ON LINEAR STRUCTURE AND
PHASE ROTATION INVARIANT PROPERTIES OF
BLOCK $2^l$-PSK MODULATION CODES

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ABSTRACT

In this correspondence, we investigate two important structural properties of block $2^t$-ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN channel, the error performance of a modulation code depends on its squared Euclidean distance distribution. Linear structure of a code makes the error performance analysis much easier. Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization. It is desirable for a modulation code to have as many phase symmetries as possible. In this paper, we first represent a $2^t$-ary modulation code as a code with symbols from the integer group, $S_{2^t,PSK} = \{0, 1, 2, \ldots, 2^t - 1\}$, under the modulo-$2^t$ addition. Then we define the linear structure of block $2^t$-ary PSK modulation codes over $S_{2^t,PSK}$ with respect to the modulo-$2^t$ vector addition, and derive conditions under which a block $2^t$-ary PSK modulation code is linear. Once the linear structure is developed, we study phase symmetry of a block $2^t$-ary PSK modulation code. In particular, we derive a necessary and sufficient condition for a block $2^t$-ary PSK modulation code, which is linear as a binary code, to be invariant under $180^\circ/2^{t-h}$ phase rotation, for $1 \leq h \leq t$. Finally, a list of short 8-PSK and 16-PSK modulation codes is given together with their linear structure and the smallest phase rotation for which a code is invariant.
1. Introduction

As the application of coded modulation in bandwidth-efficient communications grows, there is a need of better understanding of the structural properties of modulation codes, especially those properties which are useful in: error performance analysis, implementation of optimum (or suboptimum) decoders, efficient resolution of carrier-phase ambiguity, and construction of better codes. In this paper, we investigate two important structural properties of block $2^t$-ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN channel, the error performance of a modulation code depends on its squared Euclidean distance distribution [1-4]. Linear structure of a code makes the error performance analysis much easier [2, 4]. Furthermore, it may lead to a simpler implementation of encoder and decoder. Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization [1, 5-8]. It is desirable for a modulation code to have as many phase symmetries as possible.

Suppose the integer group \{0, 1, 2, \ldots, 2^t - 1\} under the modulo-$2^t$ addition, denoted $S_{2^t}\text{PSK}$, is chosen to represent a two-dimensional $2^t$-PSK signal set. Then a block $2^t$-ary PSK modulation code $C$ of length $n$ may be regarded as a block code of length $n$ over the integer group $S_{2^t}\text{PSK}$, and a codeword in $C$ is simply an $n$-tuple over $S_{2^t}\text{PSK}$. If each integer in $S_{2^t}\text{PSK}$ is represented by its binary expression of $t$ bits, then a block code of length $n$ over $S_{2^t}\text{PSK}$ can be considered as a binary block code of length $tn$. The resultant binary code is linear if it is closed under the component-wise modulo-$2$ addition. Most of the known block $2^t$-ary PSK modulation codes are linear as binary codes. A linear code in this sense is not necessarily closed under the component-wise modulo-$2^t$ addition. For two integers $s$ and $s'$ in $S_{2^t}\text{PSK}$, the squared Euclidean distance between two signal points represented by $s$ and $s'$ respectively depends only on $s - s'$ (modulo $2^t$), but is not always determined by the Hamming distance between the binary expressions of $s$ and $s'$. For an additive white Gaussian noise (AWGN) channel, error performance of a modulation code is determined by its squared
Euclidean distance distribution. If a code $C$ over $S_{2^t,PSK}$ is either closed under the component-wise modulo-$2^t$ addition or a union of relatively small number of cosets of a subcode which is closed under the component-wise modulo-$2^t$ addition, then the error performance analysis of $C$ is much easier than a code without such a property [2, 4]. In this paper, we present a condition for a code over $S_{2^t,PSK}$, which is linear as a binary code, to be closed under the component-wise modulo-$2^t$ addition. In particular, we present a necessary and sufficient condition for a basic multilevel block code over $S_{2^t,PSK}$, which is linear as a binary code, to be closed under the component-wise modulo-$2^t$ addition.

An important issue in coded modulation is the resolution of carrier-phase ambiguity. Several methods have been proposed to resolve the carrier-phase ambiguity for coded PSK modulations [6, 8, 9]. In these methods, the phase-rotation invariant property of a code over $S_{2^t,PSK}$ plays the central role. Tanner [8] has proposed a simple phase ambiguity resolution method for $2^t$-ary PSK modulation codes which are invariant under $360^\circ/2^t$ phase shift. In this paper, we present a necessary and sufficient condition for a code over $S_{2^t,PSK}$, which is linear as a binary code, to be invariant under $180^\circ/2^t$ phase shift with $1 \leq h \leq t$.

Finally, we give a list of short block 8-PSK and 16-PSK modulation codes together with their closure (or linear) properties under the component-wise modulo-$2^t$ addition, the smallest phase shifts for which these codes are invariant, and other parameters.

2. Linear Block $2^t$-PSK Modulation Codes

Let $t$ be a positive integer. Suppose the integer group $\{0, 1, 2, \ldots, 2^t - 1\}$ under the modulo-$2^t$ addition, denoted $S_{2^t,PSK}$, is used to represent a two-dimensional $2^t$-PSK signal set. We define the distance between two integers $s$ and $s'$ in $S_{2^t,PSK}$, denoted $d(s, s')$, as the squared Euclidean distance between the two $2^t$-PSK signal points represented by $s$ and $s'$ respectively. Then $d(s, s')$ is given below:

$$d(s, s') = 4 \sin^2 \left(2^{-t} \pi (s - s') \right). \tag{2.1}$$

Let $d_i$ denote $d(2^{t-1}, 0)$. From (2.1), we see that

$$d_i = 4 \sin^2 (2^{t-1-1} \pi).$$
For a positive integer \( n \), let \( S_{2^t-PSK}^n \) denote the set of all \( n \)-tuples over \( S_{2^t-PSK} \). Define the distance between two \( n \)-tuples \( \bar{v} = (v_1, v_2, \ldots, v_n) \) and \( \bar{v}' = (v'_1, v'_2, \ldots, v'_n) \) over \( S_{2^t-PSK} \), denoted \( d(\bar{v}, \bar{v}') \), as follows:

\[
d(\bar{v}, \bar{v}') = \sum_{j=1}^{n} d(v_j, v'_j)
\]

Then it follows from (2.1) and (2.2) that

\[
d(\bar{v}, \bar{v}') = d(\bar{v} - \bar{v}', \vec{0})
\]

where \(-\) denotes the component-wise modulo-2\(^t\) subtraction and \( \vec{0} \) denotes the all-zero \( n \)-tuple over \( S_{2^t-PSK} \). For an \( n \)-tuple \( \bar{v} \) over \( S_{2^t-PSK} \), define \( |\bar{v}|_d \) as follows:

\[
|\bar{v}|_d \triangleq d(\bar{v}, \vec{0}).
\]

We may regard that \( |\bar{v}|_d \) is the squared Euclidean weight of \( \bar{v} \).

Consider a block code \( C \) of length \( n \) over \( S_{2^t-PSK} \). The minimum distance of \( C \), denoted \( D[C] \), with respect to the distance measure \( d(\cdot, \cdot) \) given by (2.2) is defined as follows:

\[
D[C] \triangleq \min \{ d(\bar{v}, \bar{v}') : \bar{v}, \bar{v}' \in C \text{ and } \bar{v} \neq \bar{v}' \}.
\]

If each component of a codeword \( \bar{v} \) in \( C \) is mapped into the corresponding signal point in the two-dimensional \( 2^t \)-PSK signal set, we obtain a block \( 2^t \)-PSK modulation code with minimum squared Euclidean distance \( D[C] \). The effective rate of this code is given by

\[
R[C] = \frac{1}{2n} \log_2 |C|,
\]

which is simply the average number of information bits transmitted per dimension.

Let \( \bar{u} = (u_1, u_2, \ldots, u_n) \) and \( \bar{v} = (v_1, v_2, \ldots, v_n) \) be two \( n \)-tuples over \( S_{2^t-PSK} \). Let \( \bar{u} + \bar{v} \) denote the following \( n \)-tuple over \( S_{2^t-PSK} \):

\[
\bar{u} + \bar{v} \triangleq (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n),
\]

where \( u_i + v_i \) is carried out in modulo-2\(^t\) addition. A code over the integer group \( S_{2^t-PSK} \) is said to be linear with respect to (w.r.t.) "+", if \( C \) is closed under the component-wise modulo-2\(^t\) addition, i.e., for any \( \bar{u} \) and \( \bar{v} \) in \( C \), \( \bar{u} + \bar{v} \) is also in \( C \). It follows from (2.3) to (2.5) that, for a linear code \( C \) w.r.t. +, we have

\[
D[C] = \min \{ |\bar{v}|_d : \bar{v} \in C - \{ \vec{0} \} \}.
\]
As a result, for a linear code \( C \) over \( S_{2^t,PSK} \) w.r.t. +, the error performance analysis of \( C \) based on the distance measure \( d(\cdot, \cdot) \) is reduced to that of \( C \) in terms of the weight measure \( | \cdot |_d \). This simplifies the error performance analysis and computation of code \( C \) [2, 4].

Let \( (b_1, b_2, \ldots, b_t) \) be the binary representation of an integer \( s \) in \( S_{2^t,PSK} \), where \( b_1 \) and \( b_t \) be the least and most significant bits respectively. Then \( s = \sum_{i=1}^{t} b_i 2^{i-1} \). Let \( \bar{v} = (v_1, v_2, \ldots, v_n) \) be an \( n \)-tuple over \( S_{2^t,PSK} \) with \( v_j = \sum_{i=1}^{t} v_{ij} 2^{i-1} \) and \( v_{ij} \in \{0, 1\} \) for \( 1 \leq i \leq \ell \) and \( 1 \leq j \leq n \). Then \( \bar{v} \) can be expressed as the following sum:

\[
\bar{v} = \bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{t-1}\bar{v}^{(t)},
\]

(2.8)

where \( \bar{v}^{(i)} = (v_{i1}, v_{i2}, \ldots, v_{in}) \) is a binary \( n \)-tuple, for \( 1 \leq i \leq \ell \). We call \( \bar{v}^{(i)} \) the \( i \)-th binary component \( n \)-tuple of \( \bar{v} \). The sum of (2.8) may be regarded as the binary expansion of the \( n \)-tuple \( \bar{v} \). For \( 1 \leq i \leq \ell \), let \( C_i \) be a binary \((n, k_i)\) code with minimum Hamming distance \( \delta_i \).

Define the following block code \( C \) over \( S_{2^t,PSK} \),

\[
C = C_1 + 2C_2 + \cdots + 2^{t-1}C_t
\]

\[
\Delta \{ \bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{t-1}\bar{v}^{(t)} : \bar{v}^{(i)} \in C_i \text{ for } 1 \leq i \leq \ell \}.
\]

(2.9)

The code \( C \) defined by (2.9) is called a basic multi-level code. Basic multilevel codes were first introduced by Imai and Hirakawa [10] and then studied by other [3, 11, 12]. For \( 1 \leq i \leq \ell \), \( C_i \) is called the \( i \)-th binary component code of \( C \). The minimum distance of \( C \) is

\[
D[C] = \min_{1 \leq i \leq t} \delta_i d_i.
\]

(2.10)

where \( d_i = d(2^{i-1}, 0) \). If every component of a codeword in \( C \) is mapped into a signal point in a two-dimensional \( 2^{t}\)-PSK signal constellation, then \( C \) is a basic multi-level \( 2^{t}\)-PSK modulation code with a minimum squared Euclidean distance,

\[
D[C] = \min_{1 \leq i \leq t} \{ 4\delta_i, \sin^2(2^{i-1}\pi) \}.
\]

For \( n \)-tuples \( \bar{u} \) and \( \bar{v} \) over \( S_{2^t,PSK} \), let \( \bar{u} \oplus \bar{v} \) denote the \( n \)-tuple over \( S_{2^t,PSK} \), such that the \( i \)-th binary component \( n \)-tuple of \( \bar{u} \oplus \bar{v} \) is the modulo-2 vector sum of the \( i \)-th binary component \( n \)-tuple of \( \bar{u} \) and the \( i \)-th binary component \( n \)-tuple of \( \bar{v} \). A code \( C \) over \( S_{2^t,PSK} \) is said to be linear w.r.t. \( \oplus \), if \( C \) is closed under addition \( \oplus \). Most of the known block codes for
$2^t$-PSK modulation are linear w.r.t. $\oplus$. A linear code w.r.t. $\oplus$ is not necessarily linear w.r.t. $\oplus$. In the following, we will derive a condition for a linear code w.r.t. $\oplus$ to be linear w.r.t. $\oplus$.

Let $\bar{u}$ and $\bar{v}$ be two $n$-tuples over $S_{2^t.\text{PSK}}$, and let $\bar{w}$ denote $\bar{u} + \bar{v}$. For $1 \leq i \leq \ell$, let the $i$-th binary component $n$-tuples of $\bar{u}$, $\bar{v}$ and $\bar{w}$ be represented as $\bar{u}^{(i)} = (u_{1i}, u_{2i}, \ldots, u_{ni})$, $\bar{v}^{(i)} = (v_{1i}, v_{2i}, \ldots, v_{ni})$, and $\bar{w}^{(i)} = (w_{1i}, w_{2i}, \ldots, w_{ni})$, respectively. Then the following recursive equations hold [13]:

\begin{align*}
    w_{ji} &= u_{ji} \oplus v_{ji} \oplus x_{ji}, \quad \text{for } 1 \leq i \leq \ell, \quad (2.11) \\
    x_{ji} &= u_{j-1} v_{j-1} \oplus (u_{j-1} \oplus v_{j-1}) x_{j-1}, \quad \text{for } 1 < i \leq \ell, \quad (2.12) \\
    x_{j1} &= 0. \quad (2.13)
\end{align*}

For $1 \leq i \leq \ell$, let $c^{(i)}(\bar{u}, \bar{v})$ be defined as

\begin{equation}
    c^{(i)}(\bar{u}, \bar{v}) \triangleq (x_{1i}, x_{2i}, \ldots, x_{ni}). \quad (2.14)
\end{equation}

For two binary $n$-tuples, $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$, let $\bar{a} \cdot \bar{b}$ be defined as

\begin{equation}
    \bar{a} \cdot \bar{b} \triangleq (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n),
\end{equation}

where $a_j \cdot b_j$ denotes the logical product of $a_j$ and $b_j$.

It follows from (2.11) to (2.14) that for $1 \leq i < \ell$,

\begin{equation}
    c^{(i+1)}(\bar{u}, \bar{v}) = \bar{u}^{(i)} \cdot \bar{v}^{(i)} \oplus (\bar{u}^{(i)} \oplus \bar{v}^{(i)}) \cdot c^{(i)}(\bar{u}, \bar{v}). \quad (2.15)
\end{equation}

Let $c(\bar{u}, \bar{v})$ be defined as

\begin{equation}
    c(\bar{u}, \bar{v}) \triangleq c^{(1)}(\bar{u}, \bar{v}) + 2c^{(2)}(\bar{u}, \bar{v}) + \ldots + 2^{\ell-1}c^{(\ell)}(\bar{u}, \bar{v}). \quad (2.16)
\end{equation}

Then,

\begin{equation}
    \bar{u} + \bar{v} = \bar{u} \oplus \bar{v} \oplus c(\bar{u}, \bar{v}). \quad (2.17)
\end{equation}

Now consider a block code $C$ over $S_{2^t.\text{PSK}}$ which is linear w.r.t. $\oplus$. Let $\bar{u}$ and $\bar{v}$ be two codewords in $C$. Then it follows from (2.17) that $\bar{u} + \bar{v} \in C$ if and only if

\begin{equation}
    c(\bar{u}, \bar{v}) \in C. \quad (2.18)
\end{equation}

For $1 \leq i \leq \ell$, let $C^{(i)}$ and $C_i$ be defined as

\begin{align*}
    C^{(i)} &\triangleq \{ \bar{v}^{(i)} : \bar{v}^{(1)} + \ldots + 2^{i-1}\bar{v}^{(i)} + \ldots + 2^{\ell-1}\bar{v}^{(\ell)} \in C \}, \quad (2.19) \\
    C_i &\triangleq \{ \bar{v}^{(i)} : 2^{i-1}\bar{v}^{(i)} \in C \}. \quad (2.20)
\end{align*}
By definition

\[ C_i \subseteq C^{(i)}. \]  \hspace{1cm} (2.21)

Since \( C \) is linear w.r.t. \( \oplus \), \( C^{(i)} \) and \( C_i \) are also linear w.r.t. \( \oplus \) and

\[ C_1 + 2C_2 + \cdots + 2^t-1C_t \subseteq C_i, \]  \hspace{1cm} (2.22)

where the equality holds if \( C \) is a basic multilevel code. For binary codes \( C \) and \( C' \) of the same length, let \( C \cdot C' \) be defined as

\[ C \cdot C' \overset{\Delta}{=} \{ \bar{u} \cdot \bar{v} : \bar{u} \in C \text{ and } \bar{v} \in C' \}. \]

Now we present two lemmas regarding to the closure property of a \( 2^t \)-PSK code.

**Lemma 1:** Suppose that \( C \) is a linear code over \( S_{2^t,PSK} \) w.r.t. \( \oplus \) and for \( 1 \leq i \leq \ell \),

\[ C^{(i)} \cdot C^{(i)} \subseteq C_{i+1}. \]  \hspace{1cm} (2.23)

Then \( C \) is closed under the component-wise modulo-\( 2^t \) addition, and hence is linear w.r.t. \( + \).

**Proof:** By induction, we show that for \( 1 \leq i \leq \ell \)

\[ c^{(i)}(\bar{u}, \bar{v}) \in C_i. \]  \hspace{1cm} (2.24)

Since \( c^{(i)}(\bar{u}, \bar{v}) = 0, c^{(i)}(\bar{u}, \bar{v}) \in C_1 \). Suppose that \( c^{(j)}(\bar{u}, \bar{v}) \in C_j \) for \( 1 \leq j \leq i < \ell \). Since \( C^{(i)} \) and \( C_{i+1} \) are linear w.r.t. \( \oplus \), it follows from (2.15), (2.21) and (2.23) that \( c^{(i+1)}(\bar{u}, \bar{v}) \in C_{i+1} \). Consequently (2.18) follows from (2.16), (2.22) and (2.24), and this lemma holds.

**Lemma 2:** Suppose that \( C \) is a linear basic multilevel code over \( S_{2^t,PSK} \) w.r.t. \( \oplus \). Then \( C(= C_1 + 2C_2 + \cdots + 2^t-1C_t) \) is closed under the component-wise modulo-\( 2^t \) addition, if and only if

\[ C_i \cdot C_i \subseteq C_{i+1}, \quad \text{for } 1 \leq i < \ell. \]  \hspace{1cm} (2.25)

**Proof:** Only if part: Let \( \bar{u} \) (or \( \bar{v} \)) denote the \( n \)-tuple over \( S_{2^t,PSK} \) whose \( i \)-th binary component \( n \)-tuple is \( \bar{u}^{(i)} \in C_i \) (or \( \bar{v}^{(i)} \in C_i \)) and whose other binary component \( n \)-tuples are the all-zero \( n \)-tuple \( \bar{0} \). Assume that \( \bar{u} + \bar{v} \in C \). It follows from (2.11) to (2.13) that for these specific \( \bar{u} \) and \( \bar{v} \),

\[ z_{j,i+1} = u_j v_j, \quad \text{for } 1 \leq i \leq \ell. \]  \hspace{1cm} (2.26)
From (2.14), (2.18) and (2.26), we see that
\[ c^{(i+1)}(\tilde{u}, \varphi) = \tilde{u}^{(i)} \cdot \varphi^{(i)} \in C_{i+1}. \]
That is, \( C_i \cdot C_i \subseteq C_{i+1} \).

If part: Since \( C \) is a basic multilevel code, \( C_i = C^{(i)} \) for \( 1 \leq i \leq \ell \). Then if part follows from Lemma 1.

3. A Necessary and Sufficient Condition for a \( 2^\ell \)-PSK Modulation Code to be Invariant Under \( 180^\circ / 2^{\ell-h} \) Phase Shift with \( 1 \leq h \leq \ell \)

Now we consider the phase symmetry of a block \( 2^\ell \)-ary PSK modulation code. To determine the phase symmetry of a code, we need to know the smallest rotation under which the code is invariant.

For \( 1 \leq h \leq \ell \), let \( 2^{h-1} \bar{1} \) denote the \( n \)-tuple over \( S_{2^\ell, PSK} \) whose \( h \)-th binary component \( n \)-tuple is the all-one \( n \)-tuple and whose other binary component \( n \)-tuples are the all-zero \( n \)-tuple. A code \( C \) of length \( n \) over \( S_{2^\ell, PSK} \) is said to be invariant under \( 180^\circ / 2^{\ell-h} \) phase shift if for any codeword \( \bar{v} \) in \( C \),
\[ \bar{v} + 2^{h-1} \bar{1} \in C. \tag{3.1} \]

By letting \( \tilde{u} = 2^{h-1} \bar{1} \) in (2.11) to (2.16), we obtain the following equations:

1. \( w_{j_1} = v_{j_1} \oplus x_{j_1}, \quad \text{for} \quad 1 \leq i \leq \ell. \tag{3.2} \)

2. If \( h < \ell \), then \( x_{j_i} = v_{j_{i-1}} x_{j_{i-1}} \), for \( h < i \leq \ell. \tag{3.3} \)

3. \( x_{j_h} = 1. \tag{3.4} \)

4. If \( 1 < h \), then \( x_{j_i} = 0, \quad \text{for} \quad 1 \leq i < h. \tag{3.5} \)

It follows from (3.2) to (3.5) that we have Lemma 3.
Lemma 3: For \(1 \leq h \leq \ell\), a linear code \(C\) over \(S_{2^\ell, PSK}\) w.r.t. \(\oplus\) is invariant under \(180^\circ/2^{\ell-h}\) phase shift if and only if for any codeword \(\tilde{\mathbf{v}}^{(1)} + 2\tilde{\mathbf{v}}^{(2)} + \cdots + 2^{\ell-1}\tilde{\mathbf{v}}^{(\ell)}\) in \(C\),

\[
2^{h-1}\mathbf{1} + 2^h\tilde{\mathbf{v}}^{(h)} + 2^{h+1}(\tilde{\mathbf{v}}^{(h)} \cdot \tilde{\mathbf{v}}^{(h+1)}) + \cdots + 2^{\ell-1}(\tilde{\mathbf{v}}^{(h)} \cdot \tilde{\mathbf{v}}^{(h+1)} \cdot \cdots \tilde{\mathbf{v}}^{(\ell-1)}) \in C,
\]

where \(\mathbf{1}\) denotes the all-one \(n\)-tuple.

\[\Delta\Delta\]

If \(C\) is a linear basic \(\ell\)-level code w.r.t. \(\oplus\), denoted \(C_1 + 2C_2 + \cdots + 2^{\ell-1}C_{\ell}\), then the necessary and sufficient condition (3.6) is expressed as follows:

(1) \(\tilde{\mathbf{1}} \in C_h\), and

(2) if \(h < \ell\), then \(C_h \cdot C_{h+1} \cdot \cdots \cdot C_{j-1} \subseteq C_j\), for \(h + 1 < j \leq \ell\).

Obviously, a linear code \(C\) over \(S_{2^\ell, PSK}\) w.r.t. \(\oplus\) is invariant under \(180^\circ/2^{\ell-h}\) phase shift, if and only if \(\tilde{\mathbf{1}}_h \in C\).

4. Code Examples

In Table 1, seven basic multilevel block codes [3] and four nonbasic block codes for 8-PSK and 16-PSK modulations are given. The number of states of a trellis diagram for each basic multilevel block code is computed based on the numbers of states of trellis diagrams for its binary component codes [14]. Among four nonbasic codes, two zero-tail Ungerboeck trellis codes for 8-PSK modulation [1] are shown. In Table 1, \(V_n\), \(P_n\), \(P_n^\perp\), \(RM_{2^h}\), \(s\cdot RM_{2^h}\), and ex-Golay denote the set of all the binary \(n\)-tuples, the set of all even weight binary \(n\)-tuples, the dual code of \(P_n\) which consists of the all-zero and all-one \(n\)-tuples, the \(j\)-th order Reed-Muller code of length \(2^h\), a shortened \(j\)-th order Reed-Muller code of original length \(2^\ell\), and the extended (24,12) code of binary Golay code. \(F_1\) and \(F_2\) denote two codes over \(\{0, 1, 2, 3\}\) which are defined as following [4]. Let \(p(x_1, x_2, \cdots, x_h)\) be a boolean polynomial which is used to represent the binary \(2^h\)-tuple whose \(i\)-th bit is given by \(p(i_1, i_2, \cdots, i_h)\) where \((i_1, i_2, \cdots, i_h)\) is the binary representation of the integer \(i - 1\), i.e. \(i - 1 = \sum_{j=1}^{h} i_j 2^{j-1}\). Let \(g_{h,i}\) denote the
Next we consider the phase rotation invariant property of codes given in Table 1. Since codes $C[1], C[4], C[5], C[6]$ and $C[11]$ are linear w.r.t. $+$ and $\bar{1}$ is contained in $P_n^\perp$, $R_{M+},$ or ex-Golay, these codes are invariant under $180^\circ/2^{t-1}$ phase shift. It follows from the properties (i) and (ii) of Reed-Muller codes that codes $C[8], C[9]$ with $n \equiv 0 \bmod 4$ and $C[10]$ are readily shown to meet the conditions given by (3.7) and (3.8) with $h = 1$. Code $C[2]$ is shown to contain $2\bar{1}$, and therefore is invariant under $90^\circ$ phase shift. Code $C[3]$ contains $2^2\bar{1}$ only and is invariant only under $180^\circ$ phase shift, and code $C[7]$ does not contain even $2^2\bar{1}$. 
References


Table 1: Some Short 8-PSK, 16-PSK Codes

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<tr>
<td>8-PSK</td>
<td>$C[1] \triangleq P_8^1 + 2P_8 + 4V_8$</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>$2^2$</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[2] \triangleq F_1 + 4V_8$</td>
<td>8</td>
<td>1</td>
<td>4</td>
<td>$2^2$</td>
<td>Yes</td>
<td>90°</td>
</tr>
<tr>
<td></td>
<td>$C[3] \triangleq$ zero-tail Ungerboeck code</td>
<td>$n$</td>
<td>$\frac{n-1}{n}$</td>
<td>4</td>
<td>$2^2$</td>
<td>No</td>
<td>180°</td>
</tr>
<tr>
<td></td>
<td>$C[4] \triangleq RM_{4,1} + 2P_{16} + 4V_{16}$</td>
<td>16</td>
<td>$\frac{9}{8}$</td>
<td>4</td>
<td>$2^4$</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[5] \triangleq F_2 + 4V_{16}$</td>
<td>16</td>
<td>$\frac{9}{8}$</td>
<td>4</td>
<td>$2^4$</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[6] \triangleq$ ex-Golay+2$P_{24} + 4V_{24}$</td>
<td>24</td>
<td>$\frac{59}{48}$</td>
<td>4</td>
<td>$2^7$</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[7] \triangleq$ zero-tail Ungerboeck code</td>
<td>$n$</td>
<td>$\frac{2n-3}{2n}$</td>
<td>4</td>
<td>$2^3$</td>
<td>No</td>
<td>360°</td>
</tr>
<tr>
<td></td>
<td>$C[8] \triangleq P_{19}^1 + 2RM_{4,2} + 4P_{16}$</td>
<td>16</td>
<td>$\frac{27}{32}$</td>
<td>8</td>
<td>$2^6$</td>
<td>No</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[9] \triangleq P_n^1 + 2s-RM_{5,3} + 4P_n$</td>
<td>$16 &lt; n \leq 32$</td>
<td>$\frac{n-3}{n}$</td>
<td>8</td>
<td>$2^6$</td>
<td>No</td>
<td>$45^\circ$ for $n \equiv 0 \pmod{4}$</td>
</tr>
<tr>
<td></td>
<td>$C[10] \triangleq RM_{5,1} + 2RM_{5,3} + 4P_{32}$</td>
<td>32</td>
<td>$\frac{63}{64}$</td>
<td>8</td>
<td>$2^6$</td>
<td>No</td>
<td>45°</td>
</tr>
<tr>
<td>16-PSK</td>
<td>$C[11] \triangleq P_{32}^1 + 2RM_{5,2} + 4P_{32} + 8V_{32}$</td>
<td>32</td>
<td>$\frac{3}{4}$</td>
<td>4</td>
<td>$2^8$</td>
<td>Yes</td>
<td>22.5°</td>
</tr>
</tbody>
</table>