INTEGRATION OF SYSTEM IDENTIFICATION AND ROBUST CONTROLLER DESIGNS FOR FLEXIBLE STRUCTURES IN SPACE

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Abstract
A novel approach is developed using experimental data from the structural testing of a physical system to identify a reduced order model and its error for a robust controller design. There are three steps involved in the approach. First, an approximately balanced model is identified using the Eigen-system Realization Algorithm which is an identification algorithm. Second, the model error is calculated and described in frequency domain in terms of the $H_\infty$ norm. Third, a pole placement technique in combination with a $H_\infty$ control method is applied to design a controller for the considered system. A set of experimental data from an existing setup, namely the Mini-Mast system, is used to illustrate and verify the approach developed in this paper.

Introduction
Constructing a suitable mathematical model for a dynamic system is a major task in the fields of structure analysis and control design for vibration suppression. The difficulty of estimating and describing the model error,\(^1\) which is a necessity for a robust controller design,\(^2\) is even more severe than that of constructing a mathematical model. In the past, system identification techniques such as the Eigen-system Realization Algorithm (ERA)\(^13\)\textsuperscript{--}18 have been developed, using experimental data from the modal testing of large space structures, for constructing a mathematical model either for a controller design or for modal parameter identification including frequencies, damping and mode shapes. Yet, the model error issue has rarely been addressed.

The multi-input/multi-output time domain ERA technique was originally developed for modal parameter identification and successfully applied to the modal testing of large space structures. The ERA technique\(^13,14\) uses the singular values of a finite Hankel matrix formed by impulse response functions to determine the system order and further reduce the noise effect on the realized system model by truncating some small singular values. The retained singular values are dominated by the system signals and the order of the system is determined by the number of the retained singular values. On the other hand, the truncated singular values are dominated by noise and hence may be used to estimate a noise-related model error bound. Defining the model error bound to be the maximum truncated singular value is equivalent to defining a $H_\infty$ error bound for the realized system model. Once the $H_\infty$ error bound is obtained, a $H_\infty$ control method\(^3\) may be readily applied to derive a robust controller design for the system represented by the realized model. General speaking, the $H_\infty$ control methods are optimal frequency domain algorithms which compute the feedback control law by minimizing the maximum singular value of the system transfer function. It is known that the $H_\infty$ control methods address the full range of stability margin, sensitivity, and robust optimization. This paper is motivated by the strong connection between the ERA system identification method and the $H_\infty$ control methods. It seems natural to integrate together the system identification method with the control methods so that a robust controller design can be achieved directly using the experimental data.

The objective of this paper is to develop a method using experimental data from measurements of a structural testing to identify a reduced order model and its error for a robust controller design to suppress the vibrational motion of the structure. The approach is to integrate the Eigensystem Realization Algorithm with the $H_\infty$ robust control methods. This paper is written to outline the fundamental concepts from different disciplines and then integrate them together, while including advanced materials. In the first section, Hankel singular values of a linear system in frequency domain are discussed and the relation with the ERA Finite-Hankel singular values is described. In the second section, the ERA algorithm is briefly introduced and the convergence of its model realization to the balanced realization\(^19,20\) is described and discussed. The balanced realization has been widely used for model reduction\(^20\) and robust controller designs. In the third section, the $H_\infty$ control methods are briefly described. In the fourth section, the ERA/$H_\infty$ control algorithm is shown...
including two numerical examples. The ERA reduced real-
ized model is chosen as the $\hat{H}_\infty$ control design model and
the estimated model error is used to compute the weighting
matrices for a robust controller design. In the fifth section,
an example is given using an existing experimental setup, the
NASA Langley Mini-Mast system, to illustrate the ERA/$H_\infty$
control algorithm.

Hankel and Finite-Hankel Singular Values
of a Linear System

Computation of the model error bound is a key element
in developing robust control systems. The model error bound
can be expressed and described in many different ways
including the Hankel singular values of the system transfer-
function matrix. In this section, the Hankel and Finite-
Hankel singular values of a linear, time-invariant dynamical
system are discussed. Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) \quad (2)$$

where $x$ is an $n \times 1$ state vector, $u$ is an $m \times 1$ input vector,
$y$ is a $l \times 1$ output vector and $A$, $B$, and $C$ are $n \times n$, $n \times m$,
and $l \times n$ system matrices, respectively. The transfer function
of this system is

$$G(s) = C(sI - A)^{-1}B. \quad (3)$$

If the system is stable, i.e. all the eigenvalues of $A$ are in the
left half-plane, then the controllability and the observability
grammians are defined as

$$P = \int_0^\infty \exp(At)BB^*\exp(A^*t)dt$$

and

$$Q = \int_0^\infty \exp(A^*t)C^*Ce\exp(At)dt \quad (4)$$

where $P$ and $Q$ satisfy the Lyapunov equations

$$AP + PA^* + BB^* = 0$$

and

$$A^*Q + QA + C^*C = 0. \quad (5)$$

Here the superscript * is used herein to indicate the complex
conjugate of the variable so marked.

Hankel Singular Values

The Hankel singular values of this system are defined as

$$\sigma_i(G(s)) = (\lambda_i(PQ))^{1/2}, \quad \lambda_i \geq \lambda_{i+1} \quad (6)$$

where $\lambda_i$ is the $i$th eigenvalue. Now, consider the discrete-
time state-space model

$$\dot{x}(t+1) = \hat{A}x(t) + \hat{B}u(t) \quad (7)$$

$$y(t) = C\hat{x}(t). \quad (8)$$

If this system is stable then the discrete controllability and
observability grammians are defined as

$$P = \sum_{0 \leq k < \infty} A^kBB^*A^k$$

and

$$Q = \sum_{0 \leq k < \infty} A^kC^*C\hat{A}^k \quad (9)$$

which satisfy the discrete Lyapunov equations

$$P - A^*PA = BB^*$$

and

$$Q - A^*QA = C^*C. \quad (10)$$

The $z$ transform of this discrete system is given as

$$G(z) = C(zI - \hat{A})^{-1}\hat{B} \quad (11)$$

or in series expansion

$$G(z) = \sum_{i=0}^\infty Y_i z^{-i-1}$$

where $Y_i = C\hat{A}^i\hat{B}$, $i = 0, 1, 2, ...$ are impulse response
functions and usually referred to as the Markov parameters
associated with the model ($\hat{A}$, $\hat{B}$, $C$). The Hankel singular
values of $G(z)$ are defined as

$$\sigma_i(G(z)) = (\lambda_i(PQ))^{1/2}. \quad (12)$$

Because a discrete-time model is related to a continuous-time
model by an invariant step time response, the grammians in
the discrete time domain are equivalent to the grammians in
the continuous time domain.\textsuperscript{1,20} Therefore the Hankel
singular values for the continuous-time and discrete-time
models are identical.

Next it will be shown how the Hankel singular values
are related to the singular values of a Hankel matrix in
the discrete-time case. Consider a discrete-time infinite
Hankel matrix whose $(i,j)$th block is the Markov parameter
$Y_{i+j-2} = C\hat{A}^{i+j-2}\hat{B}$. The Hankel matrix is written as

$$H = W_6W_6^T = \begin{pmatrix}
Y_0 & Y_1 & \cdots & Y_{r-1} & \cdots \\
Y_1 & Y_2 & \cdots & Y_r & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
Y_{r-1} & Y_r & \cdots & Y_{r+s-2} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots
\end{pmatrix} \quad (13)$$
where
\[ W_o = [C^*, A^* C^*, \ldots, A^k C^*, \ldots]^* \]
and
\[ W_c = [B, A B, \ldots, A^k B, \ldots]. \]  
(14)
It can be shown that \( P = W_o W_c^* \) and that \( Q = W_c W_o \) satisfy Eq. (10). The singular values of \( H \) are
\[ \sigma_i(H) = (\lambda_i(W_o^* W_c^* W_c W_o))^1/2 = (\lambda_i(P Q))^1/2. \]  
(15)
In this equation, \( \lambda_i(W_o^* W_c^* W_c W_o) \) means that
\[ W_o^* W_c^* W_c W_o \zeta_i = \lambda_i \zeta_i \]  
(16)
where \( \zeta_i \) is the eigenvector corresponding to the eigenvalue \( \lambda_i \). Pre-multiplying Eq. (16) by \( W_c \) and defining \( \tilde{x}_i = W_c \zeta_i \) yields
\[ W_c W_o^* W_c^* W_o \tilde{x}_i = \lambda_i \tilde{x}_i. \]  
(17)
Because Eqs. (16) and (17) produce the same eigenvalues, Eq. (15) can be written as
\[ \sigma_i(H) = (\lambda_i(W_c W_o^* W_c^* W_o))^1/2 = (\lambda_i(P Q))^1/2. \]  
(18)
Hence the Hankel singular values of \( G(z) \), Eq. (11), which is formed from impulse response functions, are the singular values of \( H \), Eq. (13). Again note that the Hankel matrix \( H \) in (13) is infinite dimensional. It is, of course, impractical to solve for the Hankel singular values from an infinite dimensional Hankel matrix \( H \). If system matrices \( A, B, \) and \( C \) are exactly known, the Hankel singular values can be obtained by solving Eq. (12). In practice, however, flexible structures are subjected to uncertainties to a certain extent. Experiments are conducted to verify the matrices \( A, B \) and \( C \). In general, experimental data of inputs and outputs are stored in terms of the system transfer function (frequency domain) which is equivalent to the impulse response functions (Markov parameter) (discrete-time domain). Therefore, from a practical point of view, the Hankel singular values can be approximately computed from a Hankel matrix that is formed by a finite number of Markov parameters. The question arises how much error one expects to have if the Hankel singular values are computed from a finite-dimensional Hankel matrix. This question will be answered in the following section.

**Finite-Hankel Singular Values**

In this section, the singular values obtained from a Finite-Hankel matrix are compared with the Hankel singular values. First a Finite-Hankel matrix is defined as
\[ H(0) = \begin{pmatrix} Y_0 & Y_1 & \ldots & Y_{s-1} & 0 & \ldots \\ Y_1 & Y_2 & \ldots & Y_s & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ Y_{r-1} & Y_r & \ldots & Y_{r+s-2} & 0 & \ldots \end{pmatrix} \]  
(19)
where \( Y_i \) is the Markov parameter. An extended matrix of the Finite-Hankel matrix \( H(0) \) is defined as
\[ H_{rs}^0 = \begin{pmatrix} Y_0 & Y_1 & \ldots & Y_{s-1} & 0 & \ldots \\ Y_1 & Y_2 & \ldots & Y_s & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ Y_{r-1} & Y_r & \ldots & Y_{r+s-2} & 0 & \ldots \end{pmatrix}. \]  
(20)
It is obvious that
\[ \sigma_i(H_{rs}^0) = \sigma_i(H(0)). \]  
(21)
The Hankel matrix \( H \) in Eq. (13) can be written as
\[ H = H_{rs}^0 + H_{rs}', \]  
(22)
where
\[ H_{rs}' = \begin{pmatrix} 0 & \ldots & 0 & Y_s & \ldots \\ 0 & \ldots & 0 & Y_{s+1} & \ldots \\ \vdots & \ldots & \ddots & \vdots & \ddots \\ 0 & \ldots & 0 & Y_{r+s-1} & Y_{r+s} \end{pmatrix}. \]  
(23)
the left upper \( r \times s \) blocks of \( H_{rs}' \) are 0 and the other blocks are the same as \( H \). Next it will be shown that the Finite-Hankel singular values converges to the Hankel singular values. Consider the system with the single-input and single-output transfer function \( G(s) \) that is real rational, scalar-valued, strictly proper and analytic in \( \text{Re } s > 0 \). Then the Markov parameter \( Y_i \) can be expressed as
\[ Y_i = \sum_{j=1}^{k} c_j e^{\lambda_j \Delta t} \]  
(24)
where \( k \) is the system order, \( \lambda_j \) is the \( j \)-th eigenvalue with negative real part and \( \Delta t \) is the measurement time increment. The norm of \( Y_i \) satisfies the following inequality
\[ |Y_i| \leq ce^{-\lambda_i \Delta t}, \quad t_i = i \Delta t, \quad i \geq n_i \]  
(25)
where \( c \) and \( \lambda \) are positive real constants, and \( n_i \) is a positive integer. Using singular value inequalities\(^{21}\) yields
\[ \sigma_i(H) \leq \sigma_i(H_{rs}^0) + \sigma_i(H_{rs}'). \]  
(26)
Let $r > n_i + 1$ and $s > n_i + 1$ then

$$\sigma_1(H'_r) \leq \|H'_r\| = \max_{j > s} \sum_{i=1}^{\infty} |Y_{i+j-1}|$$

$$= \sum_{i=s}^{\infty} |Y_i| \leq \sum_{i=n_i+1}^{\infty} \sum_{i=n_i+1}^{\infty} |ce^{-\lambda t_i}|$$

(27)

$$\leq \frac{1}{\lambda^2 \Delta} \int_{n_i}^{\infty} \frac{e^{-\lambda t dt}}{\lambda^2} \leq \frac{1}{\lambda^2 \Delta} \int_{n_i}^{\infty} e^{-\lambda t} dt = \frac{e}{\lambda^2 \Delta} e^{-\lambda n_i} \leq \epsilon$$

where $\epsilon$ is a positive real number. For any small $\epsilon > 0$, there exists an $n_i$ which makes Eq. (27) true. Combination of Eqs. (21), (26) and (27) yields

$$\sigma_i(H) - \sigma_i(H'_r) \leq \sigma_i(H'_r) \leq \sigma_i(H) + \sigma_i(-H'_r)$$

or

$$\sigma_i(H) - \epsilon \leq \sigma_i(H(0)) \leq \sigma_i(H) + \epsilon.$$  (28)

Equation (28) shows that the Finite-Hankel singular values converge to the Hankel singular values. The proof can be easily extended for the multi-input/multi-output system.

The finite-Hankel matrix is the basis for the Eigensystem Realization Algorithm (ERA) which can be used to identify a system model for modal parameter (system eigenvalues) identification or controller designs. The Markov parameters $Y_i$ are the impulse response functions which may be obtained experimentally.

**Eigensystem Realization Algorithm (ERA) and Model Reduction**

The problem of constructing a suitable mathematical model for a dynamic system is a major task for a control engineer as well as a structural engineer. For structural analyses and controller designs experimental data from structural testing are used to either verify an analytical model or directly determine a mathematical model. The ERA is an algorithm which computes a mathematical model directly from experimental data. In this section, the basic ERA formulations are briefly discussed and the relation between the ERA realization and the balanced realization is derived. Note that balanced realization has been frequently used for model reduction,

**Basic ERA Formulations**

The ERA algorithm uses the singular value decomposition of a Hankel matrix to determine a system model under test. Consider the discrete-time state-space model as given by Eq. (8). Let the input $y$ be the impulse input at an initial time such that its components satisfy $u_i(0) = 1$, ($i = 1, \ldots, m$) and $u_i(k) = 0$, ($k = 1, \ldots$). The time domain description is given by the Markov parameters $Y_i$, i.e. impulse response functions,

$$Y_i = C \hat{A}^i \hat{B}.$$  (29)

Note that there are many other ways to obtain the impulse response functions experimentally. For example, inverting a transfer function from the frequency domain to the time domain will generate the impulse response functions. The algorithm begins by forming the $r \times s$ block matrix (generalized Finite-Hankel matrix)

$$H(k) = \begin{bmatrix}
Y_k & Y_{k+1} & \cdots & Y_{k+s-1} \\
Y_{k+1} & Y_{k+2} & \cdots & Y_{k+s} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{k+r-1} & Y_{k+r} & \cdots & Y_{k+r+s-2}
\end{bmatrix}.$$  (30)

The Finite-Hankel matrix $H(k)$ can be expressed as

$$H(k) = V_c \hat{A}^k W_s.$$  (31)

where

$$V_c = \begin{bmatrix} C \\ C \hat{A} \\ \vdots \end{bmatrix}, \quad W_s = [\hat{B} \hat{A} \hat{B} \cdots \hat{A}^{s-1} \hat{B}],$$

$V_c$ and $W_s$ are the observability and controllability matrices, respectively. Assume that there exists a matrix $H'$, such that

$$W_s H' V_c = I_n$$  (32)

where $I_n$ is an identity matrix of order $n$. Equations (31) and (32) imply

$$H(0) H' H(0) = V_c W_s H' V_c W_s = V_c W_s = H(0).$$  (33)

Thus $H'$ is the pseudo inverse of $H(0)$. A solution for $H'$ can be obtained as follows. Use singular value decomposition to the matrix $H(0)$

$$H(0) = U \Sigma V^T$$  (34)

where the columns of $U$ and $V$ are orthonormal and $\Sigma$ is diagonal with positive elements $[\sigma_1, \sigma_2, \ldots, \sigma_n]$. The pseudo inverse can be computed as

$$H'(0) = [V][\Sigma^{-1} U^T].$$  (35)

Define $0_m$ as the null matrix of order $m$, $I_m$ an identity matrix, $E_m^T = [I_m, 0_m, \ldots, 0_m]$. From Eqs. (31), (32), and
The observability and controllability grammians can thus be written as

\[
Q = W_0^T W_0 \\
= \nu_1^T \nu_1 + (A^T)^T V_1^T V_1 \nu_1 + (A^T)^T V_1^T V_1 \nu_1 + \ldots \\
= \Sigma + (A^T)^T \Sigma A^T + (A^T)^T \Sigma A^T + \ldots
\]

and

\[
P = W_s^r W_s^T \\
= W_s^r W_s^T + \lambda^2 W_s^r W_s^T (A^T) + \lambda^2 W_s^r W_s^T (A^T) + \ldots \\
= \Sigma + \lambda^2 \Sigma (A^T) + \lambda^2 \Sigma (A^T) + \ldots
\]

If the system is stable, all the eigenvalues of \( \tilde{A} \) should be inside the unit circle, i.e., \( |\lambda(A)| < 1 \). Therefore, for a stable system, there exists a value of \( n \) such that \( |\lambda(A^n)| \) is less than \( \varepsilon \) for a given \( \varepsilon \). Equation (28) implies

\[
\lim_{r \to \infty, s \to \infty} \Sigma = \Sigma_{\infty}
\]

where \( \Sigma_{\infty} \) is the \( \Sigma \) in Eq. (34) when \( r \) and \( s \) approach \( \infty \), which, in turn from Eq. (40), yields

\[
\lim_{r \to \infty, s \to \infty} P = \lim_{r \to \infty, s \to \infty} \Sigma = \Sigma_{\infty}
\]

and

\[
\lim_{r \to \infty, s \to \infty} Q = \lim_{r \to \infty, s \to \infty} \Sigma = \Sigma_{\infty}.
\]

Thus the ERA realization converges to the balanced realization. So far the realization order is assumed to be identical to the true one. If the realization system order \( \gamma \) is lower than the true one \( n \), Eq. (36) implies that \( C_n \) is the \( l \times \gamma \) left block of \( C_n \), \( A_\gamma \) is the \( \gamma \times \gamma \) left-upper block of \( A_n \) and \( B_\gamma \) is the \( \gamma \times m \) upper block of \( B_n \). Since the ERA realization \( \{A_n, B_n, C_n\} \) converges to the balanced realization \( \{A_\gamma, B_\gamma, C_\gamma\} \), \( A_\gamma \), \( B_\gamma \) and \( C_\gamma \) should converge to the corresponding blocks of \( A_n \), \( B_n \) and \( C_n \), respectively. From Eq. (42) and the balanced realization, one obtains

\[
\lim_{r \to \infty, s \to \infty} P_\gamma = \Sigma_{\infty} \quad \lim_{r \to \infty, s \to \infty} Q_\gamma = \Sigma_{\infty}
\]

where \( P_\gamma \) and \( Q_\gamma \) are the controllability and observability grammians of \( \{A_\gamma, B_\gamma, C_\gamma\} \) and \( \Sigma_{\infty} \) is the \( \gamma \times \gamma \) left-upper block matrix of \( \Sigma_{\infty} \). The \( \gamma \)-order ERA realization converges to the \( \gamma \)-order balanced realization. It is concluded that the reduced ERA model converges to the reduced balanced model.

**H∞ Robust Controller Designs**

The \( \infty \) control methods are optimal frequency domain algorithms\(^2\-^4\), which address the full range of stability margin, sensitivity, and robust criteria. In this section, a brief description of \( \infty \) control methods is given.
A block diagram of the $H_\infty$ design algorithm is presented in Fig. 1. The closed-loop transfer function matrices ($u_1$ to three outputs $y_2$, $u_2$ and $y$) are expressed as

$$S(s) = (I + L(s))^{-1} \quad (44)$$
$$R(s) = F(s)(I + L(s))^{-1} \quad (45)$$
$$T(s) = L(s)(I + L(s))^{-1} \quad (46)$$

where $L(s) = G(s)F(s)$, $G(s)$ is defined in Eq. (3), and $F(s)$ is the controller transfer function matrix. The matrices $S(s)$ and $T(s)$ are known as the sensitivity and complementary sensitivity, respectively. The singular values of these matrices can be used to quantify the stability margins and performance of the system. In the $H_\infty$ controller design, the weighting function matrix $W_1(jw)$ is used to weight the sensitivity so that

$$\sigma(S(jw)) \leq |W_1^{-1}(jw)| \quad \text{(47)}$$

where $\sigma(\cdot)$ means the maximum singular values of ($\cdot$). The quantity $|W_1^{-1}(jw)|$ is the desired disturbance attenuation.

The plant uncertainty can be considered as additive uncertainty $\Delta_A$ or multiplicative uncertainty $\Delta_M$. The stability margin of the additive uncertainty must satisfy

$$\sigma(\Delta_A(jw))\sigma(R(jw)) \leq 1. \quad \text{(48)}$$

If the weighting function matrix $W_2(jw)$ is chosen such that

$$|W_2(jw)| \geq \sigma(\Delta_A(jw))$$

and

$$|W_2(jw)|\sigma(R(jw)) \leq 1 \quad \text{(49)}$$

then the inequality Eq. (48) holds. The disturbance attenuation and additive stability margin designs can be combined into a single $H_\infty$ problem as

$$\|T_{y_1u_1}\|_\infty \leq 1 \quad \text{(50)}$$

where

$$T_{y_1u_1} = \begin{pmatrix} W_1S \\ W_2R \end{pmatrix} \quad \text{(51)}$$

and this combination is called the $H_\infty$ additive perturbation control design. Similarly, the $H_\infty$ multiplicative perturbation control design can be derived as

$$\|T_{y_1u_1}\|_\infty \leq 1 \quad \text{(52)}$$

where

$$T_{y_1u_1} = \begin{pmatrix} W_1S \\ W_3T \end{pmatrix} \quad \text{(53)}$$

A detailed examination of $H_\infty$ design methods can be found in Ref. 7. In the next section, a combined ERA/$H_\infty$ control algorithm is developed and discussed including some numerical examples.

**ERA/$H_\infty$ Control Algorithm**

This section shows a control design algorithm which combines the features of both ERA and $H_\infty$ control methods. The flowchart of this algorithm is shown in Fig. 2.

To use the ERA/$H_\infty$ algorithm, a set of experimental input and output data is required. The model error between the reduced ERA model and the full ERA model is used for a $H_\infty$ controller design. The full ERA model is defined as the model of maximum order which can be obtained by the ERA. For noise-free measurements, if not impossible in practice, the full ERA model is the exact system model. For noisy measurements; the order of the full ERA model is equal to either the number of columns or the number of rows in the finite-Hankel matrix whichever is less. The reduced ERA model is the model reduced from the full model such as the triple [$A_1$, $B_1$, $C_1$] shown in the last paragraph of the ERA Section.

It has been shown that the ERA models converge to the balanced models. The $\gamma$-th order reduced balanced model $G_\gamma(s)$ of an $n$-th full order system $G(s)$ satisfies the following error bound

$$\sigma(G(jw) - G_\gamma(jw)) \leq 2 \sum_{i=k+1}^{n} \sigma_i \quad \text{(54)}$$

where $\sigma(\cdot)$ is the maximum singular value of ($\cdot$) and $\sigma_i$ is the $i$th Hankel singular value of $G(s)$. This equation provides the $H_\infty$ error bound for the reduced model. There are two different $H_\infty$ control designs, namely the additive perturbation control design and the multiplicative perturbation control design. Numerical examples are used in the following to illustrate the ERA/$H_\infty$ control algorithm for the two different designs.

**ERA/$H_\infty$ Additive Perturbation Control Design**

In this design the closed-loop transfer function must satisfy the inequality

$$\|T_{yu}\|_\infty \leq 1 \quad \text{(55)}$$

where

$$T_{yu} = \begin{pmatrix} W_1S \\ W_2R \end{pmatrix} \quad \text{(56)}$$

and $u$ is the input, $y$ is the output, and $W_1$ and $W_2$ are appropriate weighting functions. The sensitivity matrix $S$ tends to be dominated by the sensitivity of low frequency modes (or dominant modes). An appropriate weighting function $W_1$ is the inverse of the desired sensitivity. The weighting function matrix $W_2$ is chosen to penalize the control action at high frequencies (or residual modes) as well as incorporate other robustness constraints. One of the design
goals is to assure that unmodelled high frequency dynamics will not destabilize the closed-loop systems. Some examples of choosing weighting functions can be found in Refs. 10 and 11.

Let the system impulse response measurement be generated by the following equation

\[ y(t_i) = \sum_{i=1}^{4} a_i \exp(\zeta_i t_i) \sin(\omega_i t_i) \]  

(57)

where the system parameters are listed in Table 1.

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<th>3</th>
<th>4</th>
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<td>-0.5</td>
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For illustration, the measurements are assumed to be noise free. The Finite-Hankel matrix size is chosen as 40 \(\times\) 45. Figure 3 shows the singular-value Bode plots of the full model transfer function \(G\), the reduced model transfer function \(G_r\) and the model error \(\Delta G = G - G_r\). Here \(G_r\) of order 4 is the transfer function obtained by the ERA algorithm. The model error \(\Delta G\) is an additive model error. The additive robustness weighting function \(W_2\) and the sensitivity weighting function \(W_1\) are chosen as

\[ W_1 = \frac{0.005s + 1}{0.4s + 0.01}, \quad W_2 = \frac{0.04s + 0.005}{0.001s + 1}. \]

Figure 4 shows the Bode plots of \(\Delta G\) and \(W_2\), whereas Fig. 5 shows the Bode plots of \(W_1\) and \(W_2\). The Bode plot of \(T_{yu}\) is shown in Fig. 6, in which the gain margin is 5.2 db and the phase margin is 56.6 degree.

**ERA/H∞ Multiplicative Perturbation Control Design**

In this design, the closed-loop transfer function must satisfy

\[ \|T_{yu}\| \leq 1 \]  

(58)

where

\[ T_{yu} = \begin{pmatrix} W_1 S \\ W_3 T \end{pmatrix} \]  

(59)

and \(W_1\) and \(W_3\) are appropriate weighting functions. The system measurements and the Finite-Hankel matrix are the same as in the above example. Figure 7 shows the design robustness \(W_3 G_r\) and model error. The design weighting matrices are chosen as

\[ W_1 = \frac{0.005s + 1}{0.2s + 0.01}, \quad W_2 = 1.0 \times 10^{-7}, \quad W_3 = \frac{0.04s + 0.01}{0.001s + 1}. \]

Figure 8 shows the Bode plots of the weighting matrices. The Bode plot of the closed-loop transfer function \(T_{yu}\) is shown in Figure 9, in which the gain margin is 6.0 db and the phase margin is 54.2 degree.

The ERA/H∞ control algorithm provides robust control designs directly from experimental input and output data. First, the ERA algorithm is used to compute an appropriate reduced balanced model and its error in terms of the \(H_\infty\) norm. Then, the model error norm is used to select the \(H_\infty\) control weighting matrices for a robust controller design. In the next section, the ERA/H∞ algorithm is applied to an experimental setup, i.e. the Mini-Mast system\(^{22}\), for further illustration of the concept developed in this paper.

**ERA/H∞ Application: The Mini-Mast System**

In this section, the ERA/H∞ algorithm is used to design robust controllers for the Mini-Mast system\(^{22}\) with two inputs (torque wheels) and two outputs (displacement sensors). Mini-Mast is a 20-meter-long, deployable/retractable truss located in the Structural Dynamics Research Laboratory at NASA Langley Research Center. Mini-Mast was designed and built to high standards typical of spacecraft hardware which was constructed from graphite-epoxy tubes, titanium joints, and precision fabrication techniques.\(^{23}\) It is used as a ground test article for research in the areas of structural analysis, system identification,\(^{24}\) and control of large space structures. The sinusoid inputs with the frequencies close to the system natural frequencies are used to excite the system. The system has five modes less than 7 Hz and the lowest frequency is about 0.86 Hz. The design strategy is as follows:

1. A time domain least squares technique\(^{25}\) is used to compute Markov parameters from input and output measurement data.
2. A Finite-Hankel matrix is formed from the Markov parameters and the ERA is used to compute a reduced model and its model error.
3. A \(H_\infty\) control method is applied to design a robust controller. When applying the ERA, the discrete time increment should be adjusted to compute an accurate model within the frequency range of interest. Figure 10 shows the singular value Bode plot of the ERA model transfer function. The singular value Bode plots of the model error (solid lines) and the designed additive robustness (dashed lines) are shown in Fig. 11.

In MATLAB\(^7\), the \(H_\infty\) control computations are performed by using the recent Glover-Doyle "2-Riccati" formulae.\(^{12}\) After the \(H_\infty\) control design is executed, the closed-loop system keeps the open-loop stable complex eigenvalue pairs and reflects the open-loop unstable complex eigenvalue pairs. Because the identified open-loop model of the Mini-Mast system has small positive damping (system
modes) or negative damping (noise modes) for most complex eigenvalues, the \( H_{\infty} \) closed-loop system will then have low damping. To improve this condition, a multivariable bilinear transform\(^8\) can be used to shift the open-loop eigenvalues before applying the \( H_{\infty} \) control methods. The process of the bilinear transform is described below:

1. The system model and the design weighting functions are shifted with a real value \( \alpha \).
2. The \( H_{\infty} \) design methods are used to design a controller for the shifted system.
3. Either the closed-loop system is shifted back by \(-\alpha\) or the controller is used in the original nonshifted system.
4. Check the singular values of the closed-loop transfer function matrix \( T_{\text{vu}} \), Eq. (56), which should be less than one.

In the process of applying the bilinear transform, several problems listed below are found.

1. The original system has a \( H_{\infty} \) control solution with appropriate design weighting functions, but the shifted system with the shifted weighting functions does not always have a \( H_{\infty} \) control design solution.
2. The singular values of the closed-loop system increase significantly when the imaginary axis has a positive shift.

To improve the above problems, a different technique which is equivalent to a pole placement technique is developed yielding freedom for assigning each closed-loop eigenvalue independently. Let the system state matrix be diagonalized such that

\[
\bar{A} = \Psi^{-1} A \Psi
\]  

(60)

where \( \Psi \) is the eigenvector matrix. Let the closed-loop state matrix be represented as

\[
\bar{A} = \Psi (\bar{A} + A1) \Psi^{-1}
\]  

(61)

where \( A1 \) is a semipositive diagonal matrix. The design strategy is as follows.

1. Use \( \bar{A} \) as the state matrix for the \( H_{\infty} \) controller design.
2. Apply the controller to the original system.
3. Check whether this control design satisfies the \( H_{\infty} \) constraint equation, Eq. (55).

Figure 12 shows the singular value Bode plot of a \( H_{\infty} \) design transfer function \( T_{\text{vu}} \) with a specified matrix \( A1 \) which is not shown. The matrix \( A1 \) provides more than 10% damping for the first mode. Next, the computational simulation is used to check the performance of the above design. The solid lines in Fig. 13 are the first 5-sec experimental outputs. In these plots, \( y_{ij} \) represents the \( i \)th output with the \( j \)th input. The outputs from the ERA reduced model using the same inputs as the experimental data are plotted in dashed lines. The inputs are turned off after 5 seconds. The open-loop and closed-loop responses are compared in Fig. 14. The solid lines in Fig. 14 are the open-loop free decay responses, whereas the closed-loop free responses are plotted in dashed lines. The first closed-loop mode has more than 10% damping.

In this ERA/\( H_{\infty} \) control design, the following items are observed.

1. This algorithm can be used to design a robust controller directly from a set of experimental input and output data.
2. The controller order is the sum of the model order and weighting function order. Economical controllers may be obtainable through the use of appropriate model reduction techniques.
3. If the identified low frequency modes are somewhat accurate, the robustness weighting can cover significant high frequency model errors.
4. The additive matrix \( A1 \) can be used to assign the closed-loop dampsings and frequencies for the modes of interest.
5. An analytical model is not required.

**Concluding Remarks**

In this paper, the Eigensystem Realization Algorithm is integrated with the \( H_{\infty} \) control methods to develop a technique for controller designs of flexible structures. The technique starts with a set of input and output data obtained from vibration tests of a physical system. The set of data is then used to identify a reduced model and assess its error bound. The reduced model describes the physical system, whereas the error bound characterizes the system and measurement uncertainties. Both reduced model and error bound are used to design a robust controller to attenuate the vibrational motion of the physical system. Since experimental data are used, actuator dynamics, sensor dynamics and filter dynamics are included in the identification of the reduced model and thus in the controller designs. This approach is believed to provide the reduced model with fidelity. Numerical simulations for control of an experimental setup (the Mini-Mast system) indicate that this approach is very promising for use in control of flexible structures. Note that experimental models are used in these simulations. The controllers which are successfully demonstrated in the simulations will be tested in the Mini-Mast system to verify the approach developed in this paper.

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References


Figure 1 Weighted sensitivity and complementary sensitivity problem

Figure 2 ERA/Hᵦ Design Flowchart
Figure 3  Full Model, Reduced Model and Model Error

Figure 4  Design Robustness and Model Error

Figure 5  Weighting Functions

Figure 6  Closed-loop Transfer Function $T_{yy}$
Singular Value (DB)

Figure 7  Design Robustness and Model Error

Singular Value (DB)

Figure 8  Weighting Functions

Singular Value (DB)

Figure 9  Closed-loop Transfer Function $T_{yx}$
Figure 10 System Model

Figure 11 Design Robustness and Model Error

Figure 12 Closed-loop Transfer Function $T_{yu}$
Figure 13 Experimental and Identified System Outputs
Figure 14 Closed-loop and Open-loop Free Decay Responses
A novel approach is developed using experimental data to identify a reduced-order model and its model error for a robust controller design. There are three steps involved in the approach. First, an approximately balanced model is identified using the Eigensystem Realization Algorithm, which is an identification algorithm. Second, the model error is calculated and described in frequency domain in terms of the $H_\infty$ norm. Third, a pole placement technique in combination with a $H_\infty$ control method is applied to design a controller for the considered system. A set of experimental data from an existing setup, namely the Mini-Mast system, is used to illustrate and verify the approach developed in this paper.