WASHINGTON UNIVERSITY

DEPARTMENT OF PHYSICS

LABORATORY FOR ULTRASONICS

St. Louis, Missouri 63130

"An Approach for Relating the Results of Quantitative Nondestructive Evaluation to Intrinsic Properties of High-Performance Materials"

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Principal Investigator:

Dr. James G. Miller

Professor of Physics

The NASA Technical Officer for this grant is:

Dr. Joseph S. Heyman

NASA Langley Research Center

Hampton, Virginia

I. Introduction

One of the most difficult problems the manufacturing community has faced during recent years has been to accurately assess the physical state of anisotropic high-performance materials by nondestructive means. Destructive testing of materials, although it supplies useful information for the design engineers in the development stage of a material, is inappropriate for ascertaining the state of the material during assembly testing and routine maintenance procedures. Conventional nondestructive ultrasonic measurement techniques, currently employed with much success, can be improved to yield even more information about the physical state of a material. In the past, communication between the physical science and engineering communities has been somewhat inadequate. Each community has valuable information to contribute towards the assessment of material integrity. Measured ultrasonic parameters can be related to the common engineering parameters to yield information useful to both communities.

In order to advance the design of ultrasonic nondestructive testing systems, a more fundamental understanding of how ultrasonic waves travel and interact within the anisotropic material is needed. The relationship between the ultrasonic and engineering parameters needs to be explored to understand their mutual dependence. One common denominator is provided by the elastic constants. Accurate measurements of these physical parameters can yield information useful in determining the physical integrity of the material. With this in mind, advanced ultrasonic measurement systems can be designed with the internal nature of the material in mind. The type of material and the very question the investigator is trying to answer should play a major role when setting up a measurement system.

In Section II of this Progress Report we discuss our preparation of specific graphite/epoxy samples to be used in the experimental interrogation of the anisotropic properties (through the measurement of the elastic stiffness constants). Accurate measurements of these constants will depend upon knowledge of refraction effects as well as the direction of group velocity propagation. Section III discusses our continuing effort for the development of improved visualization techniques for physical parameters. Group velocity images are presented and discussed. In order to fully understand the relationship between the ultrasonic and the common engineering parameters, Section IV discusses the physical interpretation of the linear elastic coefficients (the quantities that relate applied stresses to resulting strains). This discussion builds a more intuitional understanding of how the ultrasonic parameters are related to the traditional engineering parameters.

II. Sample Preparation and Preliminary Velocity Measurements

As discussed in previous Progress Reports, uniaxial graphite/epoxy laminates exhibit hexagonal symmetry. Five distinct elastic stiffness coefficients are required to describe the elastic properties of this material. Four of the five constants may be measured by propagating longitudinal and shear waves perpendicular and parallel to the fiber orientation. The fifth elastic constant must be measured at an angle that is neither parallel nor perpendicular to the fiber direction. In our Progress Report of 9/86 a discussion of the existence of a possible optimum angle at 77° with respect to the fiber orientation, for measurement of this fifth elastic constant, was predicted based on analysis of the propagation of errors.

We have prepared three uniaxial samples, received from NASA Langley Research Center, for the measurements. Two of the samples were surface ground so that their sides were parallel and perpendicular to the fiber orientation as illustrated in Figure 1.

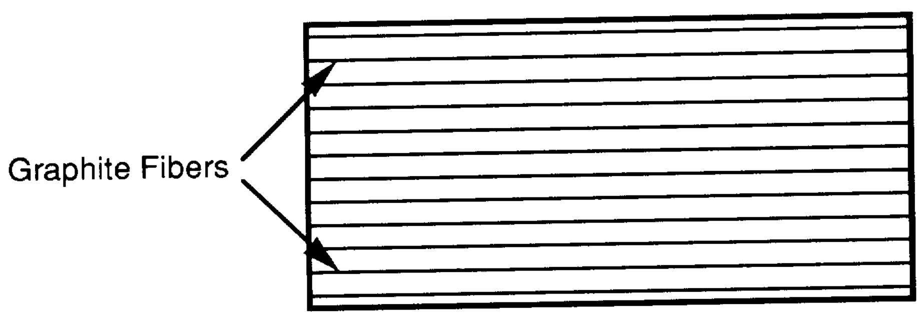


Figure 1: Two samples prepared for the propagation of longitudinal and shear waves parallel and perpendicular to the fiber orientation.

The third sample was prepared so that insonification normal to the surface will produce ultrasonic waves whose phase velocity will propagate inside the sample at an angle of 77° with respect to the fiber orientation. Figure 2 illustrates the two steel stages machined and used to prepare the third sample of graphite/epoxy. Also illustrated in Figure 2 is the finished sample.

Time resolution of the received ultrasonic signals requires an understanding of refraction effects and possible mode conversion at interfaces. An understanding of the directions of phase and group velocity propagation inside the material is also required (see Section III of this Progress Report). Using results obtained from linear elastic theory we are evaluating a preliminary design of the measurement system. A full report of these investigations will be forth coming.

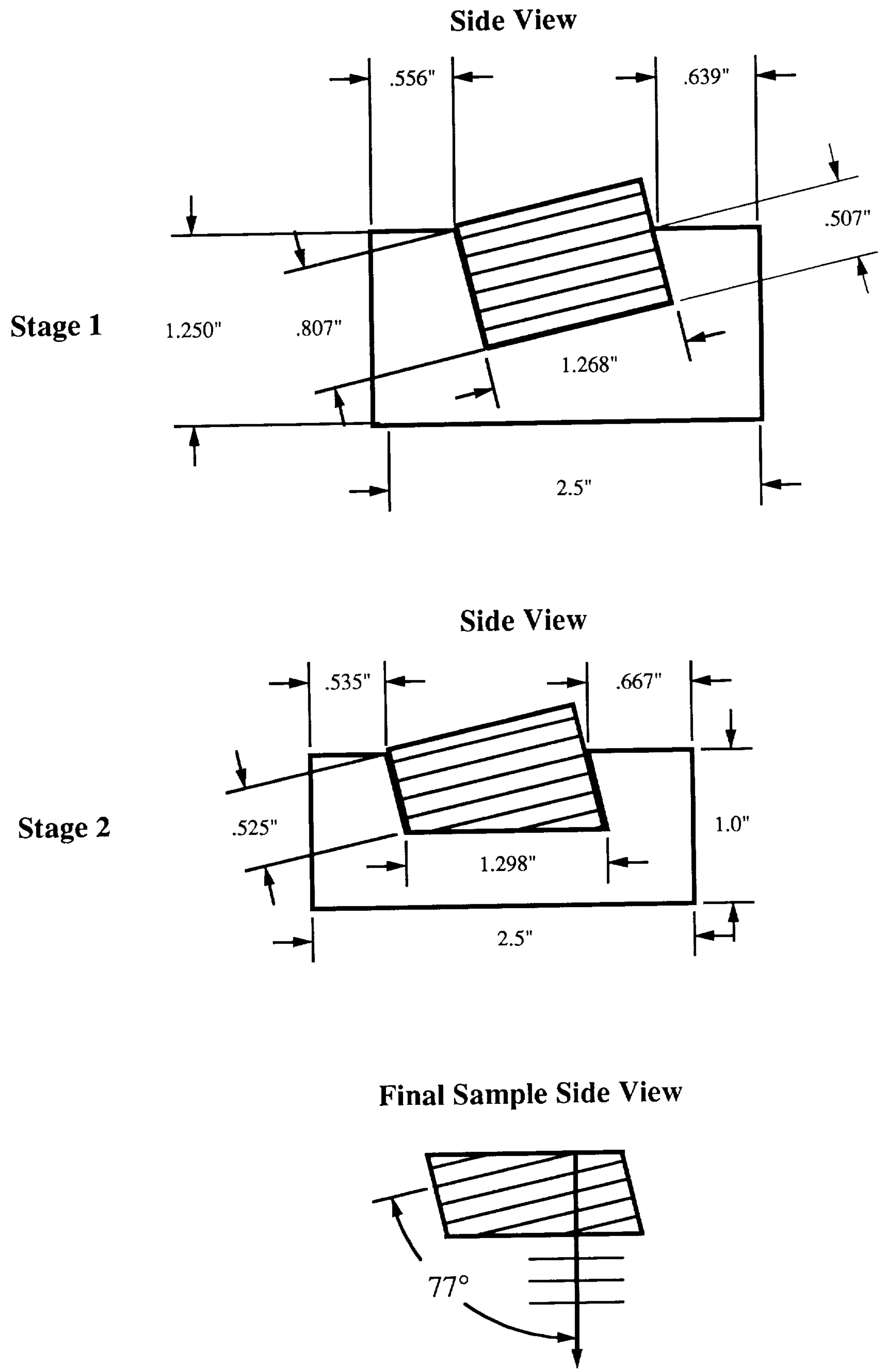


Figure 2: Preparation of a uniaxial graphite/epoxy sample for measurement of the fifth elastic constant

III. Visualization of Group Velocity Surfaces for Uniaxial Graphite/Epoxy Composites

For anisotropic materials the phase and group velocity need not be collinear. The group velocity is defined in terms of the modulation of the wavefronts. Consider an ultrasonic wave emitted from a conventional planar cylindrical cross section transducer. The wavepacket is restricted in two dimensions by the diameter of the transmitting transducer and in the third dimension by the ultrasonic pulse length. The wavefronts propagate in the direction of **k** (along the axis of the transducer) while the wavepacket propagates along the direction of group velocity (modulation envelope). If the phase and group velocities are not collinear, the receiving transducer will have to be offset with respect to the phase velocity direction in order to intercept the ultrasonic pulse. By using the information obtained from linear elastic theory we are able to predict the group velocity direction for a given initial phase velocity direction. This information will enable the investigator to incorporate such knowledge into the design of the measurement system. Although numerical calculations produce useful numbers, they do very little to raise the physical intuition of the investigator about the intrinsic nature of the material with which he is working. Our hypothesis is that a three dimensional representation of the group velocity surfaces for each allowed ultrasonic mode of propagation would greatly improve the insight one has when designing an advanced measurement system.

Using the fact that the fiber direction is an axis of six-fold symmetry, which implies that orthogonal planes to this axis are transversely isotropic, we have produced three dimensional group velocity surfaces for uniaxial graphite/epoxy composites. Figure 3 displays the group velocity surfaces for the three propagation modes.

The way to interpret these surfaces, for each propagation mode, is to place the observer inside the surface at the origin. When he/she looks in any direction, the length of the vector from the origin to a point on the surface represents the magnitude for the group velocity that propagates in that direction. In general for anisotropic materials, this does not necessarily result from the propagation of the phase velocity in this direction. We are currently working on a representation which will enable the viewer to more easily visualize the relationship between the initial phase velocity direction (i.e., the direction in which the wave is launched) and the resulting group velocity direction.

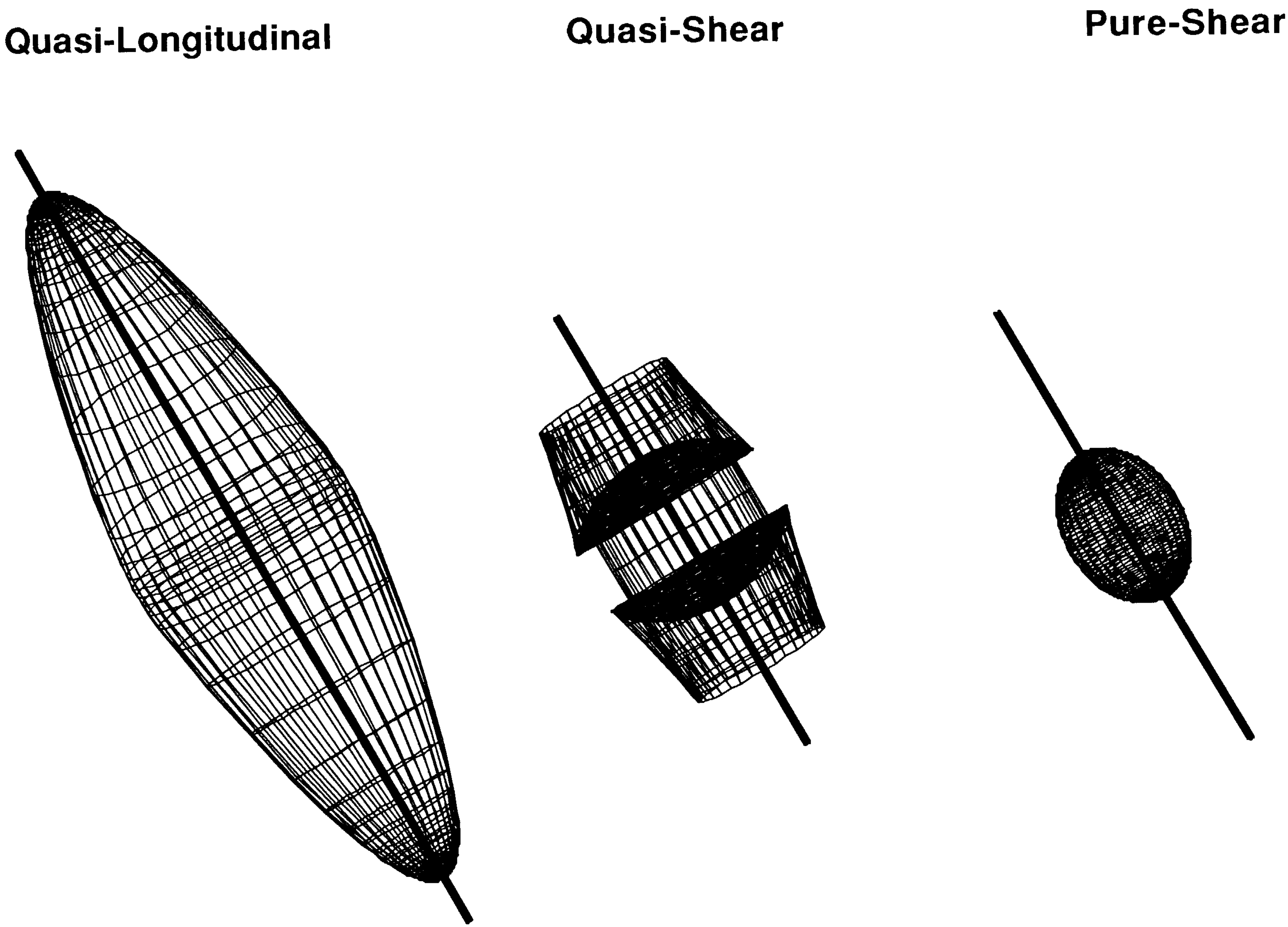


Figure 3: Uniaxial graphite/epoxy group velocity surfaces

IV. Physical Interpretation of the Elastic Matrices

In the previous Sections of this Progress Report we have stressed the importance of understanding the intrinsic nature of the material that is to be evaluated. The more we understand about the intrinsic physical properties of a material, the more intelligent the design of the measurement system can be. The ultimate goal is the assessment of material integrity by a nondestructive means. Ultrasonic measurement techniques have proven to be a very attractive approach for solving this problem. Information about the strengths and weaknesses of a material can be derived from ultrasonic parameters with the added feature that these parameters can be directly related to the traditional engineering parameters, such as Young's moduli, Poisson's ratios and the like.

In this Section the relationship between the stress and strain in a linear elastic material, via the elastic stiffness and compliance matrices, is discussed. A physical interpretation of the form of the elastic matrix as well as the individual components is presented. The relationship between the elastic coefficients and the Young's moduli, Poisson's ratios, and the pure-shear moduli is also discussed.

The Constitutive Relation: Hooke's Law

The constitutive relation, relating stress and strain, in the most general case requires 81 coefficients to describe the elastic properties of a linear elastic solid. For a lossless material the stress (T_{ij}) is proportional to the strain (S_{kl}) via the real-valued elastic stiffness coefficients (c_{ijkl}).

$$T_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} c_{ijkl} S_{kl} \qquad (i,j = 1,2,3)$$

$$\text{or} \quad \mathbf{T} = \mathbf{c} : \mathbf{S} \tag{1}$$

Stress denotes a force acting on an area. It is useful to consider the stress tensor in dyadic form to explicitly show the direction of the applied force and the surface area to which it is applied. This representation, common in engineering literature, yields a more physical interpretation of the stress tensor. A dyad is defined as the direct multiplication of two vector quantities to produce a nine component 2nd-rank tensor quantity. We can write the stress tensor in dyadic form as

$$\mathbf{T} = \mathbf{F}\frac{\hat{\mathbf{n}}}{\text{Area}} = \langle F_1, F_2, F_3\rangle \left\langle \frac{1}{A_1}, \frac{1}{A_2}, \frac{1}{A_3} \right\rangle \quad \text{(a direct product)} \quad , \tag{2}$$

or in matrix form as

$$[T_{ij}] = \begin{bmatrix} \frac{F_1}{A_1} & \frac{F_1}{A_2} & \frac{F_1}{A_3} \\ \frac{F_2}{A_1} & \frac{F_2}{A_2} & \frac{F_2}{A_3} \\ \frac{F_3}{A_1} & \frac{F_3}{A_2} & \frac{F_3}{A_3} \end{bmatrix}$$

where F_i is the component of force in the ith direction and A_j is the area of the surface whose normal is in the jth direction . (3)

In Figure (4) the notation for the applied stresses on a unit cube is illustrated. The first subscript denotes the direction the force is acting; the second denotes the direction of the normal vector to the face of the cube on which the force is applied.

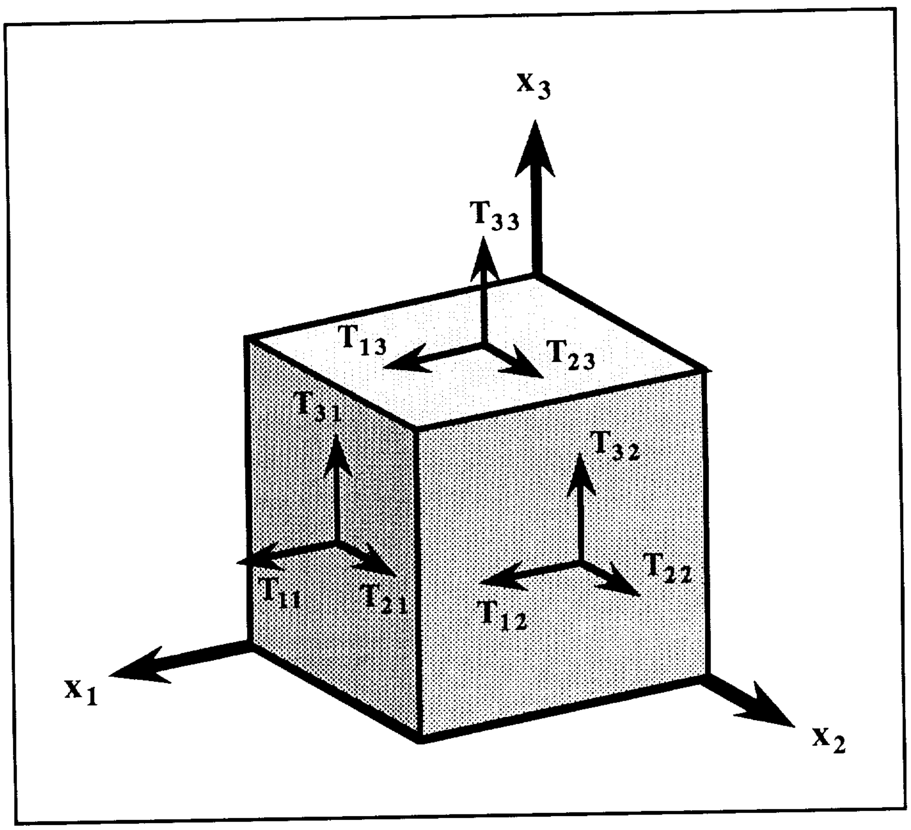


Figure 4: The stress components acting on each face of a unit cube within a stressed body (ijkl system).

Figure 4 along with the previous discussion demonstrates the necessity for the stress being a double subscript quantity.

From classical linear elastic theory, the maximum number of distinct elastic constants for a material is reduced from 81 to 36 due to the symmetry of the stress and strain tensors. That is,

$$T_{ij} = T_{ji} \quad \text{and} \quad S_{ij} = S_{ji}$$

from which it necessarily follows that

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{jilk} \quad \text{and} \quad s_{ijkl} = s_{jikl} = s_{ijlk} = s_{jilk} ,$$

which is obtained by permuting the indices in pairs.

Abbreviated Subscript Notation:

Due to the symmetry of both the stress and strain tensors, there are at most 36 distinct elastic coefficients. Since many of the 81 coefficients are equivalent a reduced notation is possible. Using the abbreviated subscript notation developed by Voigt, we can rewrite the double indexed quantities as single indexed quantities as displayed in Table 1.

Voigt Notation	
$T_1 \equiv T_{11}$	$T_4 \equiv T_{23} = T_{32}$
$T_2 \equiv T_{22}$	$T_5 \equiv T_{13} = T_{31}$
$T_3 \equiv T_{33}$	$T_6 \equiv T_{12} = T_{21}$
$S_1 \equiv S_{11}$	$S_4 \equiv 2\,S_{23} = 2\,S_{32}$
$S_2 \equiv S_{22}$	$S_5 \equiv 2\,S_{13} = 2\,S_{31}$
$S_3 \equiv S_{33}$	$S_6 \equiv 2\,S_{12} = 2\,S_{21}$

Table 1

Following the common convention, we can write the elastic coefficients in reduced notation, taking into account the numerical factors, as

$$c_{IJ} \Leftrightarrow c_{ijkl} \qquad \text{and} \qquad s_{IJ} \Leftrightarrow s_{ijkl} \times \begin{Bmatrix} 1 \text{ for } I \text{ and } J = 1,2,3 \\ 2 \text{ for } I \text{ or } J = 4,5,6 \\ 4 \text{ for } I \text{ and } J = 4,5,6 \end{Bmatrix} . \tag{4}$$

In the following discussion, quantities with lower case subscripts will refer to an 81 element c_{ijkl} system while upper case subscripts, c_{IJ}, will refer to the Voigt notation. Figure 5 illustrates the stress components acting on each face of a unit cube within a stressed body in the Voigt notation.

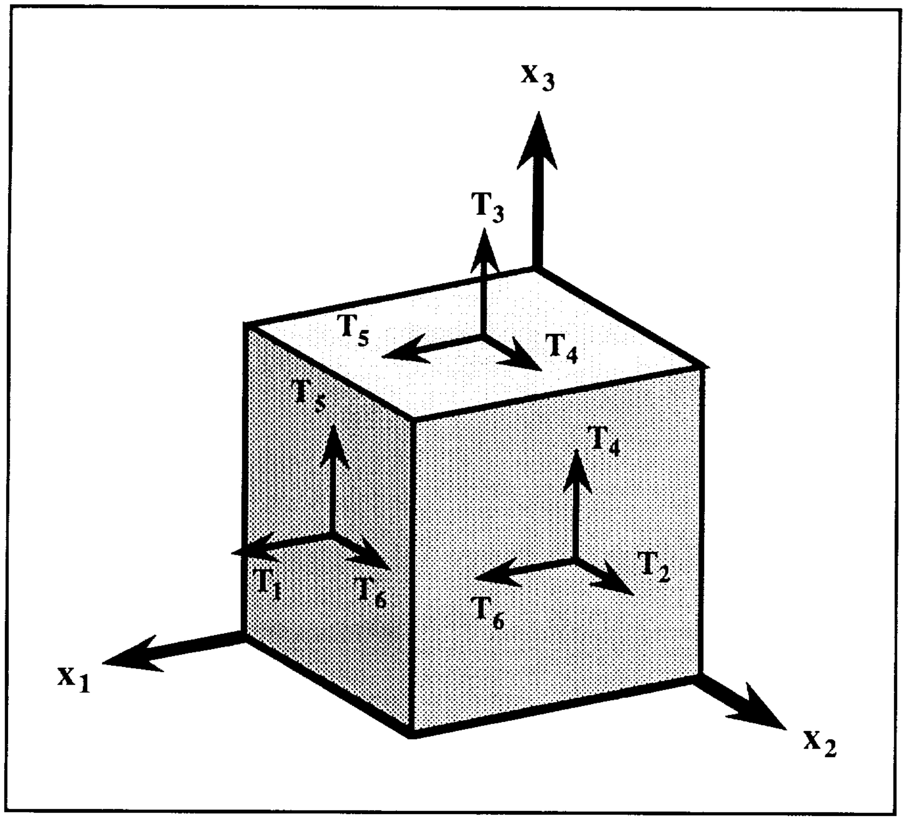


Figure 5: The stress components acting on each face of a unit cube within a stressed body (IJ Voigt system).

The constitutive relation may be rewritten as

$$T_I = \sum_{J=1}^{6} c_{IJ} S_J \qquad (I = 1,2,3,4,5,6) \quad , \tag{5}$$

where c_{IJ} represents the elastic stiffness coefficients.

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \tag{6}$$

If a localized strain-energy function exists then the elastic matrix that describes the solid is symmetric. Thus, the 36 elastic stiffness constants can be reduced to 21 distinct constants. A material which has an elastic matrix of this form is called a triclinic system and represents a very general class of materials which exhibit no elastic symmetry.

Reduction of Elastic Constants

The number of distinct elastic coefficients needed to describe the elastic properties of a material can be reduced if the material exhibits elastic symmetries. For example, a large class of real-world materials exhibit orthotropic symmetry. These materials possess three mutually orthogonal planes of elastic symmetry and have at most 12 distinct elastic coefficients. Orthorhombic, tetragonal, hexagonal, cubic, and isotropic systems all exhibit orthotropic symmetry. We should note that the number of distinct coefficients will be reduced only if the coordinate axes chosen are aligned properly with the high-symmetry directions. Thus, an improper choice of coordinate axes will generally result in an increase in the apparent number of distinct elastic coefficients needed to describe the elastic properties of the material. In reality many of the coefficients will be linear combinations of the smallest set of coefficients obtained when the coordinate axes are chosen to be aligned with the principal axes. In Table 2 information is given for 7 linear elastic systems.

The following discussion will demonstrate how a material possessing orthotropic symmetry reduces the number of elastic coefficients and produces an elastic matrix having a specific characteristic form when the material is aligned with respect to the principal axes of the system.

Orthotropic System Derivation:

The reduction of the number of distinct elastic coefficients can be illustrated by first considering a general rotation transformation **a**,

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \qquad (7)$$

Application of this transformation to the stress and strain tensors yields the following relations between the rotated (primed) and nonrotated (unprimed) components of these tensors.

$$\mathbf{T'} = \mathbf{a\,T\,a}^{\dagger} \quad \text{and} \quad \mathbf{S'} = \mathbf{a\,S\,a}^{\dagger} \qquad (8)$$

These relations can be written in matrix form as

$$T'_{mn} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{mi}\,T_{ij}\,a_{nj} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{mi}\,a_{nj}\,T_{ij} \qquad (9)$$

and

Triclinic System: 21 coefficients (no elastic symmetries)

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix}$$

Monoclinic System: 13 coefficients (only the x_1x_3 plane is a plane of elastic symmetry)

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ c_{12} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{12} & c_{23} & c_{33} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & c_{44} & 0 & c_{46} \\ c_{15} & c_{25} & c_{35} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & c_{46} & 0 & c_{66} \end{bmatrix}$$

Orthotropic Systems

All systems below have 3 mutually orthogonal planes of elastic symmetry

General Orthotropic Form: 12 coefficients

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

Orthorhombic System: 9 coefficients (rectangular packing)

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$

Tetragonal System: 6 coefficients (square packing)

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{12} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}$$

Hexagonal System: 5 coefficients (six-fold axis aligned with x_1 axis)

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{12} & c_{23} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{c_{22} - c_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{55} \end{bmatrix}$$

Cubic System: 3 coefficients

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}$$

Isotropic System: 2 coefficients

Same form as the cubic system except

$$c_{12} = c_{11} - 2\,c_{44}$$

Table 2: Linear elastic systems for which a localized strain-energy function exists.

$$S'_{op} = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ok} S_{kl} a_{pl} = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ok} a_{pl} S_{kl} \quad . \tag{10}$$

Since we want the elastic stiffness coefficients to be invariant under the transformation, we make use of the constitutive relation and impose the condition that

$$\mathbf{T'} = \mathbf{c'} : \mathbf{S'} \quad \Rightarrow \quad \mathbf{T'} = \mathbf{c} : \mathbf{S'} \quad . \tag{11}$$

Equation (11) yields two sets of equations which when compared will determine which coefficients are to be eliminated.

The following algorithm sketches the necessary steps to be performed in order to reduce the number of distinct coefficients needed to describe an orthotropic system.

Step 1:

Write out the primed (transformed) stress and strain elements in 3x3 form as a function of the unprimed variables.

$$T'_{mn} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{mi} T_{ij} a_{nj} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{mi} a_{nj} T_{ij} \tag{12}$$

$$S'_{op} = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ok} S_{kl} a_{pl} = \sum_{k=1}^{3} \sum_{l=1}^{3} a_{ok} a_{pl} S_{kl} \tag{13}$$

Step 2:

Rewrite the sets of equations obtained in Step 1 using the Voigt notation.

Step 3:

Write the primed constitutive relation in component form.

$$T'_{I} = \sum_{J=1}^{6} c_{IJ} S'_{J} \tag{14}$$

Replace the S'_I's with equivalent S_J terms as given in Step 2.
Replace the T'_I's with equivalent T_J terms as given in Step 2.
We now have the six primed equations totally in terms of the unprimed variables.

Step 4:

Compare the set of equations from Step 3 with the original unprimed constitutive equations to determine which c_{IJ} are to be set to zero.

$$T_I = \sum_{J=1}^{6} c_{IJ} S_J \tag{15}$$

Planes of Elastic Symmetry:

A plane of elastic symmetry exists at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane. The axes of such coordinate systems are referred to as *equivalent elastic directions*. If the x_1x_3 plane is one of elastic symmetry, the constants c_{IJ} are invariant under the coordinate transformation

$$x'_1 = x_1, \qquad x'_2 = -x_2, \qquad x'_3 = x_3, \tag{16}$$

which maps a right-handed coordinate system into a left-handed system as illustrated in Figure 6.

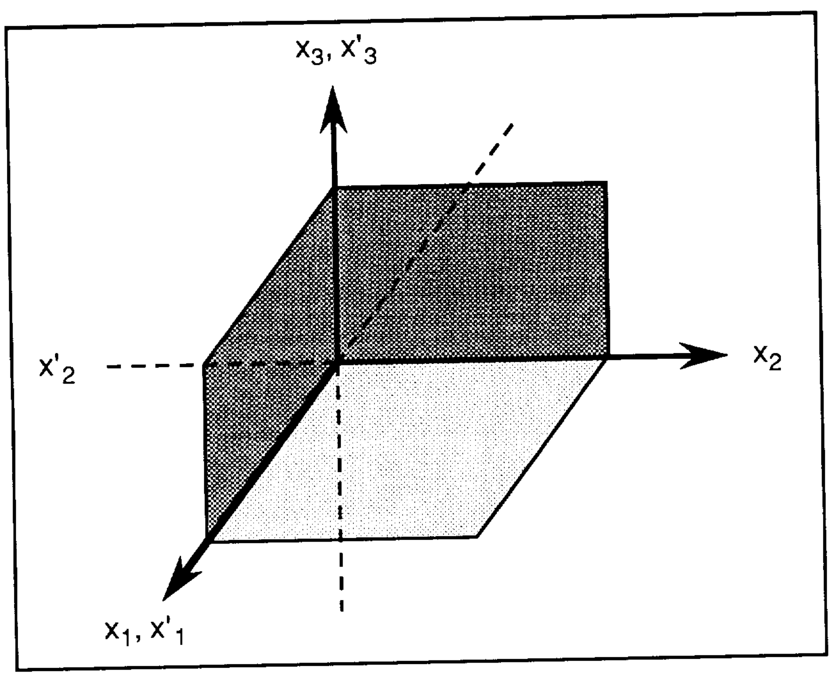


Figure 6: Geometry for illustration of an x_1x_3 plane of elastic symmetry

Let the x_1x_3 (or equivalently, $x'_1x'_3$) plane be a plane of elastic symmetry. Then $T_I = c_{IJ} S_J$ and $T'_I = c_{IJ} S'_J$ (J = 1,2,3,4,5,6). The transformation matrix for mapping x_2 to x'_2 is given by

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \qquad (17)$$

The stress and strain transformation laws yield the following relations,

$$\begin{array}{lll} T'_{11} = T_{11}, & T'_{12} = -T_{12}, & T'_{13} = T_{13} \\ T'_{21} = -T_{21}, & T'_{22} = T_{22}, & T'_{23} = -T_{23} \\ T'_{31} = T_{31}, & T'_{32} = -T_{32}, & T'_{33} = T_{33} \end{array} \qquad (18)$$

$$\begin{array}{lll} S'_{11} = S_{11}, & S'_{12} = -S_{12}, & S'_{13} = S_{13} \\ S'_{21} = -S_{21}, & S'_{22} = S_{22}, & S'_{23} = -S_{23} \\ S'_{31} = S_{31}, & S'_{32} = -S_{32}, & S'_{33} = S_{33} \end{array} \qquad (19)$$

which imply

$$\left\{ \begin{array}{l} T'_I = T_I \\ S'_I = S_I \end{array} \right\} \text{ for } I = 1,2,3,5 \quad \text{and} \quad \left\{ \begin{array}{l} T'_I = -T_I \\ S'_I = -S_I \end{array} \right\} \text{ for } I = 4,6 \; . \qquad (20)$$

Using the information in Equation (20) we are now able to write the primed variables totally in terms of the unprimed variables.

Primed Constitutive Equations:

$$\begin{array}{lll} T'_1 & \Rightarrow & T_1 = c_{11} S_1 + c_{12} S_2 + c_{13} S_3 - c_{14} S_4 + c_{15} S_5 - c_{16} S_6 \\ T'_2 & \Rightarrow & T_2 = c_{21} S_1 + c_{22} S_2 + c_{23} S_3 - c_{24} S_4 + c_{25} S_5 - c_{26} S_6 \\ T'_3 & \Rightarrow & T_3 = c_{31} S_1 + c_{32} S_2 + c_{33} S_3 - c_{34} S_4 + c_{35} S_5 - c_{36} S_6 \\ T'_4 & \Rightarrow & T_4 = -c_{41} S_1 - c_{42} S_2 - c_{43} S_3 + c_{44} S_4 - c_{45} S_5 + c_{46} S_6 \\ T'_5 & \Rightarrow & T_5 = c_{51} S_1 + c_{52} S_2 + c_{53} S_3 - c_{54} S_4 + c_{55} S_5 - c_{56} S_6 \\ T'_6 & \Rightarrow & T_6 = -c_{61} S_1 - c_{62} S_2 - c_{63} S_3 + c_{64} S_4 - c_{65} S_5 + c_{66} S_6 \end{array} \qquad (21)$$

The relations given in Equation (21) must be equivalent to the unprimed stress/strain relations in the nonrotated coordinate system.

Unprimed Constitutive Equations:

$$\begin{aligned}
T_1 &= c_{11} S_1 + c_{12} S_2 + c_{13} S_3 + c_{14} S_4 + c_{15} S_5 + c_{16} S_6 \\
T_2 &= c_{21} S_1 + c_{22} S_2 + c_{23} S_3 + c_{24} S_4 + c_{25} S_5 + c_{26} S_6 \\
T_3 &= c_{31} S_1 + c_{32} S_2 + c_{33} S_3 + c_{34} S_4 + c_{35} S_5 + c_{36} S_6 \\
T_4 &= c_{41} S_1 + c_{42} S_2 + c_{43} S_3 + c_{44} S_4 + c_{45} S_5 + c_{46} S_6 \\
T_5 &= c_{51} S_1 + c_{52} S_2 + c_{53} S_3 + c_{54} S_4 + c_{55} S_5 + c_{56} S_6 \\
T_6 &= c_{61} S_1 + c_{62} S_2 + c_{63} S_3 + c_{64} S_4 + c_{65} S_5 + c_{66} S_6
\end{aligned} \tag{22}$$

Therefore, we obtain from Equations (21) and (22) that

$$\begin{aligned}
T'_1 &\Rightarrow c_{14} = c_{16} = 0 \\
T'_2 &\Rightarrow c_{24} = c_{26} = 0 \\
T'_3 &\Rightarrow c_{34} = c_{36} = 0 \\
T'_4 &\Rightarrow c_{41} = c_{42} = c_{43} = c_{45} = 0 \\
T'_5 &\Rightarrow c_{54} = c_{56} = 0 \\
T'_6 &\Rightarrow c_{61} = c_{62} = c_{63} = c_{65} = 0
\end{aligned} \tag{23}$$

and the elastic stiffness matrix reduces to

$$[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ c_{12} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{12} & c_{23} & c_{33} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & c_{44} & 0 & c_{46} \\ c_{15} & c_{25} & c_{35} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & c_{46} & 0 & c_{66} \end{bmatrix} . \tag{24}$$

Inspection of the elastic matrix representation for a material with one plane of elastic symmetry (x_1x_3 plane) yields the matrix form for a monoclinic system as displayed in Table 2.

By repeating the procedure, this time requiring the x_1x_2 plane to be a plane of elastic symmetry, the following elastic stiffness coefficients can be shown to be zero.

$$
\begin{aligned}
T''_1 &\Rightarrow c_{15} = c_{16} = 0 \\
T''_2 &\Rightarrow c_{25} = c_{26} = 0 \\
T''_3 &\Rightarrow c_{35} = c_{36} = 0 \\
T''_4 &\Rightarrow c_{45} = c_{46} = 0 \\
T''_5 &\Rightarrow c_{51} = c_{52} = c_{53} = c_{54} = 0 \\
T''_6 &\Rightarrow c_{61} = c_{62} = c_{63} = c_{64} = 0
\end{aligned}
\tag{25}
$$

The elastic stiffness matrix for a medium having an x_1x_3 and x_1x_2 elastic plane of symmetry reduces to

$$
[c_{IJ}] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} . \tag{26}
$$

The elastic symmetry with respect to the (third) x_2x_3 plane is identically satisfied by this matrix. That is, if we have two orthogonal planes of elastic symmetry with respect to principal axes, we must necessarily have the third. Inspection of Equation (26) shows that for an orthotropic material aligned along the principal axes, the elements of the upper-right and lower-left quadrants, and the off-diagonal elements of the lower-right quadrant are zero for the elastic matrix. The physical significance of these elements being zero can be understood by again returning to the constitutive relation.

Inversion of the Constitutive Relation:

To understand the physical role played by the elastic coefficients we must first transform the constitutive relation into a more convenient form. That is, we must obtain a form where we express the strain as a function of stress. This will allow us to systematically apply stresses along specific directions in order to ascertain which coefficients are involved in producing the strains and what types of strains result from a specific applied stress. We can invert the constitutive relation by multiplying each side of Equation (1) by the inverted stiffness matrix

$$
[\mathbf{c}]^{-1} : \mathbf{T} = [\mathbf{c}]^{-1}\, \mathbf{c} : \mathbf{S} = \mathbf{S} \qquad \Rightarrow \qquad \mathbf{S} = \mathbf{s} : \mathbf{T} , \tag{27}
$$

where $[s] \equiv [c]^{-1}$ is defined as the compliance coefficients. Equation (27) is written in matrix form as

$$S_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} s_{ijkl}\, T_{kl} \qquad (i,j = 1,2,3)$$

$$S_I = \sum_{J=1}^{6} s_{IJ}\, T_J \qquad (I = 1,2,3,4,5,6) \quad , \tag{28}$$

where for the most general case

$$[s_{IJ}] = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} . \tag{29}$$

Elucidating the Relationship Between Physical and Engineering Parameters

We can obtain a better understanding of the physical significance each of the elements of the compliance matrix plays by investigating their dependence on the common engineering parameters for an orthotropic system.

Young's Moduli:

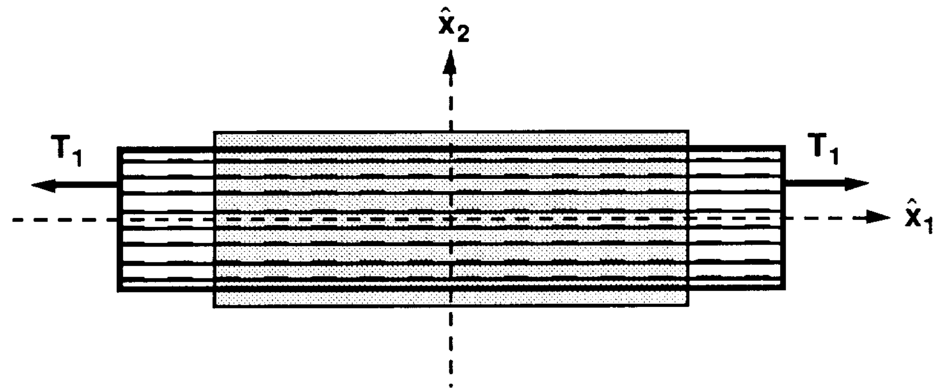
A Young's modulus describes a physical phenomenon resulting from a single direction operation. That is, we apply a tensile/compressive normal stress along a given principal axis direction and observe the resulting strain along the same axis direction (see Figure 7).

$$\textbf{Young's Modulus} \equiv \frac{\text{applied tensile/compressive normal stress}}{\text{resulting tensile/compressive normal strain}} \tag{30}$$

(both along the same principal axis direction)

Consider, for example, the application of a tensile stress T_1 (T_{11}) along the x_1 direction, with all other stresses zero. Then

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} T_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{31}$$



$$S_{11} = s_{1111}\, T_{11} \Rightarrow S_1 = s_{11}\, T_1$$
$$S_{22} = s_{2211}\, T_{11} \Rightarrow S_2 = s_{21}\, T_1$$
$$S_{33} = s_{3311}\, T_{11} \Rightarrow S_3 = s_{31}\, T_1$$

Figure 7: Resulting strains produced by a tensile normal stress T_1 (T_{11}).

By inspection we have

$$S_1 = s_{11}\, T_1 \tag{32.a}$$

$$S_2 = s_{21}\, T_1 \tag{32.b}$$

$$S_3 = s_{31}\, T_1 \tag{32.c}$$

and $S_4 = S_5 = S_6 = 0$. The first of these equations allows us to express the Young's modulus in terms of the compliance matrix elements.

$$E_1 \equiv \frac{T_1}{S_1}\Big|_{T_1} = \frac{1}{s_{11}} \tag{33}$$

By an analogous procedure we obtain

$$E_I \equiv \frac{T_I}{S_I}\Big|_{T_I} = \frac{1}{s_{II}} \quad \Rightarrow \quad s_{II} = \frac{1}{E_I}, \quad (I = 1,2,3) \tag{34}$$

As seen from Equation (34) the Young's moduli are the reciprocals of the diagonal elements of the upper-left quadrant of the compliance matrix.

Poisson Ratios:

The Poisson ratios describe an operation which involves two orthogonal principal axes. We apply a stress along a given principal axis and observe the ratio of the resulting strain along a principal axis perpendicular to the stress axis to the resulting strain along the stress axis (see Figure 7).

$$\textbf{Poisson Ratio} \equiv \left[- \frac{\text{resulting lateral strain}}{\text{resulting tensile/compressive normal strain}} \right]$$

$$\text{(due to an applied tensile/compressive normal stress)} \qquad (35)$$

From Equation (32.b) we obtain

$$\nu_{21} \equiv - \frac{S_2}{S_1}\Big|_{T_1} = - \frac{s_{21}\, T_1}{s_{11}\, T_1} = - \frac{s_{21}}{s_{11}} = - s_{21}\, E_1$$

$$\Rightarrow \quad s_{21} = - \frac{\nu_{21}}{E_1} \qquad . \qquad (36)$$

A general relationship between the Poisson ratios and the compliance elements can be written as

$$\nu_{JI} \equiv - \frac{S_J}{S_I}\Big|_{T_I} = - \frac{s_{JI}\, T_I}{s_{II}\, T_I} = - \frac{s_{JI}}{s_{II}} = - s_{JI}\, E_I$$

$$\Rightarrow \quad s_{JI} = - \frac{\nu_{JI}}{E_I}, \qquad (I,J = 1,2,3; \; I \neq J) \qquad . \qquad (37)$$

From Equation (37) we see that a specific Poisson ratio is obtained from a multiplication of the corresponding compliance element and the appropriate Young's modulus.

Shear Moduli:

A pure-shear strain results from the application of two diametrically opposing planar tangential stresses of equal magnitude acting on adjacent orthogonal surfaces to produce a pure distortion (see Figure 8). The direction of the applied forces are tangential to the surfaces to which they are applied. A shear strain is defined as

$$\textbf{Shear Modulus} \equiv \frac{\text{applied planar tangential stress}}{\text{resulting planar tangential strain}}, \qquad (38)$$

where the applied stress and the resulting strain are in the same plane.

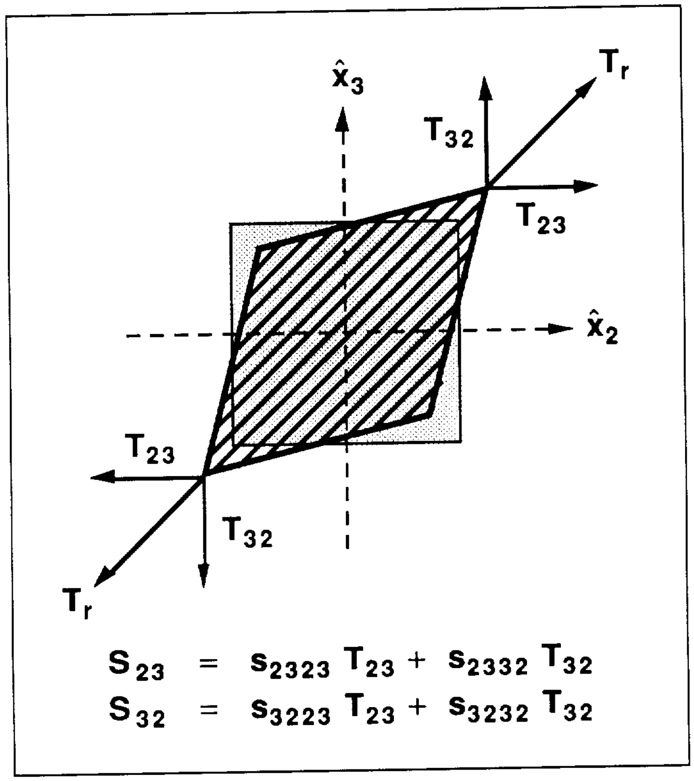


Figure 8: Pure-shear strain in the x_2x_3 plane

Consider, for example, the application of planar tangential stresses T_4 (T_{23}, T_{32}) in the x_2x_3 plane, with all other stresses zero.

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ T_4 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

Multiplication of the matrices yields the following stress/strain relation for a pure-shear strain in the x_2x_3 plane,

$$S_4 = s_{44}\,T_4\,, \tag{40}$$

where $S_1 = S_2 = S_3 = S_5 = S_6 = 0$. From Equation (40) we see that the pure-shear modulus for the x_2x_3 plane is equal to

$$G_{23} \equiv \frac{T_4}{S_4}\bigg|_{T_4} = \frac{1}{s_{44}}\,. \tag{41}$$

By an analogous procedure we see that the diagonal elements of the lower-right quadrant of the compliance matrix are related to the pure-shear strains in a reciprocal manner.

$$G_{23} \equiv \frac{T_4}{S_4}\Big|_{T_4} = \frac{1}{s_{44}} \quad \Rightarrow \quad s_{44} = \frac{1}{G_{23}} \tag{42}$$

$$G_{13} \equiv \frac{T_5}{S_5}\Big|_{T_5} = \frac{1}{s_{55}} \quad \Rightarrow \quad s_{55} = \frac{1}{G_{13}} \tag{43}$$

$$G_{12} \equiv \frac{T_6}{S_6}\Big|_{T_6} = \frac{1}{s_{66}} \quad \Rightarrow \quad s_{66} = \frac{1}{G_{12}} \tag{44}$$

Combining the information obtained above we can rewrite the compliance matrix of an orthotropic system in terms of the engineering parameters as

$$[s_{IJ}] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} . \tag{45}$$

For orthotropic materials we have seen that many of the elastic coefficients are zero when the system is aligned with the principal axes. Some of these coefficients may not be zero if the material is not aligned. But since these coefficients will be linear combinations of the smallest set of distinct coefficients, the response of the material will also be a linear combination of the responses described above.

For example, consider an orthotropic material prepared in the shape of a cube for which one side is aligned with the x_2 principal axis while the two adjacent orthogonal surfaces are at an angle of 45° degrees with respect to the x_1 and x_3 principal axes of the system. Figure 9 illustrates the geometry of the material. By application of the Bond transformation matrix we obtain the following matrix for s'_{IJ}.

$$[s'_{IJ}] = \begin{bmatrix} s'_{11} & s'_{12} & s'_{13} & 0 & s'_{15} & 0 \\ s'_{21} & s'_{22} & s'_{23} & 0 & s'_{25} & 0 \\ s'_{31} & s'_{32} & s'_{33} & 0 & s'_{35} & 0 \\ 0 & 0 & 0 & s'_{44} & 0 & s'_{46} \\ s'_{51} & s'_{52} & s'_{53} & 0 & s'_{55} & 0 \\ 0 & 0 & 0 & s'_{64} & 0 & s'_{66} \end{bmatrix} \tag{46}$$

Each of the s'_{IJ} coefficients can be written as a linear combination of the smallest set of distinct compliance coefficients that describe the material when aligned with the principal axes.

$$\boxed{\begin{array}{ll}
s'_{11} = \frac{1}{4}(s_{11} + s_{13} + s_{31} + s_{33} + 4\,s_{55}) & s'_{41} = 0 \\
s'_{12} = \frac{1}{2}(s_{12} + s_{32}) & s'_{42} = 0 \\
s'_{13} = \frac{1}{4}(s_{11} + s_{13} + s_{31} + s_{33} - 4\,s_{55}) & s'_{43} = 0 \\
s'_{14} = 0 & s'_{44} = \frac{1}{2}(s_{44} + s_{66}) \\
s'_{15} = \frac{1}{4}(s_{11} - s_{13} + s_{31} - s_{33}) & s'_{45} = 0 \\
s'_{16} = 0 & s'_{46} = \frac{1}{2}(s_{66} - s_{44}) \\
s'_{21} = \frac{1}{2}(s_{21} + s_{23}) & s'_{51} = \frac{1}{4}(s_{11} + s_{13} - s_{31} - s_{33}) \\
s'_{22} = s_{22} & s'_{52} = \frac{1}{2}(s_{12} - s_{32}) \\
s'_{23} = \frac{1}{2}(s_{21} + s_{23}) & s'_{53} = \frac{1}{4}(s_{11} + s_{13} - s_{31} - s_{33}) \\
s'_{24} = 0 & s'_{54} = 0 \\
s'_{25} = \frac{1}{2}(s_{21} - s_{23}) & s'_{55} = \frac{1}{4}(s_{11} - s_{13} - s_{31} + s_{33}) \\
s'_{26} = 0 & s'_{56} = 0 \\
s'_{31} = \frac{1}{4}(s_{11} + s_{13} + s_{31} + s_{33} - 4\,s_{55}) & s'_{61} = 0 \\
s'_{32} = \frac{1}{2}(s_{12} + s_{32}) & s'_{62} = 0 \\
s'_{33} = \frac{1}{4}(s_{11} + s_{13} + s_{31} + s_{33} + 4\,s_{55}) & s'_{63} = 0 \\
s'_{34} = 0 & s'_{64} = \frac{1}{2}(s_{66} - s_{44}) \\
s'_{35} = \frac{1}{4}(s_{11} - s_{13} + s_{31} - s_{33}) & s'_{65} = 0 \\
s'_{36} = 0 & s'_{66} = \frac{1}{2}(s_{44} + s_{66})
\end{array}} \tag{47}$$

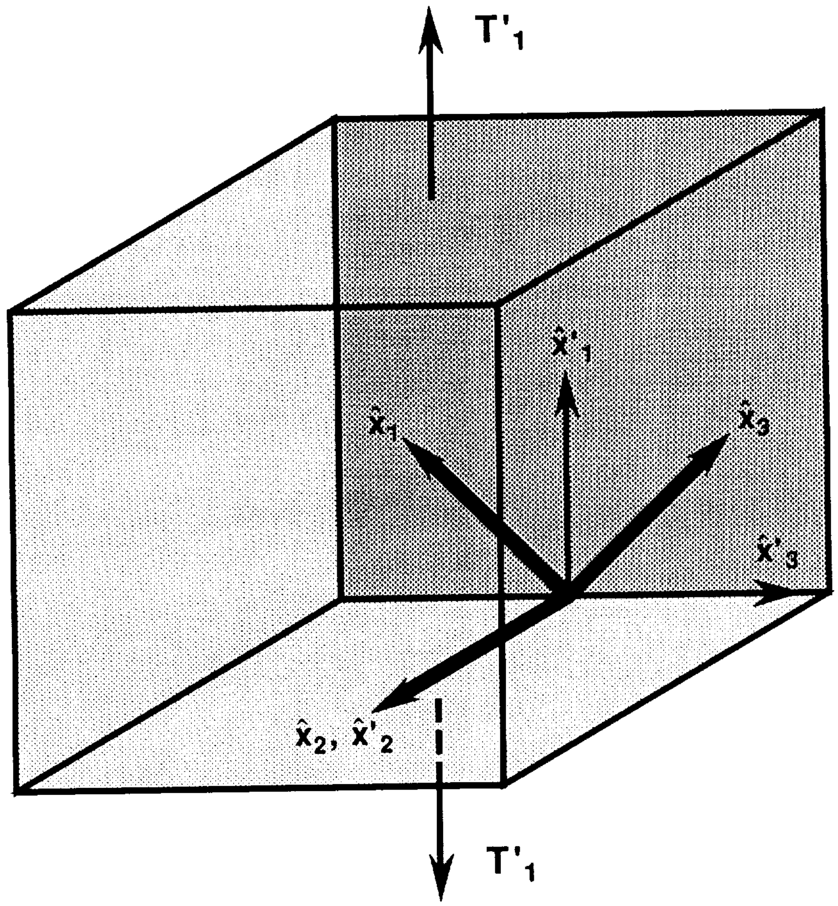


Figure 9: Application of a tensile normal stress T'_1 (T'_{11}) to an orthotropic material. The principal axes for the material is represented by the bold unprimed set of axes.

Now consider the application of a tensile normal stress, T'_1 (T'_{11}), along the x'_1 axis with all the other stresses zero (see Figure 9).

$$\begin{bmatrix} S'_1 \\ S'_2 \\ S'_3 \\ S'_4 \\ S'_5 \\ S'_6 \end{bmatrix} = \begin{bmatrix} s'_{11} & s'_{12} & s'_{13} & 0 & s'_{15} & 0 \\ s'_{12} & s'_{22} & s'_{23} & 0 & s'_{25} & 0 \\ s'_{13} & s'_{23} & s'_{33} & 0 & s'_{35} & 0 \\ 0 & 0 & 0 & s'_{44} & 0 & s'_{46} \\ s'_{15} & s'_{25} & s'_{35} & 0 & s'_{55} & 0 \\ 0 & 0 & 0 & s'_{46} & 0 & s'_{66} \end{bmatrix} \begin{bmatrix} T'_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{48}$$

Expanding Equation (48) the resultant strains can be explicitly written as

$$S'_1 = s'_{11}\ T'_1 = \frac{1}{4}\,(s_{11} + s_{13} + s_{31} + s_{33} + 4\,s_{55})\ T'_1\ , \tag{49}$$

$$S'_2 = s'_{21}\,T'_1 = \frac{1}{2}\,(s_{21} + s_{23})\,T'_1 \quad , \tag{50}$$

$$S'_3 = s'_{31}\,T'_1 = \frac{1}{4}\,(s_{11} + s_{13} + s_{31} + s_{33} - 4\,s_{55})\,T'_1 \quad , \tag{51}$$

$$S'_5 = s'_{51}\,T'_1 = \frac{1}{4}\,(s_{11} + s_{13} - s_{31} - s_{33})\,T'_1 \quad , \tag{52}$$

and $S'_4 = S'_6 = 0$. Equations (49-52) can also be written in terms of the engineering moduli which more vividly illustrates that the response of the material is a combination of bulging, constricting, and shearing effects.

$$S'_1 = s'_{11}\,T'_1 = \frac{1}{4}\left(\frac{1}{E_1} - \frac{\nu_{13}}{E_3} - \frac{\nu_{31}}{E_1} + \frac{1}{E_3} + \frac{4}{G_{13}}\right)T'_1 \tag{53}$$

$$S'_2 = s'_{21}\,T'_1 = -\frac{1}{2}\left(\frac{\nu_{21}}{E_1} + \frac{\nu_{23}}{E_3}\right)T'_1 \tag{54}$$

$$S'_3 = s'_{31}\,T'_1 = \frac{1}{4}\left(\frac{1}{E_1} - \frac{\nu_{13}}{E_3} - \frac{\nu_{31}}{E_1} + \frac{1}{E_3} - \frac{4}{G_{13}}\right)T'_1 \tag{55}$$

$$S'_5 = s'_{51}\,T'_1 = \frac{1}{4}\left(\frac{1}{E_1} - \frac{\nu_{13}}{E_3} + \frac{\nu_{31}}{E_1} - \frac{1}{E_3}\right)T'_1 \tag{56}$$

The final shape of the material is determined by the superposition of the contributions of all the nonzero strains (S'_1, S'_2, S'_3, S'_5). Inspection of Equations (53-55) illustrate that the sole application of a tensile normal stress T'_1 along the x'_1 axis produces normal tensile/compressive strain components along the x'_1, x'_2, and x'_3 axes. The amount of the elongation or constriction is determined by the linear combination of the engineering moduli measured with respect to the principal axes. Equations (52) and (56) reveal that a shear strain also results from the application of a tensile normal stress under these conditions.

As a second example, we will consider the case when all the stresses are zero except for planar tangential stresses T'_5 (T'_{13}, T'_{31}) in the $x'_1x'_3$ plane (see Figure 10).

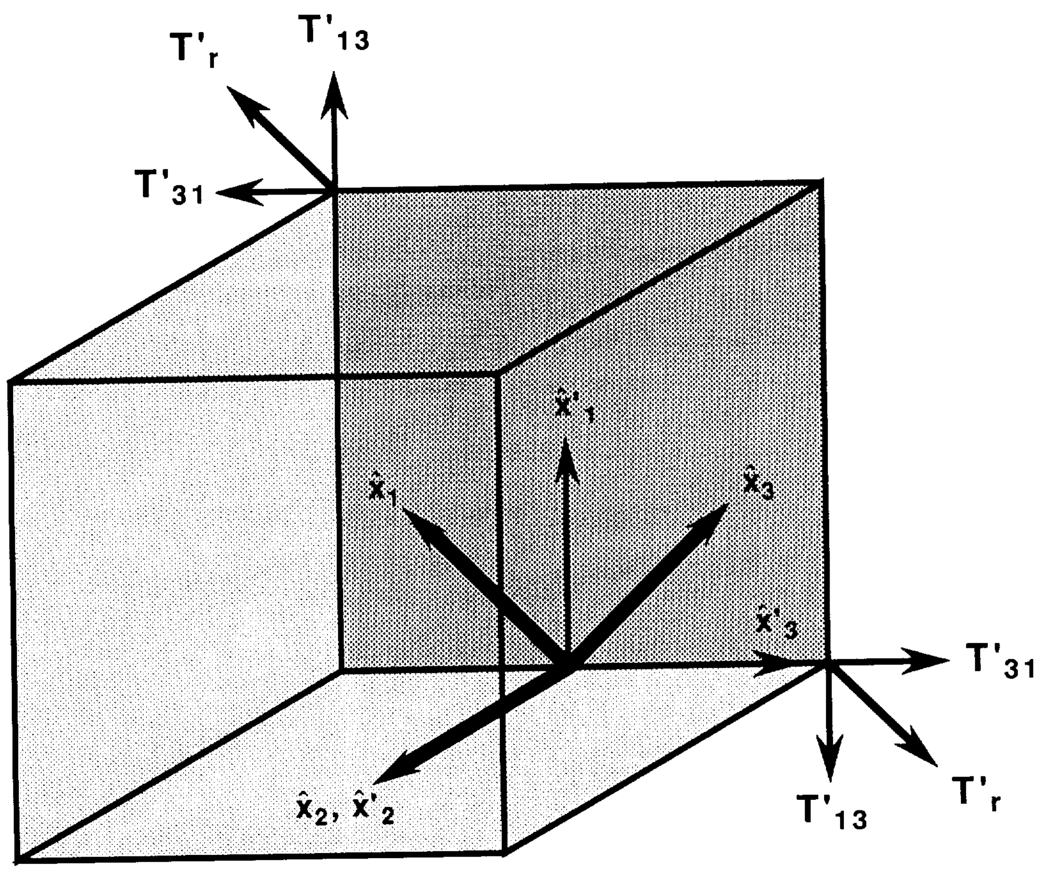


Figure 10: Application of a planar tangential stress T'_5 (T'_{13}, T'_{31}) in an $x'_1x'_3$ plane. The principal axes for the material is represented by the bold unprimed set of axes.

The stress/strain relation is given by

$$\begin{bmatrix} S'_1 \\ S'_2 \\ S'_3 \\ S'_4 \\ S'_5 \\ S'_6 \end{bmatrix} = \begin{bmatrix} s'_{11} & s'_{12} & s'_{13} & 0 & s'_{15} & 0 \\ s'_{12} & s'_{22} & s'_{23} & 0 & s'_{25} & 0 \\ s'_{13} & s'_{23} & s'_{33} & 0 & s'_{35} & 0 \\ 0 & 0 & 0 & s'_{44} & 0 & s'_{46} \\ s'_{15} & s'_{25} & s'_{35} & 0 & s'_{55} & 0 \\ 0 & 0 & 0 & s'_{46} & 0 & s'_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ T'_5 \\ 0 \end{bmatrix} . \tag{57}$$

From Equation (47) we can write the primed compliance coefficients in terms of the unprimed coefficients.

$$S'_1 = s'_{15}\,T'_5 = \frac{1}{4}\,(\,s_{11} - s_{13} + s_{31} - s_{33}\,)\,T'_5 \tag{58}$$

$$S'_2 = s'_{25}\,T'_5 = \frac{1}{2}\,(\,s_{21} - s_{23}\,)\,T'_5 \tag{59}$$

$$S'_3 = s'_{35}\, T'_5 = \frac{1}{4}\,(s_{11} - s_{13} + s_{31} - s_{33}\,)\, T'_5 \tag{60}$$

$$S'_5 = s'_{55}\, T'_5 = \frac{1}{4}\,(s_{11} - s_{13} - s_{31} + s_{33}\,)\, T'_5 \tag{61}$$

Equations (58-61) can also be written in terms of the engineering moduli.

$$S'_1 = s'_{15}\, T'_5 = \frac{1}{4}\left(\frac{1}{E_1} + \frac{\nu_{13}}{E_3} - \frac{\nu_{31}}{E_1} - \frac{1}{E_3} \right) T'_5 \tag{62}$$

$$S'_2 = s'_{25}\, T'_5 = -\frac{1}{2}\left(\frac{\nu_{21}}{E_1} - \frac{\nu_{23}}{E_3} \right) T'_5 \tag{63}$$

$$S'_3 = s'_{35}\, T'_5 = \frac{1}{4}\left(\frac{1}{E_1} + \frac{\nu_{13}}{E_3} - \frac{\nu_{31}}{E_1} - \frac{1}{E_3} \right) T'_5 \tag{64}$$

$$S'_5 = s'_{55}\, T'_5 = \frac{1}{4}\left(\frac{1}{E_1} + \frac{\nu_{13}}{E_3} + \frac{\nu_{31}}{E_1} + \frac{1}{E_3} \right) T'_5 \tag{65}$$

Again the superposition of the contributions from all the nonzero strains (S'_1, S'_2, S'_3, S'_5) will determine the final shape of this material. Inspection of Equations (62-64) illustrate that the application of planar tangential stresses in the $x'_1x'_3$ plane (for this material) produces normal tensile/compressive strain components along the x'_1, x'_2, and x'_3 axes. The amount of the elongation or constriction along a particular axis is determined by a linear combination of the engineering moduli measured with respect to the principal axes. Equation (65) illustrates explicitly how the shear strain contribution is dependent on the Poisson's ratios and Young's moduli for a sample, subjected to a shear stress, that is not aligned with the principal axes.

Reciprocity Relation:

Materials for which a symmetric strain-energy function exists exhibit a symmetric compliance matrix.

$$s_{IJ} = s_{JI} \qquad (\text{for } I,J = 1,2,3,4,5,6;\ I \neq J) \tag{66}$$

From Equation (36) for all stresses zero except T_1, we obtain

$$s_{21} = \frac{S_2}{T_1}\Big|_{T_1} = \frac{S_2}{S_1\, E_1}\Big|_{T_1} = -\frac{\nu_{21}}{E_1}\,. \tag{67}$$

In a similar manner, if all the applied stresses are zero except T_2, then

$$s_{12} \;=\; \frac{S_1}{T_2}\Big|_{T_2} \;=\; \frac{S_1}{S_2\,E_2}\Big|_{T_2} \;=\; -\frac{\nu_{12}}{E_2}\;. \tag{68}$$

Therefore, if $s_{21} = s_{12}$ then this implies

$$\frac{\nu_{21}}{E_1} = \frac{\nu_{12}}{E_2}\;. \tag{69}$$

Thus, in general the equivalence of $s_{IJ} = s_{JI}$ reveals that

$$\frac{\nu_{JI}}{E_I} = \frac{\nu_{IJ}}{E_J} \qquad (I,J = 1,2,3;\;\; I \neq J)\;. \tag{70}$$

Physical Interpretation of the Most General Compliance Matrix:

The role of the compliance coefficients is to relate the applied stresses to the resulting strains for all possible combinations. Initially, we were required to have 81 compliance coefficients to relate uniquely the 9 stress components to the 9 strain components. In order to illustrate the role each compliance coefficient plays, we will systematically determine the type of stress to the type of strain each element uniquely relates. The most general form for the constitutive relation, in the Voigt notation, is

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{56} & s_{66} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}. \tag{71}$$

Thus, the total resulting strain experienced by a material, due to the application of stresses, is the linear superposition of the contributions of S_1, S_2, S_3, S_4, S_5, and S_6.

$$\begin{aligned} S_1 &= s_{11}\,T_1 + s_{12}\,T_2 + s_{13}\,T_3 + s_{14}\,T_4 + s_{15}\,T_5 + s_{16}\,T_6 \\ S_2 &= s_{21}\,T_1 + s_{22}\,T_2 + s_{23}\,T_3 + s_{24}\,T_4 + s_{25}\,T_5 + s_{26}\,T_6 \\ S_3 &= s_{31}\,T_1 + s_{32}\,T_2 + s_{33}\,T_3 + s_{34}\,T_4 + s_{35}\,T_5 + s_{36}\,T_6 \\ S_4 &= s_{41}\,T_1 + s_{42}\,T_2 + s_{43}\,T_3 + s_{44}\,T_4 + s_{45}\,T_5 + s_{46}\,T_6 \\ S_5 &= s_{51}\,T_1 + s_{52}\,T_2 + s_{53}\,T_3 + s_{54}\,T_4 + s_{55}\,T_5 + s_{56}\,T_6 \\ S_6 &= s_{61}\,T_1 + s_{62}\,T_2 + s_{63}\,T_3 + s_{64}\,T_4 + s_{65}\,T_5 + s_{66}\,T_6 \end{aligned} \tag{72}$$

Inspection of Equation (72) reveals that the compliance elements of the first column

$$s_{I1} \quad \text{for } I = 1,2,3,4,5,6$$

relates the applied tensile/compressive normal stresses along the x_1 axis to the different types of strains. Table 3 summarizes the possible strain contributions resulting from the application of a tensile/compressive normal stress T_1 (T_{11}) along the x_1 axis.

Stress/Strain Contribution	Resulting Physical Phenomenon
$S_{11} = s_{1111} T_{11} \Rightarrow S_1 = s_{11} T_1$	Tensile/compressive strain along the x_1 axis
$S_{12} = s_{1211} T_{11} \Rightarrow S_6 = s_{61} T_1$	Shear strain in the x_1x_2 plane
$S_{13} = s_{1311} T_{11} \Rightarrow S_5 = s_{51} T_1$	Shear strain in the x_1x_3 plane
$S_{21} = s_{2111} T_{11} \Rightarrow S_6 = s_{61} T_1$	Shear strain in the x_2x_1 plane
$S_{22} = s_{2211} T_{11} \Rightarrow S_2 = s_{21} T_1$	Tensile/compressive strain along the x_2 axis
$S_{23} = s_{2311} T_{11} \Rightarrow S_4 = s_{41} T_1$	Shear strain in the x_2x_3 plane
$S_{31} = s_{3111} T_{11} \Rightarrow S_5 = s_{51} T_1$	Shear strain in the x_3x_1 plane
$S_{32} = s_{3211} T_{11} \Rightarrow S_4 = s_{41} T_1$	Shear strain in the x_3x_2 plane
$S_{33} = s_{3311} T_{11} \Rightarrow S_3 = s_{31} T_1$	Tensile/compressive strain along the x_3 axis

Table 3: Contributions to the total resulting strain due to the compliance coefficients that involve T_1 stresses.

We see from Table 3 that the s_{I1} elements for $I = 4,5,6$ imply that the tensile/compressive normal stress T_1 would produce a shearing strain. For orthotropic systems these elements are zero when referenced to the principal axes for the material. Thus, a shearing strain cannot result from a tensile/compressive normal stress applied along a principal axis for orthotropic materials.

From Equation (72) we see that the compliance elements of the sixth column

$$s_{I6} \quad \text{for } I = 1,2,3,4,5,6$$

relate an applied planar tangential stress in the x_1x_2 plane to the different types of strains. Table 4 summarizes the possible strain contributions resulting from the application of a planar tangential shearing stress T_6 (T_{12}, T_{21}) in the x_1x_2 plane.

Stress/Strain Contributions	Resulting Physical Phenomenon
$S_{11} = s_{1112}\,T_{12} + s_{1121}\,T_{21}$ $\Rightarrow S_1 = s_{16}\,T_6$	Tensile/compressive normal strain along the x_1 axis
$S_{12} = s_{1212}\,T_{12} + s_{1221}\,T_{21}$ $\Rightarrow S_6 = s_{66}\,T_6$	Shear strain in the x_1x_2 plane
$S_{13} = s_{1312}\,T_{12} + s_{1321}\,T_{21}$ $\Rightarrow S_5 = s_{56}\,T_6$	Shear strain in the x_1x_3 plane
$S_{21} = s_{2112}\,T_{12} + s_{2121}\,T_{21}$ $\Rightarrow S_6 = s_{66}\,T_6$	Shear strain in the x_1x_2 plane
$S_{22} = s_{2212}\,T_{12} + s_{2221}\,T_{21}$ $\Rightarrow S_2 = s_{26}\,T_6$	Tensile/compressive normal strain along the x_2 axis
$S_{23} = s_{2312}\,T_{12} + s_{2321}\,T_{21}$ $\Rightarrow S_4 = s_{46}\,T_6$	Shear strain in the x_2x_3 plane
$S_{31} = s_{3112}\,T_{12} + s_{3121}\,T_{21}$ $\Rightarrow S_5 = s_{56}\,T_6$	Shear strain in the x_1x_3 plane
$S_{32} = s_{3212}\,T_{12} + s_{3221}\,T_{21}$ $\Rightarrow S_4 = s_{46}\,T_6$	Shear strain in the x_2x_3 plane
$S_{33} = s_{3312}\,T_{12} + s_{3321}\,T_{21}$ $\Rightarrow S_3 = s_{36}\,T_6$	Tensile/compressive normal strain along the x_3 axis

Table 4: Contributions to the total resulting strain due to the compliance coefficients that involve T_6 stresses.

We see from Table 4 that s_{66} relates a planar tangential shearing stress to a shear strain in the same plane. The s_{46} and s_{56} coefficients relate an applied shear stress in the x_1x_2 plane to shear strains in orthogonal planes, while the s_{16}, s_{26}, and s_{36} coefficients relate a shearing stress to a tensile/compressive normal strain along the principal axes. For orthotropic systems aligned with the principal axes $s_{46} = s_{56} = s_{16} = s_{26} = s_{36} = 0$. Therefore, the strains involving these coefficients are not allowed.

We can also view the compliance matrix in terms of quadrants (see Table 5), which groups together the coefficients by types of applied stresses and types of resulting strain contributions.

	Tensile/Compressive Stress	Planar Tangential Shear Stress
Tensile/Compressive Strain Contribution	s_{11} s_{12} s_{13} s_{21} s_{22} s_{23} s_{31} s_{32} s_{33}	s_{14} s_{15} s_{16} s_{24} s_{25} s_{26} s_{34} s_{35} s_{36}
Planar Tangential Shear Strain Contribution	s_{41} s_{42} s_{43} s_{51} s_{52} s_{53} s_{61} s_{62} s_{63}	s_{44} s_{45} s_{46} s_{54} s_{55} s_{56} s_{64} s_{65} s_{66}

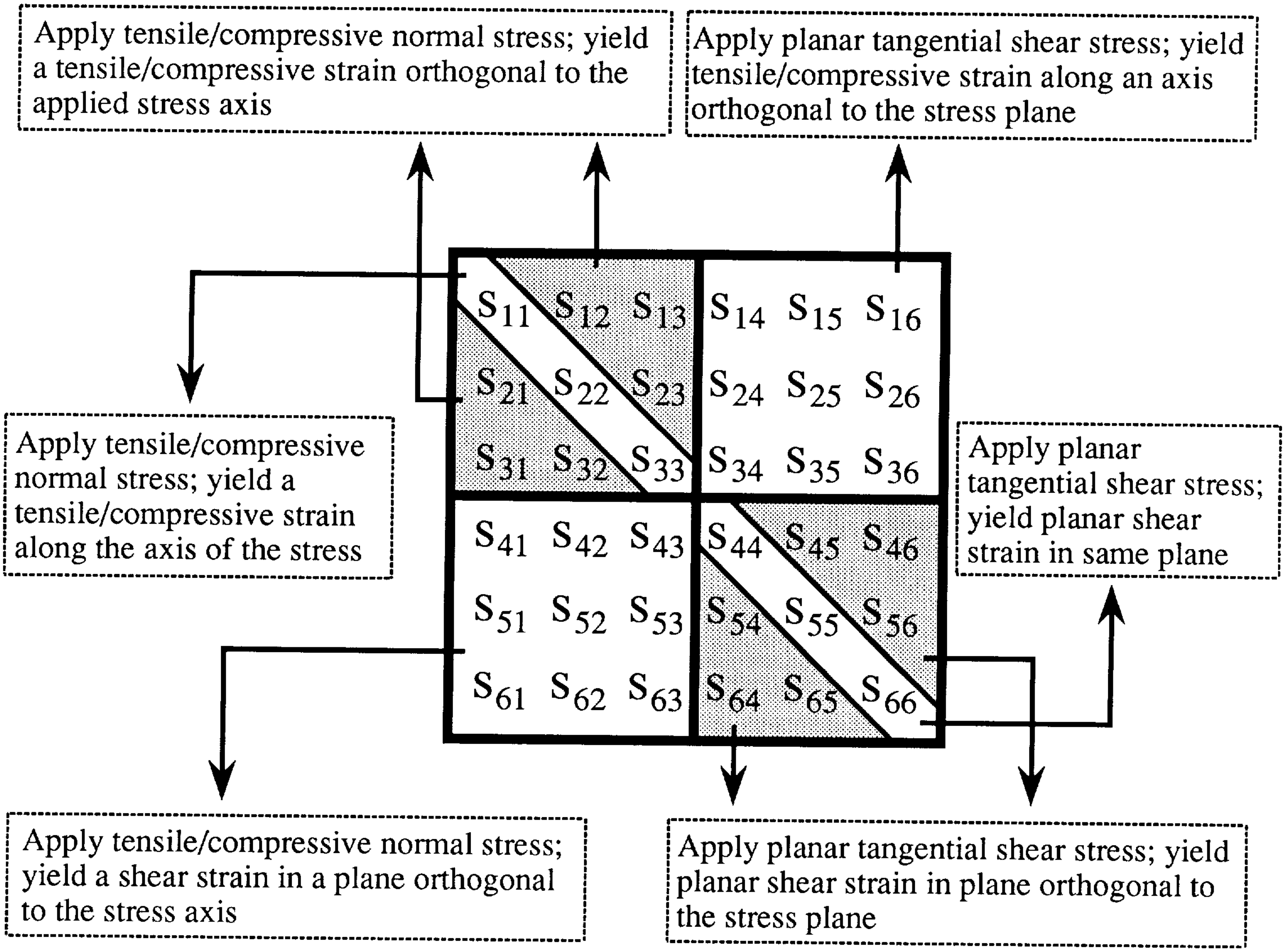
Table 5: The compliance coefficients grouped by types of applied stresses and the types of resulting strain contributions they uniquely relate.

Figure 11 and Table 6 also illustrate the role each of the compliance coefficients plays in relating an applied stress to the resulting strain.

As has been shown above, an orthotropic system has an elastic matrix which contains many coefficients equal to zero. If the material is not aligned with the principal axes, then these matrix elements may not be zero. These matrix elements are now linear combinations of the smallest set of distinct coefficients. (The Bond rotation formalism is a useful tool under these circumstances). Application of a tensile/compressive stress along the Cartesian axes of the misaligned system may produce both tensile/compressive strains and shearing strains. Thus, information on how the elements in the upper-right, lower-left and off-diagonal elements of the lower-right quadrants contribute to the resulting strains is important for materials whose macroscopic surfaces are not aligned with the principal axes for the material.

In this Section, we have reviewed the formalism of linear elasticity in order to build a more solid foundation upon which an investigator can raise his/her physical intuition about the intrinsic nature of the physical properties of materials. We presented a physical interpretation for the role the elastic compliance coefficients play in uniquely relating applied stresses to resulting strains and the dependence these coefficients have on the traditional engineering moduli. Such information will be extremely useful in the design of advanced ultrasonic measurement systems.

Physical Interpretation of Elastic Coefficients



$$\mathbf{S} = \mathbf{s} : \mathbf{T}$$

Figure 11

Applied Stress Involved	Compliance Coefficient	Valid Indices	Resulting Strain Contribution
Tensile/compressive normal stress along the x_J axis	s_{JJ}	J = 1, 2, 3	Tensile/compressive strain along the x_J axis, $\Delta x_J/x_J$
Tensile/compressive normal stress along the x_J axis	s_{IJ}	I, J = 1, 2, 3 $I \neq J$	Tensile/compressive strain along the x_I axis, $\Delta x_I/x_I$, an orthogonal axis
Tensile/compressive normal stress along the x_J axis	s_{IJ}	I = 4, 5, 6 J = 1, 2, 3	Planar shear strain in the plane denoted by the I index, orthogonal to the x_J axis
Planar tangential shear stress in the J plane	s_{JJ}	J = 4, 5, 6	Planar shear strain in the same plane (J) as the applied stress
Planar tangential shear stress in the J plane	s_{IJ}	I, J = 4, 5, 6 $I \neq J$	Planar shear strain in the plane I, an orthogonal plane
Planar tangential shear stress in the J plane	s_{IJ}	I = 1, 2, 3 J = 4, 5, 6	Tensile/compressive strain along the x_I axis, orthogonal to the stress plane, $\Delta x_I/x_I$

Table 6: Physical interpretation of the elements of the elastic compliance matrix

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