N90-20521



# Geopotential Coefficient Determination and the Gravimetric Boundary Value Problem - A New Approach

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ABSTRACT

New integral formulas to determine geopotential coefficients from terrestrial gravity and satellite altimetry data are given. The formulas are based on the integration of data over the non-spherical surface of the Earth. The effect of the topography to low degrees and orders of coefficients is estimated numerically. Formulas for the solution of the gravimetric boundary value problem are derived.

## 1. INTRODUCTION

The long-wavelength features of the Earth's gravity field is best determined from dynamic satellite observations. For short-wavelength information terrestrial gravity anomalies ($\Delta g$) and satellite altimetry data play essential roles. A well-known method to determine geopotential coefficients from $\Delta g$ is the integral method used by Rapp e. g. (1977). We present this method in section 2 followed by a new integral method in section 3.

## 2. RAPP'S INTEGRAL FORMULA

The determination of the harmonic coefficients according to Rapp requires that the data ($\Delta g$) is distributed continuously all over the surface of the Earth. Also it is assumed that the Earth's ( disturbing) potential (T) can be developed into a harmonic series convergent down to the Earth's surface:

$$T = r_o \gamma \sum_{n=2}^{\infty} \left(\frac{r_o}{r}\right)^{n+1} \sum_{m=-n}^{n} A_{nm} Y_{nm}(\phi,\lambda), \tag{2.1}$$

where $\gamma$ is the mean surface gravity of the Earth, $(r,\phi,\lambda)$ are the geocentric, spherical coordinates of the computation point, $Y_{nm}(\phi,\lambda)$ is a fully normalized spherical harmonic and, for $r = r_o$,

$$A_{mn} = \frac{1}{4\pi r_o \gamma} \iint_{\sigma} T\, Y_{mn}\, d\sigma, \tag{2.2}$$

where $\sigma$ is the unit sphere. Considering the "boundary condition" in spherical approximation (Heiskanen and Moritz 1967, p 87) one obtains the corresponding expansion for the gravity anomaly ($\Delta g$) :

$$\Delta g = \gamma \sum_{n=2}^{\infty} (n-1)\left(\frac{r_o}{r}\right)^{n+2} \sum_{m=-n}^{n} A_{nm} Y_{nm}(\phi,\lambda), \tag{2.3}$$

from which formula the coefficients are given for $r = r_o$:

$$A_{nm} = \frac{1}{4\pi\gamma(n-1)} \iint_{\sigma} \Delta g\, Y_{nm}\, d\sigma . \tag{2.4}$$

In principle $r_o$ is arbitrary, but is usually chosen as the equatorial radius of the selected reference ellipsoid. A practical obstacle in applying the integral formulas (2.2) and (2.4) is the necessity of reducing T and $\Delta g$ to the sphere of radius $r_o$. For (2.4) Rapp (1984) used the Taylor series

$$\Delta g_{reduced} = \Delta g_{observed} - \frac{\partial \Delta g}{\partial h}\Delta h - \frac{1}{2}\frac{\partial^2 \Delta g}{\partial h^2}(\Delta h)^2 - \dots, \tag{2.5}$$

where $\Delta h = r - r_o$, and the derivatives were estimated from available series of harmonic coefficients.

## 3. The New Integral Approach

Rewriting formula (2.1) on the form

$$T = R\gamma \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} C_{nm} Y_{nm}(\phi,\lambda), \tag{3.1}$$

where R is the radius of the minimum bounding sphere of the Earth, and considering Green's second identity for the functions T and U (Heiskanen and Moritz, 1967, p. 11):

$$\iiint_{\nu} (U\Delta T - T\Delta U) d\nu = \iint_{S} \left( U\frac{\delta T}{\delta n'} - T\frac{\delta U}{\delta n'} \right) dS, \tag{3.2}$$

where $\nu$ is the volume bounded by the surface S, which consists of the surface of the Earth (E) and the surface of the bounding sphere of radius R, and n' is the external normal to S with respect to $\nu$, and setting U to

$$U = \left\{ \left(\frac{R}{r}\right)^{n+1} + a_n \left(\frac{r}{R}\right)^{n} \right\} Y_{nm}(\phi,\lambda), \tag{3.3}$$

where $a_n$ is a non-zero constant with respect to position, the following general integral formula is derived in Sjöberg (1988):

$$C_{nm} = \frac{1}{4\pi R^2 \gamma (2n+1) a_n} \iint_{E} \left[ U \left\{ \overline{\Delta g} - \frac{1}{\gamma} \frac{\delta g}{\delta r} T \right\} + T \frac{\delta U}{\delta r} \right] r^2 d\sigma, \tag{3.4}$$

where $\overline{\Delta g} = \Delta g - \gamma(\xi \tan\beta_1 + \eta \tan\beta_2)$, $\beta_1$ and $\beta_2$ are the terrain inclinations, and $\zeta$ and $\eta$ are the corresponding deflections of the vertical. The integral is taken over the Earth's surface (E). Choosing $r = r_0 + \Delta h$ and $a_n = (R/r_0)^{2n+1} (n-1)/(n+2)$ one obtains (to order $\Delta h/r_0$):

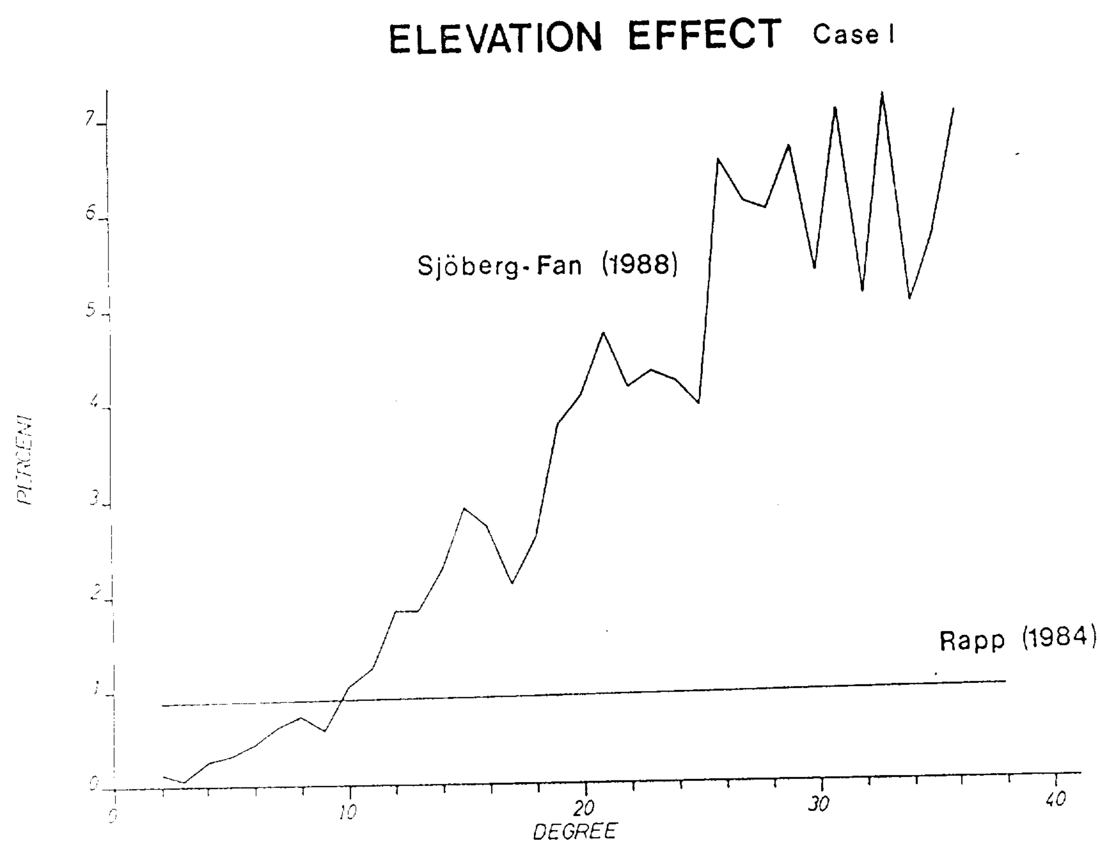


Fig. 1. The percentage terrain elevation effect by degree as computed by formula (3.6) and by Rapp (1984).

$$A_{nm} = \left(\frac{R}{r_o}\right)^{n+2} C_{nm} = A_{nm}^{0} + \Delta A_{nm}, \tag{3.5}$$

where $A_{nm}^0$ is given by (2.4) and

$$\Delta A_{nm} = \frac{n+2}{4\pi\gamma R r_o} \iint_E T\, \Delta h\, Y_{nm}\, d\sigma. \tag{3.6}$$

The percentage terrain corrections by degree determined by formula (3.6) are illustrated in Fig. 1. Details on the computations are given in Sjöberg and Fan (1988). For comparison the figure includes also terrain corrections from Rapp (1984).

Choosing $a_n = -(R/r_o)^{2n+1}$ one arrives at formula (3.5), but now with $A_{nm}^0$ given by (2.2) and

$$\Delta A_{nm} = \frac{1}{4\pi\gamma} \left(\frac{R}{r_o}\right) \iint_\sigma \left(\overline{\Delta g} + \frac{2T}{R}\right) \frac{\Delta h}{r_o} Y_{nm}(\phi,\lambda)\, d\sigma. \tag{3.7}$$

Numerical computations show that this correction is within 2% for $n \leq 50$. Again we refer to Sjöberg and Fan (1988) for details.

## 4. Solutions to Boundary Value Problems

Inserting (3.4) into (3.1) and assuming that summation and integration may change order one arrives at the following solutions for the height anomaly $\zeta_P = T_P/\gamma$ :

**Case I:** $a_n = (R/r_o)^{2n+1} (n-1)/(n+2)$ :

$$\zeta_P = \frac{R}{4\pi\gamma} \iint_\sigma [\, \{ S(\psi_{PQ},r_Q,r_P) - \Lambda(\psi_{PQ},r_Q,r_P) \} \overline{\Delta g_Q} + K(\psi_{PQ},r_Q,r_P)\, T_Q \,]\, d\sigma_Q, \tag{4.1}$$

where

$$S(\psi_{PQ},r_Q,r_P) = \frac{r_Q}{R} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{r_Q}{r_P}\right)^{n+1} P_n(\cos\psi_{PQ})$$

$$\Lambda(\psi_{PQ},r_Q,r_P) = \frac{r_Q}{R} \sum_{n=2}^{\infty} \frac{n+2}{n-1} \left\{ 1 - \left(\frac{r_o}{r_Q}\right)^{2n+1} \right\} \left(\frac{r_Q}{r_P}\right)^{n+1} P_n(\cos\psi_{PQ})$$

and

$$K(\psi_{PQ},r_Q,r_P) = \frac{1}{R} \sum_{n=2}^{\infty} (n+2) \left(\frac{r_Q}{r_P}\right)^{n+1} \left\{ 1 - \left(\frac{r_o}{r_Q}\right)^{2n+1} \right\} P_n(\cos\psi_{PQ})$$

**Case II:** $a_n = -(R/r_o)^{2n+1}$ :

$$\zeta_P = \frac{1}{4\pi\gamma} \iint_\sigma [\, \{ P(\psi_{PQ},r_Q,r_P) - L(\psi_{PQ},r_Q,r_P) \} T_Q + M(\psi_{PQ},r_Q,r_P)\, r_Q \overline{\Delta g_Q} \,]\, d\sigma_Q \;, \tag{4.2}$$

where

$$P(\psi_{PQ},r_Q,r_P) = \sum_{n=2}^{\infty} (2n+1) \left(\frac{r_Q}{r_P}\right)^{n+1} P_n(\cos\psi_{PQ})$$

$$L(\psi_{PQ},r_Q,r_P) = \sum_{n=2}^{\infty} (n-1) \left(\frac{r_Q}{r_P}\right)^{n+1} \left\{ 1 - \left(\frac{r_o}{r_Q}\right)^{2n+1} \right\} P_n(\cos\psi_{PQ})$$

and

$$M(\psi_{PQ},r_Q,r_P) = \sum_{n=2}^{\infty} \left(\frac{r_Q}{r_P}\right)^{n+1} \left\{ 1 - \left(\frac{r_o}{r_Q}\right)^{2n+1} \right\} P_n(\cos\psi_{PQ}).$$

In the limit $r_Q \to r_o$ the formulas (4.1) and (4.2) approach (the extended) Stoke's formula and the Bruns-Poisson's formula, respectively.

## 5. Discussion

As Figure 1 shows the terrain corrections of the new method (3.6) are more significant with increasing degree than those of Rapp (1984). This discrepancy might be explained by insufficient power in Rapp's derived derivatives of formula (2.5). The validity of the Earth surface integrals (4.1) and (4.2) should be further investigated. In any case they should have some interest for the determination of the external gravity field from surface data.

**Acknowledgements.** Dr. R H Rapp provided us with a global set of terrain data. This assistance is cordially acknowledged. This work was supported by the Swedish Natural Science Research Council, contract no. G-GU 4071-302.

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