{\partial x_1}
\]

and both collectively as

\[
- (a_{LT}u_i''u_j'' + a_{LT}u_i''u_j'') \frac{\partial p_i'}{\partial x_1} = \frac{u_{LT}}{\sigma_A} \frac{\partial p_i'}{\partial x_1} \frac{\partial p_j}{\partial x_1}
\]

Terms like \( a_{LT}u_i''u_j'' \) and \( a_{LT}u_i''u_j'' \) are modeled in a similar manner to that of the above terms i.e.

\[
- (a_{LT}u_i''u_j'' + a_{LT}u_i''u_j'') \frac{\partial p_i'}{\partial x_1} = \frac{u_{LT}}{\sigma_A} \frac{\partial p_i'}{\partial x_1} \frac{\partial p_j}{\partial x_1}
\]

The extra production terms (Group #7) can be written in the following form

\[
\text{Extra production} = \frac{u_{LT}}{\sigma_A} \frac{\partial p_i'}{\partial x_1} \frac{\partial p_j}{\partial x_1}
\]

The terms \( p_{LT}u_i u_j'' \) and \( p_{LT}u_i u_j'' \) in Group #8 represents the dissipation rate from large eddies \( \varepsilon_{LP} \) and transfer eddies \( \varepsilon_{LT} \). Since viscous dissipation is mainly confined to small scale eddies and to simplify the mathematical form of the multiple time scale turbulence models, Hanjalic' et al. (1979) have assumed that there is an equilibrium spectrum energy transfer between the dissipation and transfer region i.e., \( \varepsilon_{total} = \varepsilon_L \), where \( \varepsilon_L \) is the dissipation rate in the single scale scheme. Thus

\[
\rho_{LT} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_j} = - \rho_{LT} \varepsilon_{LP} = - \rho_{LT} \varepsilon_{LT} \]

where

\[
\varepsilon_{LP} = C_{DP} l_{LP}^{1.5} / \rho_L \quad \text{and} \quad \varepsilon_{LT} = C_{DT} l_{LT}^{1.5} / \rho_L
\]

In equation (47) a spectral cascading between the production and transfer eddies is considered (Hanjalic' et al. (1979)). This equation gives an additional relation between the spectrum scales.

The correlations between the fluctuating velocity component and the fluctuating friction forces (interaction terms in Groups #6 and #10) are due to fluid-fluid and fluid-solid drag force in dilute flows. The friction interaction terms due to fluid-fluid interaction and fluid-solid interaction Group #6 are given by Eq. (47) and (45). The form of the correlation between the fluctuating drag force and the velocity fluctuations depends on the expression adopted for the drag force. The viscous drag correlation (VDC) in Group #10 for Stokes flow over particles in dilute suspensions reads
VDC = \frac{18\mu}{\alpha_S^2} (\bar{u}_L - \bar{u}_S) \cdot \Delta_1 \tag{49a}

\frac{18\mu}{\alpha_S^2} (\bar{a}_S \cdot \Delta_2 + \Delta_3)

where:

\Delta_1 = \frac{\alpha_S}{\mu} (\bar{u}_L - \bar{u}_S)
\Delta_2 = \frac{\alpha_S}{\mu} (\bar{u}_L - \bar{u}_S) + \frac{\alpha_S}{\mu} (\bar{u}_L - \bar{u}_S)
\Delta_3 = \frac{\alpha_S}{\mu} (\bar{u}_S \cdot \bar{u}_L - \bar{u}_L \cdot \bar{u}_S)

The first correlation group \Delta_1 (Eq. 49a) can be approximated using the gradient type assumption. The second correlation in this expression (49b) is that due to the relative slip fluctuating motion \( \bar{u}_L (\bar{u}_L - \bar{u}_S) \) and \( \bar{u}_L (\bar{u}_S - \bar{u}_L) \). These can be modeled using similar approach to that of Elghobashi and Abou-Arab (1983) but with some modifications which allow for different particle-eden interaction according to their relative size. These modifications are based on the spectral analysis carried out by Dingg (1975) for the response of the particles to the turbulent fluctuations of the carrier fluid. For very large eddies \( \kappa \ll \kappa_S \),

\[ u_S^i = (v_S^i)_{\kappa} = (v_L^i)_{\kappa} a \cdot \exp[-(1/(\alpha_S \cdot - \beta)] \tag{50} \]

where \( \alpha \) is the wave number: \( \kappa_S \) is the Basset-Boussinesq-Ossen wave number defined as \( 1/\alpha_S \); \((v_S^i)_{\kappa} \) and \((v_L^i)_{\kappa} \) are the solid and liquid velocity components with the wave number \( \kappa \); \( \kappa_S \) is the amplitude ratio of oscillations

\[ a = [(1 + q_2^2) + q_1^2]^{0.5} \tag{51a} \]

and \( \beta \) is the phase angle of oscillation

\[ \beta = \tan^{-1} \left[ q_2 / (1 + q_1) \right] \tag{51b} \]

The expressions of \( q_1 \) and \( q_2 \) are

\[ q_1 = \frac{1}{(s + 0.5)^2} [1 - \frac{\sqrt{42(s + 0.5)}}{42} \frac{\kappa \alpha_S}{\mu} \bar{u}_S \bar{u}_L \bar{u}_S \bar{u}_L] \tag{52a} \]

\[ q_2 = \frac{1}{(s + 0.5)^2} \left[ \bar{u}_S \bar{u}_L \bar{u}_S \bar{u}_L \right] \tag{52b} \]

with the following dimensionless parameters:

\[ s = \rho_S / \rho_L \]

\[ N_S = \frac{\kappa}{u_L \kappa \alpha_S^2} = \frac{\mu}{\rho_L \kappa \alpha_S} \]

where

\( \kappa \) is the wavenumber of the most energetic eddies.

\( Re = \) Reynolds number based on \( l_T \), and

\( N_S = \) Stokes number.

For small eddies with wave numbers \( \kappa >> \kappa_S \), the particle response can be described by

\[ u_S^i = (v_S^i)_{\kappa} = (v_L^i)_{\kappa} \frac{\kappa \alpha_S}{\mu} \rho_L \tag{53} \]

For the intermediate size eddies \( \kappa = \kappa_S \) one can use (50) with

\[ a = \frac{1}{s} \quad \text{and} \quad \beta = \tan^{-1} (s) \tag{54a} \]

If the fluctuating slip velocity \( w_i \) is defined as

\[ w_i = u_S^i - u_L^i \tag{55} \]

then the ratio of the mean square \( \bar{w}_i^2 \) and \( \bar{u}_L^2 \) becomes

\[ \frac{\bar{w}_i^2}{\bar{u}_L^2} = \frac{\bar{u}_S^2 - 2\bar{u}_S \bar{u}_L \bar{u}_S + \bar{u}_L^2}{\bar{u}_L^2} \tag{56} \]

with

\[ \frac{\bar{u}_S^2 \bar{u}_L^2}{\bar{u}_L^2} = \frac{1}{2} \frac{\bar{u}_S^2 - \bar{u}_L^2}{\bar{u}_L^2} - \frac{\bar{u}_L^2 - \bar{w}_i^2}{\bar{u}_L^2} \]

\[ = \frac{1}{2} \frac{\bar{u}_L^2}{\bar{u}_L^2} [1 + \Gamma_{SL} - \Gamma_{RL}] \tag{57} \]

The values of \( \Gamma_{SL} \) and \( \Gamma_{RL} \) can be obtained by using the preceding solution (Eq. 50):

\[ \Gamma_{SL} = [ \int_{0}^{\kappa} \kappa \alpha_S^2 \bar{E}_L(\kappa) d\kappa + \int_{\kappa}^{\infty} \kappa \alpha_S^2 \bar{E}_L(\kappa) d\kappa ] / \int_{0}^{\infty} \bar{E}_L(\kappa) d\kappa \tag{58} \]

or

\[ \Gamma_{SL} = \frac{2h}{\pi (h+1)} \tan^{-1} (\kappa_S / \kappa_T) \tag{59} \]

where

\[ h = \frac{A}{\kappa_T \alpha_S} = \frac{18 u_L}{(s + 0.5)^2} \frac{\kappa_S \alpha_S}{\mu} \]

\[ \text{Re} = \frac{18 u_L}{(s + 0.5)^2} \frac{\kappa_S \alpha_S}{\mu} \]

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Figure 2. The effect of eddy-particle size ratio \( (k_S/k_T) \) on the particle response to the eddy fluctuations \( (\overline{u^2}/\overline{u_L^2}) \)

The above analysis applies equally well to the large eddies as to small eddies. The energy spectrum function \( E_L(\kappa) \) for a two-phase flow is given by Al-Taweel and Landau (1977). However, since its form is not essential (Dingguo (1967)) it is sufficient to adopt any simple form as that given above by Eq. (61). It is clear from the above analysis that the particle response to the carrier fluid fluctuations is a function of the density ratio \( k_S/k_L \), size of interacting eddy relative to particle size \( k_S/k \) and Reynolds number based on the size of the most energetic eddies of the flow. An analogous expression to that given by Eq. (62) is that based on the Chao’s solution (see Chao (1964)). This solution can be considered as a substitution of Dingguo’s solution only for \( k << k_S \) i.e. for fine particles.

The last term to be modeled in the VCD group, \( A_3 \), is separated into four correlations:

\[
A_3 = -\frac{18u_L}{S} \left[ (\frac{\partial S}{\partial u_L}) \left( \frac{\partial S}{\partial u_L} \right) + (\frac{\partial S}{\partial u_S}) \left( \frac{\partial S}{\partial u_S} \right) \right]
\]

where the triple correlations T3 and T4 are modeled in a similar manner to that used for the calculation of T1 and T2 by using Eqs. (23) and (24), thus:

\[
T_1 = \frac{\partial S}{\partial u_L} \frac{\partial S}{\partial u_L} \frac{\partial S}{\partial u_S} \frac{\partial S}{\partial u_S}
\]

T2 = \( \frac{\partial S}{\partial u_L} \frac{\partial S}{\partial u_L} \frac{\partial S}{\partial u_S} \frac{\partial S}{\partial u_S} \)

These correlations can also be collectively modeled using single velocity and length-scale. The first two and the last two terms yield, respectively:

\[
T_1 + T_2 = C \frac{2k_L}{\epsilon_S} \frac{\partial S}{\partial u_L} \left( \frac{\partial S}{\partial u_S} + \frac{\partial S}{\partial u_S} \right)
\]

\[
T_3 + T_4 = C \frac{2k_L}{\epsilon_S} \frac{\partial S}{\partial u_L} \left( \frac{\partial S}{\partial u_S} + \frac{\partial S}{\partial u_S} \right)
\]
The terms in Group 49 of Eq. A5 are of diffusive and dissipative nature. The diffusion terms as they appeared in Group 49 are multiplied by the molecular viscosity and therefore will be neglected due to their relatively small magnitude. Other higher order correlations and mixed correlations in this group are also neglected according to the modeling principles stated previously in section 4.3 and as they are also multiplied by the molecular viscosity.

By substituting all previously modeled terms into the exact form of the turbulence kinetic energy equation, and rearranging these terms, one obtains the simplified modeled form given in Appendix B (Eq. B2).

Since the present model is based on an exact equation, namely the turbulence kinetic energy equation, and the modeled form of this equation has no adjusting coefficients it is expected that the model will generally produce good results and have less limitations compared to other models. The only modeling assumption is that of the Boussinesq gradient type, which generally is accepted. The correlations that requires questionable semi-empirical modeling assumptions and introduction of empirical constant are (i) fewer in number (for $\alpha L^2 / \bar{u}_L^2$ and $\alpha L S_i L_j$), and (ii) for terms having an order of magnitude smaller (by ratio $\bar{u}_L/\bar{u}$) compared to other main terms in the $k$-equation. The only significant new correlation used in the present closure is that due to the relative motion between the phases. This is modeled with less restrictions and taking into consideration the effect of the particle diameter-eddy size ratio on the particle response to the eddying motion. The limitation of the present one-equation $k$ model closure is the algebraic formulation for the length scale. Since there are many factors affecting this length scale, the practical application to give a unique and accurate description of the scale, the use of a transport equation for the length scale in the two-equation model of Elghobashl and Abou-Arab (1983) is expected to give better results with fine particles ($d_S < \eta$). However, it should be noticed that the former model and any other similar models contain some empirical constants specific for various flow conditions, and they require the solution for an additional transport equation. It can be expected that the present model combined with an appropriate length scale equation, e.g. dissipation rate equation will simulate better most of the important features of multiphase turbulent flows, particularly the fluid particle interaction. In that case the number of closure transport equations will increase to three (if two velocity scales, $k_p$ and $l_p$, and one length scale transport equations) or four transport equations (if two velocity scales and two length scale, $l_p$ and $l_t$ are adopted).

7. SAMPLE OF RESULTS

Figures 3 to 6 compare the present predictions with LDA-measurements for single and two-phase turbulent pipe flow (see Kanda et al (1980)) and turbulent round water jet laden with uniform-size solid particles (see Parthasarathy and Faeth (1987)). Both flows are oriented vertically downward. These flows are axisymmetrical. The concentration profiles are given as input data based on experimental results.

In these flow situations the average eddy size $l_e$ was varying from about one $d_S$ to few hundreds

**Figure 3.** Computed and measured [20] mean velocity distribution of air in air-solid pipe flow (--- single-phase. --- 2 Eqs. k-$\varepsilon$ model [12]. -- 1 Eq. k-model [27]. -xx- 1 Eq. k-model [12]. ----- 1 Eq. $\bar{u}_t$-model [26]).

**Figure 4.** Predicted radial distribution of the turbulence intensity of air in air-solid pipe flow using different turbulence models (keynote as in Fig. 3).
The corresponding representative eddy size in
the transfer range was only a fraction of particle
diameter \(d_s\) in the pipe flow case. In the jet flow
case the mean size of large eddies and the
Kolmogorov length scale were also varied in a wide
range. These scales are field variables, and they
depend upon the flow configuration, location in the
flow domain and particle dimensions.

Two-phase flow solutions were obtained by
solving the flow governing equations in their
modeled form which are described in the previous
section and given in Appendix B. The numerical
procedure used for these predictions is based on
a developed version of the Genmix-Code of Spalding
(1977). However, since the main objective of this
paper is to give a complete description of a
developed turbulence closure for multiphase flows
and due to the space limitation the details of this
numerical approach will not be given here. The CPU
Time for the two considered flow cases was about 4
minutes on a VAX 780 Mini-Computer and 2 minutes on
IBM 3084.

Case I: Gas-Solid Vertical Pipe Flow:

Figure 3 shows a comparison between the
experimental data and the present predictions using
different models of turbulence namely 1.
One-equation \(k\)-model of Roco and Mahadevan (1986).
2. One-equation \(\nu_t\)-model of Roco and Balakrishnan
(1965). 3. The \(k\)-equation as given in the
two-equation \(k\)-\(\nu_t\) model of Elghobashi and Abou-Arab
(1963). 4. The two-equation model of Elghobashi and
Abou-Arab (1963), and 5. The
dependence on velocity, and turbulence
Correlation used for the
prediction of different turbulence closures are given in Figures 5 and 6.

Comparison between experimental data and
numerical predictions of different turbulence
closures are taken from
Parthasarathy & Faeth
(1967). Different axial locations within the round
jet, between eight and forty jet diameters from
the injection nozzle are
Figure 7. Radial distribution of solid phase concentration in water-solid jet flow.

The comparison shows that the two-equation two-phase k-ε model of Elghobashi and Abou-Arab (1993) describes both flows better than the one-equation mass/time averaged turbulence model. However, it is important to mention here that the presently developed closure uses only one transport equation and without adjusting any empirical coefficient. At the same time, the present closure is in its early stages and more refinements and validation tests are required, especially for coarse particles two-phase flows for which one would expect that the present k-formulation will provide improved predictions. The difference between the predictions of the mean and fluctuating flow velocity components as obtained by the present k-formulation and that of Ref. [12] depends on the particle size relative to its surrounding eddies. Figure 8 illustrates this difference at two different loading ratios for the pipe flow at a radial distance \((R - r)/R\) equal 0.1.

6. CONCLUDING REMARKS

The turbulence closure presented in this paper for dilute suspension flow is based on the fluid turbulence kinetic energy equation. The main features of this model are:

i) Two velocity scales are adopted in computation for large and medium size eddies, corresponding to the turbulence production and transfer range, respectively. They are expressed into the governing equations by a specific local mass/time averaging. On this basis the spatial and temporal transfer rates of the thermodynamic quantities and the particle-eddy interaction are better estimated.

ii) Spectral analysis of the interaction mechanism between particles and most energetic eddies provide analytical correlations for closure. The particle response and the modulation of turbulent eddying motion is given as a function of the particle-fluid density and size ratios.

iii) To keep the number of transport equations of the turbulence closure and the number of empirical constants as minimum as possible, the length scales \(l_p\) and \(l_T\) are described using algebraic expressions. Relations between these scales (Eqs. 40 and 47) are suggested.

iv) The model does not introduce additional empirical constants to the closure of the velocity scale equation.
REFERENCES


APPENDIX - A

Mass/Time Averaged conservation Equations

Mass averaging the conservation equations of mass and momentum over a flow component (K) one obtains a new system of equations for mean velocity, concentration and kinetic energy of turbulence.

The point instantaneous conservation equation can be written for any flow component (K) or for the entire mixture in the following general form

\[ \frac{\partial}{\partial t}(\rho u) + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) + \nabla \cdot \mathbf{S} = 0 \]  (A1)

where:  
\( \rho \) = density  
\( \mathbf{u} \) = velocity vector  
\( \mathbf{S} \) = source term

Let assume \( \mathbf{S} = \mathbf{u} \mathbf{K} \). By splitting each flow property into mean and turbulent fluctuating component \( (\mathbf{u} \mathbf{K} + \mathbf{u} \mathbf{t} \mathbf{K}) \) and mass/time or double time averaging the equation (A1), one obtains the following momentum equations in the i-th direction for an incompressible phase (K) without mass exchange with other flow components (see Roco and Shook (1985)):

\[ \rho_k \frac{\partial}{\partial t} \left( \bar{u}_i \mathbf{u}_i \mathbf{K}_k \right) + \rho_k \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) = \rho_k \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) + \rho_k \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) \]  (A2)

where

\[ \mathbf{S} \] = phase (or generally a flow component)  
\( \bar{\mathbf{u}}_i \) = mass/time average of \( \mathbf{u}_i \) over \( \mathbf{K} \)

This drag term takes a simple form for Reynolds numbers less than unity

\[ (1_L^2)_{KL} = (1_L^2)_{KL} - \frac{C_{D_L}}{S_L} \int dS \frac{u_{L_i} - u_{S_i}}{(1 - a_S)^{1/2}} \]  (A3)

The transverse effects caused by the presence of other solid particles, Saffman force and Ho and Leal inertial force are neglected. Their importance is small in dilute suspension turbulent flows with fine particles.

Equation (A2) contains terms due to the unsteady flow, mean flow convection, diffusion, pressure, body force, as well as frictional, inertial and pressure force. The mean form of the turbulence kinetic energy governing equation is obtained by subtracting from the steady state instantaneous momentum equation for a component \( \mathbf{K} = \mathbf{L} \) the corresponding mean flow equations, and then multiplying the resulting difference equation by \( (1_L^2)_{KL} \). By averaging one obtains the kinetic energy equation for a flow component \( \mathbf{K} = \mathbf{L} \). This equation reads

\[ \frac{\partial}{\partial t} \left( \bar{u}_i^2 / 2 \right) + \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) - \rho_k \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) = \rho_k \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) \]

Group #1 (Convection)  
Group #2 (Velocity Diffusion)

\[ \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) + \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) + \frac{\partial}{\partial x_j} \left( \bar{u}_i \mathbf{u}_j \mathbf{K}_k \right) \]

Group #3 (Higher Order Correlations)

\[ \text{other 4th order and minor terms} \]
The Modeled Form of the Turbulence Kinetic Energy Equation

The steady-state turbulence kinetic energy equation for the liquid phase \( (K = L) \) is:

\[
\frac{\partial v}{\partial t} + \nabla \cdot (K_v \nabla v) = \nabla \cdot \left( \rho \frac{\partial v}{\partial t} \nabla \mu \frac{\partial v}{\partial x} \right) + a_L \left[ \frac{\partial u}{\partial x} \right] + a_{KL} \left[ \frac{\partial u}{\partial x} \right] + a_{KL} \left[ \frac{\partial u}{\partial x} \right] \]

where the expressions for \( r_{SL}, r_{RL}, T_1, T_2, T_3 \) and \( T_4 \) are given in the text.
**Constitutive Relationships and Models in Continuum Theories of Multiphase Flows**

**Abstract**

During the first week of April 1989, a workshop, entitled "Constitutive Relationships and Models in Continuum Theories of Multiphase Flows," was convened at NASA's Marshall Space Flight Center. The purpose of this workshop was to open a dialogue on the topic of constitutive relationships for the partial or per phase stresses, including the concept of solid phase "pressure" and the models used for the exchange of mass, momentum, and energy between the phases in a multiphase flow. This volume is the result of the stated objective of the workshop. The program, abstracts, and text of the presentations made at the workshop are included.