ON LINEAR STRUCTURE AND
PHASE ROTATION INVARIANT PROPERTIES OF
BLOCK $2^k$-PSK MODULATION CODES

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ABSTRACT

In this correspondence, we investigate two important structural properties of block 2^t-
ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN
channel, the error performance of a modulation code depends on its squared Euclidean distance
distribution. Linear structure of a code makes the error performance analysis much easier.
Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring
rapid carrier-phase resynchronization after temporary loss of synchronization. It is desirable
for a modulation code to have as many phase symmetries as possible. In this paper, we
first represent a 2^t-ary modulation code as a code with symbols from the integer group,
S_{2^t,PSK} = \{0, 1, 2, ..., 2^t - 1\}, under the modulo-2^t addition. Then we define the linear
structure of block 2^t-ary PSK modulation codes over S_{2^t,PSK} with respect to the modulo-2^t
vector addition, and derive conditions under which a block 2^t-ary PSK modulation code is
linear. Once the linear structure is developed, we study phase symmetry of a block 2^t-ary
PSK modulation code. In particular, we derive a necessary and sufficient condition for a block
2^t-ary PSK modulation code, which is linear as a binary code, to be invariant under 180°/2^t-h
phase rotation, for 1 \leq h \leq \ell. Finally, a list of short 8-PSK and 16-PSK modulation codes is
given together with their linear structure and the smallest phase rotation for which a code is
invariant.
ON LINEAR STRUCTURE AND PHASE ROTATION INVARIANT PROPERTIES OF BLOCK $2^t$-PSK MODULATION CODES

1. Introduction
As the application of coded modulation in bandwidth-efficient communications grows, there is a need of better understanding of the structural properties of modulation codes, especially those properties which are useful in: error performance analysis, implementation of optimum (or suboptimum) decoders, efficient resolution of carrier-phase ambiguity, and construction of better codes. In this paper, we investigate two important structural properties of block $2^t$-ary PSK modulation codes, namely: linear structure and phase symmetry. For an AWGN channel, the error performance of a modulation code depends on its squared Euclidean distance distribution [1-4]. Linear structure of a code makes the error performance analysis much easier [2, 4]. Furthermore, it may lead to a simpler implementation of encoder and decoder. Phase symmetry of a code is important in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after temporary loss of synchronization [1, 5-8]. It is desirable for a modulation code to have as many phase symmetries as possible.

Suppose the integer group $\{0, 1, 2, \ldots, 2^t - 1\}$ under the modulo-$2^t$ addition, denoted $S_{2^t, \text{PSK}}$, is chosen to represent a two-dimensional $2^t$-PSK signal set. Then a block $2^t$-ary PSK modulation code $C$ of length $n$ may be regarded as a block code of length $n$ over the integer group $S_{2^t, \text{PSK}}$, and a codeword in $C$ is simply an $n$-tuple over $S_{2^t, \text{PSK}}$. If each integer in $S_{2^t, \text{PSK}}$ is represented by its binary expression of $\ell$ bits, then a block code of length $n$ over $S_{2^t, \text{PSK}}$ can be considered as a binary block code of length $\ell n$. The resultant binary code is linear if it is closed under the component-wise modulo-$2^t$ addition. Most of the known block $2^t$-ary PSK modulation codes are linear as binary codes. A linear code in this sense is not necessarily closed under the component-wise modulo-$2^t$ addition. For two integers $s$ and $s'$ in $S_{2^t, \text{PSK}}$, the squared Euclidean distance between two signal points represented by $s$ and $s'$ respectively depends only on $s - s'$ (modulo $2^t$), but is not always determined by the Hamming distance between the binary expressions of $s$ and $s'$. For an additive white Gaussian noise (AWGN) channel, error performance of a modulation code is determined by its squared
Euclidean distance distribution. If a code $C$ over $S_{2^t,PSK}$ is either closed under the component-wise modulo-$2^t$ addition or a union of relatively small number of cosets of a subcode which is closed under the component-wise modulo-$2^t$ addition, then the error performance analysis of $C$ is much easier than a code without such a property [2, 4]. In this paper, we present a condition for a code over $S_{2^t,PSK}$, which is linear as a binary code, to be closed under the component-wise modulo-$2^t$ addition. In particular, we present a necessary and sufficient condition for a basic multilevel block code over $S_{2^t,PSK}$, which is linear as a binary code, to be closed under the component-wise modulo-$2^t$ addition.

An important issue in coded modulation is the resolution of carrier-phase ambiguity. Several methods have been proposed to resolve the carrier-phase ambiguity for coded PSK modulations [6, 8, 9]. In these methods, the phase-rotation invariant property of a code over $S_{2^t,PSK}$ plays the central role. Tanner [8] has proposed a simple phase ambiguity resolution method for $2^t$-ary PSK modulation codes which are invariant under $360°/2^t$ phase shift. In this paper, we present a necessary and sufficient condition for a code over $S_{2^t,PSK}$, which is linear as a binary code, to be invariant under $180°/2^{t-h}$ phase shift with $1 \leq h \leq \ell$.

Finally, we give a list of short block 8-PSK and 16-PSK modulation codes together with their closure (or linear) properties under the component-wise modulo-$2^t$ addition, the smallest phase shifts for which these codes are invariant, and other parameters.

2. Linear Block $2^t$-PSK Modulation Codes

Let $\ell$ be a positive integer. Suppose the integer group $\{0, 1, 2, \ldots, 2^\ell - 1\}$ under the modulo-$2^\ell$ addition, denoted $S_{2^\ell,PSK}$, is used to represent a two-dimensional $2^\ell$-PSK signal set. We define the distance between two integers $s$ and $s'$ in $S_{2^\ell,PSK}$, denoted $d(s, s')$, as the squared Euclidean distance between the two $2^\ell$-PSK signal points represented by $s$ and $s'$ respectively. Then $d(s, s')$ is given below:

$$d(s, s') = 4 \sin^2 \left( 2^{-\ell} \pi (s - s') \right). \quad (2.1)$$

Let $d_i$ denote $d(2^{\ell-1}, 0)$. From (2.1), we see that

$$d_i = 4 \sin^2(2^{\ell-1-1} \pi).$$
For a positive integer $n$, let $S_{2^t-\text{PSK}}^n$ denote the set of all $n$-tuples over $S_{2^t-\text{PSK}}$. Define the distance between two $n$-tuples $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{v}' = (v'_1, v'_2, \ldots, v'_n)$ over $S_{2^t-\text{PSK}}$, denoted $d(\mathbf{v}, \mathbf{v}')$, as follows:

$$d(\mathbf{v}, \mathbf{v}') \triangleq \sum_{j=1}^{n} d(v_j, v'_j) \quad (2.2)$$

Then it follows from (2.1) and (2.2) that

$$d(\mathbf{v}, \mathbf{v}') = d(\mathbf{v} - \mathbf{v}', \mathbf{0}) \quad (2.3)$$

where "\(-\)" denotes the component-wise modulo-$2^t$ subtraction and $\mathbf{0}$ denotes the all-zero $n$-tuple over $S_{2^t-\text{PSK}}$. For an $n$-tuple $\mathbf{v}$ over $S_{2^t-\text{PSK}}$, define $|\mathbf{v}|_d$ as follows:

$$|\mathbf{v}|_d \triangleq d(\mathbf{v}, \mathbf{0}) \quad (2.4)$$

We may regard that $|\mathbf{v}|_d$ is the squared Euclidean weight of $\mathbf{v}$.

Consider a block code $C$ of length $n$ over $S_{2^t-\text{PSK}}$. The minimum distance of $C$, denoted $D[C]$, with respect to the distance measure $d(\cdot, \cdot)$ given by (2.2) is defined as follows:

$$D[C] \triangleq \min \{d(\mathbf{v}, \mathbf{v}') : \mathbf{v}, \mathbf{v}' \in C \text{ and } \mathbf{v} \neq \mathbf{v}'\} \quad (2.5)$$

If each component of a codeword $\mathbf{v}$ in $C$ is mapped into the corresponding signal point in the two-dimensional $2^t$-PSK signal set, we obtain a block $2^t$-PSK modulation code with minimum squared Euclidean distance $D[C]$. The effective rate of this code is given by

$$R[C] = \frac{1}{2n} \log_2 |C|, \quad (2.6)$$

which is simply the average number of information bits transmitted per dimension.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two $n$-tuples over $S_{2^t-\text{PSK}}$. Let $\mathbf{u} + \mathbf{v}$ denote the following $n$-tuple over $S_{2^t-\text{PSK}}$:

$$\mathbf{u} + \mathbf{v} \triangleq (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n),$$

where $u_i + v_i$ is carried out in modulo-$2^t$ addition. A code over the integer group $S_{2^t-\text{PSK}}$ is said to be linear with respect to (w.r.t.) "+", if $C$ is closed under the component-wise modulo-$2^t$ addition, i.e., for any $\mathbf{u}$ and $\mathbf{v}$ in $C$, $\mathbf{u} + \mathbf{v}$ is also in $C$. It follows from (2.3) to (2.5) that, for a linear code $C$ w.r.t. +, we have

$$D[C] = \min \{|\mathbf{v}|_d : \mathbf{v} \in C - \{\mathbf{0}\}\} \quad (2.7)$$

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As a result, for a linear code $C$ over $S_{2t,PSK}$ w.r.t. $+$, the error performance analysis of $C$ based on the distance measure $d(\cdot, \cdot)$ is reduced to that of $C$ in terms of the weight measure $| \cdot |_d$. This simplifies the error performance analysis and computation of code $C$ [2, 4].

Let $(b_1, b_2, \ldots, b_t)$ be the binary representation of an integer $s$ in $S_{2t,PSK}$, where $b_1$ and $b_t$ be the least and most significant bits respectively. Then $s = \sum_{i=1}^{t} b_i 2^{t-1}$. Let $\bar{v} = (v_1, v_2, \ldots, v_n)$ be an $n$-tuple over $S_{2t,PSK}$ with $v_i = \sum_{i=1}^{t} v_{ij} 2^{t-1}$ and $v_{ij} \in \{0, 1\}$ for $1 \leq i \leq \ell$ and $1 \leq i \leq n$. Then $\bar{v}$ can be expressed as the following sum:

$$\bar{v} = \bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{t-1}\bar{v}^{(t)}, \quad (2.8)$$

where $\bar{v}^{(i)} = (v_{i1}, v_{i2}, \ldots, v_{in})$ is a binary $n$-tuple, for $1 \leq i \leq \ell$. We call $\bar{v}^{(i)}$ the $i$-th binary component $n$-tuple of $\bar{v}$. The sum of (2.8) may be regarded as the binary expansion of the $n$-tuple $\bar{v}$. For $1 \leq i \leq \ell$, let $C_i$ be a binary $(n, k_i)$ code with minimum Hamming distance $\delta_i$. Define the following block code $C$ over $S_{2t,PSK}$,

$$C \triangleq C_1 + 2C_2 + \cdots + 2^{t-1}C_t$$

$$\triangleq \{ \bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{t-1}\bar{v}^{(t)} : \bar{v}^{(i)} \in C_i \text{ for } 1 \leq i \leq t \}. \quad (2.9)$$

The code $C$ defined by (2.9) is called a basic multi-level code. Basic multilevel codes were first introduced by Imai and Hirakawa [10] and then studied by other [3, 11, 12]. For $1 \leq i \leq \ell$, $C_i$ is called the $i$-th binary component code of $C$. The minimum distance of $C$ is

$$D[C] = \min_{1 \leq i \leq t} \delta_i d_i, \quad (2.10)$$

where $d_i = d(2^{t-1}, 0)$. If every component of a codeword in $C$ is mapped into a signal point in a two-dimensional $2t$-PSK signal constellation, then $C$ is a basic multi-level $2t$-PSK modulation code with a minimum squared Euclidean distance,

$$D[C] = \min_{1 \leq i \leq t} \{ 4\delta_i \sin^2(2^{t-1}\pi) \}. $$

For $n$-tuples $\bar{u}$ and $\bar{v}$ over $S_{2t,PSK}$, let $\bar{u} \oplus \bar{v}$ denote the $n$-tuple over $S_{2t,PSK}$, such that the $i$-th binary component $n$-tuple of $\bar{u} \oplus \bar{v}$ is the modulo-2 vector sum of the $i$-th binary component $n$-tuple of $\bar{u}$ and the $i$-th binary component $n$-tuple of $\bar{v}$. A code $C$ over $S_{2t,PSK}$ is said to be linear w.r.t. $\oplus$, if $C$ is closed under addition $\oplus$. Most of the known block codes for
2ℓ-PSK modulation are linear w.r.t. ⊕. A linear code w.r.t. ⊕ is not necessarily linear w.r.t. +. In the following, we will derive a condition for a linear code w.r.t. ⊕ to be linear w.r.t. +.

Let \( \mathbf{u} \) and \( \mathbf{v} \) be two \( n \)-tuples over \( S_{2\ell, \text{PSK}} \), and let \( \mathbf{w} \) denote \( \mathbf{u} + \mathbf{v} \). For \( 1 \leq i \leq \ell \), let the \( i \)-th binary component \( n \)-tuples of \( \mathbf{u} \), \( \mathbf{v} \) and \( \mathbf{w} \) be represented as \( \mathbf{u}^{(i)} = (u_{1i}, u_{2i}, \ldots, u_{ni}) \), \( \mathbf{v}^{(i)} = (v_{1i}, v_{2i}, \ldots, v_{ni}) \), and \( \mathbf{w}^{(i)} = (w_{1i}, w_{2i}, \ldots, w_{ni}) \), respectively. Then the following recursive equations hold [13]:

\[
\begin{align*}
    w_{ji} = u_{ji} \oplus v_{ji} \oplus x_{ji}, & \quad \text{for } 1 \leq i \leq \ell, \quad (2.11) \\
    x_{ji} = u_{ji-1} v_{ji-1} \oplus (u_{ji-1} \oplus v_{ji-1}) x_{ji-1}, & \quad \text{for } 1 < i \leq \ell, \quad (2.12) \\
    x_{j1} = 0. & \quad \text{for } 1 \leq i \leq \ell. \quad (2.13)
\end{align*}
\]

For \( 1 \leq i \leq \ell \), let \( c^{(i)}(\mathbf{u}, \mathbf{v}) \) be defined as

\[
    c^{(i)}(\mathbf{u}, \mathbf{v}) \triangleq (x_{1i}, x_{2i}, \ldots, x_{ni}). \tag{2.14}
\]

For two binary \( n \)-tuples, \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \) and \( \mathbf{b} = (b_1, b_2, \ldots, b_n) \), let \( \mathbf{a} \cdot \mathbf{b} \) be defined as

\[
    \mathbf{a} \cdot \mathbf{b} \triangleq (a_1 \cdot b_1, a_2 \cdot b_2, \ldots, a_n \cdot b_n),
\]

where \( a_j \cdot b_j \) denotes the logical product of \( a_j \) and \( b_j \).

It follows from (2.11) to (2.14) that for \( 1 \leq i < \ell \),

\[
    c^{(i+1)}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^{(i)} \cdot \mathbf{v}^{(i)} \oplus (\mathbf{u}^{(i)} \oplus \mathbf{v}^{(i)}) \cdot c^{(i)}(\mathbf{u}, \mathbf{v}). \tag{2.15}
\]

Let \( c(\mathbf{u}, \mathbf{v}) \) be defined as

\[
    c(\mathbf{u}, \mathbf{v}) \triangleq c^{(1)}(\mathbf{u}, \mathbf{v}) + 2c^{(2)}(\mathbf{u}, \mathbf{v}) + \cdots + 2^{\ell-1} c^{(\ell)}(\mathbf{u}, \mathbf{v}). \tag{2.16}
\]

Then,

\[
    \mathbf{u} + \mathbf{v} = \mathbf{u} \oplus \mathbf{v} \oplus c(\mathbf{u}, \mathbf{v}). \tag{2.17}
\]

Now consider a block code \( C \) over \( S_{2\ell, \text{PSK}} \) which is linear w.r.t. \( \oplus \). Let \( \mathbf{u} \) and \( \mathbf{v} \) be two codewords in \( C \). Then it follows from (2.17) that \( \mathbf{u} + \mathbf{v} \in C \) if and only if

\[
    c(\mathbf{u}, \mathbf{v}) \in C. \tag{2.18}
\]

For \( 1 \leq i \leq \ell \), let \( C^{(i)} \) and \( C_i \) be defined as

\[
    C^{(i)} \triangleq \left\{ \mathbf{v}^{(i)} : \mathbf{v}^{(1)} + \cdots + 2^{i-1} \mathbf{v}^{(i)} + \cdots + 2^{\ell-1} \mathbf{v}^{(\ell)} \in C \right\}, \tag{2.19}
\]

\[
    C_i \triangleq \left\{ \mathbf{v}^{(i)} : 2^{i-1} \mathbf{v}^{(i)} \in C \right\}. \tag{2.20}
\]
By definition
\[ C_i \subseteq C^{(i)}. \] (2.21)

Since \( C \) is linear w.r.t. \( \oplus \), \( C^{(i)} \) and \( C_i \) are also linear w.r.t. \( \oplus \) and
\[ C_1 + 2C_2 + \cdots + 2^{t-1}C_t \subseteq C, \] (2.22)

where the equality holds if \( C \) is a basic multilevel code. For binary codes \( C \) and \( C' \) of the same length, let \( C \cdot C' \) be defined as
\[ C \cdot C' \triangleq \{ \bar{u} \cdot \bar{v} : \bar{u} \in C \text{ and } \bar{v} \in C' \}. \]

Now we present two lemmas regarding to the closure property of a \( 2^t \)-PSK code.

**Lemma 1:** Suppose that \( C \) is a linear code over \( S_{2^t,PSK} \) w.r.t. \( \oplus \) and for \( 1 \leq i \leq \ell \),
\[ C^{(i)} \cdot C^{(i)} \subseteq C_{i+1}. \] (2.23)

Then \( C \) is closed under the component-wise modulo-\( 2^t \) addition, and hence is linear w.r.t. \( \oplus \).

**Proof:** By induction, we show that for \( 1 \leq i \leq \ell \)
\[ c^{(i)}(\bar{u}, \bar{v}) \in C_i. \] (2.24)

Since \( c^{(i)}(\bar{u}, \bar{v}) = \bar{0}, c^{(i)}(\bar{u}, \bar{v}) \in C_1. \) Suppose that \( c^{(j)}(\bar{u}, \bar{v}) \in C_j \) for \( 1 \leq j \leq i < \ell \). Since \( C^{(i)} \) and \( C_{i+1} \) are linear w.r.t. \( \oplus \), it follows from (2.15), (2.21) and (2.23) that \( c^{(i+1)}(\bar{u}, \bar{v}) \in C_{i+1}. \)
Consequently (2.18) follows from (2.16), (2.22) and (2.24), and this lemma holds.

**Lemma 2:** Suppose that \( C \) is a linear basic multilevel code over \( S_{2^t,PSK} \) w.r.t. \( \oplus \). Then \( C(= C_1 + 2C_2 + \cdots + 2^{t-1}C_t) \) is closed under the component-wise modulo-\( 2^t \) addition, if and only if
\[ C_i \cdot C_i \subseteq C_{i+1}, \text{ for } 1 \leq i < \ell. \] (2.25)

**Proof:** Only if part: Let \( \bar{u} \) (or \( \bar{v} \)) denote the \( n \)-tuple over \( S_{2^t,PSK} \) whose \( i \)-th binary component \( n \)-tuple is \( \bar{u}^{(i)} \in C_i \) (or \( \bar{v}^{(i)} \in C_i \)) and whose other binary component \( n \)-tuples are the all-zero \( n \)-tuple \( \bar{0} \). Assume that \( \bar{u} + \bar{v} \in C \). It follows from (2.11) to (2.13) that for these specific \( \bar{u} \) and \( \bar{v} \),
\[ x_{j,i+1} = u_j v_j, \text{ for } 1 \leq i \leq \ell. \] (2.26)
From (2.14), (2.18) and (2.26), we see that

\[ c^{(i+1)}(\bar{u}, \bar{v}) = \bar{u}^{(i)} \cdot \bar{v}^{(i)} \in C_{i+1}. \]

That is, \( C_i \cdot C_i \subset C_{i+1}. \)

If part: Since \( C \) is a basic multilevel code, \( C_i = \mathcal{C}^{(i)} \) for \( 1 \leq i \leq \ell \). Then if part follows from Lemma 1.

3. A Necessary and Sufficient Condition for a \( 2^\ell \)-PSK Modulation Code to be Invariant Under \( 180^\circ / 2^\ell - h \) Phase Shift with \( 1 \leq h \leq \ell \)

Now we consider the phase symmetry of a block \( 2^\ell \)-ary PSK modulation code. To determine the phase symmetry of a code, we need to know the smallest rotation under which the code is invariant.

For \( 1 \leq h \leq \ell \), let \( 2^{h-1} \bar{1} \) denote the \( n \)-tuple over \( S_{2^\ell \cdot \text{PSK}} \) whose \( h \)-th binary component \( n \)-tuple is the all-one \( n \)-tuple and whose other binary component \( n \)-tuples are the all-zero \( n \)-tuple. A code \( C \) of length \( n \) over \( S_{2^\ell \cdot \text{PSK}} \) is said to be invariant under \( 180^\circ / 2^\ell - h \) phase shift if for any codeword \( \bar{v} \) in \( C \),

\[ \bar{v} + 2^{h-1} \bar{1} \in C. \]  (3.1)

By letting \( \bar{u} = 2^{h-1} \bar{1} \) in (2.11) to (2.16), we obtain the following equations:

(1) \[ w_{ji} = v_{ji} \oplus x_{ji}, \quad \text{for} \quad 1 \leq i \leq \ell. \]  (3.2)

(2) If \( h < \ell \), then \[ x_{ji} = v_{ji-1} x_{ji-1}, \quad \text{for} \quad h < i \leq \ell. \]  (3.3)

(3) \[ z_{jh} = 1. \]  (3.4)

(4) If \( 1 < h \), then \[ z_{ji} = 0, \quad \text{for} \quad 1 \leq i < h. \]  (3.5)

It follows from (3.2) to (3.5) that we have Lemma 3.
Lemma 3: For $1 \leq h \leq \ell$, a linear code $C$ over $S_{2\ell, PSK}$ w.r.t. $\oplus$ is invariant under $180^\circ/2^{\ell-h}$ phase shift if and only if for any codeword $\bar{v}^{(1)} + 2\bar{v}^{(2)} + \cdots + 2^{\ell-1}\bar{v}^{(\ell)}$ in $C$,

$$2^{h-1}\bar{1} + 2^h\bar{v}^{(h)} + 2^{h+1}(\bar{v}^{(h)} \cdot \bar{v}^{(h+1)}) + \cdots + 2^{\ell-1}(\bar{v}^{(h)} \cdot \bar{v}^{(h+1)} \cdot \cdots \cdot \bar{v}^{(\ell-1)}) \in C,$$

where $\bar{1}$ denotes the all-one $n$-tuple.

If $C$ is a linear basic $\ell$-level code w.r.t. $\oplus$, denoted $C_1 + 2C_2 + \cdots + 2^{\ell-1}C_{\ell}$, then the necessary and sufficient condition (3.6) is expressed as follows:

(1) $\bar{1} \in C_h$, and

(2) if $h \leq \ell$, then $C_h \cdot C_{h+1} \cdots \cdot C_{j-1} \subseteq C_j$, for $h + 1 < j \leq \ell$.

Obviously, a linear code $C$ over $S_{2\ell, PSK}$ w.r.t. $\oplus$ is invariant under $180^\circ/2^{\ell-h}$ phase shift, if and only if $\bar{1}_h \in C$.

4. Code Examples

In Table 1, seven basic multilevel block codes [3] and four nonbasic block codes for 8-PSK and 16-PSK modulations are given. The number of states of a trellis diagram for each basic multilevel block code is computed based on the numbers of states of trellis diagrams for its binary component codes [14]. Among four nonbasic codes, two zero-tail Ungerboeck trellis codes for 8-PSK modulation [1] are shown. In Table 1, $V_n$, $P_n$, $P_n^A$, $RM_{ij}$, $s-RM_{ij}$, and ex-Golay denote the set of all the binary $n$-tuples, the set of all even weight binary $n$-tuples, the dual code of $P_n$ which consists of the all-zero and all-one $n$-tuples, the $j$-th order Reed-Muller code of length $2^n$, a shortened $j$-th order Reed-Muller code of original length $2^n$, and the extended $(24,12)$ code of binary Golay code. $F_1$ and $F_2$ denote two codes over $\{0, 1, 2, 3\}$ which are defined as following [4]. Let $p(x_1, x_2, \cdots, x_h)$ be a boolean polynomial which is used to represent the binary $2^h$-tuple whose $i$-th bit is given by $p(i_1, i_2, \cdots, i_h)$ where $(i_1, i_2, \cdots, i_h)$ is the binary representation of the integer $i - 1$, i.e. $i - 1 = \sum_{j=1}^{h} i_j 2^{i-1}$. Let $g_{h,i}$ denote the
Next we consider the phase rotation invariant property of codes given in Table 1. Since codes \( C[1], C[4], C[5], C[6] \) and \( C[11] \) are linear w.r.t. \(+\) and \( \bar{1} \) is contained in \( P_n^\perp, RM_{s,j} \) or ex-Golay, there codes are invariant under \( 180^\circ/2^{t-1} \) phase shift. It follows from the properties (i) and (ii) of Reed-Muller codes that codes \( C[8], C[9] \) with \( n \equiv 0 \mod 4 \) and \( C[10] \) are readily shown to meet the conditions given by (3.7) and (3.8) with \( h = 1 \). Code \( C[2] \) is shown to contain \( 2\bar{1} \), and therefore is invariant under \( 90^\circ \) phase shift. Code \( C[3] \) contains \( 2^2\bar{1} \) only and is invariant only under \( 180^\circ \) phase shift, and code \( C[7] \) does not contain even \( 2^2\bar{1} \).
References


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<td>$8$</td>
<td>22</td>
<td>Yes</td>
<td>90°</td>
</tr>
<tr>
<td></td>
<td>$C[3] \triangleq$ zero-tail Ungerboeck code</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$2^{n/4}$</td>
<td>22</td>
<td>No</td>
<td>45°, 90°</td>
</tr>
<tr>
<td></td>
<td>$C[4] \triangleq R_{M_4,1} + 2P_{16} + 4V_{16}$</td>
<td>$2^{n/4}$</td>
<td>$4$</td>
<td>$4$</td>
<td>$16$</td>
<td>24</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[5] \triangleq$ ex-Golay+$2P_{24} + 4V_{24}$</td>
<td>$2^{n/3}$</td>
<td>$4$</td>
<td>$4$</td>
<td>$24$</td>
<td>24</td>
<td>Yes</td>
<td>45°, 90°</td>
</tr>
<tr>
<td></td>
<td>$C[6] \triangleq$ zero-tail Ungerboeck code</td>
<td>$n$</td>
<td>$2^{n/3}$</td>
<td>$4$</td>
<td>$16$</td>
<td>24</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[7] \triangleq R_{M_4,2} + 2P_{16} + 4V_{16}$</td>
<td>$2^{n/3}$</td>
<td>$4$</td>
<td>$4$</td>
<td>$16$</td>
<td>24</td>
<td>Yes</td>
<td>45°, 90°</td>
</tr>
<tr>
<td></td>
<td>$C[8] \triangleq P_n^1 + 2R_{M_3,3} + 4P_{16}$</td>
<td>$n$</td>
<td>$2^{n/3}$</td>
<td>$4$</td>
<td>$16$</td>
<td>$16 &lt; n \leq 32$</td>
<td>No</td>
<td>45°, 90°</td>
</tr>
<tr>
<td></td>
<td>$C[9] \triangleq R_{M_4,3} + 4P_{32}$</td>
<td>$32$</td>
<td>$32$</td>
<td>$8$</td>
<td>$32$</td>
<td>4</td>
<td>Yes</td>
<td>45°</td>
</tr>
<tr>
<td></td>
<td>$C[10] \triangleq R_{M_5,1} + 2R_{M_5,3} + 4P_{32}$</td>
<td>$32$</td>
<td>$32$</td>
<td>$8$</td>
<td>$32$</td>
<td>4</td>
<td>Yes</td>
<td>45°, 90°</td>
</tr>
</tbody>
</table>

16-PSK

$C[11] \triangleq P_{32}^1 + 2R_{M_5,3} + 4P_{32} + 8V_{32}$