Three-Dimensional Analysis of Surface Crack-Hertzian Stress Field Interaction

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# TABLE OF CONTENTS

## SUMMARY

## CHAPTER ONE  INTRODUCTION

## CHAPTER TWO  BOUNDARY ELEMENT METHOD

- 2.1 Introduction
- 2.2 Derivation of Boundary Integral Equation
- 2.3 Numerical Implementation of Boundary Integral Equation
- 2.4 Subregion Technique
- 2.5 Discontinuity of Traction

## CHAPTER THREE  LINEAR ELASTIC FRACTURE MECHANICS

- 3.1 Introduction
- 3.2 Calculation of Stress Intensity Factors Using the Boundary Element Method

## CHAPTER FOUR  VERIFICATION PROBLEMS

- 4.1 Prismatic Bar Under Uniform Tension
- 4.2 Beam Subjected to Pure Bending
- 4.3 Circular Buried Crack Under Uniform Tension
- 4.4 Circular Buried Crack Inclined at 30 Degrees Under Uniform Tension

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4.5 Semi-circular Surface Crack Under Uniform Traction

CHAPTER FIVE STRESS INTENSITY FACTOR ANALYSIS OF
THE INNER RACEWAY OF HIGH SPEED
BEARINGS

5.1 Geometry and Applied loading

5.2 Stress Intensity Factor of Circular Surface Crack in the Inner
Raceway of the Engine Bearing

CHAPTER SIX CONCLUSIONS
REFERENCES

APPENDIX A SHAPE FUNCTIONS FOR ISOPARAMETRIC
ELEMENTS

APPENDIX B TRANSFORMATION FUNCTIONS
THREE-DIMENSIONAL ANALYSIS OF 
SURFACE CRACK-HERTZIAN STRESS FIELD INTERACTION

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SUMMARY

This thesis presents the results of a stress intensity factor analysis of semicircular surface cracks in the inner raceway of an engine bearing. The loading consists of a moving spherical Hertzian contact load and an axial stress due to rotation and shrink fit.

A three dimensional linear elastic Boundary Element Method code was developed to perform the stress analysis. The element library includes linear and quadratic isoparametric surface elements. Singular quarter point elements were employed to capture the square root displacement variation and the inverse square root stress singularity along the crack front. The program also possesses the capability to separate the whole domain into two subregions. This procedure enables one to solve non-symmetric fracture mechanics problems without having to separate the crack surfaces a priori.

A wide range of configuration parameters was investigated. The ratio of crack depth to bearing thickness was varied from one-sixtieth to one-fifth for several different locations of the Hertzian load. The stress intensity factors for
several crack inclinations were also investigated.

The results demonstrate the efficiency and accuracy of the Boundary Element Method. Moreover, the results can provide the basis for crack growth calculations and fatigue life prediction.
CHAPTER ONE
INTRODUCTION

Surface cracks commonly occur in machine and structural components. An example of such a component is a rotating engine bearing subjected to rolling contact. Under high speed rotation and cyclic contact loading, the surface crack initiating at the raceway of the bearing might propagate and lead to catastrophic failure. Raceway fracture is a totally unacceptable failure mechanism because it may cause serious damage to engine operations and consequently produce catastrophic engine failure. An accurate crack stress analysis of the surface-cracked component is essential in order to make a reliable prediction of fatigue life. However, due to the complexities of the nature of the surface crack problem, mathematical closed form solutions are not possible, and a numerical analysis or an experimental approach must be used to determine the stress intensity factors for surface cracks under different types of loading. The Boundary Element Method is an efficient and accurate tool for fracture mechanics analyses if singular elements and multi-domain crack modeling are employed. This method is used in this research.

Several factors will affect the growth of surface cracks in a rotational bearing under rolling contact loads. These include the geometry and inclination of the crack, the tensile hoop stress due to rotation and shrink fit, the moving Hertzian load, the pressure of the lubricant seeping into the crack, the shear stress on the raceway surface due to the sliding contact, and friction along the crack surfaces. A significant amount of research has been conducted aimed at gaining a better
understanding of the effects of each of these factors. While the surface crack is a three-dimensional problem, most of the analyses which appear in the literature are two-dimensional. These include the work of Way [1], which considers the effect of the lubricant, Fleming and Suh [2,3], which considers the effects of surface friction, Rosenfield [4], which considers the effects of crack surface friction, and Clark [5], which considers the effects of tensile hoop stresses. A recent paper by Mendelson and Ghosn [6] presents the results of fatigue life predictions of a propagating surface crack subjected to tensile hoop stresses and cyclic Hertzian contact loadings. Using a modified Forman-type crack propagation law they predicted the fatigue life of a typical bearing and compared their results with experimentally observed fatigue lives. Their predictions were conservative by a factor of 12. However, they demonstrated that the crack driving force in such problems is the alternating mixed-mode loading that occurs with each passage of the roller. Based on these results, the present research was aimed at quantifying the three-dimensional effects of the problem. Three dimensional analyses of surface cracks as applied to contact fatigue were recently performed by Murakami [7]. However, in his analysis the tensile hoop stresses were ignored. The model proposed in the present research neglects some of the factors mentioned previously. It is assumed that lubrication renders surface sliding friction negligible. The pressure on the crack surfaces which may arise from the lubricant seeping into the crack is ignored since the Hertzian loading moves past the crack very fast, thus the viscosity, compressibility, and inertia of the oil will prevent pressurization of the crack surfaces [8]. Moreover, since the radius of the Hertzian contact area is smaller than the surface length of the crack, the crack mouth will not be completely covered and the oil is allowed to squeeze out of the crack. The friction between the crack surfaces is neglected, since it tends to increase the resistance to crack growth.
This will lead to a conservative prediction of fatigue life.

Thus, the only factors assumed to be important in the present model are the mechanical loads arising from the Hertzian contact, rotation, and shrink-fit. The remaining five chapters of this paper are organized as follows:

In Chapter two the boundary integral equation is derived. The equation is reduced to a system of algebraic equations, and a procedure is described which treats the singularities which appear in the kernels. An algorithm for multi-domain analyses is also presented.

Chapter three is a brief review of linear elastic fracture mechanics. The formation of quarter point elements and traction singular elements as well as the displacement correlation method for calculating the stress intensity factors are also discussed. Several verification problems follow in Chapter four to elucidate the accuracy and efficiency of the Boundary Element Method for solving three dimensional linear elastic solid mechanics problems including crack stress analysis.

In Chapter five, the spherical Hertzian stress distribution and the hoop tensile stress due to the rotation and shrink fit of the inner raceway of the engine bearing are calculated. Results are presented for a wide range of configuration parameters. These include several different locations of the Hertzian load, different inclinations of the crack surface, several ratios of the crack depth to the raceway thickness and different intensities of the Hertzian load. A large number of the stress intensity factor versus these factors are presented.
The final chapter presents a discussion of the results and recommendations for future research.
CHAPTER TWO  
BOUNDARY ELEMENT METHOD

This chapter reviews the development of the Boundary Element Method and presents a detailed derivation of the boundary integral equation. The procedures of numerical implementation and a multi-domain technique of the Boundary Element Method are also illustrated.

2.1 Introduction

The boundary integral equation was first derived explicitly by Rizzo [9], who reduced two dimensional isotropic elastostatics problems to an integral equation by using the Betti-Somigliana formula. The equation was then discretized into line segments along the boundary over which constant displacement and traction distributions are assumed to solve the elastic problem numerically. Although much of the mathematical theory of Rizzo's formula can be traced to Kupradze [10], and several papers about integral equation methods such as Jawson [11] and Symm [12], his work presented a clear form of the integral equation relating the boundary traction and displacement which is now commonly associated with the term "Boundary Integral Equation". Following Rizzo's work, Cruse [13] extended the Boundary Element Method to three dimensional isotropic elastic problems by using triangular elements with linear variations. The accuracy of the Boundary Element Method was later improved by Lachat and Watson [14], and Rizzo and Shippy [15] using higher order isoparametric elements. The application of the Boundary Element Method to fracture mechanics was carried out by Cruse [13], [16] by modeling the crack as an open notch. The results obtained using this
approach tend to be inaccurate because as the cracks surfaces are moved close to each other the system of equations becomes singular. This problem was remedied by Blandford et al. [17], who used two subregions to model the crack. The quarter-point technique developed for the Finite Element Method by Barsoum [18] was adopted and modified by Cruse and Wilson [19] to capture the square root singularity predicted by linear elastic fracture mechanics.

The Boundary Element Method developed in this paper utilizes four kinds of isoparametric surface elements: a three-node linear triangular element, a four-node linear quadrilateral element, a six-node quadratic triangular element, and eight-node quadratic quadrilateral element. A library of Gaussian integration quadrature is installed in a subroutine which can be used to accomplish the numerical integration. The quarter point element and the traction singular element are used in the program to represent the crack tip singularity. The multi-domain technique is also applied to model the topology.

2.2 Derivation of Boundary Integral Equation

The mathematical fundations of the boundary integral equation are based on the Kelvin solution and Betti's reciprocal theorem [20].

Let P and Q be two arbitrary points in an infinite elastic body as shown in Fig 2.1. A unit concentrated load acting at point P in i direction is defined as
Figure 2.1 Domain and Boundary Geometry
\( f_i (P, Q) = \delta(P, Q) e_i \)

where \( e_i \) is the unit vector in \( i \) direction and \( \delta(P, Q) \) is the Dirac delta function which is defined as a function that is equal to zero for \( Q \) does not coincide with \( P \) and becomes infinite when \( P=Q \) in such manner that

\[
\int_\Omega \delta(P, Q) \, d\Omega(Q) = 1
\]

for any point \( P \) which lies in domain \( \Omega \). The displacement in direction \( j \) at point \( Q \) due to a unit concentrated load applied in direction \( i \) at point \( P \) in an infinite linear elastic body is given by Kelvin's solution [21]

\[
u_j = U_{ij}(P, Q) e_i
\] (2.1)

where

\[
U_{ij}(P, Q) = \frac{1}{16\pi\mu(1-\nu)r} \left[ (3-4\nu)\delta_{ij} + r_{ii}r_{jj} \right]
\] (2.2)

and \( e_i \) is the component of the unit base vector in direction \( i \), \( \mu \) is shear modulus, \( \nu \)
is Poisson's ratio, and $r$ is the distance between point P and point Q. Note that

$$
r = \sqrt{\left(x_i^Q - x_i^P\right)^2 + \left(x_i^Q - x_i^P\right)^2}^{1/2}
$$

(2.3)

and

$$
r_{ij} = \frac{\partial r}{\partial x_i} = \frac{(x_i^Q - x_i^P)}{r}
$$

$$
r_{ij} = \frac{\partial^2 r}{\partial x_j \partial x_i} = \frac{1}{r} \left( \delta_{ij} \cdot r_{ii} r_{jj} \right)
$$

(2.4)

where $\delta_{ij}$ is the Kronecker delta which is equal to 1 when $i = j$ and 0 when $i \neq j$ and comma denotes partial differentiation with respect to direction $x_i$. Cartesian tensor notation is used hereinafter with all the subscripts ranging from 1 to 3 and the convention that repeated indices are summed is employed. The stress-strain relation for an isotropic linear elastic material is [22]

$$
\sigma_{jk} = 2\mu \varepsilon_{jk} + \lambda \delta_{jk} \varepsilon_{nn}
$$

(2.5)
and the strain-displacement relation is

\[ \varepsilon_{jk} = \frac{1}{2} (u_{j,k} + u_{k,j}) \]  \hspace{1cm} (2.6)

The stress-displacement relationship for an isotropic elastic material are obtained by substituting Eq.2.6 into Eq.2.5, i.e.

\[ \sigma_{jk} = \lambda \delta_{jk} u_{m,m} + \mu (u_{j,k} + u_{k,j}) \]  \hspace{1cm} (2.7)

where \( \lambda = \frac{vE}{(1+v)(1-2v)} \), and \( E \) is Young's Modulus

Substituting Eq.2.1 into Eq.2.7, the stress field can be obtained as

\[ D_{ijk}(P,Q) = \lambda \delta_{jk} U_{im,m}(P,Q) + \mu (U_{ij,k}(P,Q) + U_{ik,j}(P,Q)) = \frac{-1}{8\pi(1-v)r^2} \left\{ (1-2v)(r_{ij}\delta_{ki} + r_{ik}\delta_{ij} - r_{ij}\delta_{jk}) + 3r_{ij}r_{ik}\delta_{jk} \right\} \]  \hspace{1cm} (2.8)
where \( D_{ijk}(P,Q) \) is interpreted as the stress component \( \sigma_{jk} \) at point \( Q \) due to a unit load in the \( i \) direction at point \( P \). If point \( Q \) is put on the boundary \( \Gamma \) of a finite body with domain \( \Omega \) cut out from the infinite body, as shown in Fig. 2.1, the tractions at point \( Q \) on the surface can be determined as

\[ t_j = \sigma_{jk} n_k \]  \hspace{1cm} (2.9)

where \( n_k \) is the \( k \)th component of the outward normal to the surface at point \( Q \).

Substitution of Eq. 2.8 into Eq. 2.9 leads to

\[ T_{ij}(P,Q) = D_{ijk}(P,Q)n_k(Q) \]

\[ = \frac{1}{8\pi (1-v) \rho^2} \left\{ (1-2v)(n_j r_i - n_i r_j) - n_k r_{ik} \left[ (1-2v)\delta_{ij} + 3r_i r_j \right] \right\} \]  \hspace{1cm} (2.10)

where \( T_{ij}(P,Q) \) is the traction at point \( Q \) in direction \( j \) on the surface with outward normal \( n_k \) due to a unit load in direction \( i \) at point \( P \). The free body cut out from the infinite body forms an equilibrium state subjected to the concentrated unit force \( f_j(P,0) \) and the boundary tractions \( T_{ij}(P,0) \delta_i \) with the corresponding boundary displacement \( U_{ij}(P,0) \delta_i \). Betti's reciprocal theorem can now be applied to derive the boundary integral equation. Suppose that there are two generalized force
systems. The first system includes body forces $b_j$, surface tractions $t_j$ and displacements $u_j$, and the second one consists of body forces $b_j^*$, boundary tractions $t_j^*$ and displacements $u_j^*$. If these two systems act simultaneously on a linear elastic body with domain $\Omega$ enveloped by the boundary surface $\Gamma$, Betti's reciprocal theorem states that [23]

$$\int_{\Gamma} t_j^* u_j^* d\Gamma + \int_{\Omega} b_j^* u_j^* d\Omega = \int_{\Gamma} t_j^* u_j d\Gamma + \int_{\Omega} b_j^* u_j d\Omega$$  \hspace{1cm} (2.11)

That is, the work done by the forces of the first system with the displacements of the second system is equal to the work performed by the forces of the second system on the displacements produced by the first system. Now let the first system be the one we are seeking a solution to with the assumption that the body forces are neglected and let the second system correspond to the fundamental solutions for the traction and displacement due to a unit concentrated load $f_j$ in an infinite body, as shown in Fig.2.2. That is,

$$b_j = 0$$

$$t_j = t_j(Q)$$

$$u_j = u_j(Q)$$

$$b_j^* = f_j = \delta(P,Q) e_j$$

$$t_j^* = T_{ij}(P,Q) e_i$$

$$u_j^* = U_{ij}(P,Q) e_i$$  \hspace{1cm} (2.12)
Figure 2.2 Generalized Force Systems: (a) First System with Traction $t_j$ and Displacement $u_j$ (b) Second System of Kelvin's Solution
Substitution of Eq.2.12 into Eq.2.11, leads to the following equation

$$\int_{\Gamma} U_{ij}(P,Q)t_j(Q)e_i d\Gamma(Q) = \int_{\Gamma} T_{ij}(P,Q)u_j(Q)e_i d\Gamma(Q) + \int_{\Omega} \delta(P,Q)e_j u_j(Q)d\Omega(Q) \quad (2.13)$$

Noting that

$$\int_{\Omega} \delta(P,Q)e_j u_j(Q)d\Omega(Q) = u_j(P) e_j = u_j(P)\delta_{ij} e_i \quad (2.14)$$

and using Eq.2.14, Eq.2.13 can be rewritten as

$$\int_{\Gamma} U_{ij}(P,Q)t_j(Q)e_i d\Gamma(Q) = \int_{\Gamma} T_{ij}(P,Q)u_j(Q)d\Gamma(Q)e_i + \delta_{ij} u_j(P)e_i \quad (2.15)$$

or

14
So far the point P is inside the boundary \( \Gamma \). In order to have the equation relate only the points on the boundary surface \( \Gamma \), we need to move the point P toward the surface. However, due to the singular nature of the kernels \( U_{ij}(P,Q) \) and \( T_{ij}(P,Q) \) as \( r \) tends to zero, a limiting process must be employed in order to obtain the boundary integral equation. Let us first choose a new boundary

\[
\Gamma_n = \Gamma_- + \Gamma_e
\]

where \( \Gamma_e \) is a surface of semi-spheric shape and \( \Gamma_- \) is the rest of the surface as shown in Fig.2.3. \( \Gamma_e \) should envelop the point P such that P is still in domain \( \Omega \) and thus Eq.2.16 is still available. With the new boundary, Eq.2.16 now becomes

\[
\delta_{ij}u_j(P) = \int_{\Gamma} U_{ij}(P,Q)t_j(Q)d\Gamma(Q) + \int_{\Gamma_e} U_{ij}(P,Q)t_j(Q)d\Gamma(Q) - \int_{\Gamma} T_{ij}(P,Q)u_j(Q)d\Gamma(Q) - \int_{\Gamma_e} T_{ij}(P,Q)u_j(Q)d\Gamma(Q)
\]  

(2.17)
As $\varepsilon$ tends to zero, the boundary $\Gamma_\varepsilon$ becomes the original boundary $\Gamma$. We also note that when point $Q$ is within the region $\Gamma_\varepsilon$ we have

$$d\Gamma(Q) = \varepsilon^2 \sin \theta \, d\theta \, d\phi$$

$$r = \varepsilon$$

$$r_i n_i = n_i$$

$$\vec{n} = \begin{cases} 
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta 
\end{cases}$$

$$n_k r^*_k = 1$$

$$r_i n_j - r_j n_i = 0$$

$$t_j(Q) = t_j(P) \text{ and } u_j(Q) = u_j(P) \quad (2.18)$$

where $t_j(P)$ and $u_j(P)$ are the tractions and displacements at point $P$ which are constants over the surface of the sphere as $\varepsilon$ tends to be zero. Substituting these relations into Eq.2.17, the second term on the right hand side of Eq.2.17 becomes

$$\int_{\Gamma_\varepsilon} U_{ij}(P,Q)t_j(Q)d\Gamma(Q) = \int_0^\theta \int_0^\phi \frac{1}{16\pi\mu(1-\nu)\varepsilon} \left[(3-4\nu)\delta_{ij} + n_i n_j\right] \varepsilon^2 \sin \theta d\theta d\phi t_j(P)$$

$$= \frac{\varepsilon t_j(P)}{16\pi\mu(1-\nu)} \int_0^\theta \int_0^\phi \left[(3-4\nu)\delta_{ij} + n_i n_j\right] \sin \theta d\theta d\phi \quad (2.19)$$
Figure 2.3 Domain and Boundary Used for Limiting Process
As $\varepsilon \to 0$, the limit of the second integral on the right hand side of Eq. 2.17 becomes

$$\lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon}} U_{ij}(P,Q)t_j(Q)d\Gamma(Q) = 0$$  \hspace{1cm} (2.20)$$

The integral of Eq.2.19 is no longer singular and thus Eq.2.20 tends to be zero as $\varepsilon$ is approaching zero. As for the fourth term on the right hand side of Eq. 2.17

$$\int_{\Gamma_{\varepsilon}} T_{ij}(P,Q)\frac{\delta}{\delta P_{ij}}d\Gamma(Q) = \int_0^\infty \frac{1}{8\pi(1-\nu)} \left[(1-2\nu)\delta_{ij} + 3n_in_j\right]e^2sin\theta d\theta d\phi u_j(P)$$

Substituting Eq.2.18 into Eq.2.21 leads to a matrix expression for the kernel inside the integration and taking the limit as $\varepsilon$ approaching zero
The integral ranges will depend on the local geometry of the point $P$. If point $P$ lies on a smooth surface,

$$\bar{\theta} = \frac{\pi}{2} \quad \text{and} \quad \bar{\phi} = 2\pi$$

and each term inside the matrix in Eq.2.22 becomes

$$\frac{-u_j(P)}{8\pi(1-\nu)} \int_{\Omega} \int_{\Omega'} \begin{bmatrix} u_1(P) \\ u_2(P) \\ u_3(P) \end{bmatrix} \sin \theta d\theta d\phi$$

$$= \int_{\Gamma_\epsilon} T_{ij}(P,Q)u_j(Q)d\Gamma(Q) = \begin{bmatrix} \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} u_1(P) \\ u_2(P) \\ u_3(P) \end{bmatrix}$$

(2.23)
or in Cartesian tensor form

\[
\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} T_{ij}(P,Q) u_j(Q) d\Gamma(Q) = \frac{1}{2} \delta_{ij} \ u_j(P)
\]  

(2.24)

Substituting Eq.2.20 and Eq.2.24 into Eq.2.17 leads to the boundary integral equation for P lying on a smooth surface

\[
\frac{1}{2} \delta_{ij} \ u_j(P) = \int_{\Gamma} U_{ij}(P,Q) t_j(Q) d\Gamma(Q) - \int_{\Gamma} T_{ij}(P,Q) u_j(Q) d\Gamma(Q)
\]  

(2.25)

If P is not on a smooth surface, the boundary integral equation is

\[
C_{ij}(P) u_j(P) = \int_{\Gamma} U_{ij}(P,Q) t_j(Q) d\Gamma(Q) - \int_{\Gamma} T_{ij}(P,Q) u_j(Q) d\Gamma(Q)
\]  

(2.26)

where
which is only a function of the local geometry in the vicinity of point P. The integration in Eq.2.27 can be carried out by using Eq.2.22 with the appropriate integral range according to the local configuration of point P. The $C_{ij}(P,Q)$ can also be calculated from the concept of rigid body motion[24]. When the body undergoes a rigid body translation, the surface is free of traction and the displacement is an arbitrary constant. By setting

\[
t_j(Q) = 0
\]
\[
u_j(Q) = \text{constant}
\]

Eq.2.26 becomes

\[
C_{ij}(P) = \int T_{ij}(P,Q)\,d\Gamma(Q)
\]

Comparing Eq.2.27 and Eq.2.28, it is apparent that the former depends on the local
geometry of each point and is different point to point. Thus it is very tedious to calculate. On the other hand, the latter is an integration on the whole boundary surface with a different kernel function and is easy to calculate for different points on the boundary. In fact, Eq.2.28 is merely a by-product of the second term on the right hand side of Eq.2.26 when it is carried out by numerical integration method. Therefore, with little effort, the term \( C_{ij}(P) \) can be easily obtained. Eq.2.28 is thus adopted in this paper to calculate the \( C_{ij}(P) \) term.

Eq.2.26 is also known as Somigliana's identity for three dimensional linear elastostatics with zero body force. In a well-posed boundary value problem, either traction or displacement in a direction on a boundary will be prescribed. Therefore, any corresponding unknown value on the boundary can be solved by the boundary integral equation of Eq.2.26.

After the unknown boundary tractions and/or displacements have been solved, the displacement of any point inside the body can be solved by Eq.2.16 from the boundary data. In order to obtain the stresses for the interior points, Eq.2.16 is differentiated and substituted into Eq.2.7. This results in

\[
\sigma_{jk}(P) = \int_{\Gamma} D_{ijk}(P,Q) t_i(Q)d\Gamma(Q) - \int_{\Gamma} S_{ijk}(P,Q) u_i(Q)d\Gamma(Q)
\]  

(2.29)
where

\[
D_{ijk}(P,Q) = \lambda \delta_{jk} U_{im,n}(P,Q) + \mu [U_{ij,k}(P,Q) + U_{ik,j}(P,Q)]
\]

\[
= \frac{-1}{8\pi(1-v)r^2} \left[ (1-2v)(r_{j,k} \delta_{ji} + r_{k,i} \delta_{ij} - r_{i,j} \delta_{jk}) + 3 r_{i} r_{j} r_{k} \right] \quad (2.30)
\]

and

\[
S_{ijk}(P,Q) = \lambda \delta_{jk} T_{im,n}(P,Q) + \mu [T_{ij,k}(P,Q) + T_{ik,j}(P,Q)]
\]

\[
= \frac{E}{8\pi(1-v^2)r^3} \left[ 3n_{m,r,n} ((1-2v)r_{j,k} \delta_{ij} + \nu(r_{j,k} \delta_{ki} + r_{k,i} \delta_{ij}) - 5 r_{i} r_{j} r_{k} \right] + 3v(n_{i,r,k} + n_{k,r,i}) r_{i} + (1-2v)(3n_{i,r,j} r_{k} + n_{j,r,i} n_{k,r,j}) - (1-4v)n_{i,j,k} \right] \quad (2.31)
\]
2.3 Numerical Implementation of Boundary Integral Equation

Owing to the difficulties of solving the boundary integral equation in closed form for practical problems with complicated geometries and boundary conditions, it is necessary to solve the boundary integral equation numerically.

The procedures for obtaining a numerical solution of the three dimensional boundary integral equation starts out with a discretization of the boundary surface \( \Gamma \) into \( m \) piecewise isoparametric surface elements. Such elements have been well developed for the Finite Element Method [25]. Each element consists of \( n_m \) nodes, the number depending on what kind of interpolation is employed in each element. The elements implemented in the present work include three-node linear triangular elements, four-node linear quadrilateral elements, six-node quadratic triangular elements, and eight-node quadratic quadrilateral elements. The shape functions for the isoparametric elements are derived in Appendix A. The cartesian coordinate of each node is given by

\[
x_i = \sum_{\alpha=1}^{n_m} N_{\alpha}^>(\xi) x_i^\alpha
\]  

(2.32)

where \( x_i^\alpha \) is the \( i \)-Cartesian coordinate of node \( \alpha \), \( N_{\alpha}^>(\xi) \) is the shape function for node \( \alpha \) which is a polynomal function of intrinsic coordinates \( \xi = (\xi_1, \xi_2) \). Eq.2.32 represents a one-to-one mapping of any point on the element from the
three dimensional Cartesian coordinate space into the two dimensional \((\xi_1, \xi_2)\) coordinate system, as shown in Fig.2.4. The isoparametric elements also use identical shape function to interpolate any function on the element, such as the displacements and tractions in the Boundary Element Method. Symbolically,

\[ u_i = \sum_{\alpha=1}^{n} N_\alpha(\xi) u^\alpha_i \]
\[ t_i = \sum_{\alpha=1}^{n} N_\alpha(\xi) t^\alpha_i \]  \hspace{1cm} (2.33)

where \(u^\alpha_i, t^\alpha_i\) are the nodal values of displacement and tractions, respectively. By discretizing the surface \(\Gamma\) into \(m\) segments and utilizing the shape function, Eq.2.26 becomes

\[ C_{ij}^P u_j^P = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} U_{ij}(P, Q(\xi)) N_\alpha(\xi) J_i(\xi) d\Gamma(\xi) t^\alpha_j \]
\[ - \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} T_{ij}(P, Q(\xi)) N_\alpha(\xi) J_i(\xi) d\Gamma(\xi) u^\alpha_j \]  \hspace{1cm} \text{P not summed} \hspace{1cm} (2.34)

where \(C_{ij}^P\) and \(u_j^P\) stand for the coefficients \(C_{ij}\) and displacement in \(j\)-direction of nodal point \(P\), respectively. The integral over the whole surface is carried out by summing up the integral over each element surface \(\Gamma_i\). The \(t^\alpha_j\) and \(u^\alpha_j\) is the
Figure 2.4 Mapping of Isoparametric Elements
traction and displacement in i-direction of node α in element 1. The term \( J_i(\xi) \) is the determinant of the Jacobian matrix of element 1 deduced from the change of the integration variables from the Cartesian coordinate to the intrinsic \( \xi \)-coordinate system. It is found to be

\[
J_i(\xi) = \begin{vmatrix}
\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} \\
\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2}
\end{vmatrix} = \begin{vmatrix}
\sum_{\alpha=1}^{n_n} \frac{\partial N^\alpha(\xi_1, \xi_2)}{\partial \xi_1} x_1^\alpha & \sum_{\alpha=1}^{n_n} \frac{\partial N^\alpha(\xi_1, \xi_2)}{\partial \xi_2} x_2^\alpha \\
\sum_{\alpha=1}^{n_n} \frac{\partial N^\alpha(\xi_1, \xi_2)}{\partial \xi_1} x_2^\alpha & \sum_{\alpha=1}^{n_n} \frac{\partial N^\alpha(\xi_1, \xi_2)}{\partial \xi_2} x_1^\alpha
\end{vmatrix}
\]

(2.35)

Similarly, Eq. 2.28 is rewritten as

\[
C_{ij}^P = \sum_{I=1}^{n_I} T_{ij}(P, Q(\xi)) J_i(\xi) d\Gamma(\xi)
\]

(2.36)

A Gaussian quadrature scheme can now be applied to accomplish the integration.

For the case when the point \( P \) is not on the element which is being integrated over, the Gaussian quadrature is straightforward, that is,
\[
\int_{\Gamma_1} \Psi_i(P, \xi) d\Gamma(\xi) = \sum_{a=1}^{n_a} \sum_{b=1}^{n_b} w_a^a w_b^b \Psi_i(P, \xi_a^a, \xi_b^b) 
\]

(2.37)

where

\[
\Psi_i(P, \xi) = U_{ij}(P, Q(\xi)) N^{\alpha}(\xi) J_i(\xi)
\]

or

\[
\Psi_i(P, \xi) = T_{ij}(P, Q(\xi)) N^{\alpha}(\xi) J_i(\xi)
\]

In Eq.2.37 \(n_a\) and \(n_b\) are the order of Gaussian quadrature and \(w_a^a\) and \(w_b^b\) are the weights of the corresponding Gaussian integration points \(\xi_1\), \(\xi_2\), respectively. When point \(P\) is one of the nodes of the element under integration, the standard Gaussian quadrature will not give accurate results because of the \(1/r\) and \(1/r^2\) singularities of \(U_{ij}(P, Q)\) and \(T_{ij}(P, Q)\), respectively. Therefore, special treatment of this singular integral must be used in order to obtain an accurate solution. The method employed here follows the work of Rizzo and Shippy [15]. The element is divided into triangles according to the position of the point \(P\) as shown in Appendix B. Inside each triangle, the \(\xi\)-coordinate system is transformed into a local polar coordinate system \(r\) and \(\theta\), and the \(d\xi\) term in the integral becomes \(rd\xi d\theta\). The additional \(r\) due to this transformation eliminates the \(1/r\) singularity of the kernel \(U_{ij}(P, Q)\). Gaussian quadrature can then be applied to the polar coordinate system. Of course, one more transformation is needed to map the \(r, \theta\) coordinate system to another polar coordinate system \(r\) and \(\theta\) so that the range
varies from -1 to 1. The details of the procedures are illustrated in Appendix B.

For the integration of the kernel \( T_{ij}(P,Q) \), a 1/r singularity remains after the transformation from the \( \xi \)-coordinate to the polar coordinate. However, by substituting Eq.2.36 into Eq.2.34, Eq.2.34 can be rewritten as

\[
\sum_{i=1}^{n} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) J_i(\xi) d\Gamma(\xi) u_j^P
\]

\[
= \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} U_{ij}(P,Q(\xi)) N^{\alpha}(\xi) j^a_\alpha J_i(\xi) d\Gamma(\xi) - \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) N^{\alpha}(\xi) u^a_\alpha J_i(\xi) d\Gamma(\xi)
\]

(2.38)

Distinguish the elements containing point \( P \) from those elements without point \( P \) for the integration involving the kernel \( T_{ij}(P,Q) \), and Eq.2.38 becomes

\[
\sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) J_i(\xi) d\Gamma(\xi) u_j^P
\]

\[
= \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} U_{ij}(P,Q(\xi)) N^{\alpha}(\xi) j^a_\alpha J_i(\xi) d\Gamma(\xi) - \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) N^{\alpha}(\xi) u^a_\alpha J_i(\xi) d\Gamma(\xi)
\]

\[
- \sum_{i=1}^{m} \sum_{\alpha=1}^{n} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) N^{\alpha}(\xi) u^a_\alpha J_i(\xi) d\Gamma(\xi) + \sum_{i=1}^{m} \int_{\Gamma_i} T_{ij}(P,Q(\xi)) J_i(\xi) d\Gamma(\xi) u_j^P
\]

(2.39)
where \( m_x \) is the total number of the elements excluding point \( P \) and \( m_p \) is the total number of the elements including point \( P \). Obviously, \( m = m_x + m_p \). Combining the last two terms on the right hand side of Eq.2.39, results in

\[
\overline{C}_{ij}^{P} a_{i}^{P} =
\sum_{l=1}^{m} \sum_{\alpha=1}^{n_{m}} \int_{\Gamma_{l}} U_{ij}(P,Q(\xi)) N^{\alpha}(\xi) J_{l}(\xi) d\Gamma(\xi) c_{i}^{\alpha} - \sum_{i=1}^{m_x} \sum_{\alpha=1}^{n_{m}} \int_{\Gamma_{l}} T_{ij}(P,Q(\xi)) N^{\alpha}(\xi) J_{l}(\xi) d\Gamma(\xi) u_{i}^{\alpha} \]

\[
- \sum_{l=1}^{m} \sum_{\alpha=1}^{n_{m}} \int_{\Gamma_{l}} T_{ij}(P,Q(\xi)) (N^{\alpha}(\xi) - \delta_{\alpha P}) J_{l}(\xi) d\Gamma(\xi) u_{i}^{\alpha} 
\]

(2.40)

where

\[
\overline{C}_{ij}^{P} = - \sum_{l=1}^{m_x} \int_{\Gamma_{l}} T_{ij}(P,Q(\xi)) d\Gamma(\xi) 
\]

(2.41)

Eq.2.41 is the negative of the integral of the kernel \( T_{ij}(P,Q) \) over the elements which do not contain the point \( P \), hence, Eq.2.41 is no longer singular. As for Eq.2.40, the remaining 1/r singularity of the kernel \( T_{ij}(P,Q) \) is removed by the special shape function \( N^{\alpha}(\xi) - \delta_{\alpha P} \) in the last term on the right hand side of equation because when \( \alpha \) coincides with point \( P \), the constant term is eliminated.
and thus the smallest order of the shape function is of order $r$. This additional $r$ can cancel the $1/r$ singularity after the coordinate has been transformed into the local polar coordinate. Using this procedure, all the singularities are eliminated and Gaussian quadrature can be used.

Eq. 2.40 represents an equation at each discrete point $P$ on the boundary surface constraining the boundary displacements and boundary tractions in the $i$-direction. For a surface including $N$ nodes, Eq. 2.40 can be expressed as

\[
\overline{C}^p_{i,j}u^p_j + \left[ T^1_{ij}T^2_{ij} \cdots T^N_{ij} \right] \begin{bmatrix}
  u^1_j \\
  u^2_j \\
  \vdots \\
  u^N_j
\end{bmatrix} = \begin{bmatrix}
  U^1_{ij} \\
  U^2_{ij} \\
  \vdots \\
  U^N_{ij}
\end{bmatrix}
\]

(2.42)

where each term of $T^n_{ij}$ for node $n$ is the integration summed up from the contribution of the elements which share the same node $n$. The same applies for $U^n_{ij}$. Eq. 2.42 can be rewritten as

\[
\overline{C}^p_{i,j}u^p_j + \sum_{n=1}^{N} T^n_{ij} u^n_j = \sum_{n=1}^{N} U^n_{ij} t^n_j
\]

(2.43)
This equation can be expressed more simply by combining the first term into the summation on the left hand side, i.e.,

\[ \sum_{n=1}^{N} T^{*n}_{ij} u^n_j = \sum_{n=1}^{N} U^{*n}_{ij} t^n_j \] (2.44)

where

\[ T^{*n}_{ij} = T^n_{ij} \quad \text{for } n \neq P \]
\[ T^{*n}_{ij} = T^n_{ij} + C^n_{ij} \quad \text{for } n = P \] (2.45)

Another way of dealing with the singularity which is worthy to mention here is through the use of rigid body motion. For a rigid body motion in direction j, Eq.2.44 reduces to

\[ \sum_{n=1}^{N} T^{*n}_{ij} = 0 \] (2.46)

This indicates that the sum of \( T^{*n}_{ij} \) from each node in a row for certain direction j should be zero. Hence the value of the singular term \( T^{*P}_{ij} \) when \( n = P \) can be
easily calculated once all the other terms are known, that is,

\[ T_{ij}^{*P} = - \sum_{n=1}^{N} T_{ij}^{*n} \]  

(2.47)

For three dimensional problems the indices i and j range from 1 to 3. Therefore, for a surface including N nodes, the dimension of the total algebraic system of equations formed by Eq.2.44 for each node in each direction is 3Nx3N. The system can be represented in matrix form as

\[ T^*u = U^*t \]  

(2.48)

The matrixes \( T^* \) and \( U^* \) are rearranged by interchanging the suitable columns on each side of equation so that all the unknown variables are contained in a column vector \( x \) and all the prescribed values of the boundary are included in the column \( y \) on the other side of the equation. Symbolically,

\[ Ax = By = f \]  

(2.49)

The system of equations can now be easily solved by the Gaussian elimination method.
After all the boundary data are obtained, the displacements and stresses at interior points are solved by substituting the boundary tractions and displacements into Eq.2.16 and Eq.2.29, respectively, and using the shape functions to carry out the integration.

2.4 Subregion Technique

Since the boundary integral equation is a constraint relation between boundary tractions and boundary displacements with the kernel functions which include the term $1/r$ and $1/r^2$, any two distinct points on the boundary can not coincide. Therefore problems for which the boundary includes two contacting surfaces can not be solved by the Boundary Element Method using a single region. Partitioning the whole boundary into subregions is necessary to deal with such problems.

Consider for simplicity the case where the body is partitioned into two subregions. Similar procedures can be followed to separate the body into more than two regions. A body with domain $\Omega$ surrounded by the boundary $\Gamma$ is partitioned into two subregions; one consists of domain $\Omega_1$ and boundary $\Gamma_1$ and the other possesses the domain $\Omega_2$ and boundary $\Gamma_2$, as was shown in Fig.2.5. The two regions share the same interface $\Gamma_1$. Each subregion can be treated as an independent body. Thus the procedure described in Section 2.1 can be used to
Figure 2.5 Domain Divided into Two Subregions
form the system of algebraic equations for each subregion. Before the compatibility and equilibrium conditions are enforced along the interface between two subregions, the corresponding columns of the matrix $U^*$ of unknown tractions and the corresponding columns of the matrix $T^*$ of known displacements in Eq. 2.48 must be interchanged so that all the known and unknown variables will be at the same side of the equations. Let $A_\text{s}$ and $B_\text{s}$ be the kernel matrices related to the unknown column vectors $U_{k_\text{e}\text{s}}$ and $K_{e\text{s}}$ on the external surface of region $s$ after rearrangement. Furthermore let $t_{1\text{s}}$, $u_{1\text{s}}$ be the unknown tractions and unknown displacements on the interface of region $s$. The system of equations for subregion one can be expressed as

$$
A_1 \begin{bmatrix} U_{k_\text{e}\text{s}}^1 \\ t_{1\text{s}}^1 \end{bmatrix} = B_1 \begin{bmatrix} K_{e\text{s}}^1 \\ u_{1\text{s}}^1 \end{bmatrix}
$$

(2.50)

After pre-multiplying each side by the inverse of $B_1$, Eq. 2.50 becomes

$$
\begin{bmatrix}
D_{ee}^1 & D_{ei}^1 \\
D_{ie}^1 & D_{ii}^1
\end{bmatrix} \begin{bmatrix} U_{k_\text{e}\text{s}}^1 \\ t_{1\text{s}}^1 \end{bmatrix} = \begin{bmatrix} K_{e\text{s}}^1 \\ u_{1\text{s}}^1 \end{bmatrix}
$$

(2.51)
where $D_1 = B_1^{-1} A_1$. By the same procedure the system for the second subregion becomes

$$
\begin{array}{cccc}
D_{\infty}^2 & D_{ei}^2 & U_k^2 & K_e^2 \\
D_{ie}^2 & D_{ii}^2 & t_1^2 & u_i^2 \\
\end{array} = 0
$$

(2.52)

Enforcing the compatibility and equilibrium condition along the interface

$$
t_1^1 = -t_2^2 = t_i
$$

$$
u_1^1 = u_2^2 = u_i
$$

(2.53)

Equations 2.51 and 2.52 can be combined to be [26]

$$
\begin{array}{cccc}
D_{\infty}^1 & D_{ei}^1 & 0 & U_k^1 \\
D_{ie}^1 & D_{ii}^1 & -D_{ii}^2 & D_{ie}^2 \\
0 & -D_{ei}^2 & D_{\infty}^2 & U_k^2 \\
\end{array} = 0
$$

$$
\begin{array}{cccc}
K_e^1 & 0 \\
U_k^2 & K_e^2 \\
\end{array}
$$

(2.54)
This is a system of $3(N_e^1 + N_e^2 + N_i)$ equations for $3N_e^1$ unknowns on the external surface of subregion one, $3N_e^2$ unknowns on the external boundary of subsurface two, and $3N_i$ unknown tractions on the interface. The unknown displacements on the interface can be obtained through either Eq.2.51 or Eq.2.52 after Eq.2.54 has been solved. The equation which is not used can be used as a check.

This method is simple and direct. However it is not usually adopted because the inverse of the matrix of kernel $B_s$ must be calculated for both subregions. This involves a tremendous amount of computer processing time and requires a lot of memory space. Therefore, this method is not used in this paper.

An alternative procedure which eliminates the need for solving the inverse of the matrix is described herein. After the appropriate columns of $T^*$ and $U^*$ have been interchanged, the system of equation of the two subregions are assembled into two big matrices and two column vectors as

\[
\begin{bmatrix}
A^1 & 0 \\
0 & A^2 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^1_k \\
\mathbf{u}^2_k \\
\end{bmatrix}
= 
\begin{bmatrix}
B^1 & 0 \\
0 & B^2 \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}^1_i \\
\mathbf{u}^2_i \\
\end{bmatrix}
\]

(2.55)
To be more specific,

\[
\begin{array}{cccc}
A_{11}^1 & A_{11}^1 & 0 & U_{11}^1 \\
A_{11}^2 & A_{22}^1 & 0 & U_{12}^2 \\
0 & A_{22}^1 & A_{22}^2 & U_{21}^2 \\
- A_{11}^2 & A_{11}^2 & A_{11}^2 & U_{11}^2 \\
\end{array} = \begin{array}{cccc}
B_{11}^1 & B_{11}^1 & 0 & K_{11}^1 \\
B_{11}^2 & B_{22}^1 & 0 & K_{11}^2 \\
0 & B_{22}^2 & B_{22}^2 & K_{22}^2 \\
0 & 0 & 0 & K_{22}^2 \\
\end{array}
\]

(2.56)

Using the compatibility and equilibrium conditions (Eq.2.53), Eq.2.56 reduces to

\[
\begin{array}{cccc}
A_{11}^1 & A_{11}^1 & 0 & B_{11}^1 \\
A_{11}^2 & A_{22}^1 & 0 & B_{12}^2 \\
0 & A_{22}^1 & A_{22}^2 & B_{21}^2 \\
- A_{11}^2 & A_{11}^2 & A_{11}^2 & B_{11}^2 \\
\end{array} = \begin{array}{cccc}
B_{11}^1 & B_{11}^1 & 0 & K_{11}^1 \\
B_{11}^2 & B_{22}^1 & 0 & K_{11}^2 \\
0 & B_{22}^2 & B_{22}^2 & K_{22}^2 \\
0 & 0 & 0 & K_{22}^2 \\
\end{array}
\begin{array}{c}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
\end{array}
\]

(2.57)

Since now the known variables appear on the right hand side of the equation and the unknown variables appear on the other side, the equations can be solved without
any difficulty. Although this method avoids solving for the inverse matrix and thus saves a lot of computer processing time, it still has the drawback of using two big matrices which occupy tremendous storage space. Therefore another algorithm is introduced next.

Let the numbering of the nodes along the interface be always arranged at the position following the external nodes for both subregions, as shown in Eq.2.50. Rewrite Eq.2.50 for the first region as follows

$$\begin{bmatrix}
A_1^e & A_1^i \\
\end{bmatrix} \begin{bmatrix}
Uk_e^1 \\
u_i^1 \\
\end{bmatrix} = \begin{bmatrix}
B_e^1 & B_i^1 \\
\end{bmatrix} \begin{bmatrix}
K_e^1 \\
t^1_i \\
\end{bmatrix}$$

(2.58)

where the dimensions of each block are as following

- $A_1^e$ and $B_1^e$ are $3(N_e^1 + N_i^1) \times 3N_e^1$
- $A_1^i$ and $B_1^i$ are $3(N_e^1 + N_i^1) \times 3N_i^1$
- $K_e^1$ and $Uk_e^1$ are $3N_e^1 \times 1$
- $t^1_i$ and $u_i^1$ are $3N_i^1 \times 1$

$B_e^1$ and $K_e^1$ can be multiplied together to form a known column vector $f^1$.
Moving the second term of the right hand side of Eq.2.58 to the left hand side leads to

\[
\begin{bmatrix}
A^v_e & A^v_i & U^e_k \\
0 & u^l_i & t^l_i
\end{bmatrix}
\begin{align*}
\begin{bmatrix}
A^v_e \\
A^v_i \\
U^e_k \\
u^l_i \\
t^l_i
\end{bmatrix}
&= \begin{bmatrix}
f^l_1
\end{bmatrix} + \\
\begin{bmatrix}
B^v_i \\
t^l_i
\end{bmatrix}
\end{align*}
\] (2.59)

Note that the left hand side of Eq.2.60 now become a matrix with dimension of 3(N_e^1 + N_i) by 3(N_e^1 + 2N_i) multiplied by an unknown vector with dimension of 3(N_e^1 + N_i) by 1. Obviously, this system can not be solved since there are 3N_i more unknown than the total number of equations. Before we proceed to the second subregion, Eq.2.60 is reduced to

\[
\begin{bmatrix}
A^v_e \\
A^v_i \\
B^v_i \\
0
\end{bmatrix}
\begin{bmatrix}
U^e_k \\
u^l_i \\
t^l_i
\end{bmatrix}
= \begin{bmatrix}
f^l_1
\end{bmatrix}
\] (2.60)

\[
\begin{bmatrix}
A^v_e \\
A^v_i \\
B^v_i \\
0
\end{bmatrix}
\begin{bmatrix}
U^e_k \\
u^l_i \\
t^l_i
\end{bmatrix}
= \begin{bmatrix}
f^*1
\end{bmatrix}
\] (2.61)
Eq.2.61 can be represented more conveniently as

\[
\begin{bmatrix}
A_e^* & G^1 \\
0 & H^1
\end{bmatrix}
\begin{bmatrix}
U_{k_e}^1 \\
{u}_i^1 \\
{t}_i^1
\end{bmatrix}
= \begin{bmatrix}
g^1 \\
h^1
\end{bmatrix}
\]  

(2.62)

Eq.2.62 can be treated as two set of equations, viz.

\[
\begin{bmatrix}
A_e^* & U_{k_e}^1 \\
0 & G^1
\end{bmatrix}
\begin{bmatrix}
{u}_i^1 \\
{t}_i^1
\end{bmatrix}
= \begin{bmatrix}
g^1 \\
h^1
\end{bmatrix}
\]  

(2.63)

\[
\begin{bmatrix}
H^1
\end{bmatrix}
\begin{bmatrix}
{u}_i^1 \\
{t}_i^1
\end{bmatrix}
= \begin{bmatrix}
h^1
\end{bmatrix}
\]

(2.64)

The same procedure can be applied to the second subregion and a similar set of
equations is obtained

\begin{align*}
A_e^2 & \begin{bmatrix} u_k^2 \end{bmatrix} + G^2 \begin{bmatrix} u_i^2 \\ \xi_i^2 \end{bmatrix} = g^2 \\
H^2 & \begin{bmatrix} u_i^2 \\ \xi_i^2 \end{bmatrix} = h^2
\end{align*} \tag{2.65}

Note that the unknowns in Eq.2.64 and Eq.2.66 now only involve the unknown displacements and unknown tractions on the interface between the two subregions. By applying the compatibility and equilibrium condition of Eq.2.53,

\begin{align*}
t_i^1 &= -t_i^2 = t_i \\
u_i^1 &= u_i^2 = u_i
\end{align*}

these two sets of equations can be assembled together as Eq.2.67 to solve for the unknown tractions and unknown displacements along the interface.
After the displacement $u_i$ and the traction $t_i$ of the interface have been solved using Eq.2.64, the unknowns on the external surfaces of each subregion can be solved through Eq.2.63 and Eq.2.65.

This method not only saves computer execution time by eliminating the need for solving for the inverse of the matrices but also saves a lot of memory storage space because the procedure can be performed on the system of equations of each subregion separately with little effort by using an index control in the computer program. It is therefore used in this paper.
2.5 Discontinuity of Traction

The data of prescribed tractions and prescribed displacements must be read into the computer according to the element rather than the node due to the possibility of a node lying on a position without an unique tangent plane. That is, when a node is at the intersection of two or more planes, the traction acting on this node depends on which plane is considered, since each surface is associated with a different normal. For example, in Fig.2.6, three elements share the same node P. For the sake of easy interpretation, we let the unit normal of these three elements coincide with the unit vectors of the Cartesian coordinate system.

\[ \hat{n}_1 = \hat{i} \]
\[ \hat{n}_2 = \hat{j} \]
\[ \hat{n}_3 = \hat{k} \]

The component of the traction of node P in direction i on element 1 is the normal stress \( \sigma_{11} \) but the traction of point P in the same direction on element 2 is the shear stress \( \sigma_{12} \) and the traction in the i-direction on element 3 is the shear stress \( \sigma_{13} \) at point P. Therefore, the prescribed traction must be input according to the element so that each different traction on the same node can be multiplied by an appropriate kernel contributed from each element surface. Also note that when the displacement in one direction is prescribed at a certain node, for a unique solution only one of
the corresponding tractions on this node can be unknown and all the others must be prescribed. For the same example in Fig. 2.6, two of three tractions on three different elements must be known if the displacement of that node is prescribed. Eq. 2.37 in fact is a concise expression for convenient interpretation. To be more specific, it should read

\[ \ldots T^p \ldots = \ldots U_1^p \ldots U_2^p \ldots U_3^p \ldots \ldots \]

(2.68)

Thus, the equation for a certain row is

\[ \ldots + T^p u^p + \ldots = \ldots + U_1^p t_1^p + U_2^p t_2^p + U_3^p t_3^p + \ldots \]

(2.69)

When the tractions \( t_1^p, t_2^p \) and \( t_3^p \) are known at point \( P \), they must be multiplied by these different kernel function \( U_1^p, U_2^p, \) and \( U_3^p \) which are calculated from each different element with different normal vector. As for the case of the displacement being prescribed, only one traction can be unknown, say \( t_2^p \), and the traction on another surface \( t_1^p \) and \( t_3^p \) must be forced to be known. Thus, the
Figure 2.6 Illustration of Different Tractions at Node P
equation becomes

\[ \ldots + T^P u^P - U_1^P t_1^P - U_3^P t_3^P + \ldots = \ldots + U_2^P t_2^P + \ldots \]  \hspace{1cm} (2.70)

so that only one unknown is left on the right hand side of the equation which is needed to be solved and all the known values are summed up on the left hand side of the equation.

One incident merit of inputting the traction element by element is that discontinuous tractions can be modeled exactly.
3.1 Introduction

The presence of cracks in a material causes a stress concentration in the vicinity of the crack tip. Consequently, plastic yielding or microcracking will occur in the region surrounding the crack tip. Linear elastic fracture mechanics assumes that the nonlinear deformations are restricted to a region whose dimensions are small compared to other characteristic dimensions, so that the elastic solution provides an accurate description of the stress and displacement fields in the vicinity of the crack tip. This is often referred to as small scale yielding (S.S.Y.).

It can be shown that the loading on a crack is in general a superposition of three independent modes as shown in Figure 3.1. The first (Fig.3.1.a) is called the crack opening mode, or mode-I, which is a result of a relative normal separation of the crack surfaces (symmetric with respect to x-z and x-y planes). Fig.3.1.b is called the crack sliding mode, or mode-II, which is associated with a relative sliding displacement in the x-direction (symmetric with respect to x-y plane and skew-symmetric with respect to the x-z plane). The tearing mode (mode-III) corresponds to relative motion in the z-direction of the two crack surfaces (skew-symmetric with respect to the x-y and x-z planes). Using Westergard’s technique, Irwin and Williams showed that the stress and displacement fields in the vicinity of a crack tip can be expressed as an infinite series whose leading term is square root singular. The coefficient of this singular term is defined as the stress intensity factor. With respect to a r-θ polar coordinate system as shown in Figure 3.2, the stresses and
Figure 3.1 The Three Basic Modes of Crack Surface Displacement
displacements near a crack tip are given by [27]

Mode I

\[
\begin{align*}
\sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] \\
\sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] \\
\sigma_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \\
\sigma_{zz} &= \nu (\sigma_{xx} + \sigma_{yy}), \quad \sigma_{xz} = 0
\end{align*}
\]

\[
\begin{align*}
u = \frac{K_I}{\mu} \left[ 1 - 2\nu + \sin^2\frac{\theta}{2} \right] \\
\begin{align*}
u = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2} & \left[ 2 - 2\nu - \cos\frac{3\theta}{2} \right] \\
u = 0 \quad \text{(3.1)}
\end{align*}
\]

Mode II

\[
\begin{align*}
\sigma_{xx} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left[ 2 + \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \right] \\
\sigma_{yy} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \\
\sigma_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[ 1 - \sin\frac{\theta}{2} \sin\frac{3\theta}{2} \right] \\
\sigma_{zz} &= \nu (\sigma_{xx} + \sigma_{yy}), \quad \sigma_{xz} = 0
\end{align*}
\]
\[ u_1 = \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2} \left[ 2 - 2v + \cos^2\frac{\theta}{2} \right] \]
\[ u_2 = \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \cos\frac{\theta}{2} \left[ -1 + 2v + \sin^2\frac{\theta}{2} \right] \]
\[ u_3 = 0 \quad (3.2) \]

Mode III

\[ \sigma_{xz} = -\frac{K_{III}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \]
\[ \sigma_{yz} = \frac{K_{III}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \]
\[ \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0 \]

\[ u_1 = u_2 = 0 \]
\[ u_3 = \frac{K_{III}}{\mu} \sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2} \quad (3.3) \]

where \( K_I, K_{II}, \) and \( K_{III} \) are the stress intensity factors corresponding to modes I, II, and III, respectively. Note that Eq.3.1-3.3 are valid only when \( r << L \), where \( L \) is another characteristic length of the geometry (may be the crack length). Also note that these equations do not contain any information about the externally applied loading, crack geometry, geometric configuration, etc, and that the stress intensity factors are not functions of the local coordinate \( r \) and \( \theta \). These factors are embedded
Figure 3.2 Coordinate and Stress Components for Crack Tip Stress Field
in $K_1$, $K_II$, and $K_III$. That is, the stress intensity factors $K_1$, $K_II$, and $K_III$, which must be determined by the boundary conditions and the loadings control the magnitudes of the stress and displacement fields around the crack tip. Thus, a given combination of values of $K_1$, $K_II$, and $K_III$ represents uniquely a crack tip stress field environment for small scale yielding. The determination of the stress intensity factors is thus the most important task in linear elastic fracture mechanics. Although Eq.3.1-3.3 are valid for the plane strain problem, they may be modified to represent the plane stress problem by letting $\sigma_z = 0$ and substituting $\nu$ with $\nu/(1+\nu)$.

Eq.3.1 and Eq.3.2 were derived for plane problems. However, it was shown by Sih and Liebowitz [28] that for an elliptical crack in a three dimensional linear elastic body, the local stress and displacement fields along a crack front are a superposition of plane strain and antiplane shear. Hence, Eq.3.1-3.3 can still be used for three dimensional crack problems as long as the coordinate system is allowed to move along the crack front with its $z$-axis tangent to the crack front and the $y$-coordinate perpendicular to the crack surface, as shown in Fig.3.3. Referring to the moving coordinate, the stress intensity factors are now also a function of the position of the origin on the crack front and the formulas are not available for a crack near a free surface because the stress singularity is not of inverse square root there.
Figure 3.3 Coordinate System on the Crack Front for Three Dimensional Crack
3.2 Calculation of Stress Intensity Factors Using the Boundary Element Method

The variation of any function in an isoparametric element is polynomial. The degree of the polynomial depends on the number and arrangement of the nodes. Therefore, if a quadratic surface element is used in the vicinity of a crack, the distributions of the displacement and traction in the element will have at most quadratic variations. Since the variation of the displacement is square root of \( r \) and the variation of the traction is inverse of square root of \( r \) around the crack, a fine mesh is needed to model the crack so that the quadratic variation can imitate the correct distribution inside each small segment. However, even this refinement cannot achieve a high degree of accuracy.

Fortunately, this problem was solved by Barsoum [18] who modified the quadratic isoparametric element by relocating appropriate midside nodes to the quarter-point to capture the inverse square root singularity. Even though he had done this for the Finite Element Method, a similar approach can be used in the Boundary Element Method. For example, consider an eight node quadrilateral element with two sides having equal length \( L \) perpendicular to the crack front (side 1-5-2), as shown in Fig.3.4. Relocate the two midside node (6 and 8) to the quarter-point near the crack front. Denoting the distance originated from the crack front to any point on the element by \( r \), then

\[
r = \sum_{i=1}^{8} N_i(\xi_1, \xi_2) r_i
\]  

(3.4)
Figure 3.4 Illustration of 8-node Quarter Point Element
Choosing \( r_1 = r_2 = r_5 = 0, \ r_3 = r_4 = r_7 = L, \) and \( r_6 = r_8 = L/4, \) Eq.3.4 becomes

\[
r = \frac{1}{4} (\xi_1 + \xi_2 - 1)(1 + \xi_1)(1 + \xi_2) L + \frac{1}{4} (\xi_1 - \xi_2 + 1)(\xi_1 - 1)(\xi_2 + 1) L
\]

\[
\frac{1}{2} (1 + \xi_1)(1 - \xi_2^2)\sqrt{\frac{L}{4}} \ + \frac{1}{2} (1 + \xi_2)(1 - \xi_1^2)\sqrt{\frac{L}{4}} \ + \frac{1}{2} (1 - \xi_1)(1 - \xi_2^2)\sqrt{\frac{L}{4}} 
\]

Simplifying.

\[
\xi_2 = 2\sqrt{\frac{r}{L}} - 1 
\quad (3.5)
\]

Substitution of Eq.3.5 into Eq.2.33, leads to the following variation of the displacement and traction in direction \( i \) versus the distance \( r \).
\( u_i(r) = \sum_{k=1}^{N} N^k u_i^k \)

\[
= \left[ (-u_i^1 - u_i^2 - u_i^3 - u_i^4 - u_i^5 + 2u_i^6 + u_i^7 + 2u_i^8) + \frac{\xi_1}{2} (3u_i^1 - 3u_i^2 - u_i^3 + u_i^4 + 4u_i^6 - 4u_i^8) \right. \\
\left. + \frac{\xi_1^2}{2} (-u_i^1 - u_i^2 + u_i^3 + u_i^4 + 2u_i^5 - 2u_i^7) \right] \frac{r}{L} \\
+ \left[ (u_i^1 + u_i^2 + u_i^3 + u_i^4 - 2u_i^6 - 2u_i^8) + \left[ u_i^5 + \frac{\xi_1}{2} (-u_i^1 + u_i^2) + \frac{\xi_1^2}{2} (u_i^1 + u_i^2 - 2u_i^5) \right] \right]
\]

(3.6)

which can be rewritten as,

\[ u_i(r) = A_i^1 + A_i^2 \frac{r}{L} + A_i^3 \frac{r}{L} \] (3.7)

Similary,

\[ \tau_i(r) = B_i^1 + B_i^2 \frac{r}{L} + B_i^3 \frac{r}{L} \] (3.8)
The displacement variation given in Eq.3.7 contains the square root of r term which is asymptotically correct. However, the traction appearing in Eq.3.8 does not include the correct inverse square root of r term needed to model the stress singularity around the crack tip. The correct singularity is obtained by multiplying the right hand side of Eq.3.8 by the factor [19]

$$\phi(r) = \sqrt{\frac{L}{r}} = \frac{2}{\xi_2 + 1}$$  (3.9)

such that the variation of traction becomes

$$t_i(r) = (B^1_i + B^2_i \sqrt{\frac{r}{L}} + B^3_i \frac{r}{L}) \sqrt{\frac{L}{r}}$$

$$= B^1_i \sqrt{\frac{L}{r}} + B^2_i + B^3_i \sqrt{\frac{r}{L}}$$  (3.10)

which possesses the correct inverse square root of r term.

For six node triangular quadratic elements, if all the sides are straight as shown in Fig.3.5, by similar procedure, the relation between r and the nature coordinate $\xi$ can be obtained as
Figure 3.5 Illustration of 6-node Quarter Point Element
\[ \xi_1 = 1 - \sqrt{\frac{r}{c}} \]  

(3.11)

where \( c \) is the ratio \( \xi_2 / \xi_3 \) which is constant for a given direction \( r \). It is easy to show that the variation of the displacement has the same form as Eq. 3.7 by substituting Eq. 3.11 into Eq. 2.30 with the help of appropriate shape function. The correct traction variation is obtained by multiplying the shape function for traction by the correction factor

\[ \phi(r) = \sqrt{\frac{c}{r}} = \frac{1}{1 - \xi_1} \]  

(3.12)

The stress intensity factors are then evaluated by the displacement correlation technique [29]. By setting \( \theta = 180^\circ \) in Eq. 3.1-3.3, the displacement fields become

\[
\begin{align*}
    u_1 &= \frac{2(1-v^2)}{E} \sqrt{\frac{2r}{\pi}} K_I \\
    u_2 &= \frac{2(1-v^2)}{E} \sqrt{\frac{2r}{\pi}} K_{II} \\
    u_3 &= \frac{2(1+v)}{E} \sqrt{\frac{2r}{\pi}} K_{III}
\end{align*}
\]  

(3.13)
Equating the square root of \( r \) term in Eq.3.13 to that in Eq.3.6 yields

\[
K_r = \frac{E}{2(1-v^2)} \sqrt{\frac{\pi}{2L}} \left[ (-u_1^1 - u_1^2 - u_1^3 - u_1^4 - u_1^5 + 2u_1^6 + u_1^7 + 2u_1^8) \\
+ \frac{\xi_1}{2} (3u_1^1 - 3u_1^2 - u_1^3 + u_1^4 + 4u_1^6 - 4u_1^8) \\
+ \frac{\xi_1^2}{2} (-u_1^1 - u_1^2 + u_1^3 + u_1^4 + 2u_1^5 - 2u_1^7) \right]
\]

\[
K_{\Pi} = \frac{E}{2(1-v^2)} \sqrt{\frac{\pi}{2L}} \left[ (-u_2^1 - u_2^2 - u_2^3 - u_2^4 - u_2^5 + 2u_2^6 + u_2^7 + 2u_2^8) \\
+ \frac{\xi_2}{2} (3u_2^1 - 3u_2^2 - u_2^3 + u_2^4 + 4u_2^6 - 4u_2^8) \\
+ \frac{\xi_2^2}{2} (-u_2^1 - u_2^2 + u_2^3 + u_2^4 + 2u_2^5 - 2u_2^7) \right]
\]

\[
K_{\text{III}} = \frac{E}{2(1+v)} \sqrt{\frac{\pi}{2L}} \left[ (-u_3^1 - u_3^2 - u_3^3 - u_3^4 - u_3^5 + 2u_3^6 + u_3^7 + 2u_3^8) \\
+ \frac{\xi_3}{2} (3u_3^1 - 3u_3^2 - u_3^3 + u_3^4 + 4u_3^6 - 4u_3^8) \\
+ \frac{\xi_3^2}{2} (-u_3^1 - u_3^2 + u_3^3 + u_3^4 + 2u_3^5 - 2u_3^7) \right]
\]

(3.14)
For non-symmetric problems both crack surfaces must be modeled. The stress intensity factors are given by

\[ K_1 = \frac{E}{4(1-v^2)} \sqrt{\frac{\pi}{2L}} \left\{ \begin{array}{l}
(\cdot u_1^3 - u_1^4 + 2u_1^6 + u_1^7 + 2u_1^8) - (\cdot u_1^{*3} - u_1^{*4} + 2u_1^{*6} + u_1^{*7} + 2u_1^{*8}) \\
+ \frac{\xi_1}{2} \left[ (- u_1^3 + u_1^4 + 4u_1^6 - 4u_1^8) - (\cdot u_1^{*3} + u_1^{*4} + 4u_1^{*6} - 4u_1^{*8}) \right] \\
+ \frac{\xi_2}{2} \left\{ (u_1^3 + u_1^4 - 2u_1^7) - (u_1^{*3} + u_1^{*4} - 2u_1^{*7}) \right\} \end{array} \right. \]

\[ K_\Pi = \frac{E}{4(1-v^2)} \sqrt{\frac{\pi}{2L}} \left\{ \begin{array}{l}
(\cdot u_2^3 - u_2^4 + 2u_2^6 + u_2^7 + 2u_2^8) - (\cdot u_2^{*3} - u_2^{*4} + 2u_2^{*6} + u_2^{*7} + 2u_2^{*8}) \\
+ \frac{\xi_1}{2} \left[ (- u_2^3 + u_2^4 + 4u_2^6 - 4u_2^8) - (\cdot u_2^{*3} + u_2^{*4} + 4u_2^{*6} - 4u_2^{*8}) \right] \\
+ \frac{\xi_2}{2} \left\{ (u_2^3 + u_2^4 - 2u_2^7) - (u_2^{*3} + u_2^{*4} - 2u_2^{*7}) \right\} \end{array} \right. \]

\[ K_{\Pi} = \frac{E}{4(1-v^2)} \sqrt{\frac{\pi}{2L}} \left\{ \begin{array}{l}
(\cdot u_3^3 - u_3^4 + 2u_3^6 + u_3^7 + 2u_3^8) - (\cdot u_3^{*3} - u_3^{*4} + 2u_3^{*6} + u_3^{*7} + 2u_3^{*8}) \\
+ \frac{\xi_1}{2} \left[ (- u_3^3 + u_3^4 + 4u_3^6 - 4u_3^8) - (\cdot u_3^{*3} + u_3^{*4} + 4u_3^{*6} - 4u_3^{*8}) \right] \\
+ \frac{\xi_2}{2} \left\{ (u_3^3 + u_3^4 - 2u_3^7) - (u_3^{*3} + u_3^{*4} - 2u_3^{*7}) \right\} \end{array} \right. \]

(3.15)

where the asterisk refers to the displacement of the node on the element opposite to the one shown in Fig.3.5. Note that Eq.3.14 and Eq.3.15 involve the natural coordinate \( \xi_1 \) up to quadratic terms. Therefore a quadratic variation of the stress intensity factor in the \( \xi_1 \) direction can also be represented.
CHAPTER FOUR

VERIFICATION PROBLEMS

Several verification problems were studied to examine the accuracy and efficiency of the Boundary Element Method. This chapter presents the results obtained using the techniques described in the previous chapters for problems whose analytical solution is known as well as for a problem whose solution was obtained numerically by other researchers using different techniques. All problems were performed using single precision on a CRAY-X-MP. The material properties are: $E = 30,000$ ksi and $v = 0.3$.

4.1 Prismatic Bar under Uniform Tension

The first verification problem is a linear elastic prismatic bar under simple tension as shown in Figure 4.1. Only one-eighth of the specimen is modeled due to the symmetry of this problem. The domain is chosen to be a cube with side length equal to 1 subjected to a uniform tension on the surface $z=1$. The planes $x=0$, $y=0$, and $z=0$ are fixed in $x$, $y$, and $z$ directions, respectively. The geometry and boundary segments are given in Figures 4.2 to 4.5. Four different elements: 3-node triangular element, 4-node quadrilateral element, 6-node triangular element, and 8-node quadrilateral element were tested. Different schemes of Gaussian quadrature were chosen to perform both the regular integral and singular integral as described in Section 2.3. The traction in direction $z$ at point A and the transverse displacement in the $x$-direction of point B in Figure 4.1 were calculated and compared with the exact solution [31]:

65
Figure 4.1 Geometry and Boundary Conditions for Prismatic Bar in Uniform Tension
Figure 4.2 3-Node Triangular Element Mesh for Uniform Tension Problem
Figure 4.3 4-Node Quadrilateral Element Mesh for Uniform Tension Problem
Number of Nodes: 26
Number of Elements: 12

Figure 4.4 6-Node Triangular Element Mesh for Uniform Tension Problem
Figure 4.5  8-Node Quadrilateral Element Mesh for Uniform Tension Problem
\[ t_A = -\sigma \]

\[ u_B = -v \sigma/E \] \hspace{1cm} (4.1)

The results are summarized in Table 4.1 to 4.4. The results indicate that as long as the singular integral is evaluated with at least a 3x3 quadrature, then the accuracy depends on the number of points used to evaluate the regular integral. For reliable results the regular integral should be evaluated using at least a 3x3 quadrature for quadrilateral elements and 6 points quadrature for triangular elements.

Tables 4.5 to 4.8 demonstrate the accuracy of the multi-domain technique. The conclusions are similar to those for the single region. However, the two-region results are more accurate since for the same number of integration points there are more nodes.
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NRI : Number of Gaussian quadrature points for regular integral
NSI : Number of Gaussian quadrature points for singular integral
Exact solution: $u=1.0E-06$, $t=-1.00$

Table 4.1 Beam in Uniform Tension, 3-node Triangular Element: Comparison of Error for Displacement and Traction, and CPU Time
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NRI : Number of Gaussian quadrature points for regular integral  
NSI : Number of Gaussian quadrature points for singular integral  
Exact solution : $u=1.E-06$,  $t=-1.00$  

Table 4.2 Beam in Uniform Tension, 4-node Quadrilateral Element: Comparison of Error for Displacement and Traction, and CPU Time
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NRI: Number of Gaussian quadrature points for regular integral
NSI: Number of Gaussian quadrature points for singular integral
Exact solution: $u=1.E-06$, $t=-1.00$

Table 4.3 Beam in Uniform Tension, 6-node Triangular Element: Comparison of Error for Displacement and Traction, and CPU Time
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NRI : Number of Gaussian quadrature points for regular integral  
NSI : Number of Gaussian quadrature points for singular integral  
Exact solution : $u=1.E-06$, $t=-1.00$

Table 4.4 Beam in Uniform Tension, 8-node Quadrilateral Element: Comparison of Error for Displacement and Traction, and CPU Time
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NRI : Number of Gaussian quadrature points for regular integral
NS I : Number of Gaussian quadrature points for singular integral
Exact solution : $u=1.E-06$, $t=-1.00$

Table 4.5 Beam in Uniform Tension, 3-node Triangular Element, Double Region: Comparison of Error for Displacement and Traction, and CPU Time
Table 4.6 Beam in Uniform Tension, 4-node Quadrilateral Element, Double Region: Comparison of Error for Displacement and Traction, and CPU Time

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NRI : Number of Gaussian quadrature points for regular integral
NSI : Number of Gaussian quadrature points for singular integral
Exact solution : $u=1.0E-06$, $t=-1.00$
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NRI: Number of Gaussian quadrature points for regular integral
NSI: Number of Gaussian quadrature points for singular integral
Exact solution: $u=1.0E-06$, $t=-1.00$

Table 4.7: Beam in Uniform Tension, 6-node Triangular Element, Double Region: Comparison of Error for Displacement and Traction, and CPU Time
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<td>-1.0000</td>
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</tr>
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</table>

NRI : Number of Gaussian quadrature points for regular integral
NSI : Number of Gaussian quadrature points for singular integral
Exact solution : $u=1.E-06$, $t=-1.00$

Table 4.8 Beam in Uniform Tension, 8-node Quadrilateral Element, Double Region: Comparison of Error for Displacement and Traction, and CPU Time
4.2 Beam Subjected to Pure Bending

The problem of a beam in pure bending was solved to investigate the convergence of each element as a function of number of elements used. The geometry and boundary conditions are shown in Figure 4.6. For each element, three different meshes were used as shown in Figures 4.7 to 4.10. The exact solution for the displacements are [31]

\[
\begin{align*}
    u_x &= \frac{vM}{EI} xy \\
    u_y &= -\frac{M}{EI} (z^2 - vx^2 + vy^2) \\
    u_z &= \frac{M}{EI} yz
\end{align*}
\]  

(4.2)

and the traction at point B in Figure 4.6 is

\[
t_z = \sigma
\]  

(4.3)

where \(M\) is the applied moment and \(I\) is the moment of inertia with respect to the \(z\) axis. The displacements at point A and the traction in direction \(z\) at point B are compared with these exact solutions. The results are listed in Tables 4.9 to 4.12. The 3-node triangular element and 4-node quadrilateral element do not converge to
Figure 4.6 Geometry and Boundary Conditions for Beam in Pure Bending

- $\sigma = 1$
- $L = 8$
- $b = 2$
- $h = 2$

○ : Fixed in $y$ and $z$ direction
□ : Fixed in $x$ and $z$ direction
Figure 4.7 3-Node Triangular Element Meshes for Beam in Pure Bending
(a) No. of Nodes: 18, No. of Elements: 16

(b) No. of Nodes: 26, No. of Elements: 24

(b) No. of Nodes: 42, No. of Elements: 40

Figure 4.8 4-Node Quadrilateral Element Meshes for Beam in Pure Bending
<table>
<thead>
<tr>
<th>Mesh (fig.)</th>
<th>NRI</th>
<th>NSI</th>
<th>$u_x \times 10^7$</th>
<th>Error (%)</th>
<th>$u_y \times 10^5$</th>
<th>Error (%)</th>
<th>$u_z \times 10^6$</th>
<th>Error (%)</th>
<th>$t_z$</th>
<th>Error (%)</th>
<th>CPU Time (second)</th>
</tr>
</thead>
<tbody>
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<td>4.7a</td>
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<td>------</td>
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<td>------</td>
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<td>4.7b</td>
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<td>1.3224</td>
<td>24.16</td>
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<td>0.8679</td>
<td>13.21</td>
<td>1.2312</td>
<td>15.42</td>
<td>-2.9395</td>
<td>10.23</td>
<td>0.5768</td>
<td>42.32</td>
<td>48.2012</td>
</tr>
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</table>

NRI: Number of Gaussian quadrature points for regular integral
NS I: Number of Gaussian quadrature points for singular integral
Exact solution: $u_x = 1.00000E-07$, $u_y = 1.06667E-05$, $u_z = -2.66667E-06$, $t_z = 1.00$

Table 4.9 Beam in Pure Bending, 3-node Triangular Element: Comparison of Error for Displacements and Traction, and CPU Time

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<tr>
<th>Mesh (fig.)</th>
<th>NRI</th>
<th>NSI</th>
<th>$u_x \times 10^7$</th>
<th>Error (%)</th>
<th>$u_y \times 10^5$</th>
<th>Error (%)</th>
<th>$u_z \times 10^6$</th>
<th>Error (%)</th>
<th>$t_z$</th>
<th>Error (%)</th>
<th>CPU Time (second)</th>
</tr>
</thead>
<tbody>
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<td>------</td>
<td>&gt;100</td>
<td>------</td>
<td>&gt;100</td>
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</table>

NRI: Number of Gaussian quadrature points for regular integral
NS I: Number of Gaussian quadrature points for singular integral
Exact solution: $u_x = 1.00000E-07$, $u_y = 1.06667E-05$, $u_z = -2.66667E-06$, $t_z = 1.00$

Table 4.10 Beam in Pure Bending, 4-node Quadrilateral Element: Comparison of Error for Displacements and Traction, and CPU Time
Figure 4.9 6-Node Triangular Element Meshes for Beam in Pure Bending
<table>
<thead>
<tr>
<th>Mesh (fig.)</th>
<th>NRI</th>
<th>NSI</th>
<th>(u_x \times 10^{-7})</th>
<th>Error (%)</th>
<th>(u_y \times 10^{-5})</th>
<th>Error (%)</th>
<th>(u_z \times 10^{-6})</th>
<th>Error (%)</th>
<th>(t_z)</th>
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<th>CPU Time (second)</th>
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</table>

NRI: Number of Gaussian quadrature points for regular integral
NSI: Number of Gaussian quadrature points for singular integral

Exact solution: \(u_x = 1.00000E-07\), \(u_y = 1.06667E-05\), \(u_z = -2.66667E-06\), \(t_z = 1.00\)

Table 4.11 Beam in Pure Bending, 6-node Triangular Element :Comparison of Error for Displacements and Traction, and CPU Time
Figure 4.10 8-Node Quadrilateral Element Meshes for Beam in Pure Bending
<table>
<thead>
<tr>
<th>Mesh (fig.)</th>
<th>NRI</th>
<th>NSF</th>
<th>( u_x \times 10^6 )</th>
<th>( u_y \times 10^5 )</th>
<th>( u_z \times 10^6 )</th>
<th>Error (%)</th>
<th>Error (%)</th>
<th>Error (%)</th>
<th>( t_z ) (sec)</th>
<th>CPU Time (sec)</th>
</tr>
</thead>
<tbody>
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<td>4.12a</td>
<td>6×6</td>
<td>1.0143</td>
<td>0.1666</td>
<td>0.9994</td>
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<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
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</tr>
<tr>
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<td>0.1966</td>
<td>0.9994</td>
<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
<td>1.0018</td>
<td>5.7333</td>
</tr>
<tr>
<td>4.12c</td>
<td>6×4</td>
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<td>0.1966</td>
<td>0.9994</td>
<td>( &gt;100 )</td>
<td>( &gt;100 )</td>
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<td>( &gt;100 )</td>
<td>1.0018</td>
<td>5.7333</td>
</tr>
</tbody>
</table>

**NRI**: Number of Gaussian quadrature points for regular integral

**NSF**: Number of Gaussian quadrature points for singular integral

**Exact solution**: \( u_x = 1.000000E-07 \), \( u_y = 1.066667E-05 \), \( u_z = 2.66667E-06 \), \( t_z = 1.00 \)

Table 4.12: Beam in Pure Bending, 8-node Quadrilateral Element. Comparison of Error for Displacements and Tracion, and CPU Time.
the exact solution even for the highest order of Gaussian quadrature and for the finest mesh. These results lead to the conclusion that the linear elements should not be used for problems in which the order of traction and displacement distributions are higher than linear. The 6-node triangular element and 8-node quadrilateral element both converge to the exact solutions either by increasing the number of Gaussian quadrature points or by refining the mesh. It is also concluded that the 8-node element is superior to the 6-node element. In order to obtain results with less than 1% error in both the traction and displacement, the 6-node element requires at least 17.9984 seconds using the 7 point Gaussian quadrature for the regular integral and $6 \times 6$ for the singular integral on the 42 node mesh but the 8-node element only needs 7.5160 seconds by choosing $4 \times 4$ Gaussian quadrature for the regular integral and $3 \times 3$ for the singular integral on the 56 node mesh.

The multi-domain technique was also applied using the 8-node element. Two different meshes were tested, as shown in Figure 4.11. The results in Table 4.13 show that the double region mesh can achieve an equivalent accuracy by consuming less CPU time than that the single domain mesh does. This is because the system matrices for the two subregions are, in fact, separated and thus for the same number of nodes the double region mesh needs less calculation than the single domain mesh.
(a) Total no. of nodes: 40 (20 nodes for each subregion)
Total no. of Elements: 12 (6 elements for each subregion)

(b) Total no. of nodes: 64 (32 nodes for each subregion)
Total no. of Elements: 20 (10 elements for each subregion)

Figure 4.11 8-Node Quadrilateral Element, Two-Subregion Meshes for Beam in Pure Bending
<table>
<thead>
<tr>
<th>Mesh (fig.)</th>
<th>NRI</th>
<th>NSI</th>
<th>( u_x \times 10^{-7} )</th>
<th>Error (%)</th>
<th>( u_y \times 10^{-5} )</th>
<th>Error (%)</th>
<th>( u_z \times 10^{-6} )</th>
<th>Error (%)</th>
<th>( t_z )</th>
<th>Error (%)</th>
<th>CPU Time (second)</th>
</tr>
</thead>
<tbody>
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<td>3x3</td>
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NRI : Number of Gaussian quadrature points for regular integral
NS I : Number of Gaussian quadrature points for singular integral
Exact solution : \( u_x = 1.00000E-07 \), \( u_y = 1.06667E-05 \), \( u_z = -2.66667E-06 \), \( t_z = 1.00 \)

Table 4.13 Beam in Pure Bending, 8-node Quadrilateral Element, Double Region: Comparison of Error for Displacements and Traction, and CPU Time
4.3 Circular Buried Crack Under Uniform Tension

The stress intensity factor of a circular crack buried in an infinite body subjected to far field uniform distributed traction is studied in this section. The exact solution is [32]

$$K_I = 2\sigma \sqrt{\frac{a}{\pi}}$$  \hspace{1cm} (4.4)

where $\sigma$ is the applied stress and $a$ is the crack radius. For this problem $K_{II}$ and $K_{III}$ are zero since the load is perpendicular to the crack surface. Six different meshes with either different number of nodes or different boundary conditions were studied. The overall dimensions of these meshes were chosen large enough to simulate the infinite medium.

The first mesh is a two-subregion model which describes the whole domain of the problem. Each subregion consists of 52 elements and 150 nodes in which 88 nodes belong to the interface which bonds the two subregions as shown in Figure 4.12. The crack face which is kept traction free. Each crack face is formed by eight 8-node quadrilateral quarter point elements and eight 6-node triangular elements. Traction singular elements are placed along the crack front. The uniform distributed tractions $\sigma$ in direction $z$ are applied on the plane parallel to the interface plane of one subregion mesh, and the same plane for another subregion is then fixed in the $z$-direction with node A fixed in direction $x$ and node B fixed in
Total number of nodes: 300 (150 for each subregion)
Total number of elements: 104 (52 for each subregion)

Figure 4.12 Element Mesh of Double Region Model for Buried Circular Crack
Fig. 4.13 Average Error in Computed KI by Different L/A for Varied Poisson Ratio
direction y, as shown in Fig.4.12, to prevent rigid body translation and rotation. Ingraffea and Mann [33] pointed out that the ratio of the length L of the crack front element to the crack length a affects the accuracy of the results. Therefore, convergence for different L/a ratios under varied Poisson's ratio v was studied first. Five different ratios of L/a varying from 0.1 to 0.5 were applied for Poisson's ratios varying from 0.0 to 0.4. The results are shown in Figure 4.13 revealing that the absolute error is confined to within 5% when the ratio of L/a is in between 0.26 and 0.34 for Poisson's ratios ranging from 0.1 to 0.4. Thus, the ratio of L/a of the models used in this paper is chosen to be 0.3.

In the second example the problem is modeled using the coarse double region mesh shown in Figure 4.14. It consists of 73 nodes and 25 elements. The ratio of L/a is 0.3. The maximum error in the calculated KI was found to be 0.23%. The CPU time for this case was 31 seconds.

The third and fourth meshes used are the same as the ones above except that only half of the subregion is taken into consideration. The nodes belonging to the interface are now constrained in the z-direction as shown in Figure 4.15 and 4.16 to simulate the correct symmetric boundary conditions. The calculated KI is almost constant along the crack front and the absolute errors were 1.6% and 0.27% for the mesh of Figure 4.15 and the mesh in Figure 4.16, respectively. The CPU times were 71 seconds and 20 seconds, respectively.

The fifth mesh is the half cut of the third mesh with an additional cut plane y=0 which is fixed in the y-direction as shown in Figure 4.17. The 6-node triangular elements around the crack tip on this cut plane are converted to be traction
Figure 4.14 Element Mesh of Double Region Model for Buried Circular Crack

Total number of nodes: 146 (73 for each subregion)
Total number of elements: 50 (25 for each subregion)
Figure 4.15 (a) Element Mesh for Half Domain of Buried Circular Crack
(b) Illustration of Elements around Crack
Figure 4.16 (a) Element Mesh for Half Domain of Buried Circular Crack
(b) Illustration of Elements around Crack
Total number of nodes: 202
Total number of elements: 72

Figure 4.17 Element Mesh of Quarter Domain for Buried Circular Crack
Figure 4.18 Element Mesh of One-Eighth Domain for Buried Circular Crack
<table>
<thead>
<tr>
<th>Mesh (fig.)</th>
<th>4.12</th>
<th>4.14</th>
<th>4.15</th>
<th>4.16</th>
<th>4.17</th>
<th>4.18</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh Type</td>
<td>Double Region</td>
<td>Double Region</td>
<td>Single Region</td>
<td>Single Region</td>
<td>Single Region</td>
<td>Single Region</td>
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<tr>
<td>No. of Node</td>
<td>300</td>
<td>146</td>
<td>150</td>
<td>73</td>
<td>202</td>
<td>110</td>
</tr>
<tr>
<td>No. of Element</td>
<td>104</td>
<td>50</td>
<td>52</td>
<td>25</td>
<td>72</td>
<td>40</td>
</tr>
<tr>
<td>$K_I$</td>
<td>1.14604</td>
<td>1.13101</td>
<td>1.14647</td>
<td>1.13141</td>
<td>1.15534</td>
<td>1.15659</td>
</tr>
<tr>
<td>Error (%)</td>
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<td>0.23</td>
<td>1.6</td>
<td>0.27</td>
<td>2.4</td>
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<tr>
<td>CPU Time (Second)</td>
<td>131</td>
<td>31</td>
<td>71</td>
<td>20</td>
<td>133</td>
<td>41</td>
</tr>
</tbody>
</table>

Exact solution: $K_I = 1.128379$
where crack radius $a = 1$, applied stress $\sigma = 1$.

Table 4.14 Circular Crack Buried in Infinite Body Under Uniform Tension: Comparison of Error for $K_I$ and CPU Time for Different Meshes
singular elements by relocating their middle nodes on the two sides to the quarter position to capture the inverse square root singularity in stresses. This mesh is composed of 202 nodes and 72 elements. The differences between the analytic solution of $K_I$ and the calculated results ranged from 1.2% to 2.3%. This mesh uses 133 seconds of CPU time.

The final mesh models one-eighth of the domain. Three planes, $x=0$, $y=0$, and $z=0$, are fixed in the x, y, and z direction, respectively, to represent the correct boundary conditions. The maximum error in $K_I$ is 2.5%.

The results of this section are summarized in Table 4.14. From the table we also observe that the last two cases which model only part of the whole domain have the largest error. This is probably due to the asymmetry of the meshes. Suprisingly, the coarsest mesh (73 nodes) leads to the most accurate result. Since the $K_I$ is constant along the crack front and the quadratic element on both side of the crack front can model the circular shape exactly, an accurate result can still be obtained by using only a few elements to model the crack. This also counts for the reason that the coarsest mesh can have such excellent results.

4.4 Circular Buried Crack Inclined at 30 Degrees Under Uniform Tension

This problem is used to ensure the ability of the Boundary Element Method to calculate stress intensity factors for mixed mode fracture problems. A circular crack deforms in three modes when the normal of the crack surface is not parallel to the direction of the applied load. The exact solutions for the stress intensity factors
Figure 4.19 Circular Crack Buried in Infinite Body Inclined at an Angle $\gamma$ With Respect to the Direction of Applied Load
are [33]

\[ K_I = \left[ \frac{2}{\pi} \sigma \sin^2 \sqrt{\pi \alpha} \right] \]

\[ K_{II} = \left[ \frac{4}{\pi(2-v)} \left( \sigma \sin \gamma \cos \gamma \right) \sqrt{\pi \alpha} \right] \cos \omega \]

\[ K_{III} = \left[ \frac{4(1-v)}{\pi(2-v)} \left( \sigma \sin \gamma \cos \gamma \right) \sqrt{\pi \alpha} \right] \sin \omega \] (4.5)

where \( \sigma \) is the applied stress, \( a \) the crack radius, \( \gamma \) the angle between the direction of applied load and the normal of the crack plane, and \( \omega \) is the angle as is shown in Figure 4.19. Only a double region mesh can model this problem appropriately since symmetry no longer exists. The mesh is composed of two parallelepiped shaped subregions as shown in Figure 4.20. Each subregion is made up of 52 quadratic elements and 150 nodes including 88 nodes belonging to the interface. Quarter point elements and traction singular elements are used along the crack front as described in the previous section. The results are shown in Figures 4.21 to 4.23 in which the stress intensity factors are normalized. The calculated \( K_I \) are almost constant along the crack front. The error ranges from 1.17% to 1.19%. The maximum error for \( K_{II} \) and \( K_{III} \) are 2.34% and 5.71%, respectively. The total CPU time is 131 seconds.
Total number of nodes: 146 (73 for each subregion)
Total number of elements: 50 (25 for each subregion)

Figure 4.20 Element Mesh of Double Region Model for Buried Circular Crack Inclined at 30 Degrees
Figure 4.21 Comparison of Calculated KI with Exact Solution for Circular Crack Inclined at 30 Degrees
Figure 4.22 Comparison of Calculated KII with Exact Solution for Circular Crack Inclined at 30 Degrees
Figure 4.23 Comparison of Calculated KIII with Exact Solution for Circular Crack Inclined at 30 Degrees
4.5 Semi-circular Surface Crack Under Uniform Traction

The purpose of this section is to check the accuracy of the method for surface crack problems. Since an exact solution for this problem does not exist, the results are compared with the work of Tada [33], and Newman and Raju [34]. Tada presents the stress intensity factor $K_I$ for a semi-circular surface crack in a semi-infinite body as shown in Figure 4.24 as

$$K_I(\theta) = \frac{2}{\pi} \sigma \sqrt{\pi a} \ F(\theta)$$ \hspace{1cm} (4.6)

where

$$F(\theta) = 1.211 - 0.186 \sqrt{\sin\theta} \hspace{1cm} (10^\circ < \theta < 170^\circ)$$

and $\sigma$ is the applied stress, $a$ the crack radius, and $\theta$ is the angle measured from the surface as shown in Figure 4.24. Note that the $K_I$ is not constant along the crack front. For the same problem Newman and Raju predict

$$K_I = \sigma \sqrt{\pi a} \left( \frac{1}{\sqrt{Q}} \right) F(\theta)$$ \hspace{1cm} (4.7)
Figure 4.24 Circular Crack in Semi-infinite Body Under Uniform Tension
where

\[ F(\theta) = 1.04 \left[ 1 + 0.1 (1 - \sin \theta)^2 \right] \]

and \( Q = 2.464 \) for a semi-circular surface crack. A double region model, as shown in Figure 4.25, is used in the problem. Each subregion consists of 72 quadratic elements and 202 nodes.

The comparison of the results with the work of Tada and Raju and Newman is shown in Figure 4.26. The present results are quite consistent with these two predictions when the \( \theta \) is between 35° and 90°. The maximum difference appears at the surface with the error of about 5.7%. This is because the \( K_I \) is calculated assuming of plane strain conditions and an inverse square root singularity of stresses. At the points where the crack intersects the free surface these conditions do not apply.
Figure 4.26 Comparison of Stress Intensity Factors for a Circular Surface Crack in a Semi-infinite Body Under Uniform Tension
CHAPTER FIVE

STRESS INTENSITY FACTOR ANALYSIS OF THE INNER RACEWAY OF HIGH SPEED BEARINGS

In this chapter, the geometry of the engine bearing is described and the loadings on the inner raceway (including the hoop stresses and the Hertzian contact load) are calculated. The stress intensity factors for several semi-circular surface cracks of different lengths and inclinations are presented as functions of the location of the indenters.

5.1 Geometry and Applied Loading

The bearing analyzed in this report is a high performance engine bearing which is used on the main shaft of an aircraft. The dimensions and the geometry of the bearing are shown in Table 5.1 and Fig.5.1, respectively.

The bearing consists of 28 ball rollers. To simulate the passage of each ball only 1/28th of the inner raceway is modeled. Since the radius of the inner raceway is large compared to the other dimensions of the part, the curvature is neglected and the mesh is modeled as a block with flat surface as shown in Fig.5.2.

The external loadings considered in the analysis are the hoop stresses and the Hertzian load. The hoop stresses due to the rotation and the shrink fit are taken from [22] and are given by
<table>
<thead>
<tr>
<th><strong>Shaft</strong></th>
<th>Inner radius $a_s = 2.0$ inch</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Outer radius $b_s = 2.30233$ inch</td>
</tr>
<tr>
<td><strong>Inner raceway</strong></td>
<td>Inner radius $a_i = 2.3$ inch</td>
</tr>
<tr>
<td></td>
<td>Outer radius $b_i = 2.6$ inch</td>
</tr>
<tr>
<td><strong>Outer raceway</strong></td>
<td>Inner radius $a_o = 3.1$ inch</td>
</tr>
<tr>
<td></td>
<td>Outer radius $b_o = 3.35$ inch</td>
</tr>
<tr>
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<tr>
<td><strong>Ball bearing</strong></td>
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<td><strong>No. of ball bearings</strong></td>
<td>28</td>
</tr>
<tr>
<td><strong>Shaft speed</strong></td>
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</tr>
<tr>
<td><strong>M50 steel material properties</strong></td>
<td>$E = 3.0 \times 10^7$ psi</td>
</tr>
<tr>
<td></td>
<td>$\rho = 0.288 \text{ lb/in}^3$</td>
</tr>
<tr>
<td></td>
<td>$v = 0.3$</td>
</tr>
<tr>
<td></td>
<td>$K_{IC} = 18 \text{ ksi\sqrt{in}}$</td>
</tr>
</tbody>
</table>

Table 5.1 Dimensions and Material Properties of Typical Ball Bearing for Aircraft Engines
Figure 5.1 Geometry of Bearing
\[
\sigma_{\theta \theta} = \frac{3+\nu}{8} \rho \omega^2 \left[ b_i^2 + a_i^2 + \frac{a_i^2 b_i^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2 \right] + P \frac{(b/r)^2 + 1}{(b/a_i)^2 - 1} \tag{5.1}
\]

where \(\omega\) is the angular velocity of the shaft and \(P\) is the pressure existing between the shaft and the inner raceway which is equal to

\[
P = \frac{E\delta_n}{a_i} \left( \frac{a_i^2 - a_i^2}{2a_i^2(b_i^2 - a_i^2)} \right) \tag{5.2}
\]

where \(\delta_n\) is the difference between the outer radius of the shaft and the inner radius of the inner raceway during rotation at speed \(\omega\). It can be obtained by

\[
\delta_n = \delta - \Delta \delta \tag{5.3}
\]

where \(\delta\) is the original shrink fit at 0 rpm which is equal to 0.00233 inch in this case and \(\Delta \delta\) is the difference in the radial displacement between the inner radius of the inner raceway, \(u_r^1\), and the outer radius of the shaft, \(u_r^S\), under the rotation at speed \(\omega\), i.e.,
\[ \Delta \delta = u_i^1 - u_i^s \]

\[ = \frac{\rho \omega^2 a_i}{4E} \left[ (3+\nu)b_i^2 + (1-\nu)a_i^2 \right] - \frac{\rho \omega^2 b_s}{4E} \left[ (3+\nu)b_s^2 + (1-\nu)a_s^2 \right] \quad (5.4) \]

Applying the data of Table 5.1 to the above equations results in

\[ \sigma_{\theta\theta} = 1973.61 \left[ 3.5986 + \frac{24.3286}{r^2} \right] \]

\[ + \omega^2 \times 10^{-5} \left[ 3.6290 - 0.1941(r^2) + \frac{9.1248}{r^2} \right] \quad \text{(psi)} \quad (5.5) \]

where \( \omega \) is in rpm. The maximum hoop stress of the inner raceway is at the inner radius. The hoop stresses range from 38 ksi to 44 ksi when the rotation speed is 25,500 rpm.

The contact region between a sphere and a flat surface is circular and the stress distribution is [30]

\[ p = p_0 \sqrt{1 - \frac{x^2}{a^2} \cdot \frac{y^2}{a^2}} \quad (5.6) \]

where \( a \) is the radius of the contact area and \( p_0 \) is the maximum stress which can be
calculated from

\[ p_0 = \frac{3}{2} \frac{P_T}{\pi a^2} \]  \hspace{1cm} (5.7)

where \( P_T \) is the total load. The radius is

\[ a = \frac{3}{2} \sqrt{\frac{3P_T R (1 - v^2)}{2E}} \]  \hspace{1cm} (5.8)

where \( R \) is the radius of the sphere. As discussed in [6] the experimental data indicates that for the bearing considered in the present analysis the maximum Hertzian stress is 285 ksi. Subsequent calculations will be based on this value.

The distribution of the Hertzian contact loading is distorted when the spherical body is rolling over a surface containing a surface breaking crack [31]. However, the influence of the subsurface crack on the Hertzian stress distribution has been shown [31] to be insignificant. The interaction effects of the subsurface crack on the Hertzian distribution are thus ignored.

Since the Hertzian stress fields can be solved in closed form, superposition is applied in order to decrease the number of nodes needed to model the Hertzian
contact load accurately. The superposition method is illustrated in Fig.5.3. The stress intensity factors of a surface crack subjected to the Hertzian load are equal to the stress intensity factors of the same crack loaded with the negative of the stresses produced by the Hertzian loading. The stresses produced in the interior of the plate by the hertzian contact loading are calculated by integrating the stresses due to a concentrated force acting on the boundary of a semi-infinite solid [22] as shown in Fig.5.4. The hoop stresses are directly applied on the the end of the plate. The total stress intensity factor is then the superposition of the stress intensity factor due to the hoop stresses and the one resulting from the Hertzian load. It should be noted that the depth of the plate is assumed to be large enough so that the distribution of loading remains Hertzian.

5.2 Stress Intensity Factor of Circular Subsurface Crack in the Inner Raceway of the Engine Bearing

This section presents the results of a quasi-static stress intensity factor analysis of a typical ball bearing which is used as a support for the main shaft of aircraft engines. All the calculations are based on the model shown in Fig.5.2 which consists of an axial hoop stress and a Hertzian load (contact radius \( a \) and maximum intensity \( p_0 \)) interacting with a semi-circular crack (radius, or length \( l \)) inclined at an angle \( \phi \). The dynamic effect is ignored and the distance \( x \), between the center of the Hertzian distribution and the crack mouth is changed incrementally to simulate the passage of each ball bearing.

The stress intensity factors for this problem vary with position along the
Figure 5.3 Illustration of Superposition Method

(a) Original problem
(a Hertzian load with a crack)

(b) Hertzian load applied on a body without crack

(c) Negative stress applied on the crack surfaces

\[ K^{(a)} = K^{(c)} \quad (K^{(b)} = 0) \]
Figure 5.4 Concentrated Force acting on the Boundary of a Semi-infinite Body
crack front ($\theta$). A typical variation of the mode I stress intensity factor at $\theta = 90^\circ$ with roller position and crack length for a vertical crack ($\phi = 0^\circ$) is shown in Fig.5.5 for $p_o = 285$ ksi ($a = 0.00679$ inch). When the roller is at a distance greater than four times the contact length from the crack mouth, the stress intensity factor is a constant which results from the axial stress. As the roller gets closer, the compressive stress arising from the Hertzian load decrease the $K_I$ stress intensity factor. When the load is on the crack mouth the stress intensity factors becomes negative for the cracks which are shorter than 0.005 inch. The negative value of $K_I$ indicates the closure of the crack. For longer cracks, the decrease of $K_I$ diminishes since the crack tip is beyond the range of the highly compressive Hertzian stress field.

The variation of the mode II stress intensity factor for the same loading condition is shown in Fig.5.6. The value of $K_{II}$ is zero when the roller is far from the crack mouth. As the roller approaches the crack, $K_{II}$ starts to increase and reaches a maximum value when the load reaches the crack mouth. As the roller crosses to the other side of the crack, $K_{II}$ abruptly changes sign and decreases to a minimum value equal in magnitude to the previous maximum. For small cracks this change is very abrupt, but for large cracks the change is more gradually. As pointed out in [6], these abrupt variations in $K_I$ and $K_{II}$ may significantly affect the propagation of short cracks.

The stress intensity factors for $\theta = 45^\circ$ are shown in Fig.5.7 to 5.9. It is observed that the variations of $K_I$ and $K_{II}$ did not differ significantly from $\theta = 90^\circ$. Another stress intensity factor $K_{III}$ is observed based on the local coordinate system moving along the crack front. The $K_{III}$ behaves as $K_{II}$. Fig.5.10 to 5.12
Figure 5.5: Variation of KI at $\theta = 90^\circ$ as a Function of Roller Position with Different Crack Length
Figure 5.6 Variation of KII at $\theta = 90^\circ$ as a Function of Roller Position with Different Crack Length
Figure 5.7 Variation of KII at $\theta = 45^\circ$ as a Function of Roller Position with Different Crack Length

Maximum Hertzian stress = 285 ksi

Crack inclined ang $\phi = 0^\circ$
Figure 5.8 Variation of $K_{II}$ at $\theta=45^\circ$ as a Function of Roller Position with Different Crack Length

Maximum Hertzian stress = 285 ksi
Crack inclined angle $\phi = 0^\circ$
Figure 5.9 Variation of KIII at $\theta = 45^\circ$ as a Function of Roller Position with Different Crack Length
show the stress intensity factors for different locations along the crack front. It can be seen from Fig.5.10 that the magnitude of $K_I$ does not change significantly. The magnitudes of $K_{II}$ and $K_{III}$ change along the crack front but however it can be calculated from Fig.5.11 and 5.12 that the square root of the sum of the squares of $K_{II}$ and $K_{III}$ are almost constant along the crack front. A comparison of the results for $\phi = 0^\circ$ obtained in the presented analysis with those presented in Mendelson and Ghosn [6] revealed that the magnitudes and variations in $K_I$ are similar. However, the mode II stress intensity factors differ significantly. As seen in Fig.5.5, for $l = 0.02$ inch the maximum value of $K_{II}$ is approximately 1.5 ksi $\sqrt{\text{in}}$, while in reference [6] it is approximately 10 ksi $\sqrt{\text{in}}$.

Fig.5.13 and 5.14 present the stress intensity factors of $K_I$ and $K_{II}$ at $\theta=90^\circ$, respectively, for cracks inclined at $\phi = 30^\circ$ for $p_o = 285$ ksi. An increase of $K_I$ is observed when the roller passes to the right hand side of the crack mouth for short cracks. This is because the Hertzian load causes the inclined crack surfaces apart when it is passing over the crack mouth. The value of $K_{II}$ before the roller crossing over the crack from the left is much greater than the $K_{II}$ after the roller moving to the right of the crack mouth since the Hertzian load is pushing the left crack surface sliding along the right crack surface when the roller is on the right hand side of the crack.

Fig.5.15 and 5.16 show $K_I$ and $K_{II}$ variations for several inclinations of a crack length $l = 0.02$ inch. High values of $K_{II}$ is observed for $\phi = 15^\circ$ and $\phi = 30^\circ$. The mode I fracture toughness of M50 steel, which is used for this type of bearing, is of the order of 18 ksi $\sqrt{\text{in}}$. Assuming the mode II toughness is of comparable magnitude, these results indicate that this applied loading would lead to large
propagation rates for cracks inclined at 30°.
Maximum Hertzian stress = 285 ksi
Crack length $a = 0.005$ inch
Crack inclined angle $\phi = 0^\circ$

Figure 5.10 Variation of $K_I$ as a Function of Roller Position with Different Angle $\theta$
Maximum Hertzian stress = 285 ksi
Crack length \( a = 0.005 \) inch
Crack inclined angle \( \phi = 0^\circ \)

Figure 5.11 Variation of \( K_{II} \) as a Function of Roller Position with Different Angle \( \theta \)
Maximum Hertzian stress = 285 ksi
Crack length $a = 0.005$ inch
Crack inclined angle $\phi = 0^\circ$

Figure 5.12 Variation of $K_{III}$ as a Function of Roller Position with Different Angle $\theta$
Figure 5.13 Variation of $K_I$ at $\theta = 90^\circ$ as a Function of Roller Position with Different Crack Length
Figure 5.14 Variation of $K_{II}$ at $\theta = 90^\circ$ as a Function of Roller Position with Different Crack Length
Figure 5.15 Variation of KI at $\theta = 90^\circ$ as a Function of Roller Position with Different Crack Length
Figure 5.16 Variation of KII at $\theta = 90^\circ$ as a Function of Roller Position with Different Inclined Angle $\phi$
A preliminary stress intensity factor analysis of a typical high speed bearing was conducted using the Boundary Element Method. The results obtained in the present three-dimensional analysis suggest lower mode $K_{II}$ stress intensity factors than those predicted by the two-dimensional analysis in [6]. This is due to the fact that the total load needed to produce the experimentally measured 285 ksi Hertzian stress using a semi-spherical contact area is much lower than that using a cylindrical contact (27.5 lbs instead of 1500 lbs). This may be the reason why the predictions of Mendelson and Ghosn's [6] analyses are overly conservative. High $K_I$ and $K_{II}$ values were observed for cracks inclined at 30°. These results indicate that the interaction of the Hertzian load would lead to large propagation rates for cracks inclined at 30°.

Although the stress intensity factor data obtained from the analysis has not been reduced to a form suitable for life prediction, these preliminary results can provide a better understanding of the complex interactions between a surface crack, a moving Hertzian load, and an axial stress.

As for the further work, an incremental crack growth analysis of elliptical cracks using the Boundary Element Method and a fatigue crack growth law would be the next step. Also, more factors which affect the stress intensity factors of a crack such as friction between the roller and the raceway, the dynamic effect, etc. could be taken into consideration.
REFERENCES


15. Rizzo, F. J. and Shippy, D. J., "An Advanced Boundary Integral Equation


APPENDIX A
SHAPE FUNCTIONS FOR ISOPARAMETRIC ELEMENTS

The shape functions for different elements are listed below corresponding to the elements shown in Fig.A.1.

(1) 3-Node Linear Triangular Element:

\[
\begin{align*}
N_1 & = \xi_1 \\
N_2 & = \xi_2 \\
N_3 & = \xi_3 \\
\end{align*}
\]

where \(\xi_1 + \xi_2 + \xi_3 = 1\)

(2) 6-Node Quadratic Triangular Element:

\[
\begin{align*}
N_1 & = \xi_1 ( 2\xi_1 - 1) \\
N_2 & = \xi_2 ( 2\xi_2 - 1) \\
N_3 & = \xi_3 ( 2\xi_3 - 1) \\
N_4 & = 4\xi_1\xi_2 \\
N_5 & = 4\xi_2\xi_3 \\
N_6 & = 4\xi_3\xi_1 \\
\end{align*}
\]

where \(\xi_1 + \xi_2 + \xi_3 = 1\)
(3) 4-Node Linear Quadrilateral Element:

\[ N_1 = (1 - \xi_1)(1 - \xi_2) / 4 \]
\[ N_2 = (1 + \xi_1)(1 - \xi_2) / 4 \]
\[ N_3 = (1 + \xi_1)(1 + \xi_2) / 4 \]
\[ N_4 = (1 - \xi_1)(1 + \xi_2) / 4 \]

(4) 8-Node Quadratic Quadrilateral Element:

\[ N_1 = -(1 + \xi_1 + \xi_2)(1 - \xi_1)(1 - \xi_2) / 4 \]
\[ N_2 = -(1 + \xi_1 - \xi_2)(1 + \xi_1)(1 - \xi_2) / 4 \]
\[ N_3 = -(1 + \xi_1 + \xi_2)(1 + \xi_1)(1 + \xi_2) / 4 \]
\[ N_4 = -(1 + \xi_1 - \xi_2)(1 - \xi_1)(1 + \xi_2) / 4 \]
\[ N_5 = (1 - \xi_1^2)(1 - \xi_2) / 2 \]
\[ N_6 = (1 + \xi_1)(1 - \xi_2^2) / 2 \]
\[ N_7 = (1 - \xi_1^2)(1 + \xi_2) / 2 \]
\[ N_8 = (1 - \xi_1)(1 - \xi_2^2) / 2 \]
Figure A.1 Isoparametric Elements

(1) 3-Node Linear Triangular Element

(2) 6-Node Quadratic Triangular Element

(3) 4-Node Linear Quadrilateral Element

(4) 8-Node Quadratic Quadrilateral Element

\[ \xi_3 = 1 - \xi_1 - \xi_2 \]
APPENDIX B

TRANSFORMATION FUNCTIONS

The integral on a boundary surface element with domain \( \Gamma \) can be expressed as

\[
I = \int_{\Gamma} k(x,y) dx dy
\]

In order to overcome the \( 1/r \) singularity of the kernel the integral is first transformed from the Cartesian coordinate system to the parametric \( \xi \)-coordinate system by using the shape functions

\[
x = \sum_{i=1}^{n_m} N_i(\xi_1, \xi_2) X_i
\]

\[
y = \sum_{i=1}^{n_m} N_i(\xi_1, \xi_2) Y_i
\]

corresponding with the Jacobian
such that the integral becomes

\[ I = \int_{\Gamma} \kappa(\xi_1,\xi_2) J_1(\xi_1,\xi_2) \, d\xi_1 \, d\xi_2 \]

The integral on the isoparametric element is then divided into several triangles according to the location of the singular node. The singular integral on each triangle is carried out by using a polar coordinate system with its origin at the singular node such that

\[ \xi_1 = f_1(r,\theta) \]
\[ \xi_2 = g_1(r,\theta) \]
\[ J_2(r, \theta) = \begin{vmatrix} \frac{\partial \xi_1}{\partial r} & \frac{\partial \xi_1}{\partial \theta} \\ \frac{\partial \xi_2}{\partial r} & \frac{\partial \xi_2}{\partial \theta} \end{vmatrix} \]

and

\[ I = \sum_{i=1}^{n_T} \int_{T_i} \kappa(r, \theta) J_1(r, \theta) J_2(r, \theta) dr d\theta \]

where \( n_T \) is the total number of triangles in which the isoparametric element is divided and \( T_i \) is the corresponding domain. The Jacobian \( J_2 = r \) can remove the \( 1/r \) singularity. In order to accomplish the integral numerically by the Gaussian quadrature the polar coordinate is again transferred to a system with both coordinates ranging from -1 to 1 by the transforming functions

\[ r = h_i(r, \theta) \]
\[ \theta = l_i(r, \theta) \]
The integral then becomes

\[ I = \sum_{i=1}^{n_T} \int_{\mathcal{Q}_i} \kappa(r, \theta) J_1(r, \theta) J_2(r, \theta) J_3(r, \theta) \, dr \, d\theta \]

where \( n_a \) and \( n_b \) are the order of the Gaussian quadrature. The transforming functions for different element are illustrated in the following pages.
\[ \xi_1 = r \cos \theta - 1 \]
\[ \xi_2 = r \sin \theta - 1 \]
\[ J_2 = r \]
\[ \theta = \frac{\pi}{8} (\theta + 1) \]
\[ r = \frac{(r + 1)}{\cos \theta} \]
\[ J_3 = \frac{\pi}{8 \cos \theta} \]

-1 \leq \xi \leq 1
-1 \leq \theta \leq 1

Figure B.1.a Illustration of Transformation of Coordinate System
\[ \xi_1 = r \cos \theta - 1 \]
\[ \xi_2 = r \sin \theta - 1 \]
\[ J_2 = r \]

\[ \theta = \frac{\pi}{8} (\theta + 3) \]
\[ r = \frac{(r + 1)}{\sin \theta} \]
\[ J_3 = \frac{\pi}{8 \sin \theta} \]

-1 \leq r \leq 1
-1 \leq \theta \leq 1

Figure B.1.b Illustration of Transformation of Coordinate System
\[ \xi_1 = r \cos \theta \]
\[ \xi_2 = r \sin \theta - 1 \]
\[ J_2 = r \]

\[ \theta = \frac{\tan^{-1} 2}{2} (\theta + 1) \]
\[ r = \frac{(r+1)}{2 \cos \theta} \]
\[ J_3 = \frac{\tan^{-1} 2}{4 \cos \theta} \]

\(-1 \leq r \leq 1\)
\(-1 \leq \theta \leq 1\)

Figure B.1.c Illustration of Transformation of Coordinate System
\[
x_1 = r \cos \theta \\
x_2 = r \sin \theta - 1 \\
J_2 = r \\
\theta = \left( \frac{\pi}{2} - \tan^{-1} 2 \right) \theta + \frac{\pi}{2} \\
r = \frac{(r + 1)}{\sin \theta} \\
J_3 = \frac{(\pi/2 - \tan^{-1} 2)}{\sin \theta} \\
-1 \leq r \leq 1 \\
-1 \leq \theta \leq 1 
\]
\[ \xi_1 = r \cos \theta \]
\[ \xi_2 = r \sin \theta - 1 \]
\[ J_2 = r \]

\[ \theta = \frac{\tan^{-1} \frac{1}{2} \cdot (\theta - 1) + \pi}{2} \]
\[ r = \frac{(r + 1)}{2 \cos (\pi - \theta)} \]
\[ J_3 = \frac{\tan^{-1} \frac{1}{2}}{4 \cos (\pi - \theta)} \]

\(-1 \leq r \leq 1\)
\(-1 \leq \theta \leq 1\)

Figure B.1.e Illustration of Transformation of Coordinate System
$\xi_1 = r \cos \theta$

$\xi_2 = r \sin \theta$

$J_2 = r$

$\theta = \frac{\pi}{4} (\theta + 1)$

$r = \frac{(r + 1)}{2\sqrt{2} \cos (\pi/4 - \theta)}$

$J_3 = \frac{\pi}{8\sqrt{2} \cos (\pi/4 - \theta)}$

$-1 \leq r \leq 1$

$-1 \leq \theta \leq 1$

Figure B.1f Illustration of Transformation of Coordinate System
\[ \xi_1 = r \cos \theta + \frac{1}{2} \]
\[ \xi_2 = r \sin \theta + \frac{1}{2} \]
\[ J_2 = r \]

\[ \theta = \frac{\pi}{4} - \left( \frac{\theta + 4}{4} \right) \]
\[ r = \frac{(r + 1)}{4 \cos (\pi - \theta)} \]
\[ J_3 = \frac{\pi}{16 \cos (\pi - \theta)} \]

-1 ≤ \( \xi \) ≤ 1
-1 ≤ \( \theta \) ≤ 1

Figure B.1.g Illustration of Transformation of Coordinate System
\[ \begin{align*}
\xi_1 &= r \cos \theta + \frac{1}{2} \\
\xi_2 &= r \sin \theta + \frac{1}{2} \\
J_2 &= r
\end{align*} \]

\[ \begin{align*}
\theta &= \frac{\pi}{4} (\theta + 6) \\
r &= \frac{(r + 1)}{4 \cos (3\pi/2 - \theta)} \\
J_3 &= \frac{\pi}{16 \cos (3\pi/2 - \theta)}
\end{align*} \]

\(-1 \leq r \leq 1\)
\(-1 \leq \theta \leq 1\)

Figure B.1.h Illustration of Transformation of Coordinate System
This thesis presents the results of a stress intensity factor analysis of semicircular surface cracks in the inner raceway of an engine bearing. The loading consists of a moving spherical Hertzian contact load and an axial stress due to rotation and shrink fit. A three-dimensional linear elastic Boundary Element Method code was developed to perform the stress analysis. The element library includes linear and quadratic isoparametric surface elements. Singular quarter point elements were employed to capture the square root displacement variation and the inverse square root stress singularity along the crack front. The program also possesses the capability to separate the whole domain into two subregions. This procedure enables one to solve nonsymmetric fracture mechanics problems without having to separate the crack surfaces a priori. A wide range of configuration parameters was investigated. The ratio of crack depth to bearing thickness was varied from one-sixtieth to one-fifth for several different locations of the Hertzian load. The stress intensity factors for several crack inclinations were also investigated. The results demonstrate the efficiency and accuracy of the Boundary Element Method. Moreover, the results can provide the basis for crack growth calculations and fatigue life prediction.