Curvature Continuity of Cubic Bezier Curves in the
Solid Modeling Aerospace Research Tools
Design Software

Interim Report

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Abstract

This report presents preliminary results of an investigation into the development of a procedure to provide curvature continuity between biparametric cubic Bezier surface patches in the computer-aided design package known as SMART (Solid Modeling Aerospace Research Tools). This initial effort was aimed at providing the designer with the ability to locally impose curvature continuity at the intersection of two Bezier curves without disrupting either the curvature or slope continuity that may exist at the ends of these curves. Such a method was found if the original Bezier control points are all coplanar. If they are not then it is possible to find a minimum deviation from exact curvature continuity. In cases where this is not sufficient, then an entire piecewise curve must be made curvature continuous simultaneously. A method was developed based on cubic splines which is very fast. The procedure returns new Bezier control points which have both slope and curvature continuity.

Introduction

The development of computer-aided design packages have advanced to the state where it is now possible to build up a geometric description of an aerospace system in a very short time using high power graphics workstations. At the same time, it becomes desirable to have the ability to apply various numerical flow solvers to the resulting shapes to test their aerodynamic efficiency. These numerical flow solvers usually need the domain partitioned into small cells to provide a computational grid suitable for the discretized equations.

The ultimate aim of this project is provide the ability of a computer-aided design package, SMART, developed at Langley Research Center, to include the generation of finite difference grids in a computational domain around the designed surfaces. Several methods for generating such grids are available but none are designed to smooth surface data. Hence if the surface has discontinuities the grid will be generated accordingly. The subsequent computation of the metrics of a coordinate transformation by using finite differences will thus have larger errors than usual. This inaccuracy of finite difference approximation to a differential equation will be larger than desired. To prevent these unwanted errors, it was necessary to begin with the requirement that the design surfaces must have curvature continuity over most of the domain.
Most computer-aided design packages describe the design surfaces by combining several surface elements called patches. These patches can be geometrically described by analytic functions of various kinds. Of particular interest are cubic polynomials. Here, the cubic polynomials are cast as biparametric Bezier curves. Cubic Bezier curves use control points as weights for a series of Bernstein polynomials and provide a relatively powerful mechanism for the designer to manipulate the shape, which is done by moving the control points. In the same manner, movement of the control points can allow for postprocessing of the designed shape as we wish to do here to impose curvature continuity. Thus, after the designer creates a shape, the resulting patches can then be smoothed by manipulating the location of the patch control points.

Unfortunately, curvature continuity, along with slope continuity, requires information to be used from more than one patch. This couples all patches together requiring the simultaneous solution of new control point locations. If the number of patches is large, then this would require the solution of a rather large linear system and would likely be time-consuming even on high speed graphics workstations. It was thus desirable to investigate whether it would be possible to provide the curvature continuity for patches individually or at least with some minimal effort.

The examination of the problem took the form of first looking at how the designer actually constructs a shape and seeing if there were some way of providing a priori curvature continuity rather than at the end of the process. It seems that a significant amount of time is spent in the generation of two dimensional cross sections of the design shape. Thus cubic Bezier curves are generated describing these cross sections.

Once the cross sections are described, they are assembled axially to build up the surface. A number of axial curves are then generated to connect the cross sections. These are also piecewise cubic curves with one cubic in between each cross section. In this manner, the surface is thus described by two families of piecewise cubic curves. The surface can be seen as a collection of four-sided patches with one cubic on each side. The task of providing curvature continuity between the patches is made much easier if there already exists curvature continuity in both families of curves.

To obtain curvature continuity on these curves two tasks were identified:

Task 1 - Provide a means of adjusting the Bezier control points at the intersection of two cubic Bezier curves to gain slope and curvature continuity without disrupting the same at neighboring intersections.

Task 2 - Provide a means of ensuring curvature continuity on an entire series of cubic Bezier curves.

Both of these tasks have been completed. The remainder of this report explains the approach, method, and presents an example for each task. The remaining effort of providing curvature continuity for entire network of patches will be the subject of the final report.
Task 1 - Curvature Continuity between adjacent Cubic Bezier Curves 
while maintaining C1 & C2 at the ends and C1 at the junction

It is desirable, in CAD systems, when using cubic curves, to be able to attain various levels of 
continuity between adjacent curves. Here the desire is to attain C2 at intersections without disturbing 
the C1 and C2 continuity at neighboring intersections. In addition, it is assumed that the C1 continuity 
which already exists at the intersection will not be disturbed. By this is meant not only that the Bezier 
control points on either side of the intersection are colinear with it, but that the ratio of their distances from 
it will remain constant. This ratio is also required at the neighboring junction points and does not allow 
for the control points nearest to neighboring junctions to be moved. It is thus allowed only to move, in 
space, the intersection point in question and the two control points adjacent to it.

If such a shift of control points is possible, this would allow the designer the flexibility of select-
ing local junctures for enhanced continuity without having to simultaneously perform the operation at 
all junctures where such continuity is desired. In the present effort, it will be shown that if adjacent Bezier 
cubics have coplanar control points, there is a unique solution to the problem. If the control points are 
not all coplanar, then it is possible to find the minimum C2 discontinuity at the intersection.

Analysis

Consider the intersection of two cubic Bezier curves shown in Fig. 1. The control points are 
coordinate vectors denoted by \( b_n \), \( b_{n-1} \), etc. Shown also are the requirements for C2 continuity at \( b_n \), 
 ie. an auxiliary point \( d \) exists which not only is the intersection of extensions from the control polygon, 
but that ratios of certain lengths must also be held.

Fig. 1  Adjacent Bezier curves, associated control points and 
auxiliary point \( d \). The \( t \) and \( 1-t \) are meant to indicate ratios only.
When C2 continuity does not exist, there are two points, \(d_1\) and \(d_2\), as shown in Fig. 2. This is the case more typically found. To gain C2 at \(b_n\), it is necessary to move one or more of the control points so that the two points \(d_1\) and \(d_2\) are coincident.

This can be done if the following are satisfied:

\[b_{n-1} = (1-t)b_{n-2} + td\]  
(1)

\[b_n = (1-t)b_{n-1} + tb_{n+1}\]  
(2)

\[b_{n+1} = (1-t)d + tb_{n+2}\]  
(3)

To maintain C2 at the ends the following must also be satisfied:

\[b_{n,3} - 2b_{n,2} + b_{n,1} = k_1(b_{n,3} - 2b_{n,2} + b_{n,1})\]  
(4)

\[b_{n,3} - 2b_{n,2} + b_{n,1} = k_2(b_{n,3} - 2b_{n,2} + b_{n,1})\]  
(5)

Where the overbar indicates the original position. Maintenance of C1 with distance ratios at the ends means that the points \(b_{n-2}\) and \(b_{n+2}\) can not be moved. Hence, only \(b_{n-1}\), \(b_n\), and \(b_{n+1}\) are to be relocated. Equations 1-3 further require that all but \(b_{n,3}\) and \(b_{n,3}\) must be coplanar for the possibility of C2 at \(b\) to exist.

The geometric interpretation of the problem is shown in Fig. 3. Equations 4 and 5 require that \(b_{n-1}\) and \(b_{n+1}\) can be moved only along lines parallel to the vectors represented by the second differences of the end control points. Thus \(k_1\) controls the location of \(b_{n-1}\) and \(k_2\) controls the location of \(b_{n+1}\). Equations 1 and 3 then locate the auxiliary points \(d_1\) and \(d_2\). Thus different values of \(k_1\) move \(b_{n-1}\) and \(d_1\) through parallel lines in space. Similarly, \(k_2\) moves \(b_{n+1}\) and \(d_2\) through two other parallel lines in space. It should now be obvious that if all control points are coplanar, then the lines swept by \(d_1\) and \(d_2\) are also coplanar and their intersection represents the solution of the problem. Mathematically, we note that there are 5 equations for the 4 unknowns (the three control points and the auxiliary point \(d\)) and two undetermined parameters \(k_1\) and \(k_2\). This shows that if all of the control points for both curves are coplanar, then each point has two independent components. Hence the two parameters \(k_1\) and \(k_2\) are sufficient to make up a 5th unknown and the system is well-posed.
Fig. 3 Geometric interpretation of the solution of equations 1-5. As \( k_1 \) and \( k_2 \) are changed, the auxiliary points \( d \) and \( d' \) move along lines parallel to the endpoint vectors. The intersection of these lines at \( d \) is the solution.

Non-Coplanar Endpoints

If \( b_{n3} \) and \( b_{n3} \) are not coplanar with the other control points then the lines swept by \( d_1 \) and \( d_2 \) are skew and no solution exists. However, there is an extremely useful interpretation of the closest approach of the two lines. The closest approach of the two skew lines represents the best possible match of the curvatures of the two cubic Bezier space curves. In fact, as the two endpoints approach being coplanar with the remaining control points, the two skew lines come closer to intersecting and the respective curvatures of the two lines approach being exactly \( C_2 \). This means that if the endpoints are not far from being coplanar with the rest of the control points then it is possible to attain near \( C_2 \) continuity.

To see why a unique solution may not exist when the endpoints are not coplanar, consider fitting two cubic splines between three points in space. Both \( C_1 \) and \( C_2 \) continuity are attained at the middle point but only specified slopes at the ends can be given. To attain specified curvatures at the ends, the middle point can be moved. There is a unique solution to this problem, even if the endpoint slopes are not coplanar. If they are, then this corresponds to the unique solution given in the last section. For the case when they are not, one need only examine the corresponding Bezier control points to see if those nearest the ends match the ones given in the problem. It is unlikely.

We next return to the idea of finding this nearest \( C_2 \) continuity. It amounts to finding the values of \( k_1 \) and \( k_2 \) for which the two skew lines have their closest approach. To find these values it is merely sufficient to find the location in space of the points on each line at closest approach. The following is an outline of the procedure and an example to show how near \( C_2 \) continuity is attained.
Procedure for Nearest C2

Consider the skew lines in space $l_1$ and $l_2$ swept out by $d_1$ and $d_2$ as the values of $k_3$ and $k_4$ are changed as shown in Fig. 4. Note that $k_3$ and $k_4$ are directly proportional to $k_1$ and $k_2$. The equations for $l_1$ and $l_2$ are given by:

$$l_1 = d_1 + k_3 c_1$$
$$l_2 = d_2 + k_4 c_2$$

Where $c_1$ and $c_2$ are the curvature vectors of the end points. Let the intersections of the line containing the shortest distance between the skew lines with the lines be called P and Q. We now seek to find P and Q and the associated values of $k_3$ and $k_4$.

![Fig. 4 Skew lines given by sweep of $d_1$ and $d_2$ due to changing $k_3$ and $k_4$.](image)

1. The cross product of the endpoint curvature vectors $c_1$ and $c_2$ defines the direction of the line containing P and Q. The distance between P and Q is found by projecting the line segment $d_1 d_2$ on the unit vector in the direction of PQ:

$$|PQ| = (d_1 d_2) \cdot \left( \frac{c_1 \times c_2}{|c_1 \times c_2|} \right)$$

2. Next $l_1$ is projected onto the plane perpendicular to PQ containing $l_2$. The equation for the projected line is:

$$l_1 = d_1 + k_3 c_1 + PQ$$

The projected line crosses $l_2$ at Q. Hence equating the projected line with $l_2$ gives an expression for solving for $k_3$ and $k_4$.

3. Having found $k_3$ and $k_4$, $b_{n-1}$ is found from equation 4, $b_{n+1}$ is found from equation 5, and $b_n$ is found from equation 2. Note that the original C1 and C2 at the ends and C1 at the intersection are maintained exactly while C2 at the intersection is approximate.
**Example (Coplanar control points)**

To insure that a coplanar set of points was generated while trying to be general, a set of points was generated all with the same z-coordinate. The set was rotated about the x-axis and then about the y-axis. The resulting set of coordinates were used:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & n-3 & n-2 & n-1 & n & n+1 & n+2 & n+3 \\
\hline
x & -1.39545 & -7.1244 & .13878 & .88660 & 2.00833 & 3.09808 & 3.08109 \\
y & .76603 & 1.63205 & 2.48076 & 2.67128 & 2.95707 & 1.63205 & .07321 \\
z & 1.58301 & 1.76603 & 1.84038 & 1.53564 & 1.07853 & -.43397 & -1.46340 \\
\hline
\end{array}
\]

The mismatch in initial curvature at the intersection is shown best by noting that equations 1-3 can also be expressed as requiring:

\[b_n - 2b_{n-1} + b_{n-2} = k(b_n - 2b_{n+1} + b_{n+2})\]  \hspace{1cm} (10)

Hence each component of the second difference vector on one side of the intersection must be the same multiple (fraction) of the components on the other side. This means that the ratios of the components must be the same. That is:

\[
\frac{b_n - 2b_{n-1} + b_{n-2}}{b_n - 2b_{n+1} + b_{n+2}} = k
\]  \hspace{1cm} (11)

In this example the initial coordinates gives:

<table>
<thead>
<tr>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>y</td>
</tr>
<tr>
<td>z</td>
</tr>
</tbody>
</table>

**Table I. Second difference ratios of coordinates at the intersection**

Thus the ratio of the second differences of the coordinates on either side of the junction are rather different. After finding the new points by the procedure above we find the ratios to be:

<table>
<thead>
<tr>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
</tr>
<tr>
<td>y</td>
</tr>
<tr>
<td>z</td>
</tr>
</tbody>
</table>

**Table II. Second difference ratios of the coordinates at the intersection after imposing curvature continuity**

Hence for the case of coplanar control points, the C2 is attained exactly.

**Example (Non coplanar)**

In this example, the first point, \(b_{n-3}\), was displaced by a distance of .8 in the z-direction before the x- and y-axis rotations. This guaranteed that all points were coplanar except the first which had coordinates of (-1.04904, .36603, 2.18301) and was 35.6 deg out of the plane containing the other control points. The initial curvatures were the same as the above example, since the end control point is not included in computing the curvatures. The final curvatures, however, are given in Table III below.
Thus, while exact $C_2$ is not attained, it can be seen that the error in matched $C_2$ at the juncture is small even though one of the ends made an angle of 35.6 deg with the plane containing the remaining control points.

Hence, the procedure outlined above allows for the designer to go to a single intersection of a composite cubic Bezier curve and locally impose curvature, along with slope, continuity if the control points of the two curves are coplanar. Since most cross sections do contain coplanar control points, this represents a rather useful capability. For those intersections where the control points on either side are not coplanar then the procedure allows to come as close as possible. A program listing is given in Appendix A which carries out the steps given above.
Task 2 - Curvature continuity (C2) of a series of cubic Bezier curves which are at first only C0

As it has been shown, if the initial control points are not coplanar, then exact curvature continuity, is locally unobtainable, though it is possible to generate the coefficients locally of that pair for Bezier curves which are as close as possible to curvature continuous. This may suffice for many applications. But if not, it becomes necessary to attain the continuity for all cubics in the piecewise curve simultaneously.

Fortunately, this can be done relatively easily. The problem is very similar to setting up an interpolant between points in space. The procedure is one where the endpoints of each Bezier curve is kept fixed in space. Thus the outer Bezier control points of each cubic are retained. We will thus show that the problem is solved by replacing the interior control points with those computed from the corresponding cubic splines placed through the end points of each piecewise cubic. The end conditions used are those which keep the original slopes at the ends of the piecewise curve.

Analysis

Consider the series of Bezier control points, control polygons, and corresponding cubic curves shown in Fig. 5 below.

![Diagram of piecewise cubic Bezier curve](image)

Fig. 5. Example of piecewise cubic Bezier curve where neither C1 nor C2 exist.

For clarity, the control polygon has not been drawn. Since the Bezier control points on either side of the three interior intersections are not collinear with the control point at that intersection, it is clear that slope continuity is absent. Table IV shows the ratio of curvature vectors for each of the three interior intersections. As they are not equal, it is clear that curvature continuity is also absent.

<table>
<thead>
<tr>
<th>Intersection</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>1.0769</td>
<td>.8000</td>
<td>1.5000</td>
<td>-2.6364</td>
</tr>
<tr>
<td>y</td>
<td>-6.4000</td>
<td>-.1250</td>
<td>.7273</td>
<td>.1250</td>
</tr>
</tbody>
</table>

Table IV. Curvature ratios at the interior points of initial set of cubic Bezier curves.

Now consider the cubic spline which has been placed through the intersection points as shown in Fig. 6. The intersection points are also the outer two control points of any one cubic. Hence, only new inner control points need be found. To place a cubic spline through these points any standard cubic spline routine can
be used. The user only need specify two end conditions. These end conditions can either be those called "natural" wherein the second derivative is zero; "slope", where the slope at the end is specified; or some combination. It seemed most appropriate in this circumstance to use the slopes existing at the ends of original Bezier cubics. With these end conditions, a piecewise parametric cubic spline was passed through the ends and the intersection points.

For this effort, an optimized parametric cubic spline routine was written where a single parameter, \( t \), varied between zero and one on each cubic. Each cubic has the form:

\[
\mathbf{x} = \alpha_i t^3 + \beta_i t^2 + \gamma_i t + \mathbf{x}_i
\]

where \( \mathbf{x} \) represents the coordinates of a point on the cubic, and \( \mathbf{x}_i \) contains the coordinates of the \( i \)th cubic when \( t=0 \). Thus, if there are \( N \) cubics, there will be \( 3N \) unknowns. To get \( 3N \) equations we note that the condition of \( C^0 \) continuity requires:

\[
\alpha_{i-1} + \beta_{i-1} + \gamma_{i-1} + \mathbf{x}_{i-1} = \mathbf{x}_i \quad i \in [1,N] \quad (13)
\]

\( C^1 \) requires:

\[
3\alpha_{i-1} + 2\beta_{i-1} + \gamma_{i-1} = \gamma_i \quad i \in [1,N-1] \quad (14)
\]

and \( C^2 \) requires:

\[
3\alpha_{i-1} + \beta_{i-1} = \beta_i \quad i \in [1,N-1] \quad (15)
\]

There are \( 3N \) equations and this leaves the two end conditions to get the remaining two equations. The usual practice is to use these equations to write a relationship between one of the coefficients at three different points. This eliminates the other two variables. Here we used eq. (13)-(15) to write and equation for \( \beta \):

\[
\beta_{i-1} + 4\beta_i + \beta_{i+1} = 3(\mathbf{x}_{i-1} - 2\mathbf{x}_i + \mathbf{x}_{i+1}) \quad i \in [2,N-1] \quad (16)
\]

The linear system is computed by adding the end conditions in terms of \( \beta \). Hence we have:

\[
\frac{2}{3} \beta_1 + \frac{1}{3} \beta_2 = 3(\mathbf{x}_2 - \mathbf{x}_1) - s_1 \quad (17)
\]

\[
\frac{2}{3} \beta_{N-1} + \frac{7}{3} \beta_N = 3(\mathbf{x}_{N+1} - \mathbf{x}_N) - 2(\mathbf{x}_N - \mathbf{x}_{N-1}) - s_2 \quad (18)
\]

where

\[
s_1 = 3(b_{11} - b_{01})
\]

\[
s_2 = 3(b_{3N} - b_{2N}) \quad (19)
\]

Equations (16)-(19) form a tridiagonal system of equations for each coordinate direction and can thus be solved very quickly. With each of the \( \beta \)’s found, the other coefficients are computed from eq. (14) and (15). This defines the cubics in the piecewise curve.
Once the cubic splines have been generated, the inner two Bezier control points for each cubic can be found from:

\[ b_{1i} = b_{0i} + \frac{\gamma_i}{3} \]
\[ b_{2i} = b_{3i} - \frac{1}{3}(3\alpha_i + 2\beta_i + \gamma_i) \]

The resulting new curves and corresponding control points are shown in Fig. 7. The dashed lines are the original cubics and the open circles are the original Bezier control points. It may be noted, as a visual check of slope continuity, that at each intersection, the two control points on either side of the intersection point are now collinear, unlike the original control points. The control points next to those at the very ends of the total curve are the same as the original ones as a consequence of retaining the original slope at the end points. The fit of the new curve is close to the old except where the old curve was rather badly matched in slope and in curvature (ie. at intersection 2).

![Fig. 7 New cubic Bezier curve with new control points.](image)

The new curvature ratios, shown below in Table V., show that curvature continuity has been achieved.

<table>
<thead>
<tr>
<th></th>
<th>Intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
</tr>
<tr>
<td>x</td>
<td>1.0000</td>
</tr>
<tr>
<td>y</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Table V. Curvature ratios at the 4 interior intersections

This completes the second task. The procedure allows for virtually any combination of cubic Bezier curves which describe a piecewise curve to be made into one which has both slope and curvature continuity. The original control points are retained at each of the curve intersections and new ones are created for the interior of each curve. The new control points are computed from the coefficients of the associated spline curve through the original intersections. The control points adjacent to the outside ends of the first and last cubic have also been retained since the end requirement of slope matching the original curve has been imposed. It may be noted that there is no requirement on the original control points to be coplanar as in task one. A program listing is given in Appendix B for a subroutine which takes the control points of a series of Bezier curves, fits a cubic spline through the intersections and endpoints, and recalculates the interior Bezier control points.
Appendix A
Program Listing for Task 1

The following program in FORTRAN will compute the values of \( k_1 \) and \( k_2 \) given an initial set of control points, and adjust the position of the control points \( b_{n-1} \), \( b_n \) and \( b_{n+1} \).

SUBROUTINE C2(X,Y,Z)

DIMENSION X(4, 2), Y(4, 2), Z(4, 2)

C------------------------ FIND CURVATURES OF ENDS ------------------------

\[
\begin{align*}
CEX1 &= X(3, 1) - 2X(2, 1) + X(1, 1) \\
CEY1 &= Y(3, 1) - 2Y(2, 1) + Y(1, 1) \\
CEZ1 &= Z(3, 1) - 2Z(2, 1) + Z(1, 1) \\
CEX2 &= X(4, 2) - 2X(3, 2) + X(2, 2) \\
CEY2 &= Y(4, 2) - 2Y(3, 2) + Y(2, 2) \\
CEZ2 &= Z(4, 2) - 2Z(3, 2) + Z(2, 2)
\end{align*}
\]

C------------------------ FIND \( t \) TO MAINTAIN C2 AT \( b_n \) ------------------------

\[
\begin{align*}
DX &= X(4, 1) - X(3, 1) \\
DY &= Y(4, 1) - Y(3, 1) \\
DZ &= Z(4, 1) - Z(3, 1) \\
S1 &= \text{SQRT}(DX^2 + DT^2 + DZ^2) \\
DX &= X(2, 2) - X(3, 1) \\
DY &= Y(2, 2) - Y(3, 1) \\
DZ &= Z(2, 2) - Z(3, 1) \\
S2 &= \text{SQRT}(DX^2 + DY^2 + DZ^2) \\
T &= S1/S2
\end{align*}
\]

C------------------------ FIND \( d_1 \) and \( d_2 \) ------------------------

\[
\begin{align*}
XD1 &= X(2, 1) + (X(3, 1) - X(2, 1)) / T \\
YD1 &= Y(2, 1) + (Y(3, 1) - Y(2, 1)) / T \\
ZD1 &= Z(2, 1) + (Z(3, 1) - Z(2, 1)) / T \\
XD2 &= (X(2, 2) - T * X(3, 2)) / (1 - T) \\
YD2 &= (Y(2, 2) - T * Y(3, 2)) / (1 - T) \\
ZD2 &= (Z(2, 2) - T * Z(3, 2)) / (1 - T)
\end{align*}
\]

C------------------------ FIND P AND Q ------------------------

\[
DX = XD2 - XD1
\]
\( \text{DY} = \text{YD2} - \text{YD1} \)
\( \text{DZ} = \text{ZD2} - \text{ZD1} \)

\[
\begin{align*}
\text{CEX} &= \text{CEY1} \times \text{CEZ2} - \text{CEY2} \times \text{CEZ1} \\
\text{CEY} &= \text{CEX2} \times \text{CEZ1} - \text{CEX1} \times \text{CEZ2} \\
\text{CEZ} &= \text{CEX1} \times \text{CEY2} - \text{CEX2} \times \text{CEY1}
\end{align*}
\]

\( \text{CL} = \sqrt{\text{CEX}^2 + \text{CEY}^2 + \text{CEZ}^2} \)
\( \text{PQ} = (\text{DX} \times \text{CEX} + \text{DY} \times \text{DEY} + \text{DZ} \times \text{CEZ}) / \text{CL} \)

\[
\begin{align*}
\text{CEX} &= \text{CEX} \times \text{PQ} / \text{CL} \\
\text{CEY} &= \text{CEY} \times \text{PQ} / \text{CL} \\
\text{CEZ} &= \text{CEZ} \times \text{PQ} / \text{CL}
\end{align*}
\]

\[
\begin{align*}
\text{DEN} &= \text{CEX1} \times \text{CEY2} - \text{CEX2} \times \text{CEY1} \\
\text{K3} &= ((\text{CEY} - \text{DY}) \times \text{CEX2} - (\text{CEX} - \text{DX}) \times \text{CEY2}) / \text{DEN} \\
\text{K4} &= ((\text{CEY} - \text{DY}) \times \text{CEX1} - (\text{CEX} - \text{DX}) \times \text{CEY1}) / \text{DEN}
\end{align*}
\]

\[
\begin{align*}
\text{PX} &= \text{XD1} + \text{K3} \times \text{CEX1} \\
\text{PY} &= \text{YD1} + \text{K3} \times \text{CEY1} \\
\text{PZ} &= \text{ZD1} + \text{K3} \times \text{CEZ1}
\end{align*}
\]

\[
\begin{align*}
\text{QX} &= \text{XD2} + \text{K4} \times \text{CEX2} \\
\text{QY} &= \text{YD2} + \text{K4} \times \text{CEY2} \\
\text{QZ} &= \text{ZD2} + \text{K4} \times \text{CEZ2}
\end{align*}
\]

C------------------------ NEW K1 AND K2 ------------------------

\[
\begin{align*}
\text{K1} &= (\text{T} \times \text{PX} - (1 + \text{T}) \times \text{X}(2, 1) + \text{X}(1, 1)) / \text{CEX1} \\
\text{K2} &= ((1 - \text{T}) \times \text{QX} + (\text{T} - 2) \times \text{X}(3, 2) + \text{X}(4, 2)) / \text{CEX2}
\end{align*}
\]

C------------------------ FIND NEW b, b , and b ------------------------

\[
\begin{align*}
\text{X}(3, 1) &= 2 \times \text{X}(2, 1) - \text{X}(1, 1) + \text{K1} \times \text{CEX1} \\
\text{Y}(3, 1) &= 2 \times \text{Y}(2, 1) - \text{Y}(1, 1) + \text{K1} \times \text{CEY1} \\
\text{Z}(3, 1) &= 2 \times \text{Z}(2, 1) - \text{Z}(1, 1) + \text{K1} \times \text{CEZ1}
\end{align*}
\]

\[
\begin{align*}
\text{X}(2, 2) &= 2 \times \text{X}(3, 2) - \text{X}(4, 2) + \text{K2} \times \text{CEX2} \\
\text{Y}(2, 2) &= 2 \times \text{Y}(3, 2) - \text{Y}(4, 2) + \text{K2} \times \text{CEY2} \\
\text{Z}(2, 2) &= 2 \times \text{Z}(3, 2) - \text{Z}(4, 2) + \text{K2} \times \text{CEZ2}
\end{align*}
\]

\[
\begin{align*}
\text{XN} &= (1 - \text{T}) \times \text{X}(3, 1) + \text{T} \times \text{X}(2, 2) \\
\text{YN} &= (1 - \text{T}) \times \text{Y}(3, 1) + \text{T} \times \text{Y}(2, 2) \\
\text{ZN} &= (1 - \text{T}) \times \text{Z}(3, 1) + \text{T} \times \text{Z}(2, 2)
\end{align*}
\]

\[
\begin{align*}
\text{X}(4, 1) &= \text{XN} \\
\text{Y}(4, 1) &= \text{YN}
\end{align*}
\]
Z(4,1) = ZN
X(1,2) = XN
Y(1,2) = YN
Z(1,2) = ZN

C--------------------------------------------------- THAT'S ALL --------------------------------------

RETURN
END
Appendix B.
Curvature Continuity on a Piecewise Cubic Curve

The following FORTRAN subroutine is designed to perform the steps listed in Task 2. That is, given an input set of N Bezier cubic control points, cubic splines are put through the intersection points and new interior control points are found.

```
SUBROUTINE BCRV(X,Y,Z,N)

DIMENSION X(4,N),Y(4,N),Z(4,N)
DIMENSION XX(N+1),YY(N+1),ZZ(N+1)
DIMENSION A(N),B(N),C(N),D(N)
DIMENSION AA(N),BB(N),CC(N)

C---------------------- SOME OF THE VARIABLES ----------------------

C
C
C   N     Number of Bezier Curves
C   X,Y,Z  Coordinates of Bezier Control points
C   XX,YY,ZZ Coordinates of the ends and intersections of the piecewise curve
C   A,B,C,D Coefficients of the cubic spline matrix
C   AA,BB,CC Coefficients of the cubic splines
C

C----------------------- FIND SLOPES AT THE ENDS ---------------------

S1X = 3. * (X(2,1) - X(1,1))
S1Y = 3. * (Y(2,1) - Y(1,1))
S1Z = 3. * (Z(2,1) - Z(1,1))

S2X = 3. * (X(4,N) - X(3,N))
S2Y = 3. * (Y(4,N) - Y(3,N))
S2Z = 3. * (Z(4,N) - Z(3,N))

C------------- GET THE INTERSECTION AND END POINTS --------------

DO 100 I = 1,N

   XX(I) = X(1,I)
   YY(I) = Y(1,I)
   ZZ(I) = Z(1,I)

100 CONTINUE

XX(N+1) = X(4,N)
YY(N+1) = Y(4,N)
ZZ(N+1) = Z(4,N)
```
C---------------------- SET UP CUBIC SPLINE MATRIX ---------------------

C-------- LOOP ON DIMENSIONS

DO 1000 K = 1,3

DO 200 I = 2,N-1

C(I) = 1  
B(I) = 4  
A(I) = 1

IF (K.EQ.1) THEN
   DD = XX(I+1) - 2. * XX(I) + XX(I-1)
ELSEIF (K.EQ.2) THEN
   DD = YY(I+1) - 2. * YY(I) + YY(I-1)
ELSE
   DD = ZZ(I+1) - 2. * ZZ(I) + ZZ(I-1)
ENDIF

D(I) = 3. * DD

200 CONTINUE

B(1) = 2. / 3.  
A(1) = 1. / 3.
IF ( K.EQ.1 ) THEN
   D(1) = 3. * (XX(2) - XX(1)) - S1X
ELSEIF ( K.EQ.2 ) THEN
   D(1) = 3. * (YY(2) - YY(1)) - S1Y
ELSE
   D(1) = 3. * (ZZ(2) - ZZ(1)) - S1Z
ENDIF

B(N) = 7. / 3.  
C(N) = 2. / 3.
IF ( K.EQ.1 ) THEN
   D(N) = 3. * (XX(N+1) - XX(N)) - 2. * (XX(N) - XX(N-1)) - S2X
ELSEIF ( K.EQ.2 ) THEN
   D(N) = 3. * (YY(N+1) - YY(N)) - 2. * (YY(N) - YY(N-1)) - S2Y
ELSE
   D(N) = 3. * (ZZ(N+1) - ZZ(N)) - 2. * (ZZ(N) - ZZ(N-1)) - S2Z
ENDIF

C------------------------- SOLVE THE MATRIX -------------------------

CALL TSOLV(A,B,C,D,BB)
C---------- FIND THE OTHER CUBIC SPLINE COEFFICIENTS ----------

IF (K.EQ.1) THEN

DO 300 I = 1,N-1
   AA(I) = (BB(I+1) - B(I)) / 3.
   CC(I) = XX(I+1) - XX(I) - AA(I) - BB(I)
300 CONTINUE
   CC(N) = 3. * AA(N-1) + 2. * BB(N-1) + CC(N-1)
   AA(N) = XX(N+1) - XX(N) - BB(N) - CC(N)

ELSEIF (K.EQ.2) THEN

DO 320 I = 1,N-1
   AA(I) = (BB(I+1) - BB(I)) / 3.
   CC(I) = YY(I+1) - YY(I) - AA(I) - BB(I)
320 CONTINUE
   CC(N) = 3.*AA(N-1) + 2.*BB(N-1) + CC(N-1)
   AA(N) = YY(N+1) - YY(N) - BB(N) - CC(N)

ELSE

DO 340 I = 1,N-1
   AA(I) = (BB(I+1) - BB(I)) / 3.
   CC(I) = ZZ(I+1) - ZZ(I) - AA(I) - BB(I)
340 CONTINUE
   CC(N) = 3.*AA(N-1) + 2.*BB(N-1) + CC(N-1)
   AA(N) = ZZ(N+1) - ZZ(N) - BB(N) - CC(N)

ENDIF

C----------------- NOW GET NEW BEZIER COEFFICIENTS -----------------

DO 500 I = 1 TO N

IF (K.EQ.1) THEN

   DX0 = CC(I)
   DX1 = 3. * AA(I) + 2. * BB(I) + CC(I)
   X(2,I) = X(1,I) + DX0 / 3.
   X(3,I) = X(4,I) - DX1 / 3.

ELSEIF (K.EQ.2) THEN

   DY0 = CC(I)
   DY1 = 3. * AA(I) + 2. * BB(I) + CC(I)

ENDIF
\[
Y(2,I) = Y(1,I) + \frac{DY0}{3}. \\
Y(3,I) = Y(4,I) - \frac{DY1}{3}.
\]

ELSE

\[
\begin{align*}
DZ0 &= CC(I) \\
DZ1 &= 3 \cdot AA(I) + 2 \cdot BB(I) + CC(I) \\
Z(2,I) &= Z(1,I) + \frac{DZ0}{3} \\
Z(3,I) &= ZX(4,I) - \frac{DZ1}{3}.
\end{align*}
\]

ENDIF

500 CONTINUE

C------------------------ END OF DIMENSION LOOP ---------------------

1000 CONTINUE

C------------------------ THAT'S ALL ----------------------

STOP
END

C

SUBROUTINE TSOLV(A,B,C,D,N,BB)

DIMENSION A(N),B(N),C(N),D(N),BB(N)

C------------------------ ELIMINATE C'S ------------------------

DO 100 I = 2,N

\[
\begin{align*}
CBI &= \frac{C(I)}{B(I-1)} \\
B(I) &= B(I) - CBI \cdot A(I-1) \\
D(I) &= D(I) - CBI \cdot D(I-1)
\end{align*}
\]

100 CONTINUE

\[
BB(N) = \frac{D(N)}{B(N)}
\]

DO 200 I = N-1,1,-1

\[
BB(I) = (D(I) - A(I) \cdot BB(I+1)) / B(I)
\]

200 CONTINUE

RETURN
END