MULTI-LEVEL TRELLIS CODED MODULATION AND MULTI-STAGE DECODING*

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• Combined Ungerboeck-type and multi-level trellis coding

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• A level-spanning, multi-level trellis coding scheme

• $90^\circ$ rotationally invariant multi-level trellis codes

• Conclusions
Introduction

• The goal in designing a coded modulation system is to achieve a good trade-off between coding gain, decoding complexity, and decoding delay.

• Multi-level coding is a powerful technique for constructing bandwidth efficient coded modulation codes. Good multi-level coding schemes can be designed by using previously known codes as component codes.

• Multi-stage decoding provides a simple decoder implementation for multi-level codes with a small loss in coding gain.

• For coded QAM, the total power gain over uncoded QAM is composed of two parts: the coding gain ($\gamma(C)$) and the shaping gain ($\gamma(S)$).
• At a bit error rate of $10^{-5} \sim 10^{-6}$, the maximum coding gain is about 7.5 dB, and the maximum shaping gain is about 1.5 dB.

• Because these gains can be achieved independently, for coded QAM we focus on coding gain only and choose the signal set to be $Z^N$ ($N$ dimensional integer lattice).

• For coded MPSK, we also focus only on coding gain, since no shaping gain is possible.

• The phase invariant property (or phase symmetry) is useful in resolving carrier-phase ambiguity and ensuring rapid carrier-phase resynchronization after a temporary loss of synchronization. It is desirable for a coded modulation system to have as much phase symmetry as possible.

• We present necessary and sufficient conditions for a QAM code to be $90^\circ$ rotationally invariant, and some $90^\circ$ rotationally invariant multi-level codes are constructed.
Multi-level Trellis Coding Based on Set Partitioning

- Figure 1 shows a multi-level trellis coding scheme based on set partitioning. $\Lambda_0$ is a signal set, $\Lambda_i$ is a subset of $\Lambda_{i-1}$, and $\Lambda_m$ is the all-zero vector. $C_1, C_2, \ldots, C_m$ represent the different component codes, and the overall multi-level code is denoted by $C$.

Fig. 1 Multi-level trellis coding based on set partitioning
Related previous work

- Leech (1964) and Leech and Sloane (1971) used a multi-level structure to construct lattices.

- Multi-level codes using "proper indexing", which is the same as Ungerboeck's "set partitioning", of two dimensional signal sets was proposed by Imai and Hirakawa (1977). They also presented a multi-stage decoding method using a posteriori probabilities based on channel statistics.


- Sayegh (1986) showed how Imai and Hirakawa's method can be combined with set partitioning to create multi-level block coded modulation systems.

- Pottie and Taylor (1989) proposed a hierarchy of codes to match the partitioning of signal sets by generalizing Imai and Hirakawa's and Ginzburg's coding schemes.
• Calderbank (1989) investigated the path multiplicity for a variety of multi-level codes.

• Tanner (1990?) studied linking subspaces of vector spaces to guarantee a large minimum separation between signals in the resulting signal set so that good multi-level codes can be designed.

Basic multi-level trellis codes

• This construction is based on two-way partition chains, where all component codes are binary codes (block or convolutional).

  • Let $\Delta_i$ be the minimum squared Euclidean distance (MSED) of $\Lambda_i$ for $i = 0, 1, \ldots, m$.

  • Let $d_i$ be the minimum Hamming distance of binary code $C_i$ for $i = 1, 2, \ldots, m$.

  • Then the MSED of the multi-level code is (Leech & Sloane, Ginzburg, Sayegh, etc.)

\[
D(C) = \min\{d_i\Delta_{i-1}, 1 \leq i \leq m\}
\]
• The normalized redundancy $\rho(C)$ is defined as (Forney) the number of redundant bits per two dimensional signal (symbol).

• The spectral efficiency $\eta(C)$ is defined as (Ungerboeck) the number of information bits per two dimensional signal (symbol).

• Basic multi-level codes with normalized redundancy $\rho(C) = 1$ bit/symbol were presented by Yamaguchi and Imai (1987).

• Basic multi-level codes with smaller normalized redundancies can be constructed by using two-way partition chains with multi-dimensional signal sets and binary convolutional or block codes. Some four and eight dimensional basic multi-level codes were constructed by Wu and Zhu (1990?).

• We present some new basic multi-level codes based on set partitioning of one and two dimensional signal sets. Some of these new codes have non-integer normalized redundancies $\rho(C)$. 
Example 1. A three-level trellis code using an 8-PSK signal set with mapping by set partitioning is shown in Figure 2.
Let \( C_1 \) be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16, 
\( C_2 \) an 8-state rate-3/4 convolutional code with free distance 4,
\( C_3 = P_n \), the \((n, n-1)\) single parity check code.

- The spectral efficiency of this multi-level code is
  \[
  \eta(C) = 1 + (n - 1)/n \text{ bits/symbol}
  \]

- The minimum free squared Euclidean distance is
  \[
  D(C) = \min\{0.586 \times 16, 2 \times 4, 4 \times 2\}
  = 8
  \]

- The nominal coding gain (Ungerboeck) over uncoded QPSK is
  \[
  \gamma(C) = 10\log_{10}\left(\frac{D(C)}{D(\text{QPSK})}\right) = 6.02\text{dB}.
  \]

- The 256-state, rate-2/3, \( C \) = 2 bits/symbol Ungerboeck code has \( D(C) = 7.515 \) and
  \( \gamma(C) = 5.75 \text{ dB} \).
Multi-Stage Decoding of Example 1

- A three-level multi-stage decoder for Example 1 is shown in Figure 3.

Fig. 3 Multi-stage decoding for Example 1
• The normalized complexity $N_D$ of multi-stage decoding is the number of required binary operations (additions and comparisons) per 2 dimensional symbol.

• For a $2^\nu$-state, $k$ input bit, $n$ output bit convolutional (trellis) code, the Add-Compare-Select (ACS) operation of the Viterbi algorithm requires $2^k$ additions and a comparison of $2^k$ numbers, or $2^k - 1$ binary comparisons, for each of the $2^\nu$ states, so its complexity is $2^{k+\nu+1} - 2^\nu$. (This number should be normalized to the complexity per 2 dimensional symbol.)
First-stage of Decoding

• For each state transition period, the symbol metrics of both QPSK subsets (see Figure 4) must be computed.

Complexity = 2 binary operations/symbol

Fig. 4
• Then the branch metrics within each state transition period must be computed by adding the four symbol metrics on each branch.
  Complexity = 6 binary operations/symbol

• The ACS operation of the Viterbi algorithm is then used to determine the surviving path at each state.
  Complexity = 12 binary operations/symbol
Second-stage of Decoding

- The decoded information from the first stage is passed on to the second-stage.

- For each state transition period, the symbol metrics of both BPSK subsets (see Figure 5) must be computed.

Complexity = 2 binary operations/symbol

![Diagram](a) BPSK signal set $aib_i=00$  
(b) BPSK signal set $aib_i=01$
• Then the branch metrics within each state transition period must be computed by adding the four symbol metrics on each branch.
  Complexity = 6 binary operations/symbol

• The ACS operation of the Viterbi algorithm is then used to determine the surviving path at each state.
  Complexity = 30 binary operations/symbol

• If the parallel transitions in the trellis are resolved by table-look-up, the complexity reduces to 14 binary operations/symbol.
Third-stage of Decoding (assume $n = 32$)

- The decoded information from the first and second stages is made available to the third stage.

- For each state transition period, the metrics of both the 0 and 1 symbols must be computed.
  Complexity $= 2$ binary operations/symbol

- In this case, the branch metrics are the symbol metrics computed above (one symbol per trellis branch).

- After 8 branches (32 symbols) in the first and second trellis are decoded, the Viterbi algorithm is used to make a decoding decision for the block code $C_3 = P_{32}$.
  Complexity $= 6$ binary operations/symbol

- The total decoding complexity is $N_D = 66$ binary operations per 2 dimensional symbol, or $N_D = 50$ not counting the parallel transitions.
• If we take $C_3$ to be $P_n$, where $n \rightarrow \infty$ (i.e., a 2-state, non-redundant, catastrophic trellis code), the multi-level code has $1 + 3 + 4 = 8$ input bits and $4 + 4 + 4 = 12$ output bits for every four 8-PSK transmitted symbols. Overall, this can be viewed as a $16 \times 8 \times 2 = 256$-state 8-dimensional trellis code.

• Without considering the computation of the symbol and branch metrics, the ACS complexity of maximum likelihood decoding of the overall trellis code is

$$\frac{2^{8+8+1} - 2^8}{4} = 2^{15} - 2^6 > 3 \times 10^4$$

binary operations/2 dimensional symbol.

• Note that the complexity of the multi-stage decoder in this example is only about 0.2% of the complexity of the overall maximum likelihood decoder.

• However, the performance of the multi-stage decoder is close to that of the maximum likelihood decoder.

• For the 256-state Ungerboeck code, the ACS complexity alone is 1792 binary operations/symbol.
Example 2. The one dimensional partition chain $\mathbb{Z}/2\mathbb{Z}/4\mathbb{Z}/\ldots$ has MSED $1/4/16/\ldots$ (see Figure 6).

Fig. 6 Set partitioning of $\mathbb{Z}$
- Let $C_1$ be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16,
- $C_2$ be an 8-state rate-3/4 convolutional code with free distance 4,
- and $C_3, C_4, \ldots$ be rate-1 codes (no coding).
- Since there are two levels of coding, this is a two-level code (see Figure 7).
- The normalized redundancy $\rho(C)$ is 2 bits per symbol.
- The MSED is
  $$D(C) = \min\{1 \times 16, \ 4 \times 4, \ 16\} = 16$$
• The nominal coding gain (Forney) is

\[ \gamma(C) = 10 \log_{10} \frac{D(C)}{2^{\rho(C)}} = 6.02 (\text{dB}) \]

• This code has the same nominal coding gain and normalized redundancy as the 24-dimensional Leech lattice \( \Lambda_{24} \) but much less decoding complexity.

• Due to a large path multiplicity, the effective coding gain of this two-level code is less than the nominal coding gain. To reduce the path multiplicity, we can choose longer convolutional codes (with larger constraint lengths and free distances).

• For example, if \( C_1 \) is a 32-state rate-1/4 convolutional code with free distance 18 and \( C_2 \) is a 32-state rate-3/4 convolutional code with free distance 5, the path multiplicity is reduced and the effective coding gain is closer to 6.02 dB.
Using multi-stage decoding, an additional loss of coding gain occurs, but the decoding complexity is less than a 64-state mother code and much less than the lattice \( \Lambda_{24} \).

Table 1. Comparison of multi-level trellis codes in other codes (spectral efficiency \( \eta(C) = 4 \) \( \gamma \) /symbol) using 8-PAM modulation

<table>
<thead>
<tr>
<th>Codes</th>
<th>#S</th>
<th>( R_i )</th>
<th>( \gamma(C) )</th>
<th>( N_D )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta )-level</td>
<td>16 &amp; 8</td>
<td>1/4 &amp; 3/4</td>
<td>5.81</td>
<td>116</td>
<td>14</td>
</tr>
<tr>
<td>( \Theta )-level</td>
<td>32 &amp; 8</td>
<td>1/4 &amp; 3/4</td>
<td>5.81</td>
<td>130</td>
<td>16</td>
</tr>
<tr>
<td>( \Theta )-level</td>
<td>32 &amp; 32</td>
<td>1/4 &amp; 3/4</td>
<td>5.81</td>
<td>350</td>
<td>20</td>
</tr>
<tr>
<td>( \Theta )-level</td>
<td></td>
<td></td>
<td>5.81</td>
<td>( \sim ) 1264</td>
<td></td>
</tr>
<tr>
<td>( \Theta )-level</td>
<td>256</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Theta )-code</td>
<td>32</td>
<td>2/3</td>
<td>4.77</td>
<td>232</td>
<td>5</td>
</tr>
<tr>
<td>( \Theta )-code</td>
<td>64</td>
<td>2/3</td>
<td>5.44</td>
<td>456</td>
<td>6</td>
</tr>
</tbody>
</table>

The decoding delay of a multi-level trellis code is proportional to

\[
D \triangleq \frac{1}{2} \sum N_i K_i,
\]

where \( N_i \) is the dimensionality of the signals (cosets) associated with a branch transition of the \( i \)th component code and \( K_i \) is the constraint length of the \( i \)th component code.
Multi-level trellis codes based on a set partition chain with strictly increasing distances

- Multi-level trellis codes using multi-dimensional signal sets can achieve higher spectral efficiencies (lower normalized redundancies) than multi-level codes based on two dimensional signal sets.

- For two-way partitioning of multi-dimensional signal sets, the MSED at successive partition levels may be equal. For example, the partition chain $\mathbb{Z}^4/D_4/R\mathbb{Z}^4/RD_4/2\mathbb{Z}^4/2D_4/\ldots$ of the four dimensional integer lattice $\mathbb{Z}^4$ has distances $1/2/2/4/4/8/\ldots$, where $R$ represents the rotation operation, $R^2 = 2$, and $D_4$ is the densest known four dimensional lattice.

- Reducing the number of component codes can reduce the decoding delay and the path multiplicity.
Some partition levels can be joined to form a new multi-way partition chain with strictly increasing distances. For example, the partition chain $Z^4/D_4/RD_4/2D_4/\ldots$ has distances $1/2/4/8/\ldots$. Since $|Z^4/D_4| = 2$ and $|R^iD_4/R^{i+1}D_4| = 4$ for $i = 0, 1, 2, \ldots$, the first component code can be a binary code, and other component codes can be binary input, 4-ary output codes or codes over $GF(4)$.

The lower bound on the MSED of these multi-level codes is given by

$$D(C) \geq \min\{d_i \Delta_{i-1}, \ 1 \leq i \leq m\}$$

where $d_i$ is now the minimum free Hamming distance of code $C_i$ (binary or 4-ary).
Example 3. This code is based on the partition chain $Z^4/D_4/RD_4/\ldots$ and includes two component codes (see Figure 8):

$C_1$ is an 8 state rate-3/4 convolutional code with free distance 4 and $C_2$ is an $(N, N-1)$ block code over GF(4) with minimum distance 2 (4 states).

- The normalized redundancy is $\rho(C) = \frac{1}{8} + \frac{1}{N}$, the MSED is 4, and the nominal coding gain is

$$\gamma(C) = 5.64 - \frac{3.01}{N} \text{ (dB)} = 5.48 \text{ dB} \ (N = 19)$$

- The decoding complexity is $N_D = 37$, and the decoding delay is $D = 24$ (excluding the decoding delay of the block code).

![Fig.8 Multi-level code of Example 3.](image_url)

- The 64-state, rate-4/5 Ungerboeck code for $Z^4$ has $\gamma(C) = 5.48 \text{ dB}$, $N_D \approx 496$, and $D = 12$.  


Combined Ungerboeck-type and multi-level trellis codes

- For the above two classes of multi-level codes, each output symbol of a component encoder corresponds to a single coset of a subset of a signal constellation (two dimensional or multi-dimensional). For Ungerboeck-type codes, all the encoder output symbols associated with a single trellis branch correspond to a single coset of a subset of a signal constellation. Ungerboeck-type codes can be used as component codes at some levels in conjunction with a multi-way partition chain.

- Instead of using several high rate codes at higher levels of partitioning, we use an Ungerboeck-type code to reduce the decoding delay and path multiplicity.

- The usual lower bound on the MSED cannot be applied to this construction. A more general lower bound on the MSED of these multi-level codes (Kasami & Lin) is given by

\[ D(C) \geq \min\{D(C_i), \ 1 \leq i \leq m\} \]

where \( D(C_i) \) is the MSED of code \( C_i \).
Example 4. The encoding structure is shown in Figure 9.

- Let $C_1$ be a 16-state rate-1/4 convolutional code with minimum free Hamming distance 16,

$C_2$ a 16-state rate-7/8 trellis code with MSED 8 (Pietrobon, Deng, et.al., 1990).

![Diagram of encoding structure](image)

Fig.9 Multi-level code of example 4.

- The 64-state, rate-4/5, $\eta(C) = 2$, $4 \times 8$-PSK code constructed by Pietrobon, Deng, et.al. (1990) has

$D(C) = 7.029$, $\gamma(C) = 5.46$ dB , $N_D \approx 496$, and $D = 12$. 
• Encoding procedure:
  The information sequence is divided into blocks of 8 bits each:
  the first bit in each block enters encoder $C_1$, and the 4 output bits specify 4 consecutive cosets of $4 \times (8\text{-}PSK/QPSK)$,
  the other 7 bits of the block enter encoder $C_2$ and the 8 output bits specify a $4 \times$ QPSK signal.

• The spectral efficiency of this multi-level code is $\eta(C) = 8/4 = 2$ bits/symbol

• The MSED is $D(C) = \min\{16 \times 0.586, 8\} = 8$
  where $D(C_1) = 16 \times 0.586$.

• The nominal coding gain over uncoded QPSK is $\gamma(C) = 6.02$ dB.

• The decoding complexity for multi-stage decoding is $N_D \approx 100$ binary operations per symbol.

• The decoding delay is only $D = 16$, which is less than a multi-level code with more stages.
Generalized multi-level trellis codes

- The previous examples were all based on Ungerboeck's set partitioning. A modified set partitioning method can be used to construct generalized multi-level trellis codes.

- **Example 5.** The four dimensional 8-state code \( C(Z^4/RD_4) \) constructed by Wei (1987) has MSED 4. Mapping the same binary code to \( RZ^4/2D_4 \) rather than \( Z^4/RD_4 \), we obtain a trellis code, denoted by \( C_2(RZ^4/2D_4) \), with MSED 8. Using an 8-state rate-1/3 convolutional code as the first component code \( C_1(Z^2/RZ^2) \) and \( C_2(RZ^4/2D_4) \) as the second component code gives the two-level trellis code shown in Figure 10.

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Fig. 10 Multi-level code of Example 5.
• Encoding procedure:
The information sequence is divided into blocks of a specified number of bits according to the desired spectral efficiency: the first 2 bits in each block enter encoder $C_1$, and the 6 output bits specify 6 consecutive cosets of $Z^2/RZ^2$, i.e., 3 consecutive cosets of $Z^4/RZ^4$; the next 6 bits of the block enter encoder $C_2$, and the 9 output bits specify 3 consecutive cosets of $RZ^4/2D_4$.
Together with uncoded bits, each coded block determines 3 consecutive four dimensional signals, i.e., 6 two dimensional signals.
• Note that the first coding level partitions a six dimensional signal set whereas the second coding level partitions a four dimensional signal set.
• The nominal coding gain is $\gamma(C) = 5.52$ dB, which is 1.00 dB greater than Wei’s code.
• The decoding delay of this two-level code is $D = 15$, whereas the delay of Wei’s code is $D = 6$.
• The decoding complexity of this two-level code is $N_D = 56$, whereas the complexity of Wei’s code is $N_D = 44$. 
Example 6. Consider the generalized multi-level trellis code shown in Figure 11.

The first component code, associated with the partition $\mathbb{Z}/2\mathbb{Z}$, is a 32 state rate-1/2 convolutional code with free distance 8.

The second component code, associated with the partition $2\mathbb{Z}^8/2D_8$, is a single parity check block code of length N.
• Encoding procedure (spectral efficiency = m bits/symbol):

• The information sequence is divided into blocks of $4Nm$ bits each, with three subsequences of length $4N$, $N - 1$, and $4Nm - 4N - (N - 1)$ corresponding to $C_1$, $C_2$, and uncoded bits, respectively.

• Each output bit of encoder $C_1$ specifies a coset of partition $Z/2Z$, i.e., $8N$ output bits specify $N$ cosets of $Z^8/2Z^8$.

• Each output bit of encoder $C_2$ specifies a coset of $2Z^8/2D_8$, i.e., $N$ output bits specify $N$ cosets of $2Z^8/2D_8$.

• Together with the $4Nm - 4N - (N - 1)$ uncoded bits, each coded block of $N$ eight dimensional signals, i.e., $4N$ two dimensional signals contains $4Nm$ bits of information and the spectral efficiency in $m$ bits/symbol.
• The normalized redundancy is \( \rho(C) = 1 + \frac{1}{4N} \) and the MSED is 8. Therefore the nominal coding gain is

\[
\gamma(C) = 6.02 - \frac{3.01}{4N} \text{ (dB)}.\]

• The decoding complexity is \( N_D = 140 \) and the decoding delay (excluding the block code) is \( D = 5 \).

• The 128-state, rate-4/5 Ungerboeck code for \( Z^8 \) has \( \gamma(C) = 5.27 \) dB, number of nearest neighbors \( N_{free} = 112 \), \( N_D \approx 992 \), and \( D = 28 \).
• In general, let $\Lambda_0$ be a signal set and $\Lambda_0^{(1)}$ be a set such that $\prod_{j_0}^{j_1} \Lambda_0 = \prod_{k_0}^{k_1} \Lambda_0^{(1)}$, for some integers $j_0, k_0 \geq 1$.

• If there are $2m$ sets $\Lambda_{i-1}^{(i)}$ and $\Lambda_i^{(i)}$, for $i = 1, 2, \ldots, m$, where $\Lambda_m^{(m)}$ is the empty set, satisfying the following conditions:

$$\prod_{j_i}^{j_{i-1}} \Lambda_{i-1}^{(i-1)} = \prod_{k_i}^{k_{i-1}} \Lambda_i^{(i-1)} = \prod_{j_i}^{j_{i-1}} \Lambda_{i-1}^{(i-1)} = \prod_{k_i}^{k_{i-1}} \Lambda_i^{(i)},$$

for $i = 2, 3, \ldots, m$, and for some $j_i, k_i \geq 1$;

(2) $\Lambda_{i-1}^{(i)} \supseteq \Lambda_i^{(i)}$, for $i = 1, 2, \ldots, m$;

then we can construct a multi-level code having the form shown in Figure 12:

$$C = C_1 \left( \Lambda_0^{(1)}/\Lambda_1^{(1)} \right) + C_2 \left( \Lambda_1^{(2)}/\Lambda_2^{(2)} \right) + \ldots + C_m \left( \Lambda_{m-1}^{(m)}/\Lambda_m^{(m)} \right)$$
• The MSED of this multi-level code is lower bounded by

\[ D(C) \geq \min \{ D[C_i(\Lambda_{i-1}^{(i)}/\Lambda_i^{(i)})], \ 1 \leq i \leq m \} \]

where \( D[C_i(\Lambda_{i-1}^{(i)}/\Lambda_i^{(i)})] \) is MSED of the \( i \)th component code.
Level spanning multi-level trellis codes

- Level spanning provides an approach to constructing rotationally invariant multi-level trellis codes.
- However, the lower bound on MSED of generalized multi-level codes may not hold for this class of codes.

Example 7. Consider the multi-level coding scheme shown in Figure 13.

![Diagram of a four dimensional multi-level trellis code with level spanning of Example 7]

*Fig. 13 Four dimensional multi-level trellis code with level spanning of Example 7*
• $C_1$ is a 16-state rate-2/3 Ungerboeck code, which has MSED 6 when used with the partition $\mathbb{Z}^2/R\mathbb{Z}^2/2\mathbb{Z}^2$.

• $C_2$ is an 8-state rate-7/8 binary convolutional code with free distance 3.

• $C_3$ is a 2-state $(8, 7)$ block code with minimum distance 2.

• Encoding procedure (spectral efficiency $= m$ bits/symbol):

  • The information sequence is divided into blocks of $16m$ bits each, with four subsequences of length 16, 7, 7, and $16m - 30$ corresponding to $C_1, C_2, C_3,$ and uncoded bits, respectively.

  • Each state transition period of encoder $C_1$ outputs three bits which specify cosets of the partitions $\mathbb{Z}^4/D_4, D_4/R\mathbb{Z}^4,$ and $RD_4/2\mathbb{Z}^4$, respectively, i.e., 24 output bits specify 8 cosets of each partition.
• Each output bit of encoder $C_2$ specifies a coset of $RZ_4/RD_4$, i.e., 8 output bits specify 8 cosets of $RZ_4/RD_4$.

• Each output bit of encoder $C_3$ specifies a coset of $2Z_4/2D_4$, i.e., 8 output bits specify 8 cosets of $2Z_4/2D_4$.

• Together with $16m - 30$ uncoded bits, each coded block of 8 four dimensional signals, i.e., 16 two dimensional signals, contains $16m$ bits of information and the spectral efficiency is $m$ bits/symbol.

• Since the MSED's of $Z_4, D_4$, and $RD_4$ are the same as $Z_2, RZ_2$, and $2Z_2$, respectively, the MSED of $C_1$ is the same as the corresponding Ungerboeck code, i.e., $D(C_1) = 6$. Therefore, assuming the lower bound on MSED holds in this case, $D(C) = \min\{6, 3 \times 2, 2 \times 4, 8\} = 6$.

• The normalized redundancy is $\rho(C) = 5/8$, the nominal coding gain is $\gamma(C) = 5.90$ dB, the decoding complexity of multi-stage decoding is $N_D = 108$, and the decoding delay is $D = 56$.

• However, due to the uncertainty regarding the bound, the actual values of $D(C)$ and $\gamma(C)$ may be less than stated above.
It can be shown that the two bits corresponding to the partition levels $D_4/RZ^4$ and $RD_4/2Z^4$ are the only ones affected by a $90^\circ$ phase rotation. So this code can be combined with a differential encoder to achieve $90^\circ$ rotational invariance as shown in Figure 14.

Fig. 14 Diagram of four dimensional $90^\circ$ rotationally invariant encoder with a differential encoder
Conclusions

• Several constructions for multi-level trellis codes are presented and many codes with better performance than previously known codes are found. These codes provide a flexible trade-off between coding gain, decoding complexity, and decoding delay.

• New multi-level trellis coded modulation schemes using generalized set partitioning methods are developed for QAM and PSK signal sets.

• New rotationally invariant multi-level trellis codes which can be combined with differential encoding to resolve phase ambiguity are presented.
Appendix B

New Multi-Level Codes over $GF(q)$