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General Rotorcraft Aeromechanical Stability Program (GRASP) Theory Manual

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General Rotorcraft Aeromechanical Stability Program (GRASP) Theory Manual

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SYMBOLS

$\underline{A}^{PF}$	acceleration of point P in coordinate system F
A^{FI}_{Fi}	acceleration component of point F (origin of coordinate system F) in coordinate system I (inertial) along $\hat{\underline{b}}_i^F$
B_1	aeroelastic beam cross section integral $= \iint E\xi_1({\xi_1}^2 + {\xi_2}^2)dA$
B_2	aeroelastic beam cross section integral $= \iint E\xi_2({\xi_1}^2 + {\xi_2}^2)dA$
B_3	aeroelastic beam cross section integral $= \iint E({\xi_1}^2 + {\xi_2}^2)^2 dA$
b	number of blades
$\hat{\underline{b}}_i^F$	basis vectors for coordinate system F
C	damping matrix or gyroscopic matrix; also aeroelastic beam direction cosines $C^{P''F''}$ of aeroelastic beam cross section principal axes relative to frame basis
C^{AB}_{ij}	direction cosines for A basis relative to B basis $= \hat{\underline{b}}_i^A \cdot \hat{\underline{b}}_j^B$
c	aeroelastic beam chord length
c_d	drag coefficient for aeroelastic beam airfoil aerodynamic center
c_i	$\cos\theta_i$
c_l	lift coefficient for aeroelastic beam airfoil aerodynamic center
c_m	moment coefficient for aeroelastic beam airfoil aerodynamic center
$\mathcal{D}$	drag force per unit blade length
D_0	aeroelastic beam cross section integral $= \iint E(\xi_2\lambda_1 - \xi_1\lambda_2)dA$
D_1	aeroelastic beam cross section integral $= \iint E\xi_1(\xi_2\lambda_1 - \xi_1\lambda_2)dA$
D_2	aeroelastic beam cross section integral $= \iint E\xi_2(\xi_2\lambda_1 - \xi_1\lambda_2)dA$
D_3	aeroelastic beam cross section integral $= \iint E(\xi_1^2 + \xi_2^2)(\xi_2\lambda_1 - \xi_1\lambda_2)dA$
D_4	aeroelastic beam cross section integral $= \iint E(\xi_2\lambda_1 - \xi_1\lambda_2)^2 dA$
E	Young's modulus
E_0	aeroelastic beam axial rigidity $= \iint E dA$
E_1	aeroelastic beam first flexural moment about local ξ_1 axis $= \iint E\xi_1 dA$
E_2	aeroelastic beam first flexural moment about local ξ_2 axis $= \iint E\xi_2 dA$
$\hat{\underline{e}}$	unit vector (other than a basis vector)
$\underline{F}^P$	force at P
G	shear modulus
g_{Fi}	component of gravity vector along $\hat{\underline{b}}_i^F$
G_j	Jacobi polynomials where $j = 0, 1, 2, \ldots$
$\underline{H}^{PI}$	angular momentum of body P in coordinate system I
$\underline{I}$	inertia dyadic

I_1	aeroelastic beam bending rigidity about local ξ_1 axis $= \iint E\xi_2^2 dA$; also, rigid-body mass principal moment of inertia for mass center about $\hat{\underline{b}}_1^{N''}$
I_2	aeroelastic beam bending rigidity about local ξ_2 axis $= \iint E\xi_1^2 dA$; also, rigid-body mass principal moment of inertia for mass center about $\hat{\underline{b}}_2^{N''}$
I_3	aeroelastic beam Young's modulus weighted polar moment of inertia $I_1 + I_2$; also rigid-body mass principal moment of inertia for mass center about $\hat{\underline{b}}_3^{N''}$
i_1	aeroelastic beam mass moment of inertia about local ξ_1 axis $= \iint \rho_s \xi_2^2 dA$
i_2	aeroelastic beam mass moment of inertia about local ξ_2 axis $= \iint \rho_s \xi_1^2 dA$
i_3	$i_1 + i_2$
J	torsion rigidity$= \iint G[(\lambda_1 - \xi_2)^2 + (\lambda_2 + \xi_1)^2] dA$
$\mathcal{L}_c$	circulatory lift
$\mathcal{L}_{nc}$	noncirculatory lift
ℓ	element length
$\mathcal{M}$	pitching moment
$\underline{M}^P$	moment at P
m	aeroelastic beam mass per unit length $= \iint \rho_s dA$; also mass of rigid-body mass
m_1	aeroelastic beam first mass moment of cross section about ξ_1 axis $= \iint \rho_s \xi_1 dA$
m_2	aeroelastic beam first mass moment of cross section about ξ_2 axis $= \iint \rho_s \xi_2 dA$
N_i	number of generalized coordinates for aeroelastic beam
Q	a generic column matrix representing generalized forces
Q_{ij}	generalized forces for aeroelastic beam
q	a generic column matrix representing generalized coordinates
q_{ij}	generalized coordinates for aeroelastic beam
$\underline{R}^{PF}$	position of point P with respect to the origin of coordinate system F
$\mathcal{R}$	constraint transformation matrix
R	rotor radius
r	perpendicular distance of the point Q'' to the axisymmetric center of the flowfield $= \sqrt{(R_{A2}^{Q''A})^2 + (R_{A3}^{Q''A})^2}$
s_i	$\sin\theta_i$
t	time
U	rectangular matrix of real eigenvectors
u_i	deflection in the ith direction
$\underline{U}^{QI}$	inertial air velocity at point Q
$\underline{V}^{PF}$	structural velocity of point P in coordinate system F
W	$\sqrt{(W_{Z''1}^{Q''})^2 + (W_{Z''2}^{Q''})^2}$

$\underline{W}^{Q}$	relative wind velocity at Q
y	lateral direction
z	vertical direction
α	angle of attack
$\underline{\alpha}^{PF}$	angular acceleration of coordinate system P in coordinate system F
β_i	C1-type shape functions
γ_{1i}^{A}	cyclic air flow velocity perturbations
γ_{1r}^{A}	radial air flow gradient
Δ	3×3 identity matrix
δ_{ij}	Kronecker delta
${}^{I}\underline{\delta R}^{BA}$	Virtual displacement of B relative to A in I
$\delta \mathcal{W}$	virtual work
$\delta\underline{\psi}^{BI}$	Virtual rotation of B in I
ϵ	Blade root cutout
ϵ_{ijk}	Levi-Civita epsilon permutation symbol
θ	pretwist angle
θ'	pretwist per unit length $\left(\frac{d\theta}{dx_3}\right)$
θ_i	Tait-Bryan angles
κ_i	moment strains
λ	cross section warp function
λ_α	$\partial\lambda/\partial\xi_\alpha \quad \alpha = 1,2$
ξ_i	cross section principal axes
ρ_a	air density
ρ_s	structural density
ϕ_i	Euler-Rodrigues parameters
ψ	azimuth angle
ψ_i	C0-type shape functions
Ω	rotor angular speed
Ω^{PF}	angular velocity of coordinate system P in coordinate system F
$\times$	cross product
$(\ )'$	$\frac{\partial}{\partial x_3}(\ )$
$(\ )''$	$\frac{\partial^2}{\partial x_3^2}(\ )$
$(\dot{\ })$	$\frac{\partial}{\partial t}(\ )$
$(\ddot{\ })$	$\frac{\partial^2}{\partial t^2}(\ )$
$(\bar{\ })$	small perturbation of $(\ )$

$(\bar{\ })$	static equlibrium value of ( )
$(\tilde{\ })$	$-\epsilon_{ijk}(\ )_k$
$\{\ \}$	column matrix
$\lfloor\ \rfloor$	row matrix
$(\ )_i$	ith component of column matrix
$(\ )_{ij}$	ijth component of matrix

SUMMARY

The Rotorcraft Dynamics Division, Aeroflightdynamics Directorate, U.S. Army Aviation Research and Technology Activity (AVSCOM) has developed the General Rotorcraft Aeromechanical Stability Program (GRASP) to calculate aeroelastic stability for rotorcraft in hovering flight, vertical flight, and ground contact conditions. In this report, GRASP is described in terms of its capabilities and the philosophy behind its modeling. The equations of motion that govern the physical system are described, as well as the analytical approximations used to derive the equations. These equations include the kinematical equation, the element equations, and the constraint equations. In addition, the solution procedures used by GRASP are described.

GRASP is capable of treating the nonlinear static and linearized dynamic behavior of structures represented by arbitrary collections of rigid-body and beam elements. These elements may be connected in an arbitrary fashion, and are permitted to have large relative motions. The main limitation of this analysis is that periodic coefficient effects are not treated, restricting rotorcraft flight conditions to hover, axial flight, and ground contact. Instead of following the methods employed in other rotorcraft programs, GRASP is designed to be a hybrid of the finite-element method and the multibody methods used in spacecraft analyses. GRASP differs from traditional finite-element programs by allowing multiple levels of substructures in which the substructures can move and/or rotate relative to others with no small-angle approximations. This capability facilitates the modeling of rotorcraft structures, including the rotating/nonrotating interface and the details of the blade/root kinematics for various rotor types. GRASP differs from traditional multibody programs by considering aeroelastic effects, including inflow dynamics (simple unsteady aerodynamics) and nonlinear aerodynamic coefficients.

1. INTRODUCTION

Previous helicopter aeroelastic stability programs have suffered from significant restrictions. The General Rotorcraft Aeromechanical Stability Program has been developed using a modern approach which overcomes these limitations.

1.1. Background

In early efforts made to calculate the aeroelastic stability of hingeless helicopter rotor blades, it was common practice to make use of simple physical models (*e.g.*, spring-restrained, centrally-hinged, rigid blades (ref. 1)). Later work treated configurations that were somewhat more complex, and included models of elastic blades (ref. 2), body degrees of freedom, and inflow dynamics (ref. 3). These simple approaches to rotorcraft aeroelastic stability calculations have been very valuable for gaining physical insight into many complicated phenomena (*e.g.*, coupled rotor-fuselage stability). They all are, however, based on a single physical model, and therefore are of limited value when more realistic rotorcraft configurations must be analyzed.

Because of the complex couplings inherent in a bent and twisted beam, the calculation of aeroelastic stability is particularly important in the analysis of rotor blades having cantilever root boundary conditions (*e.g.*, hingeless and bearingless rotors). In bearingless rotors, the blade/root kinematics demand a great deal of modeling flexibility because individual blade designs tend to have widely varying configurations. The FLAIR program (refs. 4, 5, and 6) is able to perform this type of aeroelastic stability calculation, but is limited to a configuration that has a rigid blade, a uniform flexbeam, linear aerodynamics, static induced velocity, and several different blade/root configurations. While FLAIR is currently being used in the rotorcraft community, it lacks the flexibility and generality necessary for it to be considered general-purpose analysis.

For analysis of problems involving complete rotorcraft, there exist large helicopter simulation programs such as C–81 (ref. 7) and G400 (ref. 8). These programs were designed primarily for time-history analysis of rotorcraft behavior in forward flight rather than for aeromechanical stability. Despite their generality and complexity, these programs have limitations (primarily related to aerodynamics) which are pointed out by Johnson (ref. 9) in his discussion of these and other large rotorcraft programs. While the CAMRAD program overcomes many of these limitations, all of these programs (including CAMRAD) are restricted to a fixed number of physical models, and lack the modeling flexibility needed to deal with a wide variety of blade/root geometries. Many of these programs rely on results, such as a set of modes, from other programs. This approach may present an assortment of modeling difficulties, especially for bearingless rotor blades. In particular, the mathematical and physical consistency of a combined approach is seldom examined, and the physical bases of the individual programs are likely to not be consistent. Furthermore, in stability analyses a nonlinear static equilibrium solution is needed about which to linearize — an important consideration which most of the earlier simulation programs do not address.

Therefore, it is important that a code be developed in which blade structural dynamics, isolated blade stability, and isolated rotor stability, as well as coupled rotor/airframe stability, can all be treated under a consistent set of physical assumptions.

Dynamic coupling programs, such as DYSCO (ref. 10), which have a high degree of generality, allow coupling of discrete component models and/or modal representations of flexible structures. While DYSCO has a very powerful, executive-driven system, it currently cannot treat the aeroelastic behavior of bearingless rotor systems undergoing geometrically nonlinear deformation. The problem is that it lacks a sufficiently general element in its element library.

Several recent implementations that apply the finite-element method to rotorcraft problems (refs. 11, 12, and 13) are not able to overcome these limitations because their physical models are limited to a single configuration. Simply breaking a rotating beam into a number of finite elements yields nothing more than a discretized rotating beam. This approach does not meet the requirement that the beam be coupled with an airframe, or model blade/root kinematics of an arbitrary configuration. The classical finite-element method is based on the breaking up of a single structure (*i.e.*, a beam, plate, or shell) into an arbitrary number of elements and expanding the appropriate field variables into polynomial shape functions. This approach by itself also lacks the flexibility to deal with truly arbitrary rotorcraft configurations because a helicopter is a system of structural components, some of which may be rotating and/or translating relative to one another. Because of this, rotorcraft are actually more akin to the multibody systems (refs. 14 and 15) encountered in spacecraft problems. Unfortunately, few multibody programs possess the capability to deal with flexible components, and none have the capability to deal with aeroelastic phenomena since they were developed primarily for spacecraft applications.

All previous attempts at modeling rotorcraft problems have incorporated certain restrictions that are undesirable in a truly general-purpose program. General-purpose codes that are currently under development, or will be developed in the future, should overcome the major shortcomings of existing aeroelastic analyses. Consider, for example, the following typical restrictions:

The first is a restriction to linear, small-displacement approximations of beam elastic deformation. This restriction is unacceptable in a general-purpose rotorcraft program because the rotor blade aeroelastic problem, especially for hingeless and bearingless rotor blades, has been conclusively shown to be a nonlinear problem. A consistent approach based on nonlinear kinematics is required for these configurations.

The second is a restriction to elastic blade models with ordering schemes, second-degree nonlinearity, or "moderate" rotations. These approximations are undesirable because the governing equations often have to be augmented with certain higher-order terms if the values of certain structural properties are not within some nominal range (Rosen and Friedmann (ref. 16)). Therefore, in a general-purpose analysis, the higher-order terms must be present. Ordering schemes, while still a valuable tool when used in special-purpose codes and codes where accuracy is a secondary consideration, are neither necessary nor desirable in a general-purpose context. Furthermore, a bearingless-rotor flexbeam must undergo deformation-induced rotations of the order of the collective pitch angle — a rotation too large to be classified as "moderate." Thus, bearingless rotor problems demand a large-deflection analysis without artificial restrictions on rotations due to deformation, the degree of nonlinearity, or the values of blade properties.

The third restriction is to a fixed number (usually one) of configurations (*e.g.*, isolated hingeless blade or coupled bearingless rotor and body or a single blade/root configuration). This restriction is unacceptable in a general-purpose code because the intent of such a code is to analyze different types of configurations with a single, consistent set of assumptions. Such a code should be able to treat all currently known blade/root mechanisms and, at the same time, model configurations that do not yet exist. It should be possible to construct a new configuration with simple building blocks and with no artificial limitations on the process. For maximum flexibility in treating these different configurations, the finite-element method is the preferred approach. Moreover, the existence of many different blade/hub configurations for helicopters requires a capability to analyze arbitrary configurations of structures, parts of which may be rotating. Thus, the code should employ the multibody philosophy.

1.2. Approach

To overcome the aforementioned limitations of the existing methods of aeroelastic stability analysis, the General Rotorcraft Aeromechanical Stability Program has been developed. GRASP combines the finite-element and multibody approaches, and incorporates multiple levels of substructures to provide a powerful tool for rotorcraft analysis. The design of GRASP is based on the concept of a collection of flexible and rigid bodies connected in an arbitrary manner. Libraries of elements, constraints, and solution algorithms appropriate for the helicopter aeroelastic stability problem were designed and built into the program.

The element library promotes the modeling of the blades as beams; construction of arbitrary mechanisms to treat blade/root kinematics with beam elements and rigid bodies; treatment of the fuselage as either a rigid body, a collection of beam elements, or a modal representation obtained from some other source; and treatment of both static and dynamic induced inflow by means of blade-element/momentum theory. The constraint library allows arbitrary connections between elements, includes constraints that allow for compliance in the constrained relative motion between elements, and includes constraints that allow the connection of rotating and nonrotating substructures. None of the constraints in the library use any kinematical approximations, such as small-angle assumptions. The solution procedures include nonlinear static equilibrium and linearized stability about equilibrium, both presently limited to the hovering flight condition.

It should be noted that these physical modeling assumptions and solution procedures, while adequate for aeromechanical stability analysis in axial flight and ground contact, are not adequate for a comprehensive rotorcraft dynamic analysis as defined by Johnson (ref. 9). The analysis methodology used in GRASP, although a viable approach for application to nonlinear dynamics in forward flight, would require considerable effort to be implemented in GRASP.

Several very desirable, but not required, features of a general-purpose code, have been incorporated in GRASP. 1) The accuracy of the analysis may be increased without having to add more elements. The aeroelastic beam finite element developed specifically for GRASP uses a variable-order (or p-version) approach, which is based on high-order, orthonormal, polynomial displacement functions (refs. 17 and 18). 2) As much as possible, the equations of motion are formed by the program internally, minimizing the possibility of human error in the equations. 3) The user interface is capable of handling a general problem without having to be supplied with the form of the equations of motion or even the number of degrees of freedom. 4) Both large and small problems can be modeled with the same code. Thus, the number of degrees of freedom is not fixed *a priori*. This feature not only requires a great deal of flexibility in assembling the system equations of motion, but also requires that data be structured and managed in core with a flexibility not inherent in FORTRAN (ref. 19).

2. SOLUTION APPROACH

GRASP is specifically designed to provide a tool for determining the equilibrium deflections and aeroelastic stability of arbitrary rotorcraft configurations in hover or vertical flight. A GRASP rotorcraft model is considered to be an aeroelastic system consisting of a structural system, portions of which may be rotating relative to one another, and a moving air mass with which the structure interacts. All parts of the model may be subject to forces and externally applied constraints. The position of any point on the structure or the air velocity at any point in the flow field relative to an inertial frame of reference may be determined by solving a system of partial differential and boundary value equations. These equations are obtained from the laws of fluid and structural mechanics, and from the constitutive properties of the materials in the structure and the air.

In vertical flight, hover, or ground contact a rotorcraft can assume a steady-state equilibrium configuration when the airflow, gravity, and the rotor spin axis are aligned; and when the angular velocity of the rotationally isotropic rotor is constant. In this restricted case where the structure is not subject to time-varying forces, it is possible to eliminate explicit time dependence from the equations. This steady-state equilibrium can be considered to be static when contrasted to the more general periodic equilibrium found in forward flight problems. The steady-state equilibrium configuration is characterized by a time-invariant deformation in the nonrotating portions of the rotorcraft, a steady flow of air through the rotor disk, and time-invariant deformations of the rotor blades with respect to a rotating reference frame. The steady-state solution then calculates the equilibrium values of all of the model generalized coordinates and generalized forces.

The equations of motion for the continuous-structure portions of the structure are discretized by means of variable-order, finite-element shape functions. The equations for the structure then become a system of nonlinear, ordinary differential equations. It is possible, as indicated above, to eliminate all explicit dependence on time from the equations for the restricted case of axial flight or ground contact. A linearized system of equations may then be calculated by taking small perturbations about the static equilibrium state. The stability problem is defined, therefore, by a second-order system of linear equations with constant coefficients.

For infinitesimally small perturbations about a previously-calculated, steady-state configuration, the dynamic motion of the rotorcraft can be represented as a linear combination of complex eigensolutions. Since the aeroelastic stability of the rotorcraft can be determined directly from the eigenvalues, the primary objective of GRASP can be satisfied by computing these eigensolutions. The frequency and damping information in the eigenvalues and the modal information in the eigenvectors, which can also be obtained from the eigensolutions, facilitate the user's understanding of the dynamics of the rotorcraft.

The eigensolution provides the complex eigenvalues and eigenvectors for all model degrees of freedom associated with the equations of motion $M\ddot{q} + C\dot{q} + Kq = 0$ which have been linearized about a steady-state deformation. These equations are often referred to as being "asymmetric" because of the nonsymmetry due to aerodynamics contributions to the coefficient matrices C and K. The coefficient matrix M, which is both symmetric and positive-definite, contains contributions from the mass of the structural model and from the "apparent mass" of the air. The coefficient matrix C contains contributions from structural and aerodynamic damping and inertial forces. The coefficient matrix K contains contributions from structural stiffness and effective stiffness from aerodynamic and inertial forces. Like the steady-state solution, this solution requires that the model correspond to a physical system which is not subject to time-varying forces. Currently, the asymmetric eigensolution must be computed by using the steady-state solution obtained for an identical model. This solution procedure prohibits one, for example, from obtaining the steady-state deformations of an isolated blade, then applying that solution to a coupled, rotor/fuselage configuration.

3. MODELING APPROACH

In order to form a mathematical representation of a structure that may contain bodies which are experiencing large kinematic motions relative to one another, it is necessary to be able to write the full, nonlinear equations of motion for the structure. The fundamentals of the approach used in GRASP to derive these equations are adapted from methods that were originally developed for spacecraft applications (ref. 20). For the types of structures that GRASP is designed to represent, additional emphasis has been placed on using multiple levels of substructures to model a structure.

The first step in modeling a structure in this manner is to decompose the structure (called the parent) into a set of subordinate substructures (called children), each of which in turn may also be decomposed into a set of child substructures. This decomposition process continues until every substructure has been decomposed into simple structural elements. The lowest-level substructures (*i.e.* those with no children) are called elements. The result of this method of modeling a structure is a hierarchically-ordered set (tree) of substructures (fig. 1) that has the complete structure at the root and elements at each of the leaves. Under this modeling scheme, a parent substructure may have any number (including zero) of child substructures but only one parent substructure. The only substructure without a parent is the complete structure, which is at the root of the tree.

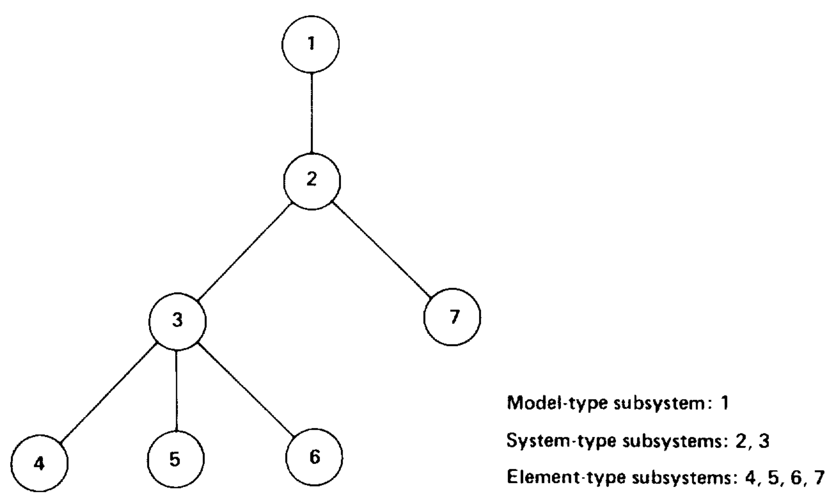


Figure 1. Hierarchical substructure tree.

The hierarchical model representation implemented in GRASP allows great generality in the types of configurations that can be analyzed, and permits essentially arbitrary kinematic motions of components relative to one another. This general framework, along with a software design that emphasizes the use of libraries for constraints, elements, solutions, and so on, means that the capabilities and limitations of the program are those associated with the members of the libraries, not with the program in general.

3.1. Subsystems

In GRASP, substructures are abstracted into subsystems. Each substructure is then represented by a subsystem, which may be classified according to its position in the hierarchy (fig. 2). The subsystem representing the complete structure (or model) is called a model-type subsystem. Substructures having no children (elements) are represented by element-type subsystems. The remaining subsystems, those having a parent and at least one child, are represented by system-type subsystems. To represent the substructures that make up the model, subsystems serve several functions. First, they contain the complete definitions of the substructures that they represent. Second, they are repositories for the generalized coordinates, generalized forces, and dynamic matrices associated with the substructures. Finally, they serve as the basic units of the hierarchical organization, which is an integral part of the computational process of transforming the parent generalized coordinates to the child generalized coordinates, and transforming the child generalized forces to the parent generalized forces.

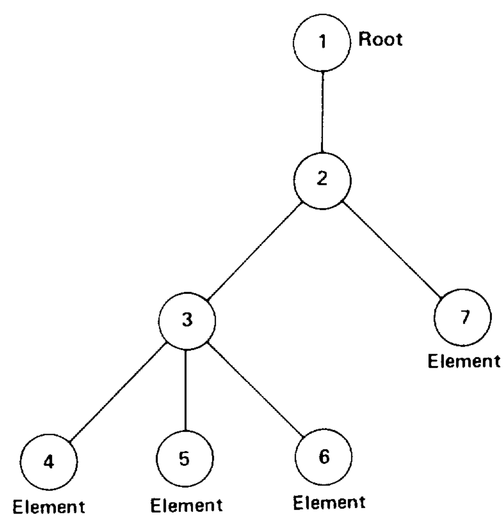


Figure 2. Hierarchical subsystem tree.

Subsystems, in general, may contain the following: a frame of reference, a set of nodes, a set of generalized coordinates, and a set of constraints. Each of these entities performs a different function within the subsystem, and will be described in the following sections.

3.1.1. Frames of Reference

Every subsystem in a GRASP model (with the sole exception of the air mass element) has a frame of reference associated with it. The frame of reference is not associated with any material point on the substructure, but instead serves as the "point of view" for the subsystem. As such, it establishes the coordinate system for that subsystem. The initial position and orientation of a reference frame may be selected to define a coordinate system that is natural for the subsystem (*e.g.*, a hub-centered frame of reference might be selected for a subsystem that contains a helicopter rotor).

Since reference frames are not physically connected to any structure, but rather are allowed to move freely, six degrees of freedom are associated with each frame. These degrees of freedom define the position and orientation of the frame of reference for the current subsystem relative to the reference frame for its parent subsystem.

In addition to serving as a reference for the subsystem, the frame of reference may be used to model the discrete motions of the substructure. This can often lead to significant simplifications in the equations of motion for subordinate subsystems. For example, if a reference frame is attached to the root of a rotating beam and used to model the rotational motion of the beam, the equations of motion of the beam itself need not explicitly include the rotational motion.

Since Newton's laws hold only in an inertial reference frame, the model-type subsystem at the root of the tree is defined to be fixed in an inertial frame of reference. Therefore, while a model-type subsystem does have a frame of reference, that reference frame has no degrees of freedom associated with it since it must be inertial. As a result, the motions of every part of the system can be related to an inertial frame of reference.

3.1.2. Nodes

Nodes are used by GRASP to introduce degrees of freedom into a model. In general, the degrees of freedom introduced by a node may be any generalized coordinates that can be associated with a physically identifiable property of the structure. For example, the set of degrees of freedom for a node could be defined to be the three rigid-body translations and the three rigid-body rotations of a point on a structure. Alternatively, there could be a node whose degrees of freedom are defined to be modal coordinates.

Currently, two different types of nodes are used by GRASP: structural nodes and air nodes. The structural nodes provide the measures for the local displacement and rotation of a structure. They move with the deformation of the structure and may be conceptualized as massless, infinitesimal, rigid bodies that are physically attached to the structure. The air nodes define the induced inflow velocity field through a helicopter rotor. The degrees of freedom for the air node are measures of the velocity distributions around the rotor disk.

3.1.3. Constraints

The constraints act as a sort of "glue" that holds a model together. Constraints are used to model both physical constraints (*e.g.*, pins, gimbals, and clamps), and to eliminate the dependent degrees of freedom that have been introduced into the model. An example of a physical constraint would be the clamped boundary condition at one end of a cantilever beam. That end of the beam is modeled by constraining the node at one end of the beam to have no translational or rotational motion. Now consider two frames of reference that are defined to move as if they are rigidly connected to one another. For this system, there are twelve degrees of freedom (six for each frame), but only six of them are independent. Therefore, a constraint must be defined to remove the dependent degrees of freedom. In general, the set of constraints for a subsystem must be sufficient to reduce the total number of degrees of freedom to only the independent degrees of freedom for that subsystem. Similarly, for the complete model, all dependent degrees of freedom must be eliminated.

All of the constraints implemented in GRASP are based on purely kinematical relationships. There are no restrictions to small or moderate displacements or rotations in any of the constraint equations. However, it is necessary to avoid the singularity that occurs for deformation-induced rotations of 180° . This singularity arises as a result of using finite-rotational kinematics that are based on Rodrigues parameters (ref. 21).

The constraints in GRASP are implemented at two levels: the program level and the user level. The constraint "primitives" are found at the program level. These simple constraints provide a basic set of connections among generalized coordinates, frames, and nodes. At the user level, these primitive constraints are combined to provide the user with physically meaningful constraints between structural elements. For example, the rigid-body mass connectivity constraint, which is used to attach a rigid-body mass element to a structure, is a combination of a primitive constraint between frames and a primitive constraint between nodes.

In order to provide a full set of constraints, the constraint library in GRASP includes several different classes of constraints. These include constraints between two frames, constraints between two nodes, constraints between generalized coordinates, and constraints between a frame and a node.

3.2. Elements

Elements are subsystems that have no children. In addition to frame and nodal degrees of freedom, they may also have additional, non-nodal generalized coordinates. Computationally, the elements are the primary source of virtual work in the structure. For steady-state problems, the elements return the generalized forces associated with a given set of generalized displacements. For perturbation problems, the elements return the element coefficient matrices. These matrices are determined from the perturbations in generalized forces resulting from perturbations in the generalized coordinates and their time derivatives.

3.2.1. Rigid-body Mass

The rigid-body mass element represents a rigid body that is subject only to inertial and gravitational forces. It has a single structural node that is located at the mass center, and its axes are aligned with the principal axes of the body. The frame of reference for the rigid-body mass element coincides with the nodal coordinates in their undeformed state.

3.2.2. Air Mass

The air mass element models the momentum air flow through an axisymmetric rotor disk. The degrees of freedom associated with this element are introduced through a single air node. Since the air mass element is defined to be fixed in inertial space, the frame degrees of freedom are suppressed. For steady-state problems, the residuals corresponding to the uniform inflow velocity and the radial velocity gradient are calculated from momentum considerations (ref. 22). For the asymmetric eigenproblem, only the momentum terms (ref. 23) involving uniform and first-harmonic, cyclic perturbations of the inflow velocity contribute to the element coefficient matrices.

3.2.3. Aeroelastic Beam

The aeroelastic beam element represents a slender, nonuniform beam (without shear deformation) that is subject to elastic, inertial, gravitational, and aerodynamic forces. The primary assumption in the derivation of the element equations (ref. 24) is that strains remain small relative to unity. There are no small-angle approximations made and all kinematically nonlinear effects are included. One current limitation is that orientation angles (ref. 21) (of type body-three: 1-2-3) are used in the description of finite rotation inside the beam element. Thus, rotations due to the deformation of beam elements may not exceed 90° .

The aeroelastic beam element degrees of freedom come from a frame of reference that coincides with the root of the element in its undeformed state, structural nodes at the root and tip, an air node, and a set of internal degrees of freedom. The internal degrees of freedom result from the higher-order polynomials that may be used to increase the accuracy of the beam deformation calculations. When no internal degrees of freedom are specified, the aeroelastic beam is an Euler-Bernoulli beam in which the axial and torsional deformations in excess of a built-in pretwist are represented by linear polynomials, while the bending deflections are represented by cubic polynomials. The method of adding internal degrees of freedom to improve the accuracy of an element is more convenient than adding elements, and is also more efficient (ref. 17) given the same number of degrees of freedom. Internal degrees of freedom may be added selectively to reflect the dynamics of the element. For example, if a beam is very stiff in bending and extension but soft in torsion, additional torsional degrees of freedom may be added without having to include any more bending or extensional degrees of freedom.

The aerodynamic forces on the beam element are calculated from quasi-steady strip theory using lift, drag, and moment coefficients that are piecewise continuous functions of the angle of attack. Spanwise scale factors for the lift, drag, and moment may be specified to allow for tip loss and other similar effects. The chord width, the pitch angle of the zero-lift-line, and the offset of the aerodynamic center from the elastic axis may also vary over the length of the element. The aeroelastic beam element also calculates the blade-element contributions to the induced velocity, which are combined with the momentum contributions from the air mass element elsewhere.

4. SOLUTION METHODS

The solutions currently implemented in GRASP allow the user to calculate the steady-state deformations of a structure under load, and then to solve for the eigenvalues and eigenvectors of the deformed structure. In order to obtain a valid eigensolution, the steady-state deformations that are used must be such that the structure is in equilibrium.

4.1. Steady-State Solution

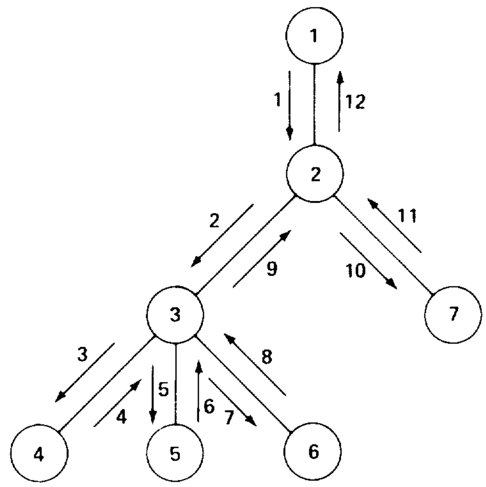
The equations for the steady-state equilibrium of the model are a set of nonlinear, algebraic equations of the form

$$Q_i = f(q_1, \ldots, q_N); \qquad i = 1, \ldots, N \tag{4.1-1}$$

where the Q_i are the generalized forces (residuals), the q_i are the generalized coordinates, and N is the number of system degrees of freedom. These equations are generated internally by GRASP at the element level, and automatically assembled by the constraints, which combine the contributions from the finite elements into the final set of equations. The solution to this set of equations is obtained through the use of the Levenberg-Marquardt algorithm. This algorithm minimizes the sum of the squares of the residuals from the steady-state equations. The implementation in GRASP uses the IMSL (ref. 25) subroutine ZXSSQ.

For problems involving the aeroelastic beam element with internal degrees of freedom, the solution algorithm is used at two levels. First, it is used in an outer iteration loop to arrive at a solution to the steady-state equations for the complete model (which excludes the aeroelastic beam internal degrees of freedom). In addition, it is used in a separate, inner iteration loop to calculate the internal degrees of freedom for each aeroelastic beam element. A full inner solution for each aeroelastic beam is calculated for each iteration of the outer solution.

In order to arrive at a steady-state solution, the residual forces on the system must be calculated, given a deformation state. The algorithm that is used to calculate the residuals for the top-level subsystem in the hierarchical organization of the model is based on a full-order tree traversal (fig. 3). When traversing down the tree (away from the root subsystem), the state vector for each child subsystem is calculated from that of its parent. Also, the inertial motion of the child subsystem reference frame is calculated from that of the parent. Upon reaching an element, the state vector for that element and the inertial motion for the element frame are used to calculate the element residuals. Traversing back up the tree (towards the root subsystem) the residuals from each child subsystem are transformed into its parent subsystem and added to the parent residuals. When the traversal is complete, the residuals corresponding to each generalized coordinate in the root subsystem are known. The complementary processes of calculating the state vectors and assembling the residual vectors are accomplished by using the constraints, which define the relationships among the degrees of freedom in the parent and child subsystems.



Calculate generalized coordinates (1, 2, 3, 5, 7, 10)

Assemble generalized forces (4, 6, 8, 9, 11, 12)

Forces calculated in subsystems 4, 5, 6, 7

Figure 3. Steady-state solution full-order traversal.

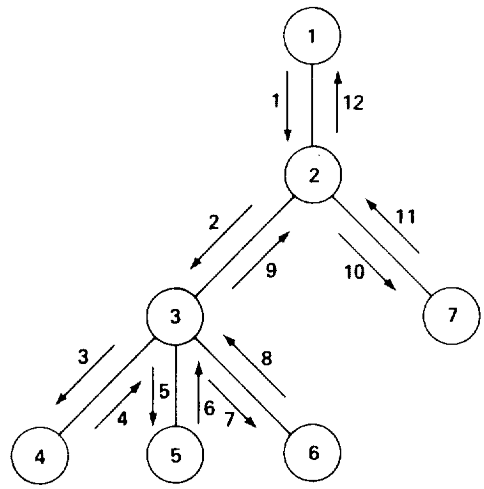
The solution methods available in the current version of GRASP are restricted in that the same model must be used for both the steady-state and asymmetric eigenproblem solutions. This creates a problem for the steady-state solution algorithm when a configuration contains unconstrained degrees of freedom. This can occur when a model having both rotating and nonrotating components is being analyzed. For such a configuration, the cyclic degrees of freedom generated by the rotating constraints are unconstrained. It can also occur in airborne configurations, which suffer from the same problem because their body degrees of freedom are unconstrained. To alleviate this problem, GRASP currently marks these unconstrained degrees of freedom during the building of the model, and eliminates them from the state vector used in the minimization algorithm.

4.2. Asymmetric Eigenproblem Solution

The system equations for the asymmetric eigenproblem can be expressed in the familiar form

$$M\ddot{\breve{q}} + C\dot{\breve{q}} + K\breve{q} = 0 \tag{4.2–1}$$

where the $\breve{q}$'s are infinitesimal perturbations of the generalized coordinates. The algorithm used to assemble the coefficient matrices for the root subsystem is very similar to that used to calculate the steady-state residuals in that it also is based on a full-order tree traversal (fig. 4). However, while traversing down the tree, no state vector calculations are required. Upon reaching an element, the coefficient matrices for that element are calculated. During the traversal back up the tree, the constraints are used to assemble the child subsystem matrices into the parent matrices. At the conclusion of the traversal, the coefficient matrices for the model subsystem are complete.



Assemble subsystem matrices (4, 6, 8, 9, 11, 12)

Calculate element matrices in subsystems 4, 5, 6, 7

Figure 4. Eigensolution full-order traversal.

The solution of this set of equations is begun by factoring matrix M using the Cholesky decomosition algorithm. The GRASP implementation uses subroutine LUCECP from the IMSL (ref. 25) library. M then becomes

$$M = LL^T \tag{4.2–2}$$

Introducing the transformation

$$\check{z} = L^T \check{q} \tag{4.2–3}$$

the mass matrix M can be reduced to an identity matrix and the system equations can be written as

$$\Delta \ddot{\check{z}} + L^{-1} C L^{-T} \dot{\check{z}} + L^{-1} K L^{-T} \check{z} = 0 \tag{4.2–4}$$

Writing this system of equations in first-order form

$$\begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} \dot{\check{y}} = \begin{bmatrix} 0 & \Delta \\ -L^{-1} K L^{-T} & -L^{-1} C L^{-T} \end{bmatrix} \check{y} \tag{4.2–5}$$

where

$$\check{y} = \begin{Bmatrix} \check{z} \\ \dot{\check{z}} \end{Bmatrix} \tag{4.2–6}$$

Time may be eliminated by the introduction of

$$\check{y} = \begin{Bmatrix} z^* \\ \lambda z^* \end{Bmatrix} e^{\lambda t} \tag{4.2–7}$$

which allows the extraction of eigenvalues and eigenvectors directly from the matrix on the right-hand side of equation (4.2–5). The dynamic matrix is balanced, converted to Hessenberg form, and then the QR algorithm is used to obtain the eigensolution. Finally, the eigenvectors are transformed back to the original coordinate system via the transformation

$$q^* = L^{-T} z^* \tag{4.2–8}$$

GRASP uses subroutine RG from the NASA/Ames Cray library to calculate the eigensolutions.

5. COORDINATE SYSTEMS

In GRASP, many different coordinate systems are used to mathematically describe the physical structure. To differentiate among them, each coordinate system is identified in its undeformed state by a capital letter (*e.g.*, A). Depending on the context, an identifier may refer either to the coordinate system itself or to a point located at the origin of the coordinate system. The addition of a prime or a double-prime to the identifier indicates that the designated coordinate system either is in a state of static equilibrium (*e.g.*, A') or is in a dynamically perturbed state (*e.g.*, A''). With these multiple coordinate systems, it is often desirable to use several types of mathematical notation when deriving and writing equations. Not only can the form of the equations be simplified, but also they can be made more readable. This section is intended as an introduction to the notation used in the sections where the equations are actually derived.

5.1. Vectors

Vectors play an important role in coordinate system mathematics. Associated with the orthogonal axes emanating from the origin of every coordinate system is a set of dextral unit vectors. These unit vectors are called the base or basis vectors of the coordinate system. In addition, vectors are used to define variables such as position, velocity, and acceleration. Three types of notation are used in writing vector expressions and operations: vector-dyadic notation, index notation, and vector notation.

5.1.1. Vector-Dyadic Notation

All vectors and dyadics used in GRASP are underlined (*e.g.*, $\underline{V}$), and all unit vectors are identified by a circumflex. The difference between a vector and a dyadic should always be clear from the context of its usage. For a coordinate system A, the basis vectors are written as $\underline{\hat{b}}_i^A$, where $i = 1, 2, 3$. Any unit vector other than a basis vector is denoted by $\underline{\hat{e}}$, and may appear either with or without superscripts.

When kinematical quantities have coordinate systems associated with them, the relationship is defined by using the appropriate superscripts. For example,

$\underline{R}^{BA} \doteq$ position of the origin of coordinate system B with respect to the origin of coordinate system A

$\underline{V}^{BA} \doteq$ velocity of the origin of coordinate system B with respect to coordinate system A

$\underline{A}^{BA} \doteq$ acceleration of the origin of coordinate system B with respect to coordinate system A

$\underline{\Omega}^{BA} \doteq$ angular velocity of coordinate system B with respect to coordinate system A

Forces and moments are significant in their point of application as well as their source. The notation adopted herein is

$$\underline{F}^A \doteq \text{force at } A$$

$$\underline{M}^A \doteq \text{moment at } A$$

For example, a force and moment at A contribute to a moment at B according to the relationship

$$\underline{M}^B = \underline{M}^A + \underline{R}^{AB} \times \underline{A}^A \tag{5.1.1–1}$$

5.1.2. Index Notation for Vectors

A vector $\underline{V}$ in the A basis may always be expressed as

$$\underline{V} = V_{Ai}\hat{\underline{b}}_i^A \tag{5.1.2–1}$$

where the summation convention adopted is that repeated indices are always summed over their range. Unless otherwise specified, Latin indices assume the values 1,2,3; Greek indices assume the values 1,2. The subscript A in V_{Ai} indicates that the measure numbers V_{Ai} are defined in the A basis.

Two symbols frequently encountered in vector operations that use index notation are the Kronecker delta δ_{ij} and the Levi-Civita epsilon ϵ_{ijk} where

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{5.1.2–2}$$

$$\epsilon_{ijk} = \begin{cases} 0 & \text{any index repeated} \\ +1 & \text{cyclic permutation} \\ -1 & \text{acyclic permutation} \end{cases} \tag{5.1.2–3}$$

The Kronecker delta consists of the components of the identity tensor in a Cartesian coordinate system, while the Levi-Civita epsilon consists of components of the permutation tensor in a Cartesian coordinate system. Some useful identities regarding both of these symbols may be found in reference 26.

5.1.3. Matrix Notation for Vectors

Using index notation, a vector $\underline{V}$ may be expressed in the A basis as shown in equation (5.1.2–1). Since the basis is identified by the subscript A, the measure numbers themselves may be viewed as a complete description of the vector. Thus, the column matrix V_A can be defined to be

$$V_A = \begin{Bmatrix} V_{A1} \\ V_{A2} \\ V_{A3} \end{Bmatrix} \tag{5.1.3–1}$$

as an alternate way of expressing the vector $\underline{V}$. The dot product $\underline{U} \cdot \underline{V}$ may then be written as

$$U_A^T V_A = \lfloor U_{A1} \; U_{A2} \; U_{A3} \rfloor \begin{Bmatrix} V_{A1} \\ V_{A2} \\ V_{A3} \end{Bmatrix} \tag{5.1.3-2}$$

The cross product of two vectors $\underline{U}$ and $\underline{V}$ may be written as

$$\begin{aligned} \underline{U} \times \underline{V} &= U_{Ai} \hat{\underline{b}}_i^A \times V_{Ai} \hat{\underline{b}}_i^A \\ &= \epsilon_{ijk} U_{Aj} V_{Ak} \hat{\underline{b}}_i^A \\ &= \tilde{U}_{Aij} V_{Aj} \hat{\underline{b}}_i^A \end{aligned} \tag{5.1.3-3}$$

This equation implies that the measure numbers of the cross product in the A basis are simply the elements of the matrix product $\tilde{U}_A V_A$ where

$$\tilde{(\;)}_{ij} = -\epsilon_{ijk} (\;)_k \tag{5.1.3-4}$$

For example,

$$\tilde{U}_A = \begin{bmatrix} 0 & -U_{A3} & U_{A2} \\ U_{A3} & 0 & -U_{A1} \\ -U_{A2} & U_{A1} & 0 \end{bmatrix} \tag{5.1.3-5}$$

There are also several useful identities that can be derived for two column matrices a and b

$$\begin{aligned} \tilde{a}^T &= -\,\tilde{a} \\ \tilde{a}b &= -\,\tilde{b}a \\ a^T\tilde{b} &= -\,b^T\tilde{a} = (\tilde{a}b)^T \\ \tilde{a}\tilde{b} &= -\,a^T b \Delta + b a^T \\ \widetilde{\tilde{a}b} &= b a^T - a b^T \\ \tilde{a}\tilde{b} - \widetilde{\tilde{a}b} &= \tilde{b}\tilde{a} \end{aligned} \tag{5.1.3-6}$$

5.2. Finite Rotations

In many kinematic analyses, rotations are assumed to be either infinitesimal or moderate in size. These assumptions allow certain simplifications in the kinematical relationships, but constrain the range of applicability of the analysis. In GRASP, no such assumptions are made and all rotations are assumed to be of arbitrary size (finite). Finite rotations are expressed in four ways in GRASP:

(1) direction cosines,

(2) Euler rotations,

(3) Tait-Bryan orientation angles, and

(4) Euler-Rodrigues parameters. Internally, GRASP expresses all finite rotations in terms of direction cosine matrices. For the convenience of the user, any of the other three methods may be used to specify the input to GRASP. Since there are significant differences in the algorithms used to compute the direction cosine matrix, all three of the other representations are also discussed in detail.

5.2.1. Direction Cosines

When a coordinate system B undergoes an arbitrary rotation relative to coordinate system A, the basis vectors are related by the equation

$$\underline{\hat{b}}_i^B = C_{ij}^{BA}\underline{\hat{b}}_j^A \tag{5.2.1–1}$$

where the superscripts are coordinate system identifiers, not indices. The matrix of direction cosines C^{BA} is orthonormal such that

$$C^{BA}C^{AB} = C^{AB}C^{BA} = \Delta \tag{5.2.1–2}$$

It should be noted that the form of the matrix of direction cosines used in this manual is the transpose of that developed in reference 21.

Similarly, with this notation it is easy to show that a basis change for any kinematical vector can be performed by changing the subscript and multiplying by the matrix of direction cosines for the bases.

$$V_B = C^{BA}V_A \tag{5.2.1–3}$$

Note that for kinematical vectors the superscripts are unaffected by these operations.

5.2.2. Euler Rotations

If coordinate system B, initially coincident with A, rotates about a unit vector $\underline{\hat{e}}$ fixed in A by an angle θ (fig. 5) then the matrix of direction cosines can be written as

$$C^{BA} = \Delta\cos\theta + e_A {e_A}^T(1-\cos\theta) - \tilde{e}_A\sin\theta \tag{5.2.2–1}$$

where

$$e_{Ai} = \underline{\hat{e}}\cdot\underline{\hat{b}}_i^A \tag{5.2.2–2}$$

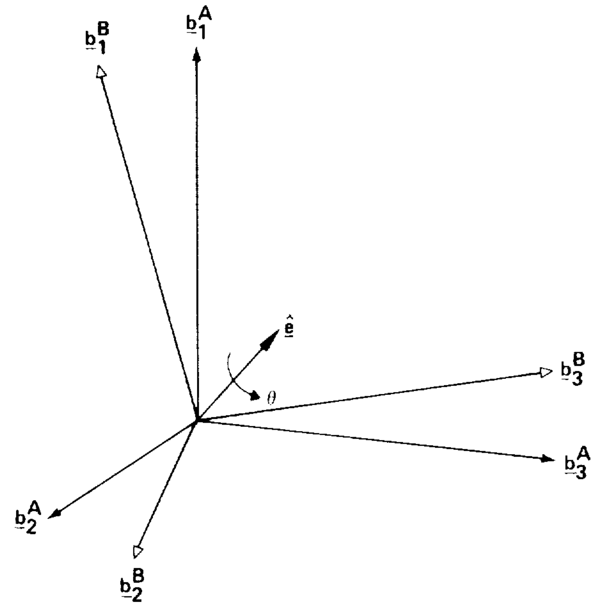


Figure 5. Euler rotation.

5.2.3. Tait-Bryan Orientation Angles

Consider two coordinate systems A and B with coincident basis vectors $\hat{\underline{b}}_i^A$ and $\hat{\underline{b}}_i^B$. Let the orientation of B with respect to A change as follows (fig. 6):

(1) Perform an Euler rotation of B about $\hat{\underline{e}} = \hat{\underline{b}}_j^B$ ($j = 1, 2$, or 3) by an angle θ_j;

(2) Perform an Euler rotation of B about $\hat{\underline{e}} = \hat{\underline{b}}_k^B$ ($k = 1, 2$, or 3, $k \neq j$) by an angle θ_k;

(3) Perform an Euler rotation of B about $\hat{\underline{e}} = \hat{\underline{b}}_l^B$ ($l = 1, 2$, or 3, $l \neq k, l \neq j$) by an angle θ_l.

The final orientation of B relative to A depends both on the magnitudes of θ_1, θ_2, and θ_3 and the sequence j-k-l. Details of this type of transformation may be found in reference 21 where Tait-Bryan angles are classified as orientation angles of type body-three. For the rotation sequence 1-2-3 the matrix of direction cosines is calculated as follows:

$$C^{BA} = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}$$

$$= \begin{bmatrix} c_2 c_3 & c_1 s_3 + s_1 s_2 c_3 & s_1 s_3 - c_1 s_2 c_3 \\ -c_2 s_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_3 - c_1 s_2 s_3 \\ s_2 & -s_1 c_2 & c_1 c_2 \end{bmatrix} \tag{5.2.3-1}$$

where

$$c_i = \cos\theta_i \qquad s_i = \sin\theta_i \tag{5.2.3-2}$$

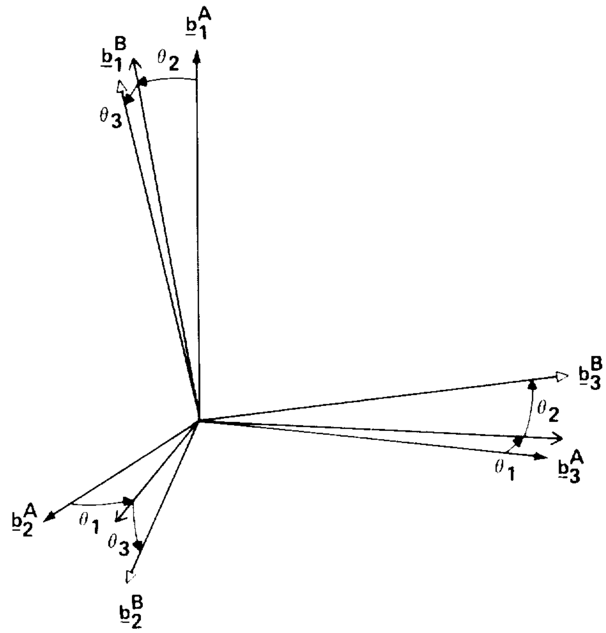


Figure 6. Tait-Bryan orientation angles (1-2-3).

5.2.4. Euler-Rodrigues Parameters

For two coordinate systems A and B, three parameters $\phi_i = 2e_{Ai}\tan(\frac{\theta}{2})$ may be used to describe a change in orientation (ref. 21). The values of ϕ_i herein are scaled by a factor of 2 relative to the Rodrigues parameters presented in reference 21, so that for infinitesimal values of $\phi_i = \check{\phi}_i$, the rotation may be regarded as a vector $\check{\phi}_i\hat{\underline{b}}_i^A = \check{\phi}_i\hat{\underline{b}}_i^B$ with $C^{BA} = \Delta - \tilde{\check{\phi}}$. The matrix of direction cosines is then simply

$$C^{BA} = \frac{(1 - \frac{\phi^T\phi}{4})\Delta + \frac{\phi\phi^T}{2} - \tilde{\phi}}{1 + \frac{\phi^T\phi}{4}} \tag{5.2.4–1}$$

The angular velocity of B relative A, expressed in the B basis, can be written as

$$\Omega_B^{BA} = \frac{(\Delta - \frac{\tilde{\phi}}{2})\dot{\phi}}{1 + \frac{\phi^T\phi}{4}} \tag{5.2.4–2}$$

These relations contain no trigonometric functions and are easily expressed in a shorthand matrix notation. Furthermore, a simple inverse transformation exists s~ that given C^{BA}, the values of ϕ may be obtained from

$$\phi_i = \frac{2\epsilon_{ijk}C_{jk}^{BA}}{1 + C_{ll}^{BA}} \tag{5.2.4–3}$$

where C_{ll}^{BA} is the trace of C^{BA}. Given ϕ and Ω_B^{BA}, $\dot{\phi}$ can be obtained from

$$\dot{\phi} = (\Delta + \frac{\tilde{\phi}}{2} + \frac{\phi\phi^T}{4})\Omega_B^{BA} \tag{5.2.4–4}$$

Transformations between Euler-Rodrigues parameters and direction cosines (or angular rates) are very simple relative to the transformations required for Tait-Bryan angles.

5.3. Angular Velocity and Virtual Rotation

The measure numbers for the angular velocity of coordinate system B relative to coordinate system A expressed in the A basis, Ω_A^{BA}, may be determined from the addition theorem discussed in reference 27. They can be related to the time derivative of the matrix of direction cosines as follows:

$$\dot{C}^{BA} = -\tilde{\Omega}_B^{BA} C^{BA} = -C^{BA} \tilde{\Omega}_A^{BA} \tag{5.3-1}$$

By virtue of the Kirchhoff kinetic analogy (ref. 28), $\dot{C}^{BA}$ in Eq. (5.3-1) may be replaced with δC^{BA}, and Ω_B^{BA} with $\underline{\delta\psi}_B^{BA}$. The expression for the components of virtual rotation of B in A then becomes

$$\delta C^{BA} = -\widetilde{\delta\psi}_B^{BA} C^{BA} = -C^{BA} \widetilde{\delta\psi}_A^{BA} \tag{5.3-2}$$

The corresponding virtual rotation vector $\underline{\delta\psi}^{BA}$ is used in determining the virtual work due to applied moments. The components of virtual rotation may be obtained from any expression involving the angular velocity in a manner identical to that used to obtain equation (5.3-2) from equation (5.3-1).

Similarly, infinitesimal perturbations of the rotation vector can be obtained by substituting $\check{C}^{BA}$ for $\dot{C}^{BA}$ and $\tilde{\theta}_B^{BA}$ for $\tilde{\Omega}_B^{BA}$ in equation (5.3-1).

$$\check{C}^{BA} = -\tilde{\theta}_B^{BA} C^{BA} = -C^{BA} \tilde{\theta}_A^{BA} \tag{5.3-3}$$

5.4. Velocity, Acceleration, and Virtual Displacement

Velocity and acceleration vectors are obtained by applying the superposition theorems discussed in reference 27. The calculation of the velocity and acceleration vectors is fundamentally nothing more than the differentiation with respect to time of a position vector in (*i.e.*, relative to) some coordinate system. It is often necessary to determine the time derivative of a vector in coordinate system B, when the derivative is known only in coordinate system A. Given an arbitrary position vector $\underline{R}$ and its first and second time derivatives in A, the first and second time derivatives of $\underline{R}$ in B may be determined from the following expressions.

$$
\begin{aligned}
{}^B\frac{d}{dt}\underline{R} &= {}^A\frac{d}{dt}\underline{R} + \underline{\Omega}^{AB} \times \underline{R} \\
{}^B\frac{d^2}{dt^2}\underline{R} &= {}^B\frac{d}{dt}\left({}^B\frac{d}{dt}\underline{R} \right) \\
&= {}^A\frac{d}{dt}\left({}^A\frac{d}{dt}\underline{R} + \underline{\Omega}^{AB} \times \underline{R} \right) + \underline{\Omega}^{AB} \times \left({}^A\frac{d}{dt}\underline{R} + \underline{\Omega}^{AB} \times \underline{R} \right) \\
&= {}^A\frac{d^2}{dt^2}\underline{R} + {}^A\frac{d}{dt}\underline{\Omega}^{AB} \times \underline{R} + 2\underline{\Omega}^{AB} \times {}^A\frac{d}{dt}\underline{R} + \underline{\Omega}^{AB} \times (\underline{\Omega}^{AB} \times \underline{R})
\end{aligned} \tag{5.4-1}
$$

The Kirchhoff kinetic analogy (ref. 28), can also be applied to equation (5.4-1) to obtain the virtual displacement vector. Time derivatives in B, ${}^{B}\frac{d}{dt}(\ )$, are replaced with ${}^{B}\delta(\ )$; velocity vectors in A, ${}^{A}\frac{d}{dt}(\ )$, are replaced with virtual displacments in A, ${}^{A}\delta(\ )$; and angular velocity vectors $\underline{\Omega}^{AB}$ are replaced with virtual rotations $\underline{\delta\psi}^{AB}$.

$$ {}^{B}\delta\underline{R} = {}^{A}\delta\underline{R} + \underline{\delta\psi}^{AB} \times \underline{R} \tag{5.4-2} $$

6. SUBSYSTEMS

As described in Section 3, the physical structure that is being modeled by GRASP is broken down into a hierarchy of substructures. Each of these substructures is represented in GRASP as a subsystem. Every subsystem in the model is in turn composed of a set of components which may include a frame of reference, a set of nodes, a set of constraints, and a set of child subsystems. It is the interrelationship among these components that allows the construction of the equations of motion for each subsystem.

6.1. Frames of Reference

The position of the frame of reference F for a child subsystem relative to the frame of reference S for the parent subsystem is defined as $\underline{R}^{FS}$, and the orientation (direction cosines) of the child subsystem's frame relative to the parent's frame is defined as C^{FS} (fig. 7). Since Newton's laws apply only in inertial frames of reference, all equations of motions must be written relative to an inertial reference frame. Therefore, it is essential to have a method of transforming back to the inertial frame from any subsystem frame in the model. If the position and orientation of the parent's reference frame S are defined relative to an inertial reference frame I, the inertial position and orientation of any child's reference frame F can be determined from the parent's reference frame S by applying the following equations recursively.

$$\begin{aligned} \underline{R}^{FI} &= \underline{R}^{FS} + \underline{R}^{SI} \\ C^{FI} &= C^{FS} C^{SI} \end{aligned} \tag{6.1-1}$$

In addition to the position and orientation of any reference frame relative to the inertial reference frame, it is necessary to know the inertial motion of every subsystem frame. A subsystem reference frame may experience accelerations relative to the inertial frame if it or any of its direct ancestors is experiencing translational accelerations or rotation motions. Thus, if the inertial motion of the parent's frame of reference S is known, then the velocity, angular velocity, and acceleration of the child's frame F can be obtained from the following equations:

$$\begin{aligned} \underline{V}^{FI} &= \underline{V}^{FS} + \underline{V}^{SI} + \underline{\Omega}^{SI} \times \underline{R}^{FS} \\ \underline{\Omega}^{FI} &= \underline{\Omega}^{SI} + \underline{\Omega}^{FS} \\ \underline{A}^{FI} &= \underline{A}^{FS} + \underline{A}^{SI} + \underline{\Omega}^{SI} \times (\underline{\Omega}^{SI} \times \underline{R}^{FS}) \end{aligned} \tag{6.1-2}$$

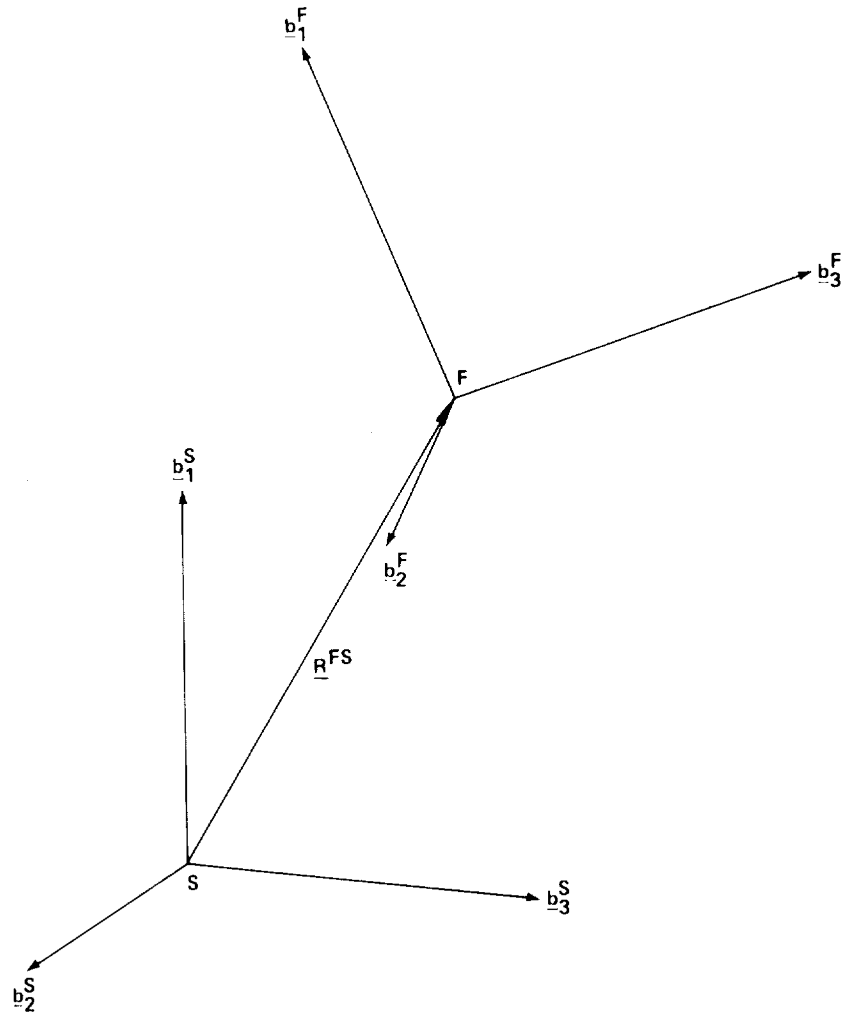


Figure 7. Frames of reference.

When expressed in the appropriate bases, these equations become (in matrix form)

$$
\begin{aligned}
V_F^{FI} &= C^{FS}(V_S^{FS} + V_S^{SI} + \tilde{\Omega}_S^{SI} R_S^{FS}) \\
\Omega_F^{FI} &= C^{FS}(\Omega_S^{SI} + \Omega_S^{FS}) \\
A_F^{FI} &= C^{FS}(A_F^{FS} + A_S^{SI} + \tilde{\Omega}_S^{SI} \tilde{\Omega}_S^{SI} R_S^{FS})
\end{aligned}
\tag{6.1-3}
$$

Note that in the current version of GRASP, it is assumed that $V_S^{FS} = A_S^{FS} = 0$.

Frames also possess six rigid-body degrees of freedom. Thus, while frames are not physically attached to the structure, they may move relative to one another in space. In the case of steady-state deformations, these six degrees of freedom include three translations along the deformed frame basis vectors and the three Euler-Rodrigues parameters for angular displacements. The steady-state displacement vector for frame F is

$$\underline{R}^{F'F} = R_{F'i}^{F'F} \hat{\underline{b}}_i^{F'} \tag{6.1-4}$$

The steady-state frame rotations are expressed in terms of $\phi_i^{F'F}$, and the direction cosines of the deformed frame coordinate axes F' with respect to the undeformed coordinate axes F are written as $C_{ij}^{F'F}$. In matrix notation, the steady-state frame state vector is

$$q_{F'} = \left\{ \begin{array}{c} R_{F'i}^{F'F} \\ \phi_i^{F'F} \end{array} \right\} \tag{6.1-5}$$

For dynamic perturbations about the steady-state condition, the displacement vector is

$$\underline{R}^{F''F'} = R_{F''i}^{F''F'} \hat{\underline{b}}_i^{F''} \tag{6.1-6}$$

The dynamic perturbations of the frame rotations are expressed in terms of infinitesimal rotations $\check{\theta}_{F''i}^{F''F'}$, for which the direction cosines (ref. 21) are

$$C^{F''F'} = \Delta - \tilde{\check{\theta}}_{F''i}^{F''F'} \tag{6.1-7}$$

In matrix notation, the dynamic perturbation frame state vector is then

$$q_{F''} = \left\{ \begin{array}{c} R_{F''i}^{F''F'} \\ \check{\theta}_{F''i}^{F''F'} \end{array} \right\} \tag{6.1-8}$$

The virtual displacements for the steady-state and dynamic formulations are simply variations of the displacement coordinates

$$\begin{aligned} \delta\underline{R}^{F'F} &= \delta R_{F'i}^{F'F} \hat{\underline{b}}_i^{F'} \\ \delta\underline{R}^{F''F'} &= \delta R_{F''i}^{F''F'} \hat{\underline{b}}_i^{F''} \end{aligned} \tag{6.1-9}$$

and the virtual rotations are variations of the rotational degrees of freedom

$$\begin{aligned} \delta\underline{\psi}^{F'F} &= \underline{\delta\psi}_{F'i}^{F'F} \hat{\underline{b}}_i^{F'} \\ \delta\underline{\psi}^{F''F'} &= \underline{\delta\psi}_{F''i}^{F''F'} \hat{\underline{b}}_i^{F''} \end{aligned} \tag{6.1-10}$$

6.2. Nodes

Nodes are used by GRASP to model the kinematics of a structure, and their degrees of freedom are representative of the physical states of that structure. The position and orientation of any node is defined relative to the frame of reference for the subsystem in which the node resides. Thus, for a node N in a subsystem with reference frame F, the position and orientation of N with respect to F are $\underline{R}^{NF}$ and C^{NF}, respectively. Two types of nodes are currently used in GRASP: structural nodes and air nodes. The kinematics of these nodes are described in the following sections.

6.2.1. Structural Nodes

A structural node represents a specified material point on a structure. Since the material point may have up to six degrees of freedom, the structural node also has six degrees of freedom. For the case of steady-state deformations, these six degrees of freedom include three translations along the undeformed nodal basis vectors $\hat{\underline{b}}_i^N$ and three Euler-Rodrigues parameters $\phi_i^{N'N}$ for angular displacements. The nodal displacement vector for node N is then

$$\underline{R}^{N'N} = R_{Ni}^{N'N}\hat{\underline{b}}_i^N \tag{6.2.1-1}$$

The direction cosines of the deformed nodal coordinate axes N' relative to the undeformed axes N are expressed as $C_{ij}^{N'N}$. Then, in matrix notation, the nodal state vector is

$$q_{N'} = \left\{ \begin{array}{c} R_{Ni}^{N'N} \\ \phi_i^{N'N} \end{array} \right\} \tag{6.2.1-2}$$

Note that the nodal steady-state degrees of freedom are referenced to the *undeformed* nodal basis, whereas the frame steady-state degrees of freedom are referenced to the *deformed* frame basis.

For dynamic perturbations about the steady-state condition, the displacement vector is

$$\underline{R}^{N''N'} = R_{Ni}^{N''N'}\hat{\underline{b}}_i^N \tag{6.2.1-3}$$

The dynamic perturbations of the nodal rotations are expressed in terms of infinitesimal rotations $\breve{\theta}_{Ni}^{N''N'}$, for which the direction cosines are

$$C^{N''N'} = \Delta - \tilde{\breve{\theta}}_{Ni}^{N''N'} \tag{6.2.1-4}$$

In matrix notation, the dynamic perturbation nodal state vector is then

$$q_{N''} = \left\{ \begin{array}{c} R_{Ni}^{N''N'} \\ \breve{\theta}_{Ni}^{N''N'} \end{array} \right\} \tag{6.2.1-5}$$

Note that the nodal dynamic degrees of freedom are referenced to the *undeformed* nodal basis, whereas the frame dynamic degrees of freedom are referenced to the *dynamically perturbed* frame basis.

The virtual displacements for the steady-state and dynamic formulations are simply variations of the displacement coordinates

$$\begin{aligned} \underline{\delta R}^{N'N} &= \delta R_{Ni}^{N'N} \hat{\underline{b}}_i^N \\ \underline{\delta R}^{N''N'} &= \delta R_{Ni}^{N''N'} \hat{\underline{b}}_i^N \end{aligned} \tag{6.2.1-6}$$

and the virtual rotations are variations of the rotational degrees of freedom

$$\begin{aligned} \underline{\delta\psi}^{N'N} &= \underline{\delta\psi}_{Ni}^{N'N} \hat{\underline{b}}_i^N \\ \underline{\delta\psi}^{N''N'} &= \underline{\delta\psi}_{Ni}^{N''N'} \hat{\underline{b}}_i^N \end{aligned} \tag{6.2.1-7}$$

6.2.2. Air Nodes

The generalized coordinates representing the axisymmetric flowfield associated with a helicopter rotor are introduced into GRASP by means of the air node. The generalized coordinates are defined relative to an inertial frame of reference I, and determine the inertial air velocity at a point Q as

$$\underline{U}^{QI} = -(U_1^A + r\gamma_{1r}^A + R_{A2}^{QA}\gamma_{12}^A + R_{A3}^{QA}\gamma_{13}^A)\hat{\underline{b}}_1^A \tag{6.2.2-1}$$

where $\hat{\underline{b}}_1^A$ is an inertially fixed unit vector and A is a coordinate system whose origin is located at the center of the axisymmetric flowfield. The distance from the center of flow r can be calculated from

$$r^2 = (R_{A2}^{QA})^2 + (R_{A3}^{QA})^2 \tag{6.2.2-2}$$

For the case of steady-state inflow, U_1^A and γ_{1r}^A represent the uniform inflow velocity and the radial velocity gradient, respectively. The other two generalized coordinates have no physical meaning under these conditions, and therefore are not used. The air node state vector for steady-state inflow is then

$$q_{A'} = \begin{Bmatrix} U_1^A \\ \gamma_{1r}^A \end{Bmatrix} \tag{6.2.2-3}$$

To model dynamic inflow, generalized coordinates U_1^A, γ_{12}^A, and γ_{13}^A represent the collective and two cyclic velocity perturbations. The air node state vector for dynamic inflow is then

$$q_{A''} = \begin{Bmatrix} U_1^A \\ \gamma_{12}^A \\ \gamma_{13}^A \end{Bmatrix} \tag{6.2.2-4}$$

7. CONSTRAINTS

The purpose of a constraint is to create a dependency among generalized coordinates. In GRASP, the dependencies among the generalized coordinates are used to eliminate dependent generalized coordinates in favor of independent generalized coordinates. In the following sections, the general formulation of a primitive constraint will be presented, followed by the specific applications in GRASP. Then, the composite constraints that have been constructed from the primitive constraints will be discussed.

7.1. Primitive Constraints

Consider a set of generalized coordinates that are related to one another through a constraint. The constraint relationship g may be written in the special form

$$q_{e_i} = g_i\left(q_{r_1}, \ldots, q_{r_{N^r}}\right), \qquad (i = 1, \ldots, N^e) \tag{7.1-1}$$

Thus, the generalized coordinates related by the constraint can be partitioned into two sets: a set to be eliminated, q_e, and a set to be retained, q_r. Using the constraint relationship, the set to be eliminated can be obtained directly from the constraint functions which depend only on the set to be retained.

The virtual work for the generalized coordinates associated with the constraint is

$$\delta W = \sum_{i=1}^{N^e} \delta q_{e_i} Q_{e_i} + \sum_{i=1}^{N^r} \delta q_{r_i} Q_{r_i} \tag{7.1-2}$$

The sum of the generalized forces Q associated with a generalized coordinate may differ from zero for two reasons. First, during the process of seeking an equilibrium solution, equilibrium may not always be satisfied. In this case, the sum of the generalized forces is residual force that is a measure of the error in the approximate solution. Second, even if the complete system is in equilibrium, individual subsystems may not be in equilibrium. The generalized forces for these subsystems will be nonzero.

Taking the variation of equation (7.1-1)

$$\delta q_{e_i} = \sum_{j=1}^{N^r} \frac{\partial g_i}{\partial q_{r_j}} \delta q_{r_j}, \qquad (i = 1, \ldots, N^e) \tag{7.1-3}$$

and substituting Eq. (7.1-3) into Eq. (7.1-2)

$$\delta W = \sum_{j=1}^{N^r} \delta q_{r_j} \left(Q_{r_j} + \sum_{i=1}^{N^e} \frac{\partial g_i}{\partial q_{r_j}} Q_{e_i} \right) \tag{7.1-4}$$

This relationship is used by GRASP to incorporate the contributions of the generalized forces associated with the eliminated generalized coordinates into the retained generalized forces. During calculation of steady-state residuals, the residuals associated with the eliminated generalized coordinates are transformed and added to the appropriate residuals in the parent substructure's residual vector.

The treatment of constraints for small perturbations about an equilibrium state is a little more involved. For this problem, each generalized coordinate is assumed to be the sum of an equilibrium value and an infinitesimal perturbation from that value (*i.e.*, $q = \bar{q} + \check{q}$). Equations (7.1–1), (7.1–3) and the generalized forces Q can all be expanded in Taylor series about the equilibrium value. Noting that equation (7.1–1) is valid when $q = \bar{q}$, expansion of equation (7.1–1) yields

$$\check{q}_{e_i} = \sum_{j=1}^{N^r} \overline{\frac{\partial g_i}{\partial q_{r_j}}} \check{q}_{r_j} + \cdots, \qquad (i = 1, \ldots, N^e) \tag{7.1–5}$$

Expansion of equation (7.1–3) yields

$$\delta q_{e_i} = \sum_{j=1}^{N^r} \delta q_{r_j} \left(\overline{\frac{\partial g_i}{\partial q_{r_j}}} + \sum_{k=1}^{N^r} \overline{\frac{\partial^2 g_i}{\partial q_{r_j} \partial q_{r_k}}} \check{q}_{r_k} + \cdots \right), \qquad (i = 1, \ldots, N^e) \tag{7.1–6}$$

Expansion of the generalized force, Q, for both eliminated and retained terms yields

$$\begin{aligned} Q_{e_i} &= \overline{Q}_{e_i} + \sum_{j=1}^{N^e} \overline{L}_{e_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N^r} \overline{L}_{e_i r_j} \check{q}_{r_j}, \qquad (i = 1, \ldots, N^e) \\ Q_{r_i} &= \overline{Q}_{r_i} + \sum_{j=1}^{N^e} \overline{L}_{r_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N^r} \overline{L}_{r_i r_j} \check{q}_{r_j}, \qquad (i = 1, \ldots, N^r) \end{aligned} \tag{7.1–7}$$

where the linear operator, $\overline{L}$, contains the terms normally associated with the mass, damping, and stiffness matrices, $-M\frac{d^2}{dt^2} - C\frac{d}{dt} - K$. Note that the minus signs are present in the definition of $\overline{L}$ because the generalized force is generally regarded as positive on the right-hand side of the dynamical equation, whereas the linear coefficient matrices are regarded as positive on the left-hand side.

GRASP calculates the M, C, and K matrices for a subsystem by adding the contributions of each of its children. The rows and columns of the child subsystem's matrices correspond to all of the generalized coordinates of the child. The constraints are used to eliminate dependent generalized coordinates, resulting in matrices whose rows and columns correspond to only the retained generalized coordinates of the child. The matrices elements are then added to the elements of the parent's matrices that correspond to the child's independent degrees of freedom. The required transformations can be found using the virtual work for the subsystem.

An expression for the virtual work from small perturbations about the equilibrium state may be obtained by substituting equation (7.1–6) and the eliminated and retained subsets from equations (7.1–7) into the virtual work expression in equation (7.1–2).

$$\delta\mathcal{W} = \sum_{i=1}^{N^e}\sum_{j=1}^{N^r} \delta q_{r_j} \left(\overline{\frac{\partial g_i}{\partial q_{r_j}}} + \sum_{k=1}^{N^r} \overline{\frac{\partial^2 g_i}{\partial q_{r_j}\partial q_{r_k}}} \check{q}_{r_k} + \ldots \right) \left(\overline{Q}_{e_i} + \sum_{k=1}^{N^e} \overline{L}_{e_i e_k} \check{q}_{e_k} + \sum_{k=1}^{N^r} \overline{L}_{e_i r_k} \check{q}_{r_k} \right)$$
$$+ \sum_{i=1}^{N^r} \delta q_{r_i} \left(\overline{Q}_{r_i} + \sum_{j=1}^{N^e} \overline{L}_{r_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N^r} \overline{L}_{r_i r_j} \check{q}_{r_j} \right) \tag{7.1-8}$$

After discarding terms of second or higher order, the expression for virtual work consists of a constant part and two first-order parts in $\check{q}$. The constant part is the same as equation (7.1–4), except that it is evaluated for the equilibrium state.

$$\sum_{i=1}^{N^e}\sum_{j=1}^{N^r} \delta q_{r_j} \left(\overline{\frac{\partial g_i}{\partial q_{r_j}}} \overline{Q}_{e_i} + \overline{Q}_{r_i} \right) \tag{7.1-9}$$

The first linear portion of the virtual work is the single term

$$\sum_{j=1}^{N^r}\sum_{k=1}^{N^r} \delta q_{r_j} \left(\sum_{i=1}^{N^e} \overline{\frac{\partial^2 g_i}{\partial q_{r_j}\partial q_{r_k}}} \overline{Q}_{e_i} \right) \check{q}_{r_k} =$$
$$\sum_{j=1}^{N^r}\sum_{k=1}^{N^r} \delta q_{r_j} \left(-K^G_{r_j r_k} \right) \check{q}_{r_k} \tag{7.1-10}$$

The matrix K^G represents the geometric stiffness associated with the constraint. During assembly of the matrices for the parent substructure, GRASP calculates this geometric stiffness and adds it to the stiffness matrix in the parent substructure. This extremely important term is often overlooked. For instance, a pendulum, modeled as a rigid-body mass constrained to rotate about an offset axis (using a screw constraint) derives *all* of its stiffness from this geometric stiffness term.

The remainder of the linear terms are

$$\sum_{j=1}^{N^r}\sum_{k=1}^{N^e} \delta q_{r_j} \left(\overline{L}_{r_j e_k} + \sum_{i=1}^{N^e} \overline{\frac{\partial g_i}{\partial q_{r_j}}} \overline{L}_{e_i e_k} \right) \check{q}_{e_k} +$$
$$\sum_{j=1}^{N^r}\sum_{k=1}^{N^r} \delta q_{r_j} \left(\overline{L}_{r_j r_k} + \sum_{i=1}^{N^e} \overline{\frac{\partial g_i}{\partial q_{r_j}}} \overline{L}_{e_i r_k} \right) \check{q}_{r_k} \tag{7.1-11}$$

After substituting equation (7.1–5) into equation (7.1–11) for the eliminated perturbation coordinates these terms become

$$\sum_{j=1}^{N^r}\sum_{k=1}^{N^r}\delta q_{r_j}\left(\overline{L}_{r_j r_k}+\sum_{i=1}^{N^e}\overline{\frac{\partial g_i}{\partial q_{r_j}}}\,\overline{L}_{e_i r_k}+\sum_{l=1}^{N^e}\overline{L}_{r_j e_l}\overline{\frac{\partial g_l}{\partial q_{r_k}}}+\sum_{i=1}^{N^e}\sum_{l=1}^{N^e}\overline{\frac{\partial g_i}{\partial q_{r_j}}}\,\overline{L}_{e_i e_l}\overline{\frac{\partial g_l}{\partial q_{r_k}}}\right)\check{q}_{r_k} \tag{7.1–12}$$

The quantity within the parentheses in equation (7.1–12) can be thought of as defining a new set of M, C, and K matrices in terms of the retained and eliminated portions of the original matrices. GRASP calculates the new matrices and adds their elements to the elements of the parent substructure's matrices.

The definition of a constraint follows from the specification of the function g. To obtain a solution for a system in equilibrium, the matrix $\frac{\partial g}{\partial q}$ must be known. A perturbation solution, however, requires both the matrix $\frac{\partial g}{\partial q}$ and the geometric stiffness matrix K^G. In the following constraint derivations, matrix $\frac{\partial g}{\partial q}$ will be denoted by $\mathcal{R}$.

7.1.1. Fixed Frame

The fixed frame constraint describes a rigid connection between two frames of reference, F and S. Regardless of the changes in position and orientation relative to inertial space that they may undergo, their position and orientation relative to one another remains constant. The current (child) frame F will have its degrees of freedom eliminated, while the degrees of freedom for the superordinate (parent) frame S will be retained. In GRASP, this constraint is available through the user interface.

Steady-State. Consider two frames in their undeformed (S and F) states and in their steady-state (S' and F') configurations (fig. 8). The degrees of freedom of F (and F') are considered to be dependent, while those of S (and S') are independent. The frames are assumed to be connected such that

$$\underline{R}^{S'S}+\underline{R}^{SF}+\underline{R}^{FF'}+\underline{R}^{F'S'}=0 \tag{7.1.1–1}$$

where

$$R_{S'}^{F'S'}=R_S^{FS} \tag{7.1.1–2}$$

and

$$C^{S'S}C^{SF}C^{FF'}C^{F'S'}=\Delta \tag{7.1.1–3}$$

Thus,

$$\begin{aligned}\underline{R}^{F'F}&=\underline{R}^{F'S'}+\underline{R}^{S'S}-\underline{R}^{FS}\\ R_{F'}^{F'F}&=C^{F'S'}(R_{S'}^{S'S}+R_{S'}^{F'S'})-C^{F'S'}C^{S'S}R_S^{FS}\\ C^{F'F}&=C^{F'S'}C^{S'S}C^{SF}\end{aligned} \tag{7.1.1–4}$$

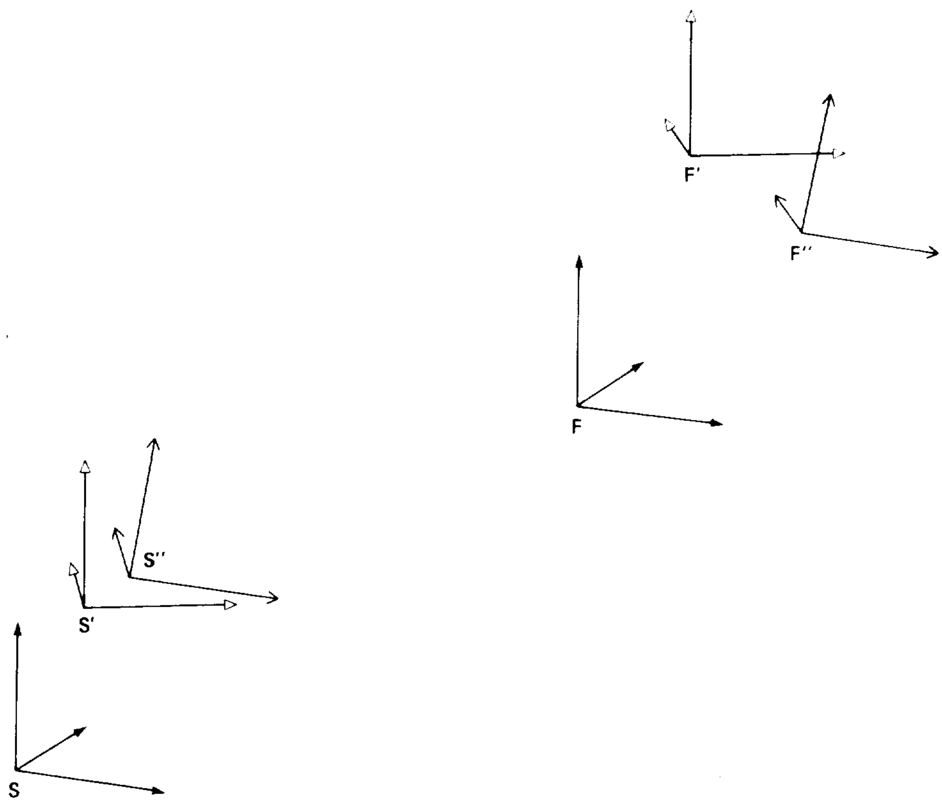


Figure 8. Fixed frame constraint.

Consider the virtual work performed in the F' frame at F'.

$$\delta \mathcal{W} = \left(\delta R_{F'}^{F'F}\right)^T F_{F'}^{F'} + \left(\delta \psi_{F'}^{F'F}\right)^T M_{F'}^{F'} \tag{7.1.1-5}$$

The virtual displacements and rotations at F are related to those at S such that

$$\begin{aligned} \delta R_{F'}^{F'F} &= C^{F'S'}(\delta R_{S'}^{S'S} - \delta C^{S'S} R_S^{FS}) \\ &= C^{F'S'}(\delta R_{S'}^{S'S} + \widetilde{\delta\psi}_{S'}^{S'S} C^{S'S} R_S^{FS}) \\ &= C^{F'S'}(\delta R_{S'}^{S'S} - \tilde{R}_{S'}^{FS} \delta\psi_{S'}^{S'S}) \\ \delta\psi_{F'}^{F'F} &= C^{F'S'} \delta\psi_{S'}^{S'S} \end{aligned} \tag{7.1.1-6}$$

so that the virtual work performed in the F' frame at S' becomes

$$\begin{aligned} \delta \mathcal{W} =& \left(\delta R_{S'}^{S'S}\right)^T C^{S'F'} F_{F'}^{F'} + \\ & \left(\delta\psi_{S'}^{S'S}\right)^T (C^{S'F'} M_{F'}^{F'} + \tilde{R}_{S'}^{FS} F_{S'}^{F'}) \end{aligned} \tag{7.1.1-7}$$

Equations (7.1.1–7) show the contributions of the force and moment acting at F to the force and moment at S for the steady-state problem.

Dynamic. Now consider the two frames in their dynamically perturbed (F'' and S'') states (fig. 8). The perturbed position and orientation are related by

$$\underline{R}^{S''S'} + \underline{R}^{S'F'} + \underline{R}^{F'F''} + \underline{R}^{F''S''} = 0 \tag{7.1.1–8}$$

where

$$R_{S''}^{F''S''} = R_{S'}^{F'S'} \tag{7.1.1–9}$$

and

$$C^{S''S'}C^{S'F'}C^{F'F''}C^{F''S''} = \Delta \tag{7.1.1–10}$$

Thus,

$$\begin{aligned} R_{F''}^{F''F'} &= C^{F''S''}(R_{S''}^{S''S'} + R_{S''}^{F''S''}) - C^{F''S''}C^{S''S'}R_{S'}^{F'S'} \\ C^{F''F'} &= C^{F''S''}C^{S''S'}C^{S'F'} \end{aligned} \tag{7.1.1–11}$$

Taking the variation of both sides yields

$$\begin{aligned} \delta R_{F''}^{F''F'} &= C^{F''S''}(\delta R_{S''}^{S''S'} - \delta C^{S''S'}R_{S'}^{F'S'}) \\ &= C^{F''S''}[\delta R_{S''}^{S''S'} + \widetilde{\delta\psi}_{S''}^{S''S'}(\Delta - \tilde{\check{\theta}}_{S'}^{S''S'})R_{S'}^{F'S'}] \\ \delta\psi_{F''}^{F''F'} &= C^{F''S''}\delta\psi_{S''}^{S''S'} \end{aligned} \tag{7.1.1–12}$$

The transformation from the F'' frame to the S'' frame can then determined in terms of $\mathcal{R}$ and K^G.

$$\mathcal{R} = \begin{bmatrix} C^{F''S''} & -C^{F''S''}R_{S'}^{F'S'} \\ 0 & C^{F''S''} \end{bmatrix} \tag{7.1.1–13}$$

where the columns of $\mathcal{R}$ are associated with variations of the generalized coordinates $\delta R_{S''}^{S''S'}$ and $\delta\psi_{S''}^{S''S'}$, and the rows are associated with $\delta R_{F''}^{F''F'}$ and $\delta\psi_{F''}^{F''F'}$. Then,

$$K^G = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{F}_{S'}^{F'}\tilde{R}_{S'}^{F'S'} \end{bmatrix} \tag{7.1.1–14}$$

where the columns of K^G are associated with generalized coordinates $\check{R}_{S''}^{S''S'}$ and $\check{\theta}_{S''}^{S''S'}$, and the rows are associated with $\delta R_{S''}^{S''S'}$ and $\delta\psi_{S''}^{S''S'}$. Equations (7.1.1–13) and (7.1.1–14) define the constraint formulation for dynamic perturbations about the steady-state configuration.

7.1.2. Structural Node Demotion

The structural node demotion constraint describes a rigid connection between a node D in the child subsystem which has a frame of reference F, and a node I in the parent subsystem which has a frame S (fig. 9). The degrees of freedom for the dependent node D will be eliminated, while the degrees of freedom for the independent node I, the superordinate frame S, and the current frame F will be retained. In GRASP, this constraint is generated internally, and is not available through the user interface.

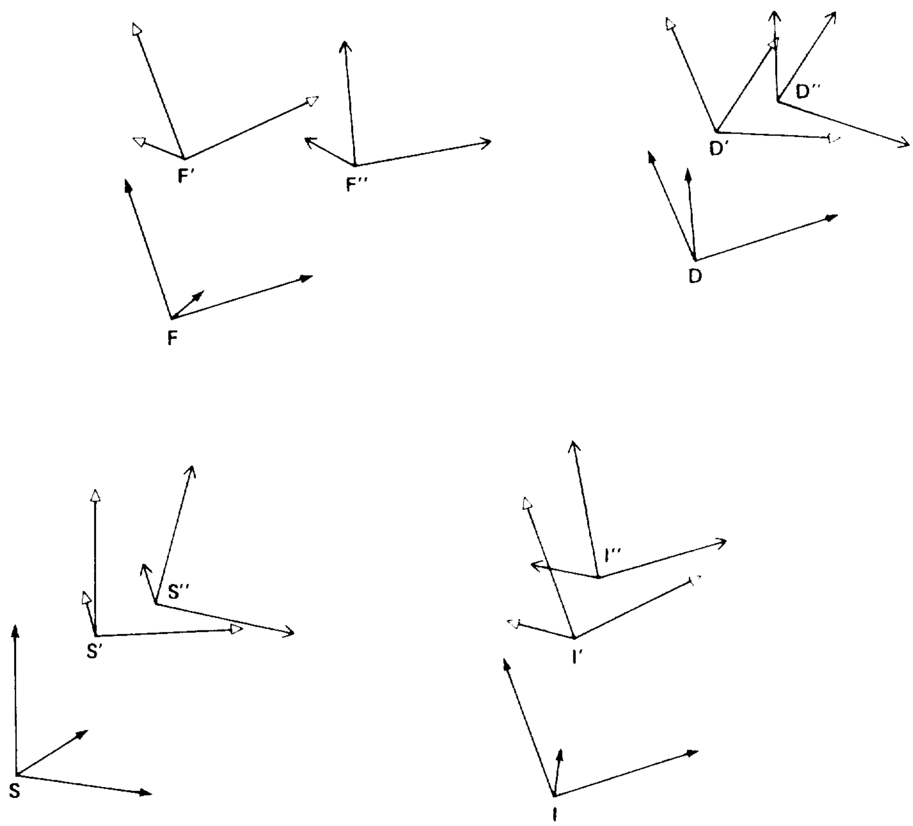


Figure 9. Structural node demotion constraint.

Steady-State. The governing equations express the displacement and orientation of node D in terms of those of I, S, and F. The basic equations for the deformations come from

$$\begin{aligned} \underline{R}^{D'D} &= \underline{R}^{D'I'} + \underline{R}^{I'I} + \underline{R}^{IS'} + \underline{R}^{S'S} + \underline{R}^{SF} + \underline{R}^{FF'} + \underline{R}^{F'D} \\ C^{D'D} &= C^{D'I'}C^{I'I}C^{IS'}C^{S'S}C^{SF}C^{FF'}C^{F'D} \end{aligned} \qquad (7.1.2\text{-}1)$$

In matrix notation the basic equation governing displacement is

$$\begin{aligned} R_D^{D'D} = {} & C^{DF'}C^{F'F}C^{FS}C^{SS'}\left[C^{S'I}(C^{II'}R_{I'}^{D'I'} + R_I^{I'I}) + R_{S'}^{IS'} + R_{S'}^{S'S}\right] - \\ & C^{DF'}(C^{F'F}R_F^{FS} + R_{F'}^{F'F} + R_{F'}^{DF'}) \end{aligned} \qquad (7.1.2\text{-}2)$$

It is also necessary to take the force and moment at D' and find their contributions to the forces and moments at I', F', and S'. For this, the virtual displacement and rotation of D' relative to D is required. The virtual displacement is

$$
\begin{aligned}
\delta R_D^{D'D} = &- C^{DF'} \widetilde{\delta\psi}_{F'}^{F'F} C^{F'F} C^{FS} C^{SS'} \left[C^{S'I} (C^{II'} R_{I'}^{D'I'} + R_I^{I'I}) + R_{S'}^{IS'} + R_{S'}^{S'S} \right] + \\
& C^{DF'} C^{F'F} C^{FS} C^{SS'} \widetilde{\delta\psi}_{S'}^{S'S} \left[C^{S'I} (C^{II'} R_{I'}^{D'I'} + R_I^{I'I}) + R_{S'}^{IS'} + R_{S'}^{S'S} \right] + \\
& C^{DF'} C^{F'F} C^{FS} C^{SS'} \left[C^{S'I} (\widetilde{\delta\psi}_I^{I'I} C^{II'} R_{I'}^{D'I'} + \delta R_I^{I'I}) + \delta R_{S'}^{S'S} \right] + \\
& C^{DF'} \widetilde{\delta\psi}_{F'}^{F'F} C^{F'F} C^{FS} R_S^{FS} - C^{DF'} \delta R_{F'}^{F'F} \\
= &- C^{DF'} \delta R_{F'}^{F'F} + C^{DS'} \delta R_{S'}^{S'S} + C^{DI} \delta R_I^{I'I} - \\
& C^{DF'} \widetilde{\delta\psi}_{F'}^{F'F} R_{F'}^{D'F} + C^{DS'} \widetilde{\delta\psi}_{S'}^{S'S} R_{S'}^{D'S} + C^{DI} \widetilde{\delta\psi}_I^{I'I} R_I^{D'I'} \\
= &- C^{DF'} (\delta R_{F'}^{F'F} - \tilde{R}_{F'}^{D'F} \delta\psi_{F'}^{F'F}) + \\
& C^{DS'} (\delta R_{S'}^{S'S} - \tilde{R}_{S'}^{D'S} \delta\psi_{S'}^{S'S}) + \\
& C^{DI} (\delta R_I^{I'I} - \tilde{R}_I^{D'I'} \delta\psi_I^{I'I})
\end{aligned}
\tag{7.1.2-3}
$$

and the virtual rotation is

$$
\delta\psi_D^{D'D} = C^{DI} \delta\psi_I^{I'I} + C^{DS'} \delta\psi_{S'}^{S'S} - C^{DF'} \delta\psi_{F'}^{F'F} \tag{7.1.2-4}
$$

The virtual work performed in D' contributes the following terms at I', S', and F'

$$
\begin{aligned}
\delta\mathcal{W} = &- (\delta R_{F'}^{F'F})^T F_{F'}^{D'} + (\delta R_{S'}^{S'S})^T F_{S'}^{D'} + (\delta R_I^{I'I})^T F_I^{D'} - (\delta\psi_{F'}^{F'F})^T (M_{F'}^{D'} + \tilde{R}_{F'}^{D'F} F_{F'}^{D'}) + \\
& (\delta\psi_{S'}^{S'S})^T (M_{S'}^{D'} + \tilde{R}_{S'}^{D'S} F_{S'}^{D'}) + (\delta\psi_I^{I'I})^T (M_I^{D'} + \tilde{R}_I^{D'I'} F_I^{D'})
\end{aligned}
\tag{7.1.2-5}
$$

Dynamic. For dynamic perturbations about the steady-state configuration, the basic kinematic relation is used to determine the matrices $\mathcal{R}$ and K^G. The basic equation governing the displacement is

$$
\begin{aligned}
\underline{R}^{D''D'} = & \underline{R}^{D''I''} + \underline{R}^{I''I'} + \underline{R}^{I'I} + \underline{R}^{IS''} + \underline{R}^{S''S'} + \underline{R}^{S'S} - \\
& \underline{R}^{FS} - \underline{R}^{F'F} - \underline{R}^{F''F'} - \underline{R}^{DF''} - \underline{R}^{D'D} \\
\underline{R}^{D''D'} = & \underline{R}^{D''I''} + \underline{R}^{I''I'} + \underline{R}^{I'I} + \underline{R}^{IS''} + \underline{R}^{S''S'} - \underline{R}^{F'S'} - \\
& \underline{R}^{F''F'} - \underline{R}^{DF''} - \underline{R}^{D'D}
\end{aligned}
\tag{7.1.2-6}
$$

Expressing the position of the perturbed state relative to the steady-state position in the D basis (fig. 9), the dependent node displacement may be obtained.

$$\begin{aligned} R_D^{D''D'} =& C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}C^{S''I}C^{II'}R_{I''}^{D''I''}+ \\ & C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}C^{S''I}(R_I^{I''I'}+R_I^{I'I})+ \\ & C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}(R_{S''}^{IS''}+R_{S''}^{S''S'})- \\ & C^{DF''}C^{F''F'}C^{F'S'}R_{S'}^{F'S'}+ \\ & C^{DF''}(R_{F''}^{F''F'}+R_{F''}^{DF''})-R_D^{D'D} \end{aligned} \tag{7.1.2-7}$$

The first variation is

$$\begin{aligned} \delta R_D^{D''D'} =& C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}C^{S''I}\delta R_I^{I''I'}+ \\ & C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}\delta R_{S''}^{S''S'}-C^{DF''}\delta R_{F''}^{F''F'}+ \\ & C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}C^{S''I}C^{II'}\delta C^{I'I''}R_{I''}^{D''I''}+ \\ & C^{DF''}C^{F''F'}C^{F'S'}\delta C^{S'S''}\big[C^{S''I}C^{II'}C^{I'I''}R_{I''}^{D''I''}+ \\ & C^{S''I}(R_I^{I''I'}+R_I^{I'I})+ \qquad R_{S''}^{IS''}+R_{S''}^{S''S'}\big]+ \\ & C^{DF''}\delta C^{F''F'}C^{F'S'}\big[C^{S'S''}C^{S''I}C^{II'}C^{I'I''}R_{I''}^{D''I''}+ \\ & \quad C^{S'S''}C^{S''I}(R_I^{I''I'}+R_I^{I'I})+C^{S'S''}(R_{S''}^{IS''}+R_{S''}^{S''S'})-R_{S'}^{F'S'}\big] \end{aligned} \tag{7.1.2-8}$$

Similarly, the relationship governing the virtual rotation is

$$\begin{aligned} \underline{\delta\psi}^{D''D'} =& \underline{\delta\psi}^{D''I''}+\underline{\delta\psi}^{I''I'}+\underline{\delta\psi}^{I'I}+\underline{\delta\psi}^{IS''}+\underline{\delta\psi}^{S''S'}- \\ & \underline{\delta\psi}^{F'S'}-\underline{\delta\psi}^{F''F'}-\underline{\delta\psi}^{DF''}-\underline{\delta\psi}^{D'D} \end{aligned} \tag{7.1.2-9}$$

which in the D basis becomes

$$\begin{aligned} \delta\psi_D^{D''D'} =& C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}C^{S''I}\delta\psi_I^{I''I'}+ \\ & C^{DF''}C^{F''F'}C^{F'S'}C^{S'S''}\delta\psi_{S''}^{S''S'}- \\ & C^{DF''}\delta\psi_{F''}^{F''F'} \end{aligned} \tag{7.1.2-10}$$

The virtual work done by a force at D'' is then

$$\begin{aligned}(\delta R_D^{D''D'})^T F_D^{D''} =&(\delta R_I^{I''I'})^T (F_I^{D'} + C^{IS'}\tilde{F}_{S'}^{D'}\theta_{S''}^{S''S'} - C^{IF'}\tilde{F}_{F'}^{D'}\theta_{F''}^{F''F'})+\\ &(\delta R_{S''}^{S''S'})^T (F_{S'}^{D'} + \tilde{F}_{S'}^{D'}\theta_{S''}^{S''S'} - C^{S'F'}\tilde{F}_{F'}^{D'}\theta_{F''}^{F''F'})-\\ &(\delta R_{F''}^{F''F'})^T F_{F'}^{D'} +\\ &(\delta \psi_I^{I''I'})^T (\tilde{R}_I^{D'I'}F_I^{D'} + \tilde{F}_I^{D'}\tilde{R}_I^{D'I'}\theta_I^{I''I'} + C^{IS'}\tilde{R}_{S'}^{D'S'}\tilde{F}_{S'}^{D'}\theta_{S''}^{S''S'} -\\ &\qquad C^{IF'}\tilde{R}_{F'}^{D'I'}\tilde{F}_{F'}^{D'}\theta_{F''}^{F''F'})+\\ &(\delta \psi_{S''}^{S''S'})^T (\tilde{R}_{S'}^{D'S'}F_{S'}^{D'} - C^{S'I}\tilde{F}_I^{D'}R_I^{I''I'} - \tilde{F}_{S'}^{D'}R_{S''}^{S''S'} +\\ &\qquad C^{S'I}\tilde{F}_I^{D'}\tilde{R}_I^{D'I'}\theta_I^{I''I'} + \tilde{R}_{S'}^{D'S'}\tilde{F}_{S'}^{D'}\theta_{S''}^{S''S'} -\\ &\qquad C^{S'F'}\tilde{R}_{F'}^{D'S'}\tilde{F}_{F'}^{D'}\theta_{F''}^{F''F'})-\\ &(\delta \psi_{F''}^{F''F'})^T (\tilde{R}_{F'}^{D'F'}F_{F'}^{D'} - C^{F'I}\tilde{F}_I^{D'}R_I^{I''I'} - C^{F'S'}\tilde{F}_{S'}^{D'}R_{S''}^{S''S'} +\\ &\qquad C^{F'I}\tilde{F}_I^{D'}\tilde{R}_I^{D'I'}\theta_I^{I''I'} + C^{F'S'}\tilde{F}_{S'}^{D'}\tilde{R}_{S'}^{D'S'}\theta_{S''}^{S''S'} -\\ &\qquad \tilde{F}_{F'}^{D'}\tilde{R}_{F'}^{D'F'}\theta_{F''}^{F''F'})\end{aligned} \tag{7.1.2-11}$$

and the virtual work done by a moment at D'' is

$$\begin{aligned}(\delta \psi_D^{D''D'})^T M_D^{D''} =&(\delta \psi_I^{I''I'})^T C^{IS''}(\Delta - \tilde{\theta}_{S''}^{S''S'})C^{S'F'}(\Delta + \tilde{\theta}_{F''}^{F''F'})M_{F''}^{D''} +\\ &(\delta \psi_{S''}^{S''S'})^T (\Delta - \tilde{\theta}_{S''}^{S''S'})C^{S'F'}(\Delta + \tilde{\theta}_{F''}^{F''F'})M_{F''}^{D''} -\\ &(\delta \psi_{F''}^{F''F'})^T M_{F''}^{D''}\\ =&(\delta \psi_I^{I''I'})^T (M_I^{D'} + C^{IS'}\tilde{M}_{S'}^{D'}\theta_{S''}^{S''S'} - C^{IF'}\tilde{M}_{F'}^{D'}\theta_{F''}^{F''F'})+\\ &(\delta \psi_{S''}^{S''S'})^T (M_{S'}^{D'} + \tilde{M}_{S'}^{D'}\theta_{S''}^{S''S'} - C^{S'F'}\tilde{M}_{F'}^{D'}\theta_{F''}^{F''F'})-\\ &(\delta \psi_{F''}^{F''F'})^T M_{F'}^{D'}\end{aligned} \tag{7.1.2-12}$$

The matrices $\mathcal{R}$ and K^G are

$$R = \begin{bmatrix} C^{DI} & -C^{DI}\tilde{R}_I^{D'I'} & C^{DS'} & -C^{DS'}R_{S'}^{D'S'} & -C^{DF'} & C^{DF'}R_{F'}^{D'F'} \\ 0 & C^{DI} & 0 & C^{DS'} & 0 & -C^{DF'} \end{bmatrix} \tag{7.1.2-13}$$

where the columns of $\mathcal{R}$ are associated with variations of the generalized coordinates $\delta R_{I''}^{I''I'}$, $\delta \psi_{I''}^{I''I'}$, $\delta R_{S''}^{S''S'}$, $\delta \psi_{S''}^{S''S'}$, $\delta R_{F''}^{F''F'}$, and $\delta \psi_{F''}^{F''F'}$, and the rows are associated with

$\delta R_{D''}^{D''D'}$ and $\delta\psi_{D''}^{D''D'}$. Then,

$$K^G = \begin{bmatrix} 0 & 0 & 0 & -C^{IS'}\tilde{F}_{S'}^{D'} & 0 & C^{IF'}\tilde{F}_{F'}^{D'} \\ 0 & -\tilde{F}_I^{D'}\tilde{R}_I^{D'I'} & 0 & -C^{IS'}(\tilde{R}_{S'}^{D'I'}\tilde{F}_{S'}^{D'} + \tilde{M}_{S'}^{D'}) & 0 & C^{IF'}(\tilde{R}_{F'}^{D'I'}\tilde{F}_{F'}^{D'} + \tilde{M}_{F'}^{D'}) \\ 0 & 0 & 0 & -\tilde{F}_{S'}^{D'} & 0 & C^{S'F'}\tilde{F}_{F'}^{D'} \\ C^{S'I}\tilde{F}_I^{D'} & -C^{S'I}\tilde{F}_I^{D'}\tilde{R}_I^{D'I'} & \tilde{F}_{S'}^{D'} & -\tilde{R}_{S'}^{D'S'}\tilde{F}_{S'}^{D'} - \tilde{M}_{S'}^{D'} & 0 & C^{S'F'}(\tilde{R}_{F'}^{D'S'}\tilde{F}_{F'}^{D'} + \tilde{M}_{F'}^{D'}) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -C^{F'I}\tilde{F}_I^{D'} & C^{F'I}\tilde{F}_I^{D'}\tilde{R}_I^{D'I'} & -C^{F'S'}\tilde{F}_{S'}^{D'} & C^{F'S'}\tilde{F}_{S'}^{D'}\tilde{R}_{S'}^{D'S'} & 0 & -\tilde{F}_{F'}^{D'}\tilde{R}_{F'}^{D'F'} \end{bmatrix} \quad (7.1.2\text{–}14)$$

where the columns of K^G are associated with perturbations of the generalized coordinates $\breve{R}_{I''}^{I''I'}$, $\breve{\theta}_{I''}^{I''I'}$, $\breve{R}_{S''}^{S''S'}$, $\breve{\theta}_{S''}^{S''S'}$, $\breve{R}_{F''}^{F''F'}$, and $\breve{\theta}_{F''}^{F''F'}$, and the rows are associated with $\delta R_{I''}^{I''I'}$, $\delta\psi_{I''}^{I''I'}$, $\delta R_{S''}^{S''S'}$, $\delta\psi_{S''}^{S''S'}$, $\delta R_{F''}^{F''F'}$, and $\delta\psi_{F''}^{F''F'}$.

7.1.3. Screw

The screw constraint describes two nodes, D and I, that are connected by a mechanism that permits translation along and rotation about a single axis which is fixed in the coordinate system of both nodes. The dependent node D will have its degrees of freedom eliminated, while the degrees of freedom of the independent node I will be retained. This constraint is available through the GRASP user interface.

To simplify the derivation, two intermediate nodes located on the screw axis $\underline{\hat{e}}^{scr}$ will be introduced (fig. 10). The "stationary" node S is rigidly connected to the independent node I, while the "moving" node M is rigidly connected to the dependent node D. Nodes M and S initially coincide in both position and orientation, but may translate along and rotate about the screw axis relative to one another.

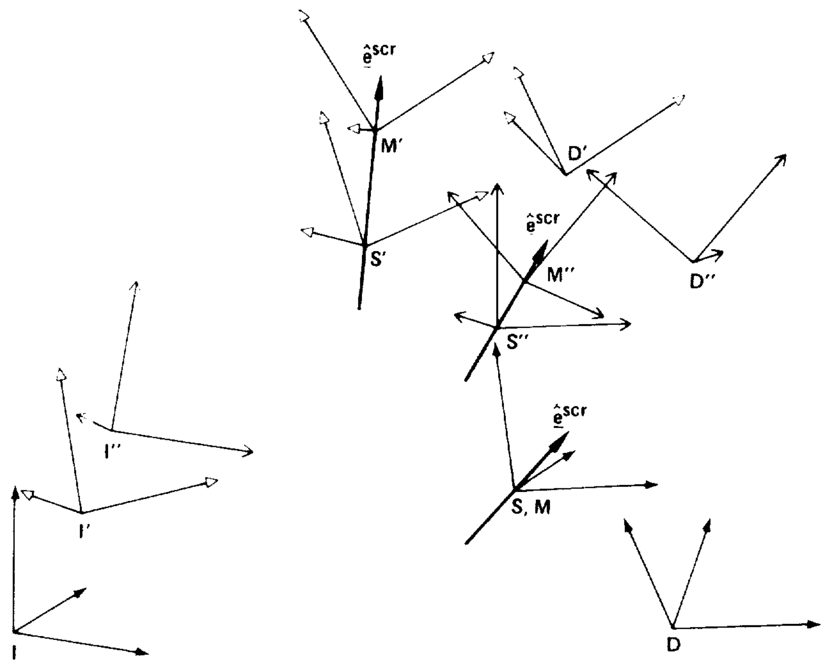


Figure 10. Screw constraint.

Steady-State. For the steady-state problem, the equations governing the degrees of freedom must be developed, as well as the equation for the contribution of the force and moment acting at D' to those at I'. The basic displacement and orientation relationships for the screw constraint are

$$\begin{aligned} \underline{R}^{D'D} &= \underline{R}^{D'M'} + \underline{R}^{M'S'} + \underline{R}^{S'I'} + \underline{R}^{I'I} + \underline{R}^{IS} + \underline{R}^{SM} + \underline{R}^{MD} \\ C^{D'D} &= C^{D'M'} C^{M'\bar{M}'} C^{\bar{M}'S'} C^{S'I'} C^{I'I} C^{ID} \end{aligned} \tag{7.1.3-1}$$

where $\bar{M}'$ indicates a node whose position and orientation relative to S' is the same as that of M relative to S. The position of D' relative to D in the D basis is then

$$\begin{aligned} R_D^{D'D} = & C^{DI} C^{II'} C^{I'S'} C^{S'\bar{M}'} C^{\bar{M}'M'} C^{M'D'} R_{D'}^{D'M'} - \\ & R_D^{DM} + C^{DI} C^{II'} C^{I'S'} u e_{S'}^{scr'} + \\ & C^{DI} (C^{II'} R_{I'}^{S'I'} + R_I^{I'I} - R_I^{SI}) \end{aligned} \tag{7.1.3-2}$$

where

$$\underline{R}^{M'S'} = u \hat{\underline{e}}^{scr'} \tag{7.1.3-3}$$

and u is the screw displacement.

These equations simplify somewhat since $C^{IS} = C^{I'S'} = C^{D'M'} = C^{DM} = \Delta$. $C^{M'\bar{M}'}$ can be easily expressed as an Euler rotation, given the screw rotation θ, and $e^{scr'}_{M'}$, the screw axis unit vector. The virtual displacement is then

$$
\begin{aligned}
\delta R^{D'D}_{D} =& C^{DI} \delta C^{II'} C^{I'S'} C^{S'\bar{M}'} C^{\bar{M}'M'} C^{M'D'} R^{D'M'}_{D'} + \\
& C^{DI} C^{II'} C^{I'S'} C^{S'\bar{M}'} \delta C^{\bar{M}'M'} C^{M'D'} R^{D'M'}_{D'} + \\
& C^{DI} \delta C^{II'} C^{I'S'} e^{scr'}_{S'} u + C^{DI} C^{II'} C^{I'S'} e^{scr'}_{S'} \delta u + \\
& C^{DI} (\delta C^{II'} R^{S'I'}_{I'} + \delta R^{I'I}_{I}) \\
=& C^{DI} [\delta R^{I'I}_{I} - (\tilde{R}^{D'M'}_{I} + u \tilde{e}^{scr'}_{I} + \tilde{R}^{S'I'}_{I}) \delta \psi^{I'I}_{I} + \\
& e^{scr'}_{I} \delta u + \tilde{e}^{scr'}_{I} R^{D'M'}_{I} \delta \theta]
\end{aligned}
\tag{7.1.3–4}
$$

and the virtual rotation is given by

$$
\delta \psi^{D'D}_{D} = C^{DI} (\delta \psi^{I'I}_{I} + e^{scr'}_{I} \delta \theta) \tag{7.1.3–5}
$$

The virtual work at the screw connection and at I' due to a force and moment at D' is

$$
\begin{aligned}
\delta \mathcal{W} =& (\delta R^{I'I}_{I})^T F^{D'}_{I} + (\delta \psi^{I'I}_{I})^T (\tilde{R}^{D'I'}_{I} F^{D'}_{I} + M^{D'}_{I}) + \\
& \delta u (e^{scr'}_{I})^T F^{D'}_{I} + \delta \theta (e^{scr'}_{I})^T (\tilde{R}^{D'M'}_{I} F^{D'}_{I} + M^{D'}_{I})
\end{aligned}
\tag{7.1.3–6}
$$

Dynamic. For the dynamics of the constraint, the equations governing the degrees of freedom are used to to find the matrices $\mathcal{R}$ and K^G. Consider the nodes and the screw axis in their perturbed states (fig. 10), an infinitesimal perturbation from their steady-state positions and orientations. The basic equations are similar to those of the steady-state case.

$$
\begin{aligned}
\underline{R}^{D''D'} =& \underline{R}^{D''M''} + \underline{R}^{M''\bar{M}''} + \underline{R}^{\bar{M}''S''} + \underline{R}^{S''I''} + \\
& \underline{R}^{I''I'} + \underline{R}^{I'D'} \\
\delta \underline{\psi}^{D''D'} =& \delta \underline{\psi}^{D''M''} + \delta \underline{\psi}^{M''\bar{M}''} + \delta \underline{\psi}^{\bar{M}''S''} + \\
& \delta \underline{\psi}^{S''I''} + \delta \underline{\psi}^{I''S'} + \delta \underline{\psi}^{I'D'}
\end{aligned}
\tag{7.1.3–7}
$$

The first, third, fourth, and sixth terms are zero in both equations. Proceeding as above, and noting that

$$
\begin{aligned}
C^{I''I'} =& \Delta - C^{I'I} \tilde{\theta}^{I''I'}_{I} C^{II'} = C^{I'I} (\Delta - \tilde{\theta}^{I''I'}_{I}) C^{II'} \\
\delta C^{I''I'} =& - C^{I'I} (\Delta - \tilde{\theta}^{I''I'}_{I}) \widetilde{\delta \psi}^{I''I'}_{I} C^{II'} \\
\delta C^{M''\bar{M}''} =& - \delta \theta \tilde{e}^{scr''}_{M''} C^{M''\bar{M}''}
\end{aligned}
\tag{7.1.3–8}
$$

the $\mathcal{R}$ and K^G matrices are then

$$\mathcal{R} = \begin{bmatrix} C^{DI} & -C^{DI}\tilde{R}_I^{D'I'} & C^{DI}e_I^{scr'} & C^{DI}\tilde{e}_I^{scr'}R_I^{D'M'} \\ 0 & C^{DI} & 0 & C^{DI}e_I^{scr'} \end{bmatrix} \tag{7.1.3–9}$$

where the columns of $\mathcal{R}$ are associated with the variations of the generalized coordinates $\delta R_I^{I''I'}$, $\delta\psi_I^{I''I'}$, δu, and $\delta\theta$, while the rows are associated with $\delta R_D^{D''D'}$ and $\delta\psi_D^{D''D'}$.

$$K^G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\tilde{F}_I^{D'}\tilde{R}_I^{D'I'} & -\tilde{e}_I^{scr'}F_I^{D'} & -\tilde{F}_I^{D'}\tilde{R}_I^{D'M'}e_I^{scr'} \\ 0 & -\left(e_I^{scr'}\right)^T\tilde{F}_I^{D'} & 0 & 0 \\ 0 & \begin{matrix}\left(R_I^{D'M'}\right)^T\tilde{e}_I^{scr'}\tilde{F}_I^{D'} \\ -\left(e_I^{scr'}\right)^T\tilde{M}_I^{D'}\end{matrix} & 0 & -\left(R_I^{D'M'}\right)^T\tilde{e}_I^{scr'}\tilde{e}_I^{scr'}F_I^{D'} \end{bmatrix} \tag{7.1.3–10}$$

where the columns of K^G are associated with the perturbations of the generalized coordinates $\breve{R}_I^{I''I'}$, $\breve{\theta}_I^{I''I'}$, $\breve{u}$, and $\breve{\theta}$ and the rows are associated with $\delta R_I^{I''I'}$, $\delta\psi_I^{I''I'}$, δu, and $\delta\theta$.

7.1.4. Copy

The copy constraint describes the relationship between generalized coordinates that are common to both parent and child subsystems, but are otherwise unconstrained. This situation most often exists when unconstrained generalized coordinates in the child subsystem are passed up to the parent subsystem. This constraint is not available through the GRASP user interface.

Steady-State. From equation (7.1–1), the constraint relationship between the child subsystem generalized coordinates q_{e_i} that will be eliminated and the parent subsystem generalized coordinates q_{r_i} that will be retained can be written as

$$q_{e_i} = q_{r_i}, \qquad (i = 1, \ldots, N) \tag{7.1.4–1}$$

Therefore, the calculation of the contributions of these generalized coordinates to the child subsystem state vector involves only copying the values of generalized coordinates from the parent subsystem state vector into the child subsystem state vector.

The variation of q_{e_i} is then

$$\delta q_{e_i} = \delta q_{r_i}, \qquad (i = 1, \ldots, N) \tag{7.1.4–2}$$

When this expression for δq_{e_i} is substituted into equation (7.1–2), the virtual work is

$$\delta\mathcal{W} = \sum_{j=1}^{N} \delta q_{r_j}\left(Q_{r_j} + Q_{e_i}\right) \tag{7.1.4–3}$$

Thus, to assemble the residual vector for the parent subsystem, the contributions of these generalized forces from the child subsystem are added to the generalized forces from the parent system.

Dynamic. The derivation of the constraint dynamics follows a similar vein. First, the perturbed generalized coordinates, the variations of the generalized coordinates, and the generalized forces are expanded in Taylor's series.

$$\check{q}_{e_i} = \check{q}_{r_j}\,, \qquad (i = 1, \ldots, N) \tag{7.1.4-4}$$

$$\delta q_{e_i} = \delta q_{r_j} \qquad (i = 1, \ldots, N) \tag{7.1.4-5}$$

$$\begin{aligned} Q_{e_i} &= \overline{Q}_{e_i} + \sum_{j=1}^{N} \overline{L}_{e_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N} \overline{L}_{e_i r_j} \check{q}_{r_j}\,, \qquad (i = 1, \ldots, N) \\ Q_{r_i} &= \overline{Q}_{r_i} + \sum_{j=1}^{N} \overline{L}_{r_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N} \overline{L}_{r_i r_j} \check{q}_{r_j}\,, \qquad (i = 1, \ldots, N) \end{aligned} \tag{7.1.4-6}$$

When these expressions are substituted into equation (7.1–2), and the resulting expression simplified, the virtual work is written as

$$\delta W = \sum_{i=1}^{N} \delta q_{r_i} \left[\overline{Q}_{e_i} + \overline{Q}_{r_i} + \sum_{j=1}^{N} \left(\overline{L}_{e_i e_j} + \overline{L}_{e_i r_j} + \overline{L}_{r_i e_j} + \overline{L}_{r_i r_j} \right) \check{q}_{r_j} \right] \tag{7.1.4-7}$$

Since the q_{e_i} generalized coordinates exist only in the child system, and the q_{r_i} exist only in the parent system, $\overline{L}_{e_i r_j}$ and $\overline{L}_{e_i r_j}$ are null. The $\mathcal{R}$ matrix is, therefore, an identity matrix. For small perturbations about the steady-state solution, the coefficients in the rows and columns associated with the copied generalized coordinates in the child subsystem dynamic matrices (M, C, and K) are simply added to coefficients in the corresponding rows and columns of the parent subsystem dynamic matrices. The geometric stiffness matrix K^G is null.

7.1.5. Prescribed

The prescribed constraint is used to describe the permanent deformation of a particular generalized coordinate. This constraint is trivial, because the steady-state value is constant. In GRASP, the prescribed constraint is available through the user interface for nodal degrees of freedom.

Steady-State. Following the derivation of a general constraint, consider a child subsystem that has N^e generalized coordinates $q_{e_i}, i = 1, \ldots, N^e$. For this constraint, one of those generalized coordinates (*e.g.*, q_{e_1}) has a prescribed, constant value.

$$\begin{aligned} q_{e_1} &= \text{constant} \\ q_{e_i} &= g_i\left(q_{r_1}, \ldots, q_{r_{N^r}}\right), \qquad (i = 2, \ldots, N^e) \end{aligned} \tag{7.1.5-1}$$

The total virtual work is

$$\begin{aligned} \delta W &= \sum_{i=1}^{N^e} \delta q e_i Q_{e_i} \sum_{i=1}^{N^r} \delta q r_i Q_{r_i} \\ &= \sum_{i=2}^{N^e} \delta q e_i Q_{e_i} \sum_{i=1}^{N^r} \delta q r_i Q_{r_i} \end{aligned} \tag{7.1.5-2}$$

since the variation of the prescribed generalized coordinate $\delta q e_1$ is zero. Therefore, this generalized coordinate makes no contribution to the virtual work of either the child or parent subsystem. In practice, degrees of freedom that are prescribed in the child subsystem may be eliminated from the parent subsystem state vector.

Dynamic. The derivation of the dynamic constraint equations for small perturbations about the steady-state solution proceeds following equations (7.1–5) through (7.1–7). The only difference is that in equations (7.1–5) and (7.1–6), $i = 2, \ldots, N^e$. When these expressions are substituted into equation (7.1–2),

$$\begin{aligned} \delta W = &\sum_{i=2}^{N^e} \sum_{j=1}^{N^r} \delta q_{r_j} \left(\overline{\frac{\partial g_i}{\partial q_{r_j}}} + \sum_{k=1}^{N^r} \overline{\frac{\partial^2 g_i}{\partial q_{r_j} \partial q_{r_k}}} \check{q}_{r_k} + \cdots \right) \left(\overline{Q}_{e_i} + \sum_{k=2}^{N^e} \overline{L}_{e_i e_k} \check{q}_{e_k} + \sum_{k=1}^{N^r} \overline{L}_{e_i r_k} \check{q}_{r_k} \right) \\ &+ \sum_{i=1}^{N^r} \delta q_{r_i} \left(\overline{Q}_{r_i} + \sum_{j=2}^{N^e} \overline{L}_{r_i e_j} \check{q}_{e_j} + \sum_{j=1}^{N^r} \overline{L}_{r_i r_j} \check{q}_{r_j} \right) \end{aligned} \tag{7.1.5-3}$$

From this equation it can be seen that the contributions to the virtual work are the same as for the general case, with one exception. The rows of $\overline{L}_{e_i e_k}$ and $\overline{L}_{e_i r_k}$, and the columns of $\overline{L}_{e_i e_k}$ and $\overline{L}_{r_i e_j}$, associated with the prescribed generalized coordinate have been eliminated. This is equivalent to removing the appropriate rows and columns from the M, C, and K matrices that are passed up to the parent subsystem.

7.1.6. Copy Air Mass

The copy air mass constraint is the constraint used to transform the air mass generalized coordinates and forces between child and parent subsystems. This constraint is a clone of the copy constraint, specialized to copy only the four air mass degrees of freedom. Due to the fact that the air mass degrees of freedom are defined in an inertial frame, and need never be transformed out of that frame, the generalized coordinates and forces are simply copied. The copy air mass constraint is not available through the GRASP user interface.

7.1.7. Periodic Frame

The periodic frame constraint describes the relationship between a superordinate (parent) frame S and three or more identical, child frames F_k (for $k = 1, 2, \ldots, b$) rigidly attached to S. Frames F_k are located at equally-spaced, azimuthal intervals about an axis fixed in S (fig. 11). The origin of S is located on the axis of symmetry, while the origins of the F_k may be located elsewhere. In GRASP, the periodic frame constraint is available through the user interface.

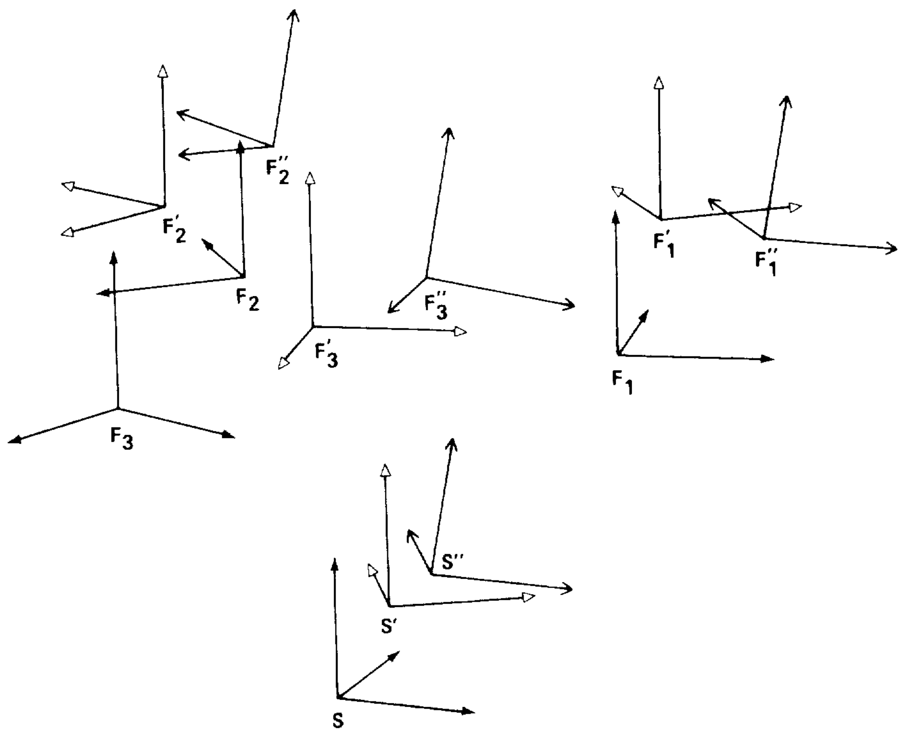


Figure 11. Periodic frame constraint.

The derivation of the periodic frame constraint is very similar to that for the fixed frame, except that it is assumed here that there are b identical frames spaced at equal azimuthal intervals around an axis. The quantity $R_{F_k}^{SF_k}$ is independent of k, and $C^{F_kS} = C^{F_1S}T_k$ where

$$T_k = T_0 + T_c \cos\phi_k + T_s \sin\phi_k \tag{7.1.7–1}$$

and where

$$T_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.1.7–2}$$

$$T_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\phi_k = \frac{2\pi}{b}(k-1), \qquad k = 1, 2, \ldots, b \tag{7.1.7–3}$$

The fixed frame equations can be easily modified to account for this configuration. It will be assumed that the axis of symmetry for the periodic, child frames is $\hat{\underline{b}}_1^S$.

Steady-State. For the steady-state problem, the equations for the deformed position and orientation of any one of the child frames can be written as

$$
\begin{aligned}
R_{F'_k}^{F'_k F_k} &= C^{F'_1 S'} T_k R_{S'}^{S' S} + R_{F'_k}^{F'_k S'} - C^{F'_k F_k} R_{F_k}^{F_k S} \\
&= C^{F'_1 S'} T_k R_{S'}^{S' S} + R_{F'_k}^{F'_k S'} - C^{F'_1 S'} T_k C^{S' S} T_k^T C^{S F_1} R_{F_k}^{F_k S} \qquad (7.1.7\text{–}4) \\
C^{F'_k F_k} &= C^{F'_1 S'} T_k C^{S' S} T_k^T C^{S F_1}
\end{aligned}
$$

In order to make the left-hand sides independent of k, let $R_{S' 2}^{S' S} = R_{S' 3}^{S' S} = 0$ and $\phi_2^{S' S} = \phi_3^{S' S} = 0$. Since the right-hand sides are equivalent for all k, all T_k can be set to T_0 to simplify the equations. The virtual displacements are then

$$
\begin{aligned}
\delta R_{F'_k}^{F'_k F_k} &= C^{F'_1 S'} T_0 \delta R_{S'}^{S' S} - C^{F'_1 S'} T_0 \delta C^{S' S} T_0^T C^{S F_1} R_{F_k}^{F_k S} \\
&= C^{F'_1 S'} T_0 \delta R_{S'}^{S' S} + C^{F'_1 S'} T_0 \widetilde{\delta\psi}_{S'}^{S' S} C^{S' S} T_0^T C^{S F_1} R_{F_k}^{F_k S}
\end{aligned}
\qquad (7.1.7\text{–}5)
$$

and the virtual rotations are

$$
\delta\psi_{F'_k}^{F'_k F_k} = C^{F'_1 S'} T_0 \delta\psi_{S'}^{S' S} \qquad (7.1.7\text{–}6)
$$

The virtual work at S' due to the b sets of forces and moments acting at F'_k is therefore

$$
\begin{aligned}
\delta \mathcal{W} &= \sum_{k=1}^{b} \left[\left(\delta R_{F'_k}^{F'_k F_k}\right)^T F_{F'_k}^{F'_k} + \left(\delta\psi_{F'_k}^{F'_k F_k}\right)^T M_{F'_k}^{F'_k} \right] \\
&= b \Big\{ \left(\delta R_{S'}^{S' S}\right)^T T_0^T C^{S' F'_1} F_{F'_1}^{F'_1} + \left(\delta\psi_{S'}^{S' S}\right)^T \Big[T_0^T C^{S' F'_1} M_{F'_1}^{F'_1} - \\
&\quad \left(T_0^T C^{S' F'_1} F_{F'_1}^{F'_1}\right)^{\sim} C^{S' S} T_0^T C^{S F_1} R_{F_1}^{F_1 S} \Big] \Big\}
\end{aligned}
\qquad (7.1.7\text{–}7)
$$

Dynamic. For small perturbations about the steady-state solution, the perturbed position and orientation (fig. 11) of any one of the child frames F_k is

$$
\begin{aligned}
R_{F''_k}^{F''_k F'_k} &= R_{F''_k}^{F''_k S''} + C^{F''_1 S''} T_k R_{S''}^{S'' S'} - C^{F''_1 S''} T_k C^{S'' S'} T_k^T C^{S' F'_1} R_{F'_k}^{F'_k S'} \\
C^{F''_k F'_k} &= C^{F''_1 S''} T_k C^{S'' S'} T_k^T C^{S' F'_1}
\end{aligned}
\qquad (7.1.7\text{–}8)
$$

To first order in the perturbation quantities, the virtual displacements and rotations are

$$\begin{aligned} \delta R_{F''_k}^{F''_k F'_k} &= C^{F''_1 S''} T_k \left[\delta R_{S''}^{S'' S'} + \widetilde{\delta\psi}_{S''}^{S'' S'} (\Delta - \tilde{\theta}_{S''}^{S'' S'} \quad) T_k^T C^{S' F'_1} R_{F'_k}^{F'_k S'} \right] \\ \delta\psi_{F''_K}^{F''_k F'_k} &= C^{F''_1 S''} T_k \delta\psi_{S''}^{S'' S'} \end{aligned} \tag{7.1.7-9}$$

Note that the geometric stiffness matrix will come from the $\tilde{\theta}_{S''}^{S'' S'}$ term.

After substituting into the expression for the virtual work, the matrix $\mathcal{R}$ is

$$\mathcal{R} = \begin{bmatrix} C^{F''_1 S''} T_k & -(T_k^T C^{S' F'_1} R^{F'_k S'_{F'_k}})^{\sim} \\ 0 & C^{F''_1 S''} T_k \end{bmatrix} \tag{7.1.7-10}$$

where the columns of $\mathcal{R}$ correspond to $\delta R_{S''}^{S'' S'}$ and $\delta\psi_{S''}^{S'' S'}$, and the rows correspond to $\delta R_{F''_k}^{F''_k F'_k}$ and $\delta\psi_{F''_r}^{F''_r F'_r}$. The matrix K^G is

$$K^G = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{k=1}^{b} T_k^T C^{S' F'_1} \tilde{F}_{F'_k}^{F'_k} \tilde{R}_{F'_k}^{F'_k S'} C^{F'_1 S'} T_k \end{bmatrix} \tag{7.1.7-11}$$

where the columns of K^G correspond to $\check{R}_{S''}^{S'' S'}$ and $\check{\theta}_{S''}^{S'' S'}$, and the rows correspond to $\delta R_{F''_k}^{F''_k F'_k}$ and $\delta\psi_{F''_r}^{F''_r F'_r}$. For evaluation of the lower-right submatrix, it should be noted that

$$\sum_{k=1}^{b} T_k^T(\ )T_k = b[T_0^T(\ )T_0 + \tfrac{1}{2} T_c^T(\ )T_c + \tfrac{1}{2} T_s^T(\ )T_s] \tag{7.1.7-12}$$

when the expression enclosed in parentheses is independent of k.

7.1.8. Periodic Node Demotion

Just as the periodic frame constraint is very similar in concept to the fixed frame constraint, the periodic node demotion constraint has a similar relationship to the structural node demotion constraint. In this case, a node belonging to a parent subsystem is replicated in the child subsystem at b equally-spaced azimuthal intervals about an axis that is fixed in the parent subsystem. The periodic node demotion constraint is not available through the GRASP user interface.

The degrees of freedom of the b child subsystem nodes D_k are expressed in terms of the degrees of freedom of F_k, S, and I. To visualize this constraint, consider figure 11 and imagine a node I associated with frame S and a node D_k associated with each frame F_k, as in figure 9. The virtual work done at all nodes D_k for $k = 1, 2, \ldots, b$ is determined at F_k. The total virtual work is summed for $k = 1, 2, \ldots, b$ and determined at S and I. For this constraint, it is assumed that the axis of symmetry is $\hat{\underline{b}}_1^S$, and that $\hat{\underline{b}}_1^I$ also lies along that axis.

Steady-State. The governing equations for the periodic node constraint are derived in a similar manner to those of the structural node demotion constraint. First, let

$$
\begin{aligned}
C^{F_k S} &= C^{F'_k S'} = C^{F_1 S} T_k = C^{F'_1 S'} T_k \\
C^{D_k I} &= C^{D'_k I'} = C^{D_1 I} T_k = C^{D'_1 I'} T_k
\end{aligned}
\tag{7.1.8–1}
$$

where T_k is defined in equation (7.1.7–1) and (7.1.7–2). When

$$
\begin{aligned}
R_{F_k}^{F_k S} &= R_{F'_k}^{F'_k S'} = \text{constant} \\
R_{D_k}^{D_k I} &= R_{D'_k}^{D'_k I'} = \text{constant}
\end{aligned}
\tag{7.1.8–2}
$$

the positions and orientations of the frames and nodes may be written as

$$
\begin{aligned}
R_{D_k}^{D'_k D_k} =& C^{D_k F'_k} C^{F'_k F_k} C^{F_1 S} T_k C^{SS'} \big[C^{S'I} (C^{II'} T_k^T C^{I'D'_1} R_{D'_k}^{D_k I'} + \\
& R_I^{I'I}) + R_{S'}^{IS'} + R_{S'}^{S'S} \big] - \\
& C^{D_k F'_k} (C^{F'_k F_k} R_{F_k}^{F_k S} + R_{F'_k}^{F'_k F_k} R_{F'_k}^{D_k F'_k}) \\
C^{D'_k D_k} =& C^{D'_1 I'} T_k C^{I'I} C^{IS'} C^{S'S} T_k^T C^{SF_1} C^{F_k F'_k} C^{F'_k D_k}
\end{aligned}
\tag{7.1.8–3}
$$

To make equation (7.1.8–3) independent of k, let $R_{S'}^{IS'} = 0$, $C^{IS'} = \Delta$ and let $T_k = T_0$ (all displacements and rotations for S and I take place along or about the axis of symmetry).

The virtual displacements and rotations are required in order to calculate the virtual work of a force and moment at D'_k for all k.

$$
\begin{aligned}
\delta R_{D_k}^{D'_k D_k} =& - C^{D_k F'_k} \widetilde{\delta\psi}_{F'_k}^{F'_k F_k} C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} \big[C^{S'I} (C^{II'} T_0^T C^{ID_1} R_{D'_k}^{D'_k I'} + \\
& R_I^{II'}) + R_{S'}^{IS'} + R_{S'}^{S'S} \big] + \\
& C^{D_k F'_k} C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} \widetilde{\delta\psi}_{S'}^{S'S} \big[C^{S'I} (C^{II'} T_0^T C^{ID_1} R_{D'_k}^{D'_k I'} + \\
& R_I^{II'}) + R_{S'}^{IS'} + R_{S'}^{S'S} \big] + \\
& C^{D_k F'_k} C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} \big[C^{S'I} (\widetilde{\delta\psi}_I^{I'I} C^{II'} T_0^T C^{I'D'_1} R_{D'_k}^{D'_k I'} + \\
& \delta R_I^{I'I}) + \delta R_{S'}^{S'S} \big] - \\
& C^{D_k F'_k} (- \widetilde{\delta\psi}_{F'_k}^{F'_k F_k} C^{F'_k F_k} R_{F_k}^{F_k S} + \delta R_{F'_k}^{F'_k F_k})
\end{aligned}
\tag{7.1.8–4}
$$

$$\delta\psi_{D_k}^{D'_k D_k} = C^{D_k F'_k} C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} (C^{S'I} \delta\psi_I^{I'I} + \delta\psi_{S'}^{S'S}) - C^{D_k F'_K} \delta\psi_{F'_k}^{F'_k F_k} \tag{7.1.8-5}$$

The virtual work due to the virtual displacements of each of the nodes D'_k is

$$\begin{aligned}\delta\mathcal{W} = &- (\delta R_{F'_k}^{F'_k F_k})^T F_{F'_k}^{D'_k} - (\delta\psi_{F'_k}^{F'_k F_k})^T (M_{F'_k}^{D'_k} + \tilde{R}_{F'_k}^{D'_k F_k} F_{F'_k}^{D'_k}) + \\ &(\delta R_{S'}^{S'S})^T C^{S'S} T_0^T C^{SF_1} F_{F_k}^{D'k} + \\ &(\delta\psi_{S'}^{S'S})^T \{C^{S'S} T_0^T C^{SF_1} M_{F_k}^{D'_k} + [C^{S'S} T_0^T C^{SF_1} C^{F_k F'_k} (R_{F'_k}^{D'_k F_k} + C^{F'_k F_k} R_{F_k}^{F_k S} + \\ &C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} R_{S'}^{S'S})]^{\sim} C^{S'S} T_0^T C^{SF_1} F_{F_k}^{D'_k}\} + \\ &(\delta R_I^{I'I})^T C^{IS'} C^{S'S} T_0^T C^{SF_1} F_{F_k}^{D'_k} + \\ &(\delta\psi_I^{I'I})^T \langle C^{IS'} C^{S'S} T_0^T C^{SF_1} M_{F_k}^{D'_k} + \{C^{IS'} C^{S'S} T_0^T C^{SF_1} C^{F_k F'_k} [R_{F'_k}^{D'_k F_k} + \\ &C^{F'_k F_k} R_{F_k}^{F_k S} + C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} (R_{S'}^{S'S} + R_{S'}^{IS'}) + \\ &C^{F'_k F_k} C^{F_1 S} T_0 C^{SS'} C^{S'I} R_I^{I'I}]\}^{\sim} C^{IS'} C^{S'S} T_0^T C^{SF_1} F_{F_k}^{D'_k}\rangle\end{aligned} \tag{7.1.8-6}$$

The summation of terms involving the virtual displacements and rotations at S' and I' involves only a multiplication by b. The corresponding terms at F_k need not be summed since only one system contributes to the virtual work there.

Dynamic. As in the case of the equations for the steady-state periodic node demotion constraint, the derivation of the equations for the dynamics is similar to the derivation of the structural node demotion constraint dynamic equations. In a manner similar to that for the static equations above, let $R_{S'}^{IS'} = 0$ and $C^{IS'} = \Delta$. Also, let

$$\begin{aligned} C^{F_k S} &= C^{F'_k S'} = C^{F''_k S''} = C^{F_1 S} T_k = C^{F'_1 S'} T_k = C^{F''_1 S''} T_k = \text{ constant} \\ C^{D_k I} &= C^{D'_k I'} = C^{D''_k I''} = C^{D_1 I} T_k = C^{D'_1 I'} T_k = C^{D''_1 I''} T_k = \text{ constant} \\ R_{F_k}^{F_k S} &= R_{F'_k}^{F'_k S'} = R_{F''_k}^{F''_k S''} = \text{ constant} \\ R_{D_k}^{D_k I} &= R_{D'_k}^{D'_k I'} = R_{D''_k}^{D''_k I''} = \text{ constant} \end{aligned} \tag{7.1.8-7}$$

The resulting equation for the matrix $\mathcal{R}$ is

$$\mathcal{R} = \begin{bmatrix} C^{D_k I} & -\tilde{R}_{D_k}^{D'_k I'} C^{D_k I} & C^{D_k S'} & -C^{D_k S'} \tilde{R}_{S'}^{D'_k S'} & -C^{D_k F'_k} & C^{D_k F'_k} \tilde{R}_{F'_k}^{D'_k F} \\ 0 & C^{D_k I} & 0 & C^{D_k S'} & 0 & -C^{D'_k F'_k} \end{bmatrix} \tag{7.1.8-8}$$

where

$$\begin{aligned} C^{D_k I} &= C^{D_k F'_k} C^{F'_1 S'} T_k C^{S' I} \\ C^{D_k S'} &= C^{D_k F'_k} C^{F'_1 S'} T_k \\ R_{S'}^{D'_k S'} &= C^{S' F'_1} T_k^T C^{F'_k D_k} R_{D_k}^{D'_k I'} + C^{S' I} R_I^{I' I} + R_{S'}^{I S'} \end{aligned} \tag{7.1.8-9}$$

The columns of $\mathcal{R}$ correspond to $\delta R_I^{I'' I'}$, $\delta\psi_I^{I'' I'}$, $\delta R_{S'}^{S'' S'}$, $\delta\psi_{S''}^{S'' S'}$, $\delta R_{F''_k}^{F''_k F'_k}$, and $\delta\psi_{F''_k}^{F''_k F'_k}$, respectively; while the rows are associated with $\delta R_{D_k}^{D''_k D'_k}$ and $\delta\psi_{D_k}^{D''_k D'_k}$. The coefficients of the geometric stiffness matrix K^G are then

$\delta R_I^{I'' I'}$ row:

$$\begin{aligned} &R_I^{I'' I'} \text{ column: } \quad 0 \\ &\theta_I^{I'' I'} \text{ column: } \quad 0 \\ &R_{S''}^{S'' S'} \text{ column: } \quad 0 \\ &\theta_{S''}^{S'' S'} \text{ column: } \quad -b(T_0^T F_I^{D'_1})^{\sim} \\ &R_{F''_k}^{F''_k F'_k} \text{ column: } \quad 0 \\ &\theta_{F''_k}^{F''_k F'_k} \text{ column: } \quad T_k^T C^{I F'_1} \tilde{F}_{F'_1}^{D'_1} \end{aligned} \tag{7.1.8-10a}$$

$\delta\psi_I^{I''I}$ row:

$R_I^{I''I'}$ column: 0

$\theta_I^{I''I'}$ column: $-bT_0^T(\tilde{F}_I^{D'_1}\tilde{R}_I^{D'_1I})T_0 - \dfrac{b}{2}T_c^T(\tilde{F}_I^{D'_1}\tilde{R}_I^{D'_1I})T_c -$

$\dfrac{b}{2}T_s^T(\tilde{F}_I^{D'_1}\tilde{R}_I^{D'_1I})T_s$

$R_{S''}^{S''S'}$ column: 0 (7.1.8–10b)

$\theta_{S''}^{S''S'}$ column: $-bT_0^T(\tilde{R}_I^{D'_1I'}\tilde{F}_I^{D'_1})T_0 - \dfrac{b}{2}T_c^T(\tilde{R}_I^{D'_1I'}\tilde{F}_I^{D'_1})T_c -$

$\dfrac{b}{2}T_s^T(\tilde{R}_I^{D'_1I'}\tilde{F}_I^{D'_1})T_s - b(T_0^T M_I^{D'_1})^{\sim}$

$R_{F''_k}^{F''_kF'_k}$ column: 0

$\theta_{F''_k}^{F''_kF'_k}$ column: $T_k^T\tilde{R}_I^{D'_1I'}C^{IF'_1}\tilde{F}_{F'_1}^{D'_1} + T_kC^{IF'_1}\tilde{M}_{F'_1}^{D'_1}$

$\delta R_{S''}^{S''S'}$ row:

$R_I^{I''I'}$ column: 0

$\theta_I^{I''I'}$ column: 0

$R_{S''}^{S''S'}$ column: 0 (7.1.8–10c)

$\theta_{S''}^{S''S'}$ column: $-b(T_0^T F_I^{D'_1})^{\sim}$

$R_{F''_k}^{F''_kF'_k}$ column: 0

$\theta_{F''_k}^{F''_kF'_k}$ column: $T_kC^{IF'_1}\tilde{F}_{F'_1}^{D'_1}$

$\delta\psi_{S''}^{S''S'}$ row:

$R_I^{I''I'}$ column: $b(T_0^T F_I^{D_1'})^{\sim}$

$\theta_I^{I''I'}$ column: $-bT_0^T(\tilde{F}_I^{D_1'}\tilde{R}_I^{D_1'I'})T_0 - \frac{b}{2}T_c^T(\tilde{F}_I^{D_1'}\tilde{R}_I^{D_1'I'})T_c -$
$\frac{b}{2}T_s^T(\tilde{F}_I^{D_1'}\tilde{R}_I^{D_1'I'})T_s$

$R_{S''}^{S''S'}$ column: $b(T_0^T F_I^{D_1'})^{\sim}$ (7.1.8–10d)

$\theta_{S''}^{S''S'}$ column: $-bT_0^T(\tilde{R}_I^{D_1'I'}\tilde{F}_I^{D_1'})T_0 - \frac{b}{2}T_c^T(\tilde{R}_I^{D_1'I'}\tilde{F}_I^{D_1'})T_c -$
$\frac{b}{2}T_s^T(\tilde{R}_I^{D_1'I'}\tilde{F}_I^{D_1'})T_s - b(T_0^T M_I^{D_1'})^{\sim}$

$R_{F_k''}^{F_k''F_k'}$ column: 0

$\theta_{F_k''}^{F_k''F_k'}$ column: $T_k^T\tilde{R}_I^{D_1'I'}C^{IF_1'}\tilde{F}_{F_1'}^{D_1'} + T_k C^{IF_1'}\tilde{M}_{F_1'}^{D_1'}$

$\delta R_{F_k''}^{F_k''F_k'}$ row:

$R_I^{I''I'}$ column: 0

$\theta_I^{I''I'}$ column: 0

$R_{S''}^{S''S'}$ column: 0 (7.1.8–10e)

$\theta_{S''}^{S''S'}$ column: 0

$R_{F_k''}^{F_k''F_k'}$ column: 0

$\theta_{F_k''}^{F_k''F_k'}$ column: 0

$\delta\psi_{F_k''}^{F_k''F_k'}$ row:

$$
\begin{aligned}
&R_I^{I''I'} \text{ column:} \quad -\tilde{F}_{F_1'}^{D_1'} C^{F_1'I} T_k \\
&\theta_I^{I''I'} \text{ column:} \quad \tilde{F}_{F_1'}^{D_1'} C^{F_1'I} \tilde{R}_I^{D_1'I'} T_k \\
&R_{S''}^{S''S'} \text{ column:} \quad -\tilde{F}_{F_1'}^{D_1'} C^{F_1'I} T_k \\
&\theta_{S''}^{S''S'} \text{ column:} \quad \tilde{F}_{F_1'}^{D_1'} C^{F_1'I} \tilde{R}_I^{D_1'I'} T_k \\
&R_{F_k''}^{F_k''F_k'} \text{ column:} \quad 0 \\
&\theta_{F_k''}^{F_k''F_k'} \text{ column:} \quad -\tilde{F}_{F_1'}^{D_1'} \tilde{R}_{F_1'}^{D_1'F_1'}
\end{aligned}
\qquad (7.1.8\text{–}10f)
$$

7.1.9. Periodic Generalized Coordinate

A rotationally isotropic structure consists of three or more identical substructures that are spaced around an axis of symmetry at equal azimuthal intervals. The periodic generalized coordinate constraint exists in order to transform generalized coordinates that belong to the rotationally isotropic structure into the generalized coordinates for a generic member of that structure. Additionally, it must transform generalized forces for a generic substructure into the generalized forces for the complete structure. In one sense, it is simply an extension of the copy constraint for periodic structures. In GRASP, the periodic generalized coordinate constraint is not available through the user interface.

Steady-State. The set of independent generalized coordinates for a rotationally isotropic structure may be grouped as collective (q_0), cosine (q_c), and sine (q_s) components. The generalized coordinates for the kth generic substructure q_k may be written as

$$q_k = q_0 + q_c \cos\phi_k + q_s \sin\phi_k \qquad (7.1.9\text{–}1)$$

where $\phi_k = \frac{2\pi}{b}(k-1)$. The variation of these coordinates is

$$\delta q_k = \delta q_0 + \delta q_c \cos\phi_k + \delta q_s \sin\phi_k \qquad (7.1.9\text{–}2)$$

Given generalized forces Q_k, the total virtual work from all of the generic substructures is

$$\delta W = \sum_{k=1}^{b} \delta q_k{}^T Q_k = \delta q_0{}^T \sum_{k=1}^{b} Q_k + \delta q_c{}^T \sum_{k=1}^{b} Q_k \cos\phi_k + \delta q_s{}^T \sum_{k=1}^{b} Q_k \sin\phi_k \qquad (7.1.9\text{–}3)$$

Since the generalized force of a generic structure is independent of k, $Q_0 = Q_k$ and

$$\delta\mathcal{W} = \delta q_0{}^T b Q_0 \tag{7.1.9–4}$$

Dynamic. The dynamic perturbations of the generalized coordinates are related in the same manner as the variations of the steady-state generalized coordinates.

$$\check{q}_k = \check{q}_0 + \check{q}_c \cos\phi_k + \check{q}_s \sin\phi_k \tag{7.1.9–5}$$

Like the generalized forces, the substructure coefficient matrices M, C, and K are independent of k and

$$\delta\mathcal{W} = \delta q_k{}^T (M\ddot{\check{q}}_k + C\dot{\check{q}}_k + K\check{q}_k) \tag{7.1.9–6}$$

The contribution to the virtual work in terms of the independent generalized coordinates is

$$\delta\mathcal{W} = \sum_{k=1}^{h} \left\lfloor \delta q_0{}^T + \delta q_c{}^T \cos\phi_k + \delta q_s{}^T \sin\phi_k \right\rfloor \left\{ [M] \left\{ \begin{matrix} \ddot{\check{q}}_0 \\ \ddot{\check{q}}_c \cos\phi_k \\ \ddot{\check{q}}_s \sin\phi_k \end{matrix} \right\} + [C] \left\{ \begin{matrix} \dot{\check{q}}_0 \\ \dot{\check{q}}_c \cos\phi_k \\ \dot{\check{q}}_s \sin\phi_k \end{matrix} \right\} + [K] \left\{ \begin{matrix} \check{q}_0 \\ \check{q}_c \cos\phi_k \\ \check{q}_s \sin\phi_k \end{matrix} \right\} \right\} \tag{7.1.9–7}$$

$$= \begin{matrix} b\delta q_0{}^T (M\ddot{\check{q}}_0 + C\dot{\check{q}}_0 + K\check{q}_0) + \frac{b}{2}\delta q_c{}^T (M\ddot{\check{q}}_c + C\dot{\check{q}}_c + K\check{q}_c) \\ + \frac{b}{2}\delta q_s{}^T (M\ddot{\check{q}}_s + C\dot{\check{q}}_s + K\check{q}_s) \end{matrix}$$

The matrices for the rotationally isotropic structure therefore have three rows and columns for every row and column in the generic substructure, and are of block diagonal structure.

7.1.10. Periodic Air Mass

The periodic air node constraint describes the transformation of the air node generalized coordinates and forces between subsystems associated with periodic structures and subsystems associated with generic substructures. Since the air node generalized coordinates describe an induced airflow velocity field that is already axially symmetric, the periodicity of the structure has no effect on them. In fact, it is assumed ***a priori*** that the flow field is interacting with a rotating, periodic structure. This constraint is not available from the GRASP user interface.

Steady-State. When a subsystem is periodic (in the sense that it consists of three or more generic, periodic members such as those described under the periodic node demotion constraint), the steady-state air node generalized coordinates $\bar{U}_1^A$ and $\bar{\gamma}_{1_r}^A$ are simply copied from the parent subsystem to the child subsystem in a manner similar to the copy air mass constraint. During the assembly of the generalized forces, the air node generalized forces from a single, generic substructure are simply multiplied by b and added to the corresponding generalized coordinates of the parent subsystem.

Dynamic. For perturbed motions, let the generalized coordinates for the kth subsystem be

$$\check{q}_k = T_k \check{q}; \qquad \delta q_k = T_k \delta q \tag{7.1.10–1}$$

where T_k is as given in equation (7.1.7–1) and where

$$\check{q}_k = \begin{Bmatrix} \check{P}^A_{1_k} \\ \check{\phi}^A_{12_k} \\ \check{\phi}^A_{13_k} \end{Bmatrix}; \qquad \check{q} = \begin{Bmatrix} \check{P}^A_1 \\ \check{\phi}^A_{12} \\ \check{\phi}^A_{13} \end{Bmatrix} \tag{7.1.10–2}$$

and

$$\delta q_k = \begin{Bmatrix} \delta P^A_{1k} \\ \delta\phi^A_{12k} \\ \delta\phi^A_{13k} \end{Bmatrix} \quad ; \quad \delta q = \begin{Bmatrix} \delta P^A_1 \\ \delta\phi^A_{12} \\ \delta\phi^A_{13} \end{Bmatrix} \tag{7.1.10–3}$$

Note that $\check{q}_k$ and $\check{q}$ will not appear in the dynamical equations. The equations now transform in exactly the same manner as the ones in the copy air mass constraint.

7.1.11. Rotating Frame

The rotating frame constraint describes a constraint that is very much similar to the fixed frame constraint, except that frame F is rotating at a constant angular speed relative to frame S (fig. 12). The axis of rotation passes through the origin of F and along $\underline{\hat{b}}_1^F$. No time-dependent terms are retained in the equations. In GRASP, this constraint is available through the user interface.

Steady-State. In moving to its steady-state, equilibrium position, the axis of rotation follows $\underline{\hat{b}}_1^{F'}$. The position vectors $\underline{R}^{SF}$ and $\underline{R}^{S'F'}$ are constant in the S and S' bases, respectively. The change in orientation is then

$$C^{F'S'}(t) = T(t)C^{F'S'}(0) = C^{FS}(t) = T(t)C^{FS}(0) \tag{7.1.11–1}$$

where

$$T(t) = T_0 + T_c \cos(\Omega t) + T_s \sin(\Omega t) \tag{7.1.11–2}$$

where T_0, T_c, and T_s are given in equations (7.1.7–2).

The kinematics for the rotating frame constraint are based on the following equations.

$$\begin{aligned} R^{F'F}_{F'} &= TC^{F'S'}(0)\left(R^{F'S'}_{S'} + R^{S'S}_{S'} - C^{S'S}R^{FS}_{S}\right) \\ C^{F'F} &= TC^{F'S'}(0)\,C^{S'S}C^{SF}(0)\,T^T \end{aligned} \tag{7.1.11–3}$$

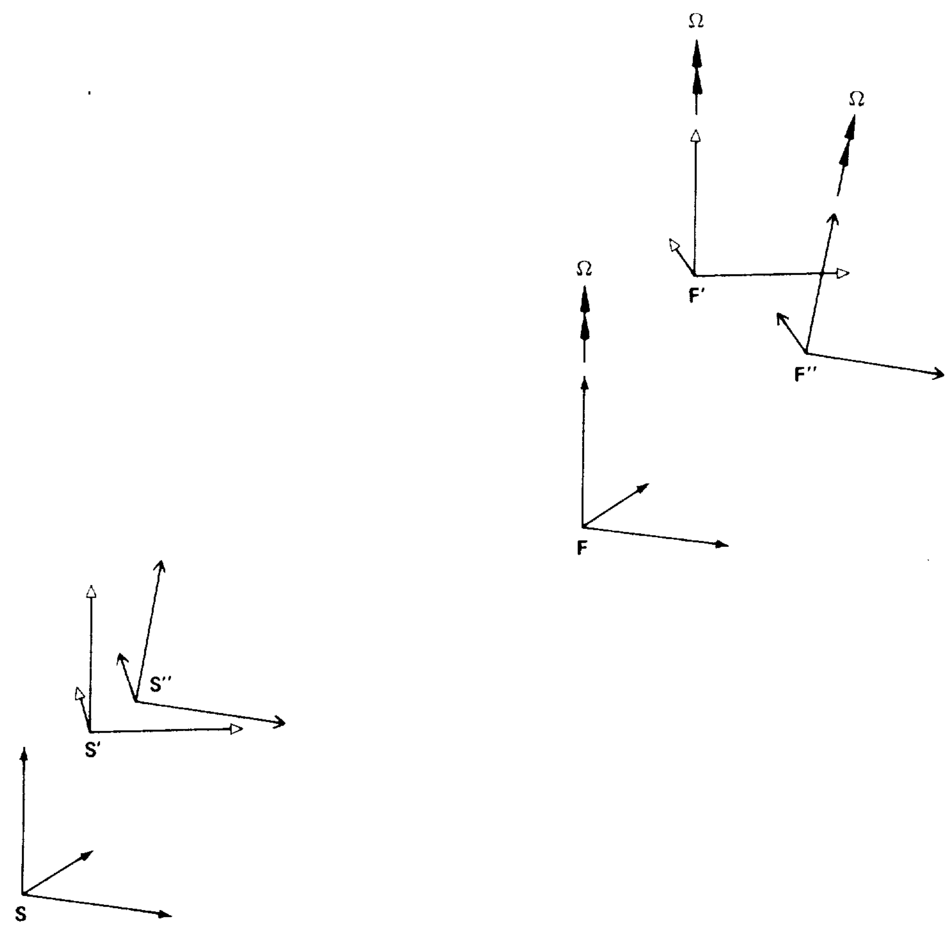


Figure 12. Rotating frame constraint.

The time-dependent terms in these equations vanish if all of the displacements and rotations of S' relative to S are along and about $\hat{\underline{b}}_1^{F'}$ (the axis of rotation). Therefore, let

$$R_{F'2}^{F'F} = R_{F'3}^{F'F} = 0$$
$$\phi_2^{F'F} = \phi_3^{F'F} = 0 \qquad (7.1.11\text{-}4)$$

The virtual displacements and rotations of the F' frame are then

$$\delta R_{F'}^{F'F} = \begin{Bmatrix} \delta R_{F'1}^{F'F} \\ 0 \\ 0 \end{Bmatrix} = T_0 C^{F'S'}(0) \left(\delta R_{S'}^{S'S} + \widetilde{\delta\psi}_{S'}^{S'S} C^{S'S} R_S^{FS} \right)$$
$$\delta\psi_{F'}^{F'F} = \begin{Bmatrix} \delta\psi_{F'1}^{F'F} \\ 0 \\ 0 \end{Bmatrix} = T_0 C^{F'S'}(0)\, \delta\psi_{S'}^{S'S} \qquad (7.1.11\text{-}5)$$

where the use of only the T_0 component of T eliminates the time-dependent terms.

The virtual work at S associated with the force and moment at F yields the following contribution at S:

$$\begin{aligned} \delta W =&(\delta R_{S'}^{S'S})^T C^{S'F'}(0)\, T_0^T F_{F'}^{F'} - \\ &(\delta\psi_{S'}^{S'S})^T \tilde{R}_{S'}^{FS} C^{S'F'}(0)\, T_0^T F_{F'}^{F'} + \\ &(\delta\psi_{S'}^{S'S})^T C^{S'F'}(0)\, T_0^T M_{F'}^{F'} \end{aligned} \tag{7.1.11-6}$$

Dynamic. The position and orientation of the perturbed frame relative to the steady-state position and orientation are related as follows:

$$\begin{aligned} R_{F''}^{F''F'} =& T C^{F''S''}(0) \left(R_{S''}^{F''S''} + R_{S''}^{S''S'} - C^{S''S'} R_{S'}^{F'S'} \right) \\ C^{F''F'} =& T C^{F''S''}(0)\, C^{S''S'} C^{S'F'}(0)\, T^T \end{aligned} \tag{7.1.11-7}$$

From these equations, the virtual displacement and rotation may be obtained. To first order in the perturbation quantities,

$$\begin{aligned} \delta R_{F''}^{F''F'} =& T C^{F''S''}(0) \left[\delta R_{S''}^{S''S'} + \widetilde{\delta\psi}_{S''}^{S''S'} (\Delta - \tilde{\check{\theta}}_{S''}^{S''S'}) R_{S'}^{F'S'} \right] \\ \delta\psi_{F''}^{F''F'} =& T C^{F''S''}(0)\, \delta\psi_{S''}^{S''S'} \end{aligned} \tag{7.1.11-8}$$

where contributions due to geometric stiffness come from the $\tilde{\check{\theta}}_{S''}^{S''S'}$ term. The matrix $\mathcal{R}$ is

$$R = \begin{bmatrix} T(t) C^{F''S''}(0) & -T(t) C^{F''S''}(0)\, \tilde{R}_{S'}^{F'S'} \\ 0 & T(t) C^{F''S''}(0) \end{bmatrix} \tag{7.1.11-9}$$

where the columns of $\mathcal{R}$ correspond to $\check{R}_{S''}^{S''S'}$ and $\check{\theta}_{S''}^{S''S'}$, and the rows correspond to $\delta R_{F''}^{F''F'}$ and $\delta\psi_{F''}^{F''F'}$.

Since $\mathcal{R}$ depends on t, the time-dependent terms must be removed from the final transformed equations. This is easily accomplished by taking the time-averaged value of the transformed equations. The only contributing (*i.e.*, nonzero) terms then are the constant terms, the $\cos^2(\Omega T)$ terms, and the $\sin^2(\Omega T)$ terms. In addition, since $\mathcal{R}$ depends on t, terms from matrix M will contribute to M, C, and K in the transformed equations and C will contribute to C and K by virtue of the following relations

$$\begin{Bmatrix} \dot{\check{R}}_{F''}^{F''F'} \\ \dot{\check{\theta}}_{F''}^{F''F'} \end{Bmatrix} = \dot{R} \begin{Bmatrix} \check{R}_{S''}^{S''S'} \\ \check{\theta}_{S''}^{S''S'} \end{Bmatrix} + R \begin{Bmatrix} \dot{\check{R}}_{S''}^{S''S'} \\ \dot{\check{\theta}}_{S''}^{S''S'} \end{Bmatrix} \tag{7.1.11-10}$$

$$\left\{\begin{matrix} \ddot{\check{R}}_{F''}^{F''F'} \\ \\ \ddot{\check{\theta}}_{F''}^{F''F'} \end{matrix}\right\} = \ddot{R}\left\{\begin{matrix} \check{R}_{S''}^{S''S'} \\ \\ \check{\theta}_{S''}^{S''S'} \end{matrix}\right\} + 2\dot{R}\left\{\begin{matrix} \dot{\check{R}}_{S''}^{S''S'} \\ \\ \dot{\check{\theta}}_{S''}^{S''S'} \end{matrix}\right\} + R\left\{\begin{matrix} \ddot{\check{R}}_{S''}^{S''S'} \\ \\ \ddot{\check{\theta}}_{S''}^{S''S'} \end{matrix}\right\} \tag{7.1.11-11}$$

K^G, the geometric stiffness matrix, is

$$K^G = \begin{bmatrix} 0 & 0 \\ 0 & [C^{S''F''}(0)\,T_0^T F_{F'}^{F'}]^{\sim}\,\tilde{R}_{S'}^{F'S'} \end{bmatrix} \tag{7.1.11-12}$$

where the columns of K^G correspond to $\check{R}_{S''}^{S''S'}$ and $\check{\theta}_{S''}^{S''S'}$, and the rows correspond to $\delta R_{S''}^{S''S'}$ and $\delta\psi_{S''}^{S''S'}$.

7.1.12. Rotating Node Demotion

The rotating node demotion constraint describes the relationship between two nodes, one of which is located in a rotating frame of reference and the other in a nonrotating frame of reference (fig. 13). This constraint combines many of the characteristics of the rotating frame and structural node demotion constraints. It is assumed that the child frame F is rotating about a fixed axis at an angular speed Ω, and that a dependent node D is defined relative to that rotating frame. The parent frame S is stationary (relative to F) and an independent node I is defined relative to S. The rotating node demotion constraint is not available through the GRASP user interface.

Steady-State. The governing equations for the steady-state condition are similar to those for structural node demotion except that

$$\begin{aligned} C^{FS}(t) &= T(t)C^{FS}(0) = C^{F'S'}(t) = T(t)C^{F'S'}(0) = C^{DI}(t) \\ &= T(t)C^{DI}(0) = C^{D'I'}(t) = T(t)C^{D'I'}(0) \end{aligned} \tag{7.1.12-1}$$

In addition, R_S^{FS} and $R_{I'}^{D'I'}$ are constants, and

$$T(t) = T_0 + T_c\cos(\Omega t) + T_s\sin(\Omega t) \tag{7.1.12-2}$$

The governing equations describing the deformed position and orientation of the dependent node are then

$$\begin{aligned} R_D^{D'D} =& C^{DF'}C^{F'F}TC^{FS}(0)C^{SS'}\left[C^{S'I}(C^{II'}R_{I'}^{D'I'} + R_I^{I'I}) + R_{S'}^{IS'} + R_S^{S'S}\right] - \\ & C^{DF'}\left[C^{F'F}TC^{FS}(0)R_S^{FS} + R_{F'}^{F'F} + R_{F'}^{DF'}\right] \\ C^{D'D} =& TC^{D'I'}(0)C^{I'I}C^{IS'}C^{S'S}C^{SF}(0)T^TC^{FF'}C^{F'D} \end{aligned} \tag{7.1.12-3}$$

In order to be independent of t, let $T = T_0$ and choose $R_{F'}^{DF'} = 0$ and $C^{DF'} = \Delta$. Thus, only displacements along and rotations of D about the axis of rotation can be nonzero.

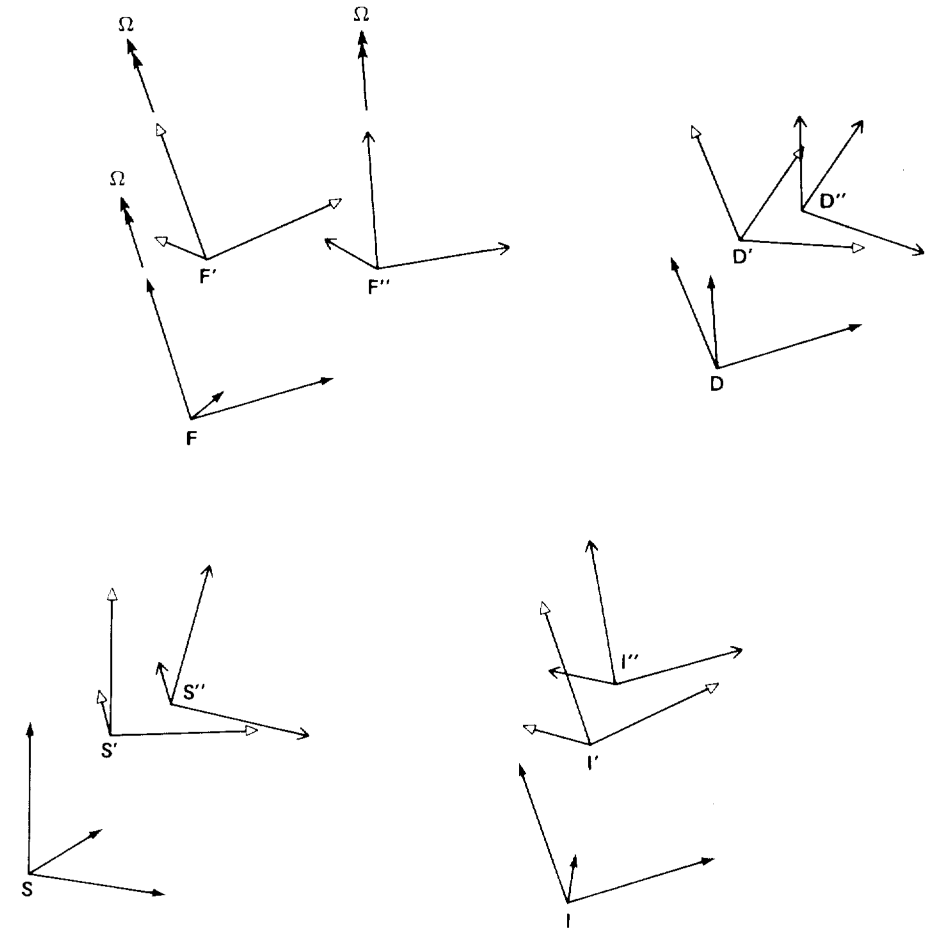


Figure 13. Rotating node demotion constraint.

The virtual work at D' due to a force and a moment acting at D' is determined in terms of the virtual displacement and rotation

$$
\begin{aligned}
\delta R_D^{D'D} = & - C^{DF'} \widetilde{\delta\psi}_{F'}^{F'F} C^{F'F} T_0 C^{FS}(0) C^{SS'} + C^{DF'} C^{F'F} T C^{FS}(0) C^{SS'} R_{S'}^{DS'} \widetilde{\delta\psi}_{S'}^{S'S} + \\
& C^{DF'} C^{F'F} T_0 C^{FS}(0) C^{SS'} \left[C^{S'I} (\widetilde{\delta\psi}_I^{I'I} C^{II'} R_{I'}^{D'I'} + \delta R_I^{I'I}) + \delta R_{S'}^{S'S} \right] - \\
& C^{DF'} \left[- \widetilde{\delta\psi}_{F'}^{F'F} C^{F'F} T_0 C^{FS}(0) R_S^{FS} + \delta R_{F'}^{F'F} \right]
\end{aligned}
$$

$$
\delta\psi_D^{D'D} = C^{DF} T_0 C^{FS}(0) C^{SI} \delta\psi_I^{I'I} + C^{DF} T_0 C^{FS}(0) C^{SS'} \delta\psi_{S'}^{S'S} - C^{DF'} \delta\psi_{F'}^{F'F} \tag{7.1.12-4}
$$

The virtual work done at D' is then

$$\begin{aligned}
\delta\mathcal{W} = &-(\delta R_{F'}^{F'F})^T F_{F'}^{D'} - (\delta\psi_{F'}^{F'F})^T (M_{F'}^{D'} + \tilde{R}_{F'}^{D'F} F_{F'}^{D'}) + \\
&(\delta R_{S'}^{S'S})^T C^{S'S} C^{SF}(0) T_0 F_F^{D'} + \\
&(\delta\psi_{S'}^{S'S})^T [C^{S'S} C^{SF}(0) T_0^T M_F^{D'} + \tilde{R}_{S'}^{D'S} C^{S'S} C^{SF}(0) T_0^T F_F^{D'}] + \\
&(\delta R_I^{I'I})^T C^{IS} C^{SF}(0) T_0 F_F^{D'} + \\
&(\delta\psi_I^{I'I})^T [C^{IS} C^{FS}(0) T_0^T M_F^{D'} + \tilde{R}_I^{D'I'} C^{IS} C^{SF}(0) T_0^T F_F^{D'}]
\end{aligned} \tag{7.1.12–5}$$

Dynamic. The governing equations are similar to those for structural node demotion except that, as in the static case, $R_{F'}^{DF'} = 0$ and $C^{DF'} = \Delta$. The governing equation for the position of the perturbed dependent node is

$$\begin{aligned}
\underline{R}^{D''D'} = &\underline{R}^{D''I''} + \underline{R}^{I''I'} + \underline{R}^{I'I} + \underline{R}^{IS''} + \underline{R}^{S''S'} + \underline{R}^{S'S} - \\
&\underline{R}^{NS} - \underline{R}^{FN} - \underline{R}^{F'F} - \underline{R}^{F''F} - \underline{R}^{DF''} - \underline{R}^{D'D}
\end{aligned} \tag{7.1.12–6}$$

where

$$\underline{R}^{NS} + \underline{R}^{FN} = \underline{R}^{S'S} + \underline{R}^{N'S'} + \underline{R}^{F'N'} - \underline{R}^{F'F} \tag{7.1.12–7}$$

Solving for the dependent node displacement,

$$\begin{aligned}
\underline{R}^{D''D'} = &\underline{R}^{D''I''} + \underline{R}^{I''I'} + \underline{R}^{I'I} + \underline{R}^{IS''} + \underline{R}^{S''S'} - \underline{R}^{N'S'} - \\
&\underline{R}^{F'N'} - \underline{R}^{F''F'} - \underline{R}^{DF''} - \underline{R}^{D'D}
\end{aligned} \tag{7.1.12–8}$$

By referring the displacements to the D basis this vector relation becomes

$$\begin{aligned}
R_D^{D''D'} = &C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} C^{S''I} C^{II'} C^{I'I''} R_{I''}^{D''I''} + \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} C^{S''I} (R_I^{I''I'} + R_I^{I'I'}) + \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} (R_{S''}^{IS''} + R_{S''}^{S''S'}) - \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} R_D^{N'S'} - \\
&C^{DF''} C^{F''F'} C^{F'N'} R_{N'}^{F'N'} - C^{DF''} (R_{F''}^{F''F'} + R_{F''}^{DF''}) - R_D^{D'D}
\end{aligned} \tag{7.1.12–9}$$

The virtual displacement is then

$$\begin{aligned}
\delta R_D^{D''D'} = &C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} C^{S''I} \delta R_I^{I''I} + \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} \delta R_{S''}^{S''S'} - C^{DF''} \delta R_{F''}^{F''F'} + \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} C^{S''I} C^{II'} \delta C^{I'I''} R_{I''}^{D''I''} + \\
&C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} \delta C^{S''S'} [C^{S''I} C^{II'} C^{I'I''} R_{I''}^{D''I''} + \\
&\quad C^{S''I} (R_I^{I''I'} + R_I^{I'I}) + R_{S''}^{IS''} + R_{S''}^{S''S'}] + \\
&C^{DF''} \delta C^{F''F'} C^{F'N'} [C^{N'S'} C^{S'S''} C^{S''I} C^{II'} C^{I'I''} R_{I''}^{D''I''} + \\
&\quad C^{N'S'} C^{S'S''} C^{S''I} (R_I^{I''I'} + R_I^{I'I}) + \\
&\quad C^{N'S'} C^{S'S''} (R_{S''}^{IS''} + R_{S''}^{S''S'}) - C^{N'S'} R_{S'}^{N'S'} - R_{N'}^{F'N'}]
\end{aligned} \tag{7.1.12–10}$$

and the virtual work done by the force at D' is

$$
\begin{aligned}
(\delta R_D^{D''D'})^T F_D^{D'} =& (\delta R_I^{I''I'})^T [C^{IN'} C^{N'F'} F_{F'}^{D'} + C^{IN'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'S'} \theta_{S''}^{S''S'} - \\
& \quad C^{IN'} C^{N'F'} \tilde{F}_{F'}^{D'} \theta_{F''}^{F''F'}] + \\
& (\delta R_{S''}^{S''S'})^T [C^{S'N'} C^{N'F'} F_{F'}^{D'} + C^{S'N'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'S'} \theta_{S''}^{S''S'} - \\
& \quad C^{S'N'} C^{N'F'} \tilde{F}_{F'}^{D'} \theta_{F''}^{F''F'}] - \\
& (\delta R_{F''}^{F''F'})^T F_{F'}^{D'} + \\
& (\delta \psi_I^{I''I'})^T [C^{IN'} \tilde{R}_{N'}^{D'I'} C^{N'F'} F_{F'}^{D'} + C^{IN'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'I'} \tilde{R}_I^{D'I'} \theta_I^{I''I'} + \\
& \quad C^{IN'} \tilde{R}_{N'}^{D'I'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'S'} \theta_{S''}^{S''S'} - C^{IN'} \tilde{R}_{N'}^{D'I'} C^{N'F'} \tilde{F}_{F'}^{D'} \theta_{F''}^{F''F'}] + \\
& (\delta \psi_{S''}^{S''S'})^T [C^{S'N'} \tilde{R}_{N'}^{D'I'} C^{N'F'} F_{F'}^{D'} - C^{S'N'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'I'} R_I^{I''I'} - \\
& \quad C^{S'N'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'S'} R_{S''}^{S''S'} + \\
& \quad C^{S'N'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'I} \tilde{R}_I^{D'I'} \theta_I^{I''I'} + \\
& \quad C^{S'N'} \tilde{R}_{N'}^{D'I'} (C^{N'F'} F_{F'}^{D'})^{\sim} C^{N'S'} \theta_{S''}^{S''S'} - \\
& \quad C^{S'N'} \tilde{R}_{N'}^{D'S'} C^{N'F'} \tilde{F}_{F'}^{D'} \theta_{F''}^{F''F'}] - \\
& (\delta \psi_{F''}^{F''F'})^T [(C^{F'N'} R_{N'}^{D'F'})^{\sim} F_{F'}^{D'} - \tilde{F}_{F'}^{D'} C^{F'N'} C^{N'I} R_I^{I''I'} - \\
& \quad \tilde{F}_{F'}^{D'} C^{F'N'} C^{N'S'} R_{S''}^{S''S'} + \tilde{F}_{F'}^{D'} C^{F'N'} C^{N'I} \tilde{R}_I^{D'I'} \theta_I^{I''I'} + \\
& \quad \tilde{F}_{F'}^{D'} C^{F'N'} C^{N'S'} \tilde{R}_{S'}^{D'S'} \theta_{S''}^{S''S'} - \tilde{F}_{F'}^{D'} (C^{F'N'} R_{N'}^{D'F'})^{\sim} \theta_{F''}^{F''F'}]
\end{aligned}
\tag{7.1.12–11}
$$

The virtual rotations can be easily obtained from the variation of the direction cosine relation

$$
\begin{aligned}
\delta C^{D''D'} =& - C^{D''D'} C^{D'D} \widetilde{\delta\psi}_D^{D''D'} C^{DD'} \\
=& C^{D''I''} \delta C^{I''I'} C^{I'I} C^{IS''} C^{S''S'} C^{S'N'} C^{N'F'} C^{F'F''} C^{F''D} C^{DD'} + \\
& C^{D''I''} C^{I''I'} C^{I'I} C^{IS''} \delta C^{S''S'} C^{S'N'} C^{N'F'} C^{F'F''} C^{F''D} C^{DD'} + \\
& C^{D''I''} C^{I''I'} C^{I'I} C^{IS''} C^{S''S'} C^{S'N'} C^{N'F'} \delta C^{F'F''} C^{F''D} C^{DD'}
\end{aligned}
\tag{7.1.12–12}
$$

thus yielding

$$
\widetilde{\delta\psi}_D^{D''D'} = C^{DI} \widetilde{\delta\psi}_I^{I''I'} C^{ID} + C^{DS''} \widetilde{\delta\psi}_{S''}^{S''S'} C^{S''D} - C^{DF''} \widetilde{\delta\psi}_{F''}^{F''F'} C^{F''D} \tag{7.1.12–13}
$$

Upon removal of the tilde the virtual rotations are

$$
\begin{aligned}
\delta\psi_D^{D''D'} =& C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} C^{S''I} \delta\psi_I^{I''I'} + \\
& C^{DF''} C^{F''F'} C^{F'N'} C^{N'S'} C^{S'S''} \delta\psi_{S''}^{S''S'} - C^{DF''} \delta\psi_{F''}^{F''F'}
\end{aligned}
\tag{7.1.12–14}
$$

and the virtual work done by the moment is

$$
\begin{aligned}
(\delta\psi_D^{D''D'})^T M_D^{D'} =&(\delta\psi_I^{I''I'})^T C^{IS''}(\Delta - \tilde{\theta}_{S''}^{S''S'})C^{S'N'}C^{N'F'}(\Delta + \tilde{\theta}_{F''}^{F''F'})M_{F'}^{D'} + \\
&(\delta\psi_{S''}^{S''S'})^T(\Delta - \tilde{\theta}_{S''}^{S''S'})C^{S'N'}C^{N'F}(\Delta + \tilde{\theta}_{F''}^{F''F'})M_{F''}^{D'} - (\delta\psi_{F''}^{F''F'})^T M_{F''}^{D'} \\
=&(\delta\psi_I^{I''I'})^T[C^{IN'}C^{N'F'}M_{F'}^{D'} + C^{IN'}(C^{N'F'}M_{F'}^{D'})^{\tilde{}}\,C^{N'S'}\theta_{S''}^{S''S'} - \\
&\quad C^{IN'}C^{N'F'}\tilde{M}_{F'}^{D'}\theta_{F''}^{F''F'}] + \\
&(\delta\psi_{S''}^{S''S'})^T[C^{S'N'}C^{N'F'}M_{F'}^{D'} + C^{S'N'}(C^{N'F'}M_{F'}^{D'})^{\tilde{}}\,C^{N'S'}\theta_{S''}^{S''S'} - \\
&\quad C^{S'N'}C^{N'F'}\tilde{M}_{F'}^{D'}\theta_{F''}^{F''F'}] - \\
&(\delta\psi_{F''}^{F''F'})^T M_{F'}^{D'}
\end{aligned}
\tag{7.1.12-15}
$$

since $C^{F'N'} = T$, let $C^{N'S'} = C^{F'S'}$, as in the structural node demotion constraint. The time-dependent terms can be eliminated when $R_{F'}^{DF'} = 0$ and $C^{DF'} = \Delta$.

Combining the virtual work due to the force and the moment at D'', the matrix $\mathcal{R}$ can then be calculated.

$$
\mathcal{R} = \begin{bmatrix}
C^{DF'}TC^{F'I} & -C^{DF'}TC^{F'I}\tilde{R}_I^{D'I'} & C^{DF'}TC^{F'S'} & -C^{DF'}TC^{F'S'}\tilde{R}_{S'}^{D'S'} & -C^{DF'} & 0 \\
0 & C^{DF'}TC^{F'I} & 0 & C^{DF'}TC^{F'S'} & 0 & -C^{DF'}
\end{bmatrix}
\tag{7.1.12-16}
$$

where the columns of $\mathcal{R}$ are associated with $\delta R_I^{I''I'}$, $\delta\psi_I^{I''I'}$, $\delta R_{S''}^{S''S'}$, $\delta\psi_{S''}^{S''S'}$, $\delta R_{F''}^{F''F}$, and $\delta\psi_{F''}^{F''F'}$, respectively; and the rows correspond to $\delta R_D^{D''D'}$ and $\delta\psi_D^{D''D'}$. The coefficients of the matrix K^G are

$\delta R_I^{I''I'^T}$ row:

$$R_I^{I''I'} \text{ column: } \quad 0$$

$$\theta_I^{I''I'} \text{ column: } \quad 0$$

$$R_{S''}^{S''S'} \text{ column: } \quad 0 \tag{7.1.12–17a}$$

$$\theta_{S''}^{S''S'} \text{ column: } \quad -C^{IF'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'S'}$$

$$R_{F''}^{F''F'} \text{ column: } \quad 0$$

$$\theta_{F''}^{F''F'} \text{ column: } \quad C^{IF'} T^T \tilde{F}_{F'}^{D'}$$

$\delta \psi_I^{I''I^T}$ row:

$$R_I^{I''I'} \text{ column: } \quad 0$$

$$\theta_I^{I''I'} \text{ column: } \quad -C^{IF'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'I}$$

$$R_{S''}^{S''S'} \text{ column: } \quad 0 \tag{7.1.12–17b}$$

$$\theta_{S''}^{S''S'} \text{ column: } \quad -[\tilde{R}_I^{D'I'} C^{IF'}(T^T F_{F'}^{D'})^{\tilde{}} + C^{IF'}(T^T M_{F'}^{D'})^{\tilde{}}]C^{F'S'}$$

$$R_{F''}^{F''F'} \text{ column: } \quad 0$$

$$\theta_{F''}^{F''F'} \text{ column: } \quad \tilde{R}_I^{D'I'} C^{IF'} T^T \tilde{F}_{F'}^{D'} + C^{IF'} T^T \tilde{M}_{F'}^{D'}$$

$\delta R_{S''}^{S''S'\,T}$ row:

$R_I^{I''I'}$ column: 0

$\theta_I^{I''I'}$ column: 0

$R_{S''}^{S''S'}$ column: 0 (7.1.12–17c)

$\theta_{S''}^{S''S'}$ column: $-C^{S'F'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'S'}$

$R_{F''}^{F''F'}$ column: 0

$\theta_{F''}^{F''F'}$ column: $C^{S'F'} T^T \tilde{F}_{F'}^{D'}$

$\delta \psi_{S''}^{S''S'\,T}$ row:

$R_I^{I''I'}$ column: $C^{S'F'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'I}$

$\theta_I^{I''I'}$ column: $-C^{S'F'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'I}$

$R_{S''}^{S''S'}$ column: $C^{S'F'}(T^T F_{F'}^{D'})^{\tilde{}}\, C^{F'S'}$ (7.1.12–17d)

$\theta_{S''}^{S''S'}$ column: $-[\tilde{R}_{S'}^{D'S'} C^{S'F'}(T^T F_{F'}^{D'})^{\tilde{}} + C^{S'F'}(T^T M_{F'}^{D'})^{\tilde{}}]C^{F'S'}$

$R_{F''}^{F''F'}$ column: 0

$\theta_{F''}^{F''F'}$ column: $\tilde{R}_{S'}^{D'S'} C^{S'F'} T^T \tilde{F}_{F'}^{D'} + C^{S'F'} T^T \tilde{M}_{F'}^{D'}$

$\delta R_{F''}^{F''F'T}$ row:

$R_I^{I''I'}$ column: 0

$\theta_I^{I''I'}$ column: 0

$R_{S''}^{S''S'}$ column: 0 (7.1.12–17e)

$\theta_{S''}^{S''S'}$ column: 0

$R_{F''}^{F''F'}$ column: 0

$\theta_{F''}^{F''F'}$ column: 0

$\delta\psi_{F''}^{F''F'T}$ row:

$R_I^{I''I'}$ column: $-\tilde{F}_{F'}^{D'}TC^{F'I}$

$\theta_I^{I''I'}$ column: $\tilde{F}_{F'}^{D'}TC^{F'I}\tilde{R}_I^{D'I'}$

$R_{S''}^{S''S'}$ column: $-\tilde{F}_{F'}^{D'}TC^{F'S'}$ (7.1.12–17f)

$\theta_{S''}^{S''S'}$ column: $\tilde{F}_{F'}^{D'}TC^{F'S'}\tilde{R}_{S'}^{D'S'}$

$R_{F''}^{F''F'}$ column: 0

$\theta_{F''}^{F''F'}$ column: 0

7.1.13. Rotating Generalized Coordinates

The rotating generalized coordinate constraint relates generalized coordinates in one subsystem to the corresponding generalized coordinates in another subsystem that is rotating at constant angular speed relative to the first. This constraint is often applied to subsystems that contain periodic structures. This constraint is not available through the GRASP user interface.

Steady-State. The general form of the transformation from rotating to nonrotating coordinates is

$$q_R = Tq_N \tag{7.1.13–1}$$

where

$$T = T_0 + T_c \cos \Omega t + T_s \sin \Omega t \tag{7.1.13–2}$$

and q_R is a set of generalized coordinates in the rotating subsystem that corresponds to a set of q_N generalized coordinates in the nonrotating subsystem.

In order to make the transformation equations independent of time, let $T = T_0$ and $\delta q_R{}^T = T_0 \delta q_N$. This eliminates any generalized forces of the lateral (cosine or sine) type. Then, the virtual work is

$$\delta\mathcal{W} = \bar{q}_R^T Q_R = \bar{q}_N^T T_0^T Q_R \tag{7.1.13–3}$$

Dynamic. In the rotating system, the virtual work can be written as

$$\delta\mathcal{W} = \check{q}_R^T (M_R \ddot{\check{q}}_R + C_R \dot{\check{q}}_R + K_R \check{q}_R) \tag{7.1.13–4}$$

where

$$\begin{aligned} \check{q}_R &= T\check{q}_N \\ \dot{\check{q}}_R &= \dot{T}\check{q}_N + T\dot{\check{q}}_N \\ \ddot{\check{q}}_R &= \ddot{T}\check{q}_N + 2\dot{T}\dot{\check{q}}_N + T\ddot{\check{q}}_N \end{aligned} \tag{7.1.13–5}$$

Substituting these relations into equation (7.1.13–4), the virtual work can be obtained in terms of the generalized coordinates of the nonrotating system.

$$\delta\mathcal{W} = \check{q}_N^T T^T \left[M_R T \ddot{\check{q}}_N + (C_R T + 2M_R \dot{T})\dot{\check{q}}_N + (K_R T + C_R \dot{T} + M_R \ddot{T})\check{q}_N \right] \tag{7.1.13–6}$$

Thus, the C_N coefficient matrix (in the nonrotating system) depends on C_R and M_R, and K_N depends on K_R, C_R, and M_R.

7.1.14. Rotating Air Mass

The rotating air mass constraint transforms the air node generalized coordinates and their associated generalized forces between a rotating subsystem and a nonrotating subsystem. As in the other air mass constraints, the air node generalized coordinates are not transformed out of the inertial frame of reference. The rotating air mass constraint is not available throught the user interface in GRASP.

Steady-State. Since only $\bar{U}_1^A$ and $\bar{\gamma}_{1r}^A$ are valid coordinates in the steady-state problem, and both are rotationally symmetric, they are treated in exactly the same manner as in the copy air mass constraint.

Dynamic. For a set of dynamically perturbed air node generalized coordinates, let

$$\check{q}_R = \begin{Bmatrix} \check{P}_{1R}^A \\ \check{\phi}_{12R}^A \\ \check{\phi}_{13R}^A \end{Bmatrix}; \qquad \delta q_R = \begin{Bmatrix} \delta P_{1R}^A \\ \delta\phi_{12R}^A \\ \delta\phi_{13R}^A \end{Bmatrix} \tag{7.1.14–1}$$

and

$$\check{q}_N = \begin{Bmatrix} \check{P}^A_{1N} \\ \check{\phi}^A_{12N} \\ \check{\phi}^A_{13N} \end{Bmatrix}; \qquad \delta q_N = \begin{Bmatrix} \delta P^A_{1N} \\ \delta\phi^A_{12N} \\ \delta\phi^A_{13N} \end{Bmatrix} \tag{7.1.14–2}$$

Then

$$\begin{aligned} \check{q}_R &= T\check{q}_N \\ \dot{\check{q}}_R &= T\dot{\check{q}}_N \\ \ddot{\check{q}}_R &= \dot{T}\dot{\check{q}}_N + T\ddot{\check{q}}_N \end{aligned} \tag{7.1.14–3}$$

The virtual work for the rotating subsystem is

$$\delta\mathcal{W} = \check{q}_R^T(M\ddot{\check{q}}_R + C\dot{\check{q}}_R) \tag{7.1.14–4}$$

which, when transformed into the nonrotating subsystem, becomes

$$\delta\mathcal{W} = \check{q}_N^T T^T \left[(MT\ddot{\check{q}}_N + (CT + M\dot{T})\dot{\check{q}}_N\right] \tag{7.1.14–5}$$

The C coefficient matrix for the transformed (nonrotating) subsystem therefore depends on the M and C coefficient matrices from the original (rotating) subsystem.

7.2. Composite Constraints

In general, a composite constraint is a constraint that is built up out of one or more of the primitive constraints that have been described in the previous sections. The bundling of primitive constraints into a single constraint is primarily done for the convenience of the user. There are many times that sets of constraints must be used together, and it makes sense to combine them internally. In the following sections, the composite constraints that have been constructed from the set of primitive constraints in GRASP will be described. All of the composite constraints are available from the GRASP user interface.

7.2.1. Aeroelastic Beam Connectivity

The purpose of the aeroelastic beam connectivity constraint is to provide a means for attaching an aeroelastic beam element to a GRASP model. The element subsystem for the aeroelastic beam consists of a frame of reference, a root node, a tip node, and an air node, all of which must be connected to their counterparts in the existing portion of the model. Therefore, the aeroelastic beam connectivity constraint must contain a fixed-frame constraint (for the frame), two structural node demotion constraints (for the root and tip nodes), and a copy air mass constraint (for the air node).

In the definition of the aeroelastic beam connectivity constraint, the position and orientation of the dependent, element root node R relative to an existing, independent node I_R must be provided. The position and orientation of I_R relative to its subsystem frame of reference (the superordinate frame S_R) is known from the definition of I_R. Therefore, the position and orientation of the dependent, root node relative to the superordinate frame can be calculated.

$$\begin{aligned} C^{RS_R} &= C^{RI_R} C^{I_R S_R} \\ R^{RS_R}_{S_R} &= C^{S_R I_R} R^{RI_R}_{I_R} + R^{I_R S_R}_{S_R} \end{aligned} \tag{7.2.1-1}$$

After locating the parent subsystem of the element subsystem in the system organization tree, the position and orientation of the parent frame relative to the superordinate frame can be calculated. Since the element frame and the element root node are coincident, the position and orientation of the element frame relative to its parent frame can then be determined.

$$\begin{aligned} C^{FP} &= C^{RP} = C^{RS_R} C^{S_R P} \\ R^{FP}_{P} &= R^{RP}_{P} = C^{PS_R}(R^{RS_R}_{S_R} + R^{S_R P}_{S_R}) \end{aligned} \tag{7.2.1-2}$$

With this information, the fixed-frame constraint can be defined. In addition, all of the position and orientation information is available to define the structural node demotion constraint for the element root node. In those cases where the superordinate frame is not the same as the parent frame, it is necessary to create copies of the independent and element root nodes in each of the subsystems leading to their nearest common ancestor. These nodes are chained together by a series of structural node demotion constraints.

The position of the element tip node T relative to the root node is defined as $\underline{\underline{R}}^{TR} = \ell \hat{\underline{b}}_3^R$, and the orientation C^{TR} is defined as an Euler rotation of magnitude $\theta'\ell$ about $\hat{\underline{b}}_3^R$. After the position and orientation of the root node relative to the independent tip node I_T has been calculated, the offset of the element tip node from the independent tip node can be determined.

$$\begin{aligned} R^{TI_T}_{I_T} &= C^{I_T S_T}(C^{S_T R} R^{TR}_{R} + R^{RS_T}_{S_T} - R^{I_T S_T}_{S_T}) \\ C^{TI_T} &= C^{TR} C^{RS_T} C^{S_T I_T} \end{aligned} \tag{7.2.1-3}$$

At this point, the structural node demotion constraint for the element tip node can be defined. In those cases where the superordinate frame is not the same as the parent frame, it is necessary to create copies of the independent and element tip nodes in each of the subsystems leading to their nearest common ancestor. This creates another chain of structural nodes, all connected together by structural node demotion constraints.

If the beam element is to be connected to an air mass element, the position and orientation of the element subsystem relative to the corresponding air node is calculated. Then, the copy air mass constraint is defined. In those cases where the air node is not defined in the parent frame, it is necessary to create copies of the independent and element air nodes in each of the subsystems leading to their nearest common ancestor. This creates a chain of air nodes, all connected together by copy air mass constraints. If, however, the beam element is not to be connected to an air mass element, the four nodal air mass degrees of freedom are constrained out using prescribed constraints.

7.2.2. Air Mass Connectivity

The purpose of the air mass connectivity constraint is to provide a means for attaching an air mass element to a GRASP model. The air mass element subsystem is unusual in that the frame serves only to establish the position and orientation of the element relative to the remainder of the model. Therefore, while the frame does exist and does need a frame constraint, it has no frame degrees of freedom. The air mass connectivity constraint is then made up of a fixed-frame constraint, a copy air mass constraint, and one or more prescribed constraints.

In the air mass connectivity constraint it is assumed that the independent air node I, the dependent (element) air node A, and the element frame F are all coincident.

$$\begin{aligned} C^{IA} &= C^{AF} = \Delta \\ \underline{R}^{IA} &= R^{AF} = \underline{0} \end{aligned} \tag{7.2.2-1}$$

After locating the parent of the element subsystem, the position and orientation of the parent frame relative to the superordinate frame can be calculated. Since the position and orientation of the independent air node relative to its subsystem frame (the superordinate frame S) is known, the position and orientation of the element frame relative to the parent frame can also be calculated.

$$\begin{aligned} C^{FP} &= C^{IP} = C^{IS}C^{SP} \\ R_P^{FP} &= R_P^{IP} = C_{PS}(R_S^{IS} + R_S^{SP}) \end{aligned} \tag{7.2.2-2}$$

These expressions provide the information necessary to define the fixed-frame constraint. In addition, the copy air mass constraint can be defined at this time. In those cases where the superordinate frame is not the same as the parent frame, it is necessary to create copies of the independent and element air nodes in each of the subsystems leading to their nearest common ancestor. These additional air nodes are also connected together using copy air mass constraints.

If the model containing the air mass connectivity constraint is to be used in a steady-state problem, the two cyclic air node degrees of freedom are meaningless. Therefore, they must be eliminated by defining two prescribed constraints in the *superordinate* subsystem. If, on the other hand, the model is to be used in an eigensolution, the gradient degree of freedom is meaningless. A single prescribed constraint is then defined in the superordinate subsystem.

7.2.3. Periodic Structure

The purpose of the periodic structure constraint is to provide a simple means for creating an axially symmetric structure. This is accomplished by replicating a single branch of the model at equal azimuth angles about an axis of symmetry. For this constraint, the parent subsystem represents the assembled periodic structure and the child subsystem represents a single component. The periodic structure constraint consists of one or more of the following: a periodic node demotion constraint, a periodic generalized coordinate constraint, and a periodic air mass constraint. Note that the periodic frame constraint must be defined separately.

When there are nodes in the component, periodic node demotion constraints are needed to transform them into the assembled structure. If the independent node corresponding to a dependent node (in the component) does not exist in the parent subsystem, a string of images of the independent node are created in the intervening subsystems and chained together with structural node demotion constraints. Similarly, if the dependent node does not exist in the child subsystem, a string of images of that node are created and chained together. Since the independent node (or its image) now exists in the parent subsystem and the dependent node (or its image) exists in the child subsystem, a periodic node demotion constraint can be defined.

One or more periodic generalized coordinate constraints are needed if there are generalized coordinates in the child subsystem. Similarly, one or more periodic air mass constraints are needed if there are air nodes in the component. A process identical to that used to connect structural nodes is used if the dependent and independent air nodes are not in the child and parent subsystems, respectively.

7.2.4. Rigid-body Connection

The purpose of the rigid-body connection constraint is to provide a simple means for connecting two nodes together rigidly. It is actually a special case of the screw constraint in which the translation and rotation degrees of freedom are both locked.

7.2.5. Rigid-body Mass Connectivity

The purpose of the rigid-body mass connectivity constraint is to provide a means for attaching a rigid-body mass element to a GRASP model. The element subsystem consists of a frame ot reference and a center-of-mass node, both of which must be connected to their counterparts in the existing portion of the model. Therefore, the rigid-body mass connectivity constraint is made up of a fixed-frame constraint and a structural node demotion constraint.

In the definition of the rigid-body mass connectivity constraint, the position and orientation of the dependent, element center-of-mass node C relative to an independent, existing node I is provided. The position and orientation of the independent node relative to its subsystem frame of reference (the superordinate frame S) is also known. Then, the position and orientation of the center-of-mass node relative to the superordinate frame can be written.

$$C^{CS} = C^{CI} C^{IS}$$
$$R_S^{CS} = C^{SI} R_I^{CI} + R_S^{IS} \tag{7.2.5–1}$$

After locating the parent subsystem of the element subsystem in the system organization tree, the position and orientation of the parent frame relative to the superordinate frame can be calculated. Since the element frame and the element center-of-mass node are coincident, the position and orientation of the element frame relative to its parent frame can then be determined.

$$C^{FP} = C^{CP} = C^{CS} C^{SP}$$
$$R_P^{FP} = R_P^{CP} = C^{PS}(R_S^{CS} + R_S^{SP}) \tag{7.2.5–2}$$

With this information, the fixed-frame constraint can be defined. In addition, all of the position and orientation information is available to define the structural node demotion constraint for the center-of-mass node. In those cases where the superordinate frame is not the same as the parent frame, it is necessary to create copies of the independent and element center-of-mass nodes in each of the subsystems leading to their nearest common ancestor. These nodes are chained together using a series of structural node demotion constraints.

7.2.6. Rotating Structure

The purpose of the rotating structure constraint is to provide a simple means for allowing one subsystem to rotate relative to another. For this constraint, the parent subsystem represents the nonrotating structure, while the child subsystem represents a rotating structure. The rotating structure constraint consists of one or more of the following: a rotating node demotion constraint, a rotating generalized coordinate constraint, and a rotating air mass constraint. Note that the rotating frame constraint must be defined separately.

When there are nodes in the rotating subsystem, rotating node demotion constraints are needed to transform them into the nonrotating subsystem. If the independent node corresponding to a dependent node (in the rotating subsystem) does not exist in the parent subsystem, a string of images of the independent node are created in the intervening subsystems and chained together with structural node demotion constraints. Similarly, if the dependent node does not exist in the rotating subsystem, a string of images of that node are created and chained together. Since the independent node (or its image) now exists in the parent subsystem and the dependent node (or its image) exists in the child subsystem, a rotating node demotion constraint can be defined.

One or more rotating generalized coordinate constraints are needed if there are generalized coordinates in the child subsystem. Similarly, one or more rotating air mass constraints are needed if there are air nodes in the component. A process identical to that used to connect structural nodes is used if the dependent and independent air nodes are not in the child and parent subsystems, respectively.

8. ELEMENTS

The GRASP element library currently contains three elements, the aeroelastic beam, the air mass, and the rigid-body mass.

8.1. Rigid-Body Mass

In GRASP, rigid bodies are modeled as being influenced only by inertial and gravitational forces.

For the purposes of modeling the motion of a rigid body in an inertial and (possibly) gravitational field, consider a rigid-body mass element B that has an inertia dyadic $\underline{I}$.

Steady-State. The rigid-body mass element (fig. 14) has a body-fixed node N and a frame of reference F. Node N is initially coincident with the deformed frame F' ($\underline{R}^{F'N} = \underline{0}$ and $C^{F'N} = \Delta$). The virtual work at the deformed node N' is

$$\delta\mathcal{W} = \underline{F}^{N'} \cdot {}^{I}\underline{\delta R}^{N'I} + \underline{M}^{N'} \cdot \underline{\delta\psi}^{N'I} \tag{8.1-1}$$

from which nodal forces and moments can be derived. The nodal virtual displacment and rotation variables for this element are $\delta R_N^{N'N}$ and $\delta\psi_N^{N'N}$, respectively.

The inertial virtual displacement and rotation of the deformed node N' are

$$\begin{aligned} {}^{I}\underline{\delta R}^{N'I} &= {}^{I}\underline{\delta R}^{F'I} + \underline{\delta\psi}^{F'I} \times \underline{R}^{N'N} + {}^{N}\underline{\delta R}^{N'N} \\ \underline{\delta\psi}^{N'I} &= \underline{\delta\psi}^{N'N} + \underline{\delta\psi}^{NF'} + \underline{\delta\psi}^{F'I} \end{aligned} \tag{8.1-2}$$

The force acting on the body at N' is

$$\underline{F}^{N'} = -m\underline{A}^{N'I} + m\underline{g} \tag{8.1-3}$$

where the inertial acceleration of N' is

$$\underline{A}^{N'I} = \underline{A}^{F'I} + \underline{\Omega}^{F'I} \times \left(\underline{\Omega}^{F'I} \times \underline{R}^{N'N}\right) \tag{8.1-4}$$

Substituting equation (8.1–4) into equation (8.1–3), and transforming from the body-fixed (N) coordinate system into the deformed-frame (F') system,

$$\begin{aligned} F_N^{N'} &= m\left(g_N - A_N^{F'I} - \tilde{\Omega}_N^{F'I}\tilde{\Omega}_N^{F'I}R_N^{N'N}\right) \\ &= m\left(g_{F'} - A_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_N^{N'N}\right) \end{aligned} \tag{8.1-5}$$

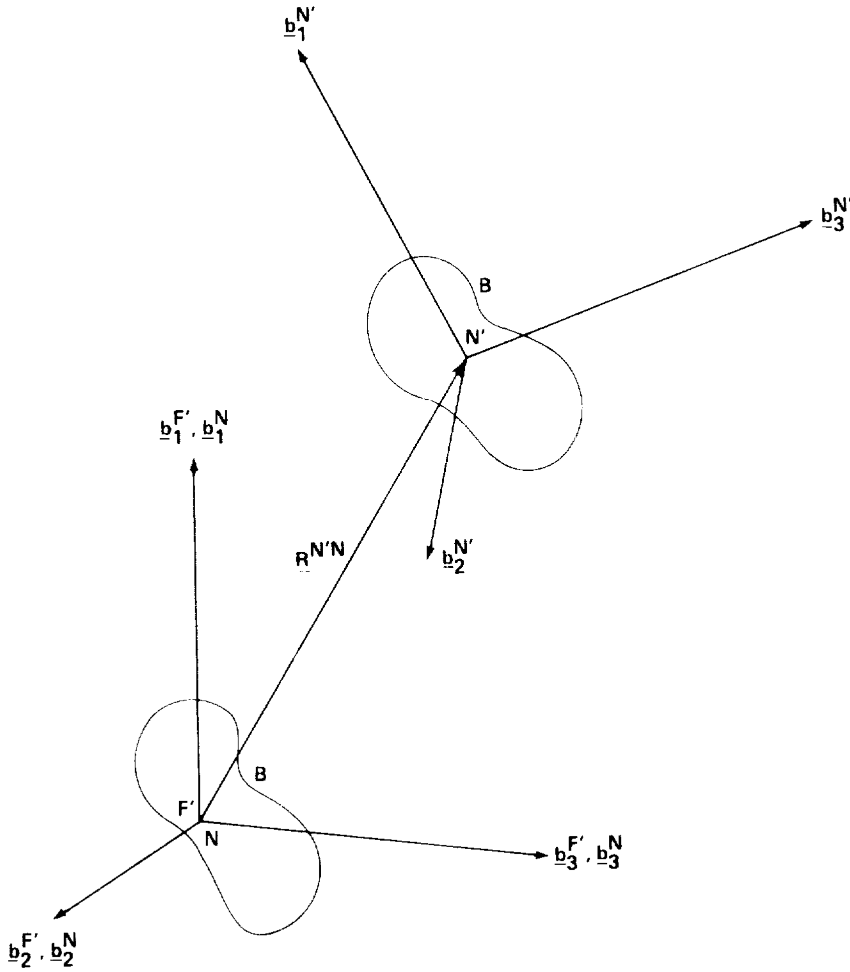


Figure 14. Rigid-body mass element.

The moment acting on the body at N' is

$$\underline{M}^{N'} = \underline{H}^{N'I} \times \underline{\Omega}^{N'I} \tag{8.1-6}$$

where the inertial angular momentum at N' is

$$\underline{H}^{N'I} = \underline{I} \cdot \underline{\Omega}^{N'I} \tag{8.1-7}$$

and the inertial angular velocity at N' is

$$\underline{\Omega}^{N'I} = \underline{\Omega}^{N'N} + \underline{\Omega}^{NF'} + \underline{\Omega}^{F'I} \tag{8.1-8}$$

Substituting,

$$M_N^{N'} = \tilde{H}_{F'}^{N'I} \Omega_{F'}^{N'I} \tag{8.1-9}$$

where $H_{F'}^{N'I} = I_{N'} \Omega_{N'}^{N'I}$.

The frame force and moment components can be derived in a similar manner. If the frame virtual displacements and rotations are $\delta R_{F'}^{F'F}$ and $\delta\psi_{F'}^{F'F}$, respectively,

$$\begin{aligned} F_{F'}^{F'} &= C^{F'N} F_N^{N'} \\ &= F_N^{N'} \\ M_{F'}^{F'} &= M_N^{N'} + \tilde{R}_N^{N'N} \tilde{F}_N^{N'} \\ &= M_N^{N'} - \tilde{F}_N^{N'} R_N^{N'N} \end{aligned} \tag{8.1–10}$$

Dynamics. Assuming that the rigid-body mass node is perturbed from its steady-state position (fig. 14), the virtual work at the node may be expressed as

$$\delta\mathcal{W} = \delta R_N^{N''I^T} F_N^{N''} + \delta\psi_N^{N''I^T} M_N^{N''} \tag{8.1–11}$$

where the force and moment are

$$\begin{aligned} F_N^{N''} &= -\,mA_N^{N''I} + mg_N \\ M_N^{N''} &= -\,I_N\alpha_N^{N''I} + \tilde{H}_N^{N''I}\Omega_N^{N''I} \end{aligned} \tag{8.1–12}$$

the angular momentum is

$$H_N^{N''I} = I_N\Omega_N^{N''I} \tag{8.1–13}$$

and the inertial angular velocity is

$$\Omega_N^{N''I} = \Omega_{F''}^{F''F'} + \Omega_N^{N''N'} + C^{F''F'}\Omega_{F'}^{F'I} \tag{8.1–14}$$

The virtual displacement is then

$$\delta R_N^{N''I} = \delta R_{F''}^{F''F'} + \delta R_N^{N''N'} + \widetilde{\delta\psi}_{F''}^{F''F'}\,[\tilde{R}_{F''}^{F''F'} + \tilde{R}_N^{N''N} + R_N^{N'N}] \tag{8.1–15}$$

and the virtual rotation is

$$\delta\psi_N^{N''I} = \delta\psi_{F''}^{F''F'} + \delta\psi_N^{N''N'} \tag{8.1–16}$$

The inertial acceleration of the node is

$$
\begin{aligned}
A_N^{N''I} =& A_{F'}^{F'I} + \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_N^{N'N} + \ddot{\check{R}}_{F''}^{F''F'} - \tilde{R}_N^{N'N}\ddot{\check{\theta}}_{F''}^{F''F'} + \ddot{\check{R}}_N^{N''N'} + 2\tilde{\Omega}_{F'}^{F'I}\dot{\check{R}}_{F''}^{F''F'} - \\
& [\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I} + (\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}} + \tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}]\dot{\check{\theta}}_{F''}^{F''F'} + \\
& 2\tilde{\Omega}_{F'}^{F'I}\dot{\check{R}}_N^{N''N'} + \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\check{R}_{F''}^{F''F'} + \\
& [\tilde{A}_{F'}^{F'I} - (\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}}\tilde{\Omega}_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}]\check{\theta}_{F''}^{F''F'} + \\
& \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\check{R}_N^{N''N'}
\end{aligned}
\tag{8.1–17}
$$

and the components of the gravitational acceleration are

$$
g_N = g_{F'} + \tilde{g}_{F'}\check{\theta}_{F''}^{F''F'} \tag{8.1–18}
$$

The components of the inertia dyadic in the nodal basis can be expressed in matrix form as

$$
I_N = (\Delta + \tilde{\check{\theta}}_N^{N''N'})C^{NN'}I_{N''}C^{N'N}(\Delta - \tilde{\check{\theta}}_N^{N''N'}) \tag{8.1–19}
$$

Finally, the angular acceleration is

$$
\alpha_N^{N''N'} = \ddot{\check{\theta}}_{F''}^{F''F'} + \ddot{\check{\theta}}_N^{N''N'} + \tilde{\Omega}_{F'}^{F'I}\dot{\check{\theta}}_{F''}^{F''F'} + \tilde{\Omega}_{F'}^{F'I}\dot{\check{\theta}}_N^{N''N'} \tag{8.1–20}
$$

The force and moment can then be obtained from the substitution of equations (8.1–12) through (8.1–20) into (8.1–11). When the virtual work is calculated, it consists of the same steady-state residuals $\{Q\}$ as were obtained in the previous section, in addition to the virtual work associated with the coefficient matrices $[M]$, $[C]$, and $[K]$.

$$
-\delta\mathcal{W} = \begin{Bmatrix} \delta R_{F''}^{F''F'} \\ \delta\psi_{F''}^{F''F'} \\ \delta R_N^{N''N'} \\ \delta\psi_N^{N''N'} \end{Bmatrix}^T \left\{ [M]\begin{Bmatrix} \ddot{\check{R}}_{F''}^{F''F'} \\ \ddot{\check{\theta}}_{F''}^{F''F'} \\ \ddot{\check{R}}_N^{N''N'} \\ \ddot{\check{\theta}}_N^{N''N'} \end{Bmatrix} + [C]\begin{Bmatrix} \dot{\check{R}}_{F''}^{F''F'} \\ \dot{\check{\theta}}_{F''}^{F''F'} \\ \dot{\check{R}}_N^{N''N'} \\ \dot{\check{\theta}}_N^{N''N'} \end{Bmatrix} + [K]\begin{Bmatrix} \check{R}_{F''}^{F''F'} \\ \check{\theta}_{F''}^{F''F'} \\ \check{R}_N^{N''N'} \\ \check{\theta}_N^{N''N'} \end{Bmatrix} - \{Q\} \right\} \tag{8.1–21}
$$

where the M coefficient matrix is defined to be

$\delta R_{F''}^{F''F'}$ row:

$$\ddot{R}_{F''}^{F''F'} \text{ column: } m$$

$$\ddot{\theta}_{F''}^{F''F'} \text{ column: } -m\tilde{R}_N^{N'N} \tag{8.1–22a}$$

$$\ddot{R}_N^{N''N'} \text{ column: } m$$

$$\ddot{\theta}_N^{N''N} \text{ column: } 0$$

$\delta\psi_{F''}^{F''F'}$ row:

$$\ddot{R}_{F''}^{F''F'} \text{ column: } m\tilde{R}_N^{N'N}$$

$$\ddot{\theta}_{F''}^{F''F'} \text{ column: } -m\tilde{R}_N^{N'N}\tilde{R}_N^{N'N} + C^{NN'}I_{N''}C^{N'N} \tag{8.1–22b}$$

$$\ddot{R}_N^{N''N'} \text{ column: } m\tilde{R}_N^{N'N}$$

$$\ddot{\theta}_N^{N''N} \text{ column: } C^{NN'}I_{N''}C^{N'N}$$

$\delta R_N^{N''N'}$ row:

$$\ddot{R}_{F''}^{F''F'} \text{ column: } m$$

$$\ddot{\theta}_{F''}^{F''F'} \text{ column: } -m\tilde{R}_N^{N'N} \tag{8.1–22c}$$

$$\ddot{R}_N^{N''N'} \text{ column: } m$$

$$\ddot{\theta}_N^{N''N} \text{ column: } 0$$

$\delta\psi_N^{N''N'}$ row:

$$\ddot{R}_{F''}^{F''F'} \text{ column: } 0$$

$$\ddot{\theta}_{F''}^{F''F'} \text{ column: } C^{NN'}I_{N''}C^{N'N} \tag{8.1-22d}$$

$$\ddot{R}_N^{N''N'} \text{ column: } 0$$

$$\ddot{\theta}_N^{N''N} \text{ column: } C^{NN'}I_{N''}C^{N'N}$$

the C coefficient matrix is defined to be

$\delta R_{F''}^{F''F'}$ row:

$$\dot{R}_{F''}^{F''F'} \text{ column: } 2m\tilde{\Omega}_{F'}^{F'I}$$

$$\dot{\theta}_{F''}^{F''F'} \text{ column: } -2m\tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N} \tag{8.1-23a}$$

$$\dot{R}_N^{N''N'} \text{ column: } 2m\tilde{\Omega}_{F'}^{F'I}$$

$$\dot{\theta}_N^{N''N} \text{ column: } 0$$

$\delta\psi_{F''}^{F''F'}$ row:

$$\dot{R}_{F''}^{F''F'} \text{ column: } 2m\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}$$

$$\begin{aligned}\dot{\theta}_{F''}^{F''F'} \text{ column: } & -2m\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N} - (C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\sim} + \\ & C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} + \tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N}\end{aligned} \tag{8.1-23b}$$

$$\dot{R}_N^{N''N'} \text{ column: } 2m\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}$$

$$\begin{aligned}\dot{\theta}_N^{N''N} \text{ column: } & \tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N} - (C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\sim} + \\ & C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I}\end{aligned}$$

$\delta R_N^{N''N'}$ row:

$\dot{\breve{R}}_{F''}^{F''F'}$ column: $2m\tilde{\Omega}_{F'}^{F'I}$

$\dot{\breve{\theta}}_{F''}^{F''F'}$ column: $-2m\tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}$ (8.1–23c)

$\dot{\breve{R}}_N^{N''N'}$ column: $2m\tilde{\Omega}_{F'}^{F'I}$

$\dot{\breve{\theta}}_N^{N''N}$ column: 0

$\delta\psi_N^{N''N'}$ row:

$\dot{\breve{R}}_{F''}^{F''F'}$ column: 0

$\dot{\breve{\theta}}_{F''}^{F''F'}$ column: $C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} + \tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N} -$

$(C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})\tilde{}$ (8.1–23d)

$\dot{\breve{R}}_N^{N''N'}$ column: 0

$\dot{\breve{\theta}}_N^{N''N}$ column: $\tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N} - (C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})\tilde{} +$

$C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I}$

and the K coefficient matrix is defined to be

$\delta R_{F''}^{F''F'}$ row:

$\breve{R}_{F''}^{F''F'}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\breve{\theta}_{F''}^{F''F'}$ column: $m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} - (\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})\tilde{}\,\tilde{\Omega}_{F'}^{F'I} -$

$\tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}]$ (8.1–24a)

$\breve{R}_N^{N''N'}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\breve{\theta}_N^{N''N}$ column: 0

$\delta\psi_{F''}^{F''F'}$ row:

$\check{R}_{F''}^{F''F'}$ column: $-m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}} - \tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}]$

$$\check{\theta}_{F''}^{F''F'} \text{ column: } m\tilde{R}_N^{N'N}[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} - (\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}}\tilde{\Omega}_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}] + \tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} - (C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\tilde{}}\tilde{\Omega}_{F'}^{F'I} \tag{8.1–24b}$$

$\check{R}_N^{N''N'}$ column: $-m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}} - \tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}]$

$\check{\theta}_N^{N''N'}$ column: $\tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}(C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\tilde{}}$

$\delta R_N^{N''N'}$ row:

$\check{R}_{F''}^{F''F'}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$$\check{\theta}_{F''}^{F''F'} \text{ column: } m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} - (\tilde{\Omega}_{F'}^{F'I}R_N^{N'N})^{\tilde{}}\tilde{\Omega}_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}\tilde{R}_N^{N'N}\tilde{\Omega}_{F'}^{F'I}] \tag{8.1–24c}$$

$\check{R}_N^{N''N'}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\check{\theta}_N^{N''N}$ column: 0

$\delta\psi_N^{N''N'}$ row:

$\check{R}_{F''}^{F''F'}$ column: 0

$$\check{\theta}_{F''}^{F''F'} \text{ column: } \tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} - (C^{NN'}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\tilde{}}\tilde{\Omega}_{F'}^{F'I} \tag{8.1–24d}$$

$\check{R}_N^{N''N'}$ column: 0

$\check{\theta}_N^{N''N}$ column: $\tilde{\Omega}_{F'}^{F'I}C^{NN'}I_{N''}C^{N'N}\tilde{\Omega}_{F'}^{F'I} - \tilde{\Omega}_{F'}^{F'I}(C^{N'N}I_{N''}C^{N'N}\Omega_{F'}^{F'I})^{\tilde{}}$

8.2. Air Mass

The air mass element models the momentum flow of air through a helicopter rotor disk. For this element, the rotor is assumed to be an actuator disk, and the flow field a cylindrical region surrounding the disk (fig. 15). The state vector for the air mass element is made up of the generalized coordinates for a single air node.

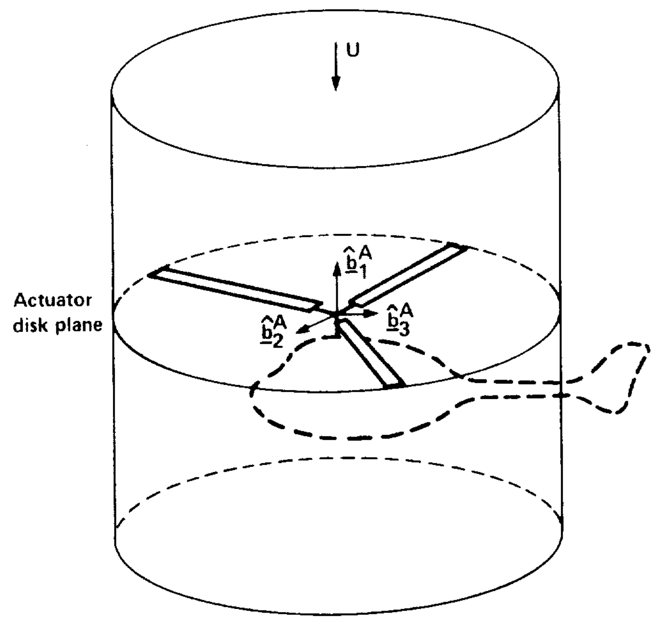


Figure 15. Air mass element flow field.

Steady-State. Consider the air flowing steadily through a rotor. Reference 22 shows that the thrust dT acting on a differential annulus of the rotor (fig. 16) is related to the induced velocity v via a momentum balance such that

$$dT = 4\pi\rho_a rv\,|V+v|\;dr \tag{8.2-1}$$

where r is the radial coordinate of the rotor and V is the velocity of the rotor relative to still air (V is positive when the rotor is moving in the positive x_1 direction). The use of the absolute value of the sum of the velocities $V+v$ assures that the differential thrust dT has the proper sign under all operating conditions. Integrating, the total rotor thrust is

$$T = 4\pi\rho_a \int_\epsilon^R v\,|V+v|\,r\,dr \tag{8.2-2}$$

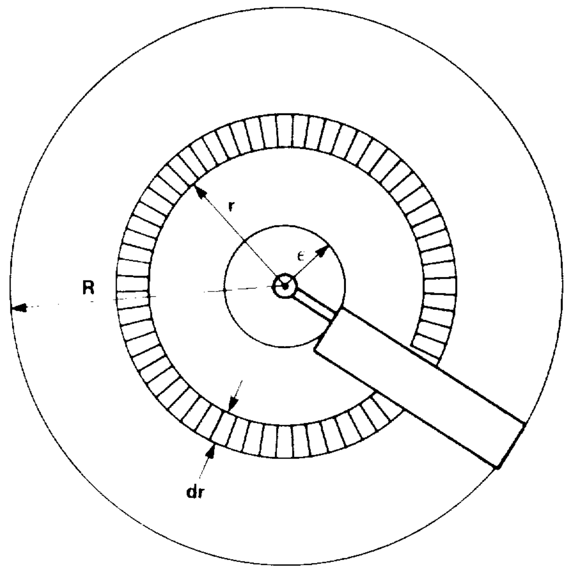


Figure 16. Air mass element differential annulus.

The virtual work done by the thrust on the air is

$$\delta\mathcal{W} = 4\pi\rho_a \int_\epsilon^R v\delta R\,|V + v|\,r\,dr \tag{8.2-3}$$

where δR is the virtual displacement of the air. The right-hand side of the equation for the virtual work can be discretized by letting $v = \bar{U}_1^A + \bar{\gamma}_{1r}^A r$ and $\delta R = \delta P_1^A + r\delta\phi_{1r}^A$. Then

$$\begin{aligned}
\delta\mathcal{W} =& 4\pi\rho_a \int_\epsilon^R (\bar{U}_1^A + \bar{\gamma}_{1r}^A r)\,\left|V + \bar{U}_1^A + \bar{\gamma}_{1r}^A r\right|\,(\delta P_1^A + r\delta\phi_{1r}^A) r\,dr \\
=& \delta P_1^A 4\pi\rho_a \int_\epsilon^R (\bar{U}_1^A + \bar{\gamma}_{1r}^A r)\,\left|V + \bar{U}_1^A + \bar{\gamma}_{1r}^A r\right|\,r\,dr + \\
& \delta\phi_{1r}^A 4\pi\rho_a \int_\epsilon^R (\bar{U}_1^A + \bar{\gamma}_{1r}^A r)\,\left|V + \bar{U}_1^A + \bar{\gamma}_{1r}^A r\right|\,r^2\,dr
\end{aligned} \tag{8.2-4}$$

Note that while the coefficient of δP_1^A in equation (8.2-4) is equal to the rotor thrust, the coefficient of $\delta\phi_{1r}^A$ has the dimensions of moment but no clear physical significance.

The contributions to the $\delta\mathcal{W}$ (applied loads) side of equation (8.2-4) are determined from blade element theory, and are obtained by summing the contributions from each of the aeroelastic beam elements that make up the rotor.

Dynamic. Simple models for the induced inflow dynamics, such as the one introduced in reference 23, have been shown to improve the accuracy of mathematical models of helicopter rotor dynamics. The velocity of the air mass is idealized as consisting of a spatially and temporally uniform freestream velocity V, which is augmented within a cylindrical region by the steady-state inflow $\bar{U}_1^A$ induced by the rotor steady-state thrust, and by the infinitesimal dynamic perturbations to the inflow induced by dynamic perturbations to the thrust, roll moment, and pitch moment of the rotor.

For a differential annulus of a rotor disk through which air is flowing unsteadily, the momentum balance can be expressed as a system of first-order, integro-differential equations.

$$\delta \mathcal{W} = \int_{\epsilon}^{R} \int_{0}^{2\pi} 2\rho_a v\,|V + v|\,\delta P\,d\psi\,dr + \iiint\limits_{V_{\text{eff}}} \rho_a \dot{v}\,\delta P\,dV_{\text{eff}} \tag{8.2-5}$$

In order to intermix the air mass terms with the structural generalized coordinates in a single set of second-order equations, the perturbed air mass generalized velocities are expressed as the time derivatives of generalized coordinates.

$$v = \bar{U}_1^A + \bar{\gamma}_{1r}^A r + \dot{P}_1^A - \dot{\phi}_{12}^A r\sin\psi + \dot{\phi}_{13}^A r\cos\psi \tag{8.2-6}$$

where $\dot{P}_1^A$ is the vertical component of the perturbation of the induced inflow velocity component at the center of flow, $\dot{\phi}_{12}^A$ and $\dot{\phi}_{13}^A$ are the flow gradients at the center of flow in the x_2 and x_3 directions, respectively, and ψ is the azimuthal coordinate of the rotor, measured as a right handed rotation about the x_1 axis from the x_3 axis. The flow direction is assumed to be positive along the x_1 axis.

In addition, virtual displacement of the air inside the cylindrical flow field is assumed to be

$$\delta P = \delta P_1^A - \delta\phi_{12}^A r\sin\psi + \delta\phi_{13}^A r\cos\psi \tag{8.2-7}$$

where δP_1^A is the vertical virtual displacement of the air at the center of flow, and $\delta\phi_{12}^A$ and $\delta\phi_{13}^A$ are the cyclic virtual displacement components at the center of flow.

Now, consider the expression $v|V + v|$, where $v = \bar{v} + \check{v}(t)$. In seeking the linearized perturbation of such an expression, if $V + \bar{v} = 0$ then $v|V + v| = (\bar{v} + \check{v})|\check{v}|$. Since there is no *linear* contribution in this expression, it may be assumed to be zero. Now, define

$$\text{sgn}(a) = \begin{cases} +1 & \text{for } a > 0 \\ 0 & \text{for } a = 0 \\ -1 & \text{for } a < 0 \end{cases} \tag{8.2-8}$$

Then,

$$v\,|V + v| = [(V + \bar{v})\bar{v} + (V + 2\bar{v})\check{v}]\text{sgn}(V + \bar{v}) \tag{8.2-9}$$

Since only the linear perturbation dynamics are pertinent to this problem, the contribution of the change in momentum per unit area term from equation (8.2–5) is

$$\int_\epsilon^R \int_0^{2\pi} 2\rho_a r(V + 2\bar{v})\mathrm{sgn}(V + \bar{v})(\dot{P}_1^A - \dot{\phi}_{12}^A r\sin\psi + \dot{\phi}_{13}^A r\cos\psi)(\delta P_1^A + \delta\phi_{12}^A r\sin\psi + \delta\phi_{13}^A r\cos\psi)\, d\psi\, dr =$$

$$\int_\epsilon^R 4\pi\rho_a r(V + 2\bar{v})\mathrm{sgn}(V + \bar{v})\left[\dot{P}_1^A \delta P_1^A + \frac{r^2}{2}(\dot{\phi}_{12}^A \delta\phi_{12}^A + \dot{\phi}_{13}^A \delta\phi_{13}^A)\right] dr = \tag{8.2–10}$$

$$\dot{P}_1^A \delta P_1^A\, 4\pi\rho_a \int_\epsilon^R (V + 2\bar{v})\mathrm{sgn}(V + \bar{v}) r\, dr +$$
$$2\pi\rho_a(\dot{\phi}_{12}^A \delta\phi_{12}^A + \dot{\phi}_{13}^A \delta\phi_{13}^A) \int_\epsilon^R (V + 2\bar{v})\mathrm{sgn}(V + \bar{v}) r^3\, dr$$

The contribution of the volume term from equation (8.2–5) is the virtual mass-virtual inertia effect as calculated in reference 23.

$$\frac{8\rho_a}{3}(R^3 - \epsilon^3)\ddot{P}_1^A \delta P_1^A + \frac{16\rho_a}{45}(R^5 - \epsilon^5)(\ddot{\phi}_{12}^A \delta\phi_{12}^A + \ddot{\phi}_{13}^A \delta\phi_{13}^A) \tag{8.2–11}$$

From this development, the coefficient matrix for the generalized accelerations may be defined to be

$$M = \frac{8\rho_a R^3}{3}\begin{bmatrix} 1 - (\frac{\epsilon}{R})^3 & 0 & 0 \\ 0 & \frac{2}{15}R^2(1 - \frac{\epsilon^5}{R^5}) & 0 \\ 0 & 0 & \frac{2}{15}R^2(1 - \frac{\epsilon^5}{R^5}) \end{bmatrix} \tag{8.2–12}$$

and the coefficient matrix for the generalized velocities may be defined to be

$$C = 2\pi\rho_a \begin{bmatrix} 2\int_\epsilon^R gr\, dr & 0 & 0 \\ 0 & \int_\epsilon^R gr^3\, dr & 0 \\ 0 & 0 & \int_\epsilon^R gr^3\, dr \end{bmatrix} \tag{8.2–13}$$

where $g = (V + 2\bar{v})\mathrm{sgn}(V + \bar{v})$ and $\bar{v} = \bar{U}_1^A + r\bar{\gamma}_{1r}^A$.

To eliminate all periodic coefficients in the equations of motion, and to assure the existence of a steady-state solution, the air mass element degrees of freedom must be inertial. In addition, the flow direction must be coincident with the steady-state spin axis of the rotor and the gravity vector, if gravity is included in the model.

8.3. Aeroelastic Beam

The aeroelastic beam element is designed to model a beam undergoing small strains and large rotations, and for which shear deformation and warping rigidity may be ignored. A model of this type is developed in reference 24, which formulates the nonlinear beam kinematics and applies them to the dynamic analysis of a pretwisted, rotating beam element. The kinematic relations that describe the orientation of the cross section during deformation are simplified by systematically ignoring the extensional strain compared to unity. The only restriction on the magnitudes of the orientation angles used in describing the cross section orientation is that they remain less than 90° . All influences of warp other than warping rigidity are retained. The beam cross section is not allowed to deform in its own plane. The static equations from reference 24 are used without simplification; the dynamical equations are linearized relative to static equilibrium. One noteworthy feature of the derivation of the equations in reference 24 is that the common practice of using an ordering scheme has been abandoned. Thus, all higher-order terms (within the assumptions above) are retained.

In the following sections, the details of the derivation of the equations for the aeroelastic beam element are presented. First, a synopsis of the basis under which the governing equations of the beam are derived is given. Next, the equations of motion for the beam element are derived in terms of the frame, air, bending, extension, and torsion degrees of freedom. These equations include contributions from beam elasticity, inertial and gravitational forces, and aerodynamic forces. Then, the discretization of the beam degrees of freedom is presented to show how the beam displacements are transformed into the beam generalized coordinates. The final two sections describe the transformation from root and tip node degrees of freedom to beam generalized coordinates, and the transformation from beam generalized forces to root and tip node forces and moments.

8.3.1. Basis of the Governing Equations

Consider the beam element shown in figure 17. The element frame is denoted by F, and the root and tip nodes are denoted by R and T, respectively. The addition of primes and double-primes signifies the static and perturbed dynamic states, respectively. It should be noted that F'' and R are coincident with each other and that their coordinates line up with the principal axes of the root end of the undeformed beam element with the undeformed beam lying along $\hat{\underline{b}}_3^R$. Similarly, T is at the tip of the undeformed beam element and its coordinate directions lie along the principal axes for the tip cross section. The air node, denoted by A, must be included in the problem so that the influence of aerodynamic forces on the air node generalized forces can be determined and so that the influence of perturbations of the air node generalized coordinates can be determined for the generalized coordinates of both the beam and the air node. The position and orientation of A are inertially fixed.

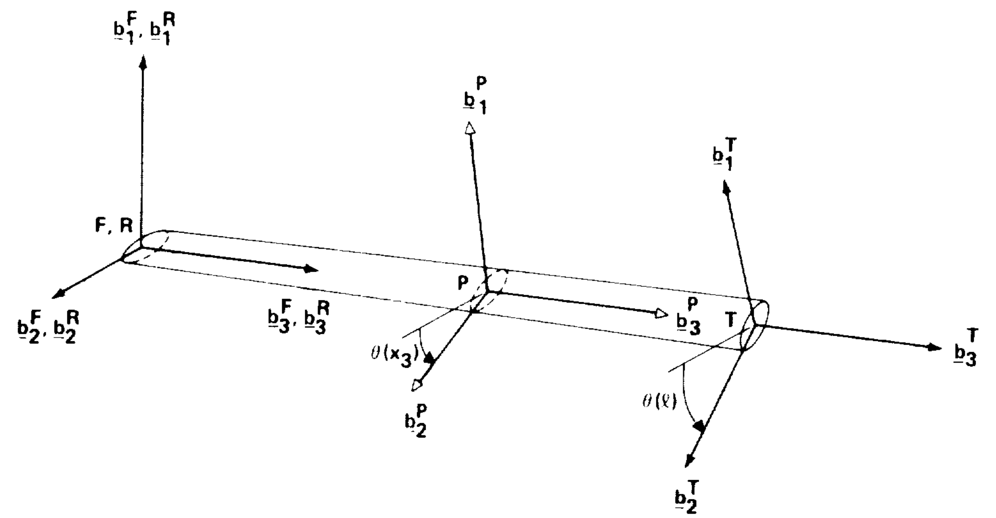


Figure 17. Aeroelastic beam element (undeformed with pretwist).

Interior displacements of the beam are represented by four functions of the axial coordinate x_3: u_i and θ_3. Bending is described by u_1 and u_2, axial displacement by u_3, and torsion by θ_3. These functions are discretized in terms of standard cubic and linear polynomials so that the generalized coordinates at the root and tip of the beam can be related to the nodal displacements and rotations. In addition, however, there are also generalized coordinates, called internal degrees of freedom, associated with higher-order polynomials.

8.3.2. Beam Elasticity

The derivation of the equations to calculate contributions of the elastic deformations of a straight, pretwisted beam follows the derivation presented in reference 24.

Steady-State. The elastic beam equations for a beam in equilibrium are derived from the variation of the strain energy

$$\delta U = \int_0^\ell \left(G\epsilon_{3\alpha}\delta\epsilon_{3\alpha} + E\epsilon_{33}\delta\epsilon_{33} \right) dx_3 \tag{8.3.2-1}$$

where

$$\begin{aligned} \epsilon_{31} &= (\lambda_1 - \xi_2)(\kappa_3 - \theta') \\ \epsilon_{32} &= (\lambda_2 + \xi_1)(\kappa_3 - \theta') \\ \epsilon_{33} &= \bar{\epsilon}_{33} + \xi_2\kappa_1 - \xi_1\kappa_2 + \frac{1}{2}\left({\xi_1}^2 + {\xi_2}^2\right)(\kappa_3 - \theta')^2 + (\xi_2\lambda_1 - \xi_1\lambda_2)(\kappa_3 - \theta')\,\theta' \end{aligned} \tag{8.3.2-2}$$

where $\theta(x_3)$ is the pretwist angle (fig. 17), with $\theta(0) = 0$, and $(\ )' = d(\ )/dx_3$. The generalized strains are

$$
\begin{aligned}
\bar{\epsilon}_{33} &= s' - 1 \\
s'^2 &= u_1'^2 + u_2'^2 + (1 + u_3')^2 \\
\kappa_1 &= (C_{12} u_1'' - C_{11} u_2'')/C_{33} \\
\kappa_2 &= (C_{22} u_1'' - C_{21} u_2'')/C_{33} \\
\kappa_3 &= \theta_3' - \left(\frac{C_{31}{}^2 C_{32} u_1''}{1 - C_{31}{}^2} + C_{31} u_2'' \right) / C_{33}
\end{aligned}
\tag{8.3.2–3}
$$

where $C = C^{P'F'}$, the direction cosines of local principal axes relative to the static frame orientation. The elements of C may be expressed in terms of Tait-Bryan orientation angles (orientation angles of type body-three: 1-2-3) as

$$
\begin{aligned}
C_{11} &= c_2 c_3 \\
C_{12} &= s_2 c_1 + s_1 s_2 c_3 \\
C_{13} &= s_3 s_1 - c_1 s_2 c_3
\end{aligned}
\tag{8.3.2–4a}
$$

$$
\begin{aligned}
C_{21} &= -c_2 s_3 \\
C_{22} &= c_3 c_1 - s_1 s_2 s_3 \\
C_{23} &= c_3 s_1 + c_1 s_2 s_3 \\
C_{31} &= u_1' \\
C_{32} &= u_2' \\
C_{33} &= (1 - u_1'^2 - u_2'^2)^{\frac{1}{2}}
\end{aligned}
\tag{8.3.2–4b}
$$

where

$$
\begin{aligned}
s_1 &= -\,u_2'(1 - {u_1'}^2)^{-\frac{1}{2}} \\
s_2 &= u_1' \\
s_3 &= \sin\theta_3 \\
c_1 &= (1 - {s_1}^2)^{\frac{1}{2}} \\
c_2 &= (1 - {s_2}^2)^{\frac{1}{2}} \\
c_3 &= \cos\theta_3
\end{aligned}
\tag{8.3.2–5}
$$

After integrating over the cross-sectional area, the variation of the strain energy is obtained in terms of the stress resultants F_3, M_1, M_2, and M_3.

$$
\delta U = \int_0^{\ell} (F_3\delta s' + M_1\delta\kappa_1 + M_2\delta\kappa_2 + M_3\delta\kappa_3)\,dx_3 \tag{8.3.2–6}
$$

where

$$
\begin{aligned}
F_3 &= E_0\bar{\epsilon}_{33} + E_2\kappa_1 - E_1\kappa_2 + \frac{I_3}{2}{\tau_3}^2 + D_0\theta'\tau_3 \\
M_1 &= E_2\bar{\epsilon}_{33} + I_1\kappa_1 + \frac{B_2{\tau_3}^2}{2} + D_2\theta'\tau_3 \\
M_2 &= -\,E_1\bar{\epsilon}_{33} + I_2\kappa_2 - \frac{B_1{\tau_3}^2}{2} - D_1\theta'\tau_3 \\
M_3 &= \left(J + I_3\bar{\epsilon}_{33} + B_2\kappa_1 - B_1\kappa_2 + \frac{B_3{\tau_3}^2}{2} + \frac{3D_3\theta'}{2}\tau_3 + D_4{\theta'}^2\right)\tau_3 + \\
&\quad (D_0\bar{\epsilon}_{33} + D_2\kappa_1 - D_1\kappa_2)\theta'
\end{aligned}
\tag{8.3.2–7}
$$

C2

where $\tau_3 = \kappa_3 - \theta'$ and the section integrals are defined as

$$
\begin{aligned}
E_0 &= \iint E dA \\
E_1 &= \iint E\xi_1 dA \\
E_2 &= \iint E\xi_2 dA \\
I_1 &= \iint E{\xi_2}^2 dA \\
I_2 &= \iint E{\xi_1}^2 dA \\
I_3 &= I_1 + I_2 \\
J &= \iint G[(\lambda_1 - \xi_2)^2 + (\lambda_2 + \xi_1)^2] dA \\
B_1 &= \iint E\xi_1({\xi_1}^2 + {\xi_2}^2) dA \\
B_2 &= \iint E\xi_2({\xi_1}^2 + {\xi_2}^2) dA \\
B_3 &= \iint E({\xi_1}^2 + {\xi_2}^2)^2 dA \\
D_0 &= \iint E(\xi_2\lambda_1 - \xi_1\lambda_2) dA \\
D_1 &= \iint E\xi_1(\xi_2\lambda_1 - \xi_1\lambda_2) dA \\
D_2 &= \iint E\xi_2(\xi_2\lambda_1 - \xi_1\lambda_2) dA \\
D_3 &= \iint E({\xi_1}^2 + {\xi_2}^2)(\xi_2\lambda_1 - \xi_1\lambda_2) dA \\
D_4 &= \iint E(\xi_2\lambda_1 - \xi_1\lambda_2)^2 dA
\end{aligned}
\tag{8.3.2–7b}
$$

Here, E_0 is the axial rigidity; E_1 and E_2 are the first flexual moments about the local ξ_1 and ξ_2 axes, respectively; I_1 and I_2 are second moments (*bending rigidities*) about the local ξ_1 and ξ_2 axes, respectively; and J is the Saint-Venant torsional rigidity.

The variations of generalized strains can be expressed in terms of the fundamental variables as

$$\begin{aligned} \delta s' &= \frac{\partial s'}{\partial u_i}\delta u_i \\ \delta \kappa_i &= \frac{\partial \kappa_i}{\partial u''_\alpha}\delta u''_\alpha + \delta_{3i}\delta\theta'_3 + \frac{\partial \kappa_i}{\partial u'_\alpha}\delta u'_\alpha + \epsilon_{3i\alpha}\kappa_\alpha\delta\theta_3 \end{aligned} \tag{8.3.2–8}$$

and the variation of strain energy as

$$\begin{aligned} \delta U = \int_0^\ell \Bigg[& \left(F_3\frac{\partial s'}{\partial u'_\alpha} + M_i\frac{\partial \kappa_i}{\partial u'_\alpha}\right)\delta u'_\alpha + F_3\frac{\partial s'}{\partial u'_\alpha}\delta u'_3 + \epsilon_{\alpha\beta3}M_\alpha\kappa_\beta\delta\theta_3 + \\ & M_i\frac{\partial \kappa_i}{\partial u''_\alpha}\delta u''_\alpha + M_3\delta\theta'_3\Bigg]\, dx_3 \end{aligned} \tag{8.3.2–9}$$

where

$$\begin{aligned} \frac{\partial s'}{\partial u'_i} &= \frac{\delta_{3i} + u'_i}{s'} \\ \frac{\partial \kappa_1}{\partial u''_1} &= \frac{C_{12}}{C_{33}} \\ \frac{\partial \kappa_1}{\partial u''_2} &= -\frac{C_{11}}{C_{33}} \\ \frac{\partial \kappa_2}{\partial u''_1} &= \frac{C_{22}}{C_{33}} \\ \frac{\partial \kappa_2}{\partial u''_2} &= -\frac{C_{21}}{C_{33}} \\ \frac{\partial \kappa_3}{\partial u''_1} &= -\frac{{C_{31}}^2 C_{32}}{C_{33}(1 - {C_{31}}^2)} \\ \frac{\partial \kappa_3}{\partial u''_2} &= -\frac{C_{31}}{C_{33}} \\ \frac{\partial \kappa_1}{\partial u'_1} &= \frac{u''_1}{{C_{33}}^2}\left(-\frac{C_{22}C_{32}}{1 - {C_{31}}^2} + \frac{C_{12}C_{31}}{C_{33}}\right) - \frac{u''_2 C_{11}C_{31}C_{32}^2}{{C_{33}}^3(1 - {C_{31}}^2)} \end{aligned} \tag{8.3.2–10a}$$

$$\frac{\partial \kappa_2}{\partial u_1'} = \frac{u_1''}{C_{33}^2}\left(\frac{C_{12}C_{32}}{1-C_{31}^2} + \frac{C_{22}C_{31}}{C_{33}}\right) - \frac{u_2''C_{21}C_{31}C_{32}^2}{C_{33}^3(1-C_{31}^2)}$$

$$\frac{\partial \kappa_3}{\partial u_1'} = \frac{-C_{31}C_{32}u_1''}{C_{33}^3(1-C_{31}^2)^2}(2C_{33}^2 + C_{31}^2 - C_{31}^4) - \frac{u_2''(1-C_{32}^2)}{C_{33}^3} \qquad (8.3.2\text{–}10b)$$

$$\frac{\partial \kappa_i}{\partial u_2'} = \frac{-C_{i1}}{C_{33}^3}(u_1''C_{31} + u_2''C_{32})$$

Dynamic. Since the explicit, analytical derivation of the elastic stiffness matrix would be exceptionally tedious and lengthy, GRASP generates it numerically. This is accomplished by taking the Jacobian of the function that calculates the steady-state elastic loads. Because of the necessity of calculating an accurate stiffness matrix, the algorithm used to calculate the Jacobian uses a two-point central difference scheme plus a generalized formulation of Richardson extrapolation.

8.3.3. Beam Inertial and Gravitational Forces

The generalized forces resulting from motion of the aeroelastic beam relative to an inertial frame are also determined following reference 24. Warping dynamics are again ignored. The derivation is based on the work done by inertial and gravitational forces moving through a virtual displacement. The work is calculated by taking the scalar product of the gravity minus the acceleration of a generic point P (fig. 17)in the beam interior (ref. 24, eq. 32), with the virtual displacement of the same point (ref. 24, eq. 34), then integrating the result over the beam length.

Steady-State. For a beam element in equilibrium, the virtual work is

$$\begin{aligned}\delta\mathcal{W} = &\int_0^\ell (\delta u^T F_{F'}^{P'} + \delta\psi_{P'}^T M_{P'}^{P'})dx_3 + \\ &\delta u_{F'}^T \int_0^\ell F_{F'}^{P'}dx_3 + \delta\psi_{F'}^T \int_0^\ell (M_{F'}^{P'} + \tilde{R}_{F'}^{P'F'}F_{F'}^{P'})dx_3\end{aligned} \qquad (8.3.3\text{–}1)$$

where

$$\begin{aligned}
F^{P'}_{F'} &= m(g_{F'} - A^{F'I}_{F'}) - \tilde{\Omega}^{F'I}_{F'}\tilde{\Omega}^{F'I}_{F'}R^{MF'}_{F'} \\
M^{P'}_{P'1} &= m_2 G_{P'3} - i_1 \Omega^{F'I}_{P'2}\Omega^{F'I}_{P'3} \\
M^{P'}_{P'2} &= -m_1 G_{P'3} + i_2 \Omega^{F'I}_{P'1}\Omega^{F'I}_{P'3} \\
M^{P'}_{P'3} &= m_1 G_{P'2} - m_2 G_{P'1} + (i_1 - i_2)\Omega^{F'I}_{P'1}\Omega^{F'I}_{P'2} \\
R^{MF'}_{F'i} &= m(x_3\delta_{3i} + u_i) + m_1 C_{1i} + m_2 C_{2i} \\
G_{F'} &= g_{F'} - A^{F'I}_{F'} - \tilde{\Omega}^{F'I}_{F'}\tilde{\Omega}^{F'I}_{F'}R^{P'F'}_{F'}
\end{aligned} \tag{8.3.3-2}$$

and

$$u = R^{P'P}_{F'}$$

$$\delta u = \delta R^{P'P}_{F'}$$

$$\delta\psi_{P'i} = \delta_{3i}\delta\theta_3 + \frac{\partial\kappa_i}{\partial u''_\alpha}\delta u'_\alpha = \delta\psi^{P'F'}_{P'i} \tag{8.3.3-3}$$

$$\delta u_{F'} = \delta R^{F'F}_{F'}$$

$$\delta\psi_{F'} = \delta\psi^{F'F}_{F'}$$

The section integrals are

$$\begin{aligned}
m &= \iint \rho_s dA \\
m_1 &= \iint \rho_s \xi_1 dA \\
m_2 &= \iint \rho_s \xi_2 dA \\
i_1 &= \iint \rho_s \xi_2^2 dA \\
i_2 &= \iint \rho_s \xi_1^2 dA
\end{aligned} \tag{8.3.3-4}$$

Dynamic. For a linearized perturbation about the equilibrium solution, it is possible to express the equations of motion in a matrix format such that the virtual work per unit beam length is given by

$$-\delta W = \begin{Bmatrix} \delta u_{F''i} \\ \delta\psi_{F''i} \\ \delta u_i(x_3) \\ \delta u'_\alpha(x_3) \\ \delta\theta_3(x_3) \end{Bmatrix}^T \left\{ [M]\begin{Bmatrix} \ddot{\breve{u}}_{F''i} \\ \ddot{\breve{\theta}}_{F''i} \\ \ddot{\breve{u}}_i(x_3) \\ \ddot{\breve{u}}'_\alpha(x_3) \\ \ddot{\breve{\theta}}_3(x_3) \end{Bmatrix} + [C]\begin{Bmatrix} \dot{\breve{u}}_{F''i} \\ \dot{\breve{\theta}}_{F''i} \\ \dot{\breve{u}}_i(x_3) \\ \dot{\breve{u}}'_\alpha(x_3) \\ \dot{\breve{\theta}}_3(x_3) \end{Bmatrix} + [K]\begin{Bmatrix} \breve{u}_{F''i} \\ \breve{\theta}_{F''i} \\ \breve{u}_i(x_3) \\ \breve{u}'_\alpha(x_3) \\ \breve{\theta}_3(x_3) \end{Bmatrix} - \{Q\} \right\} \tag{8.3.3-5}$$

where the components of the generalized force vector Q are the same as the static generalized forces (see the previous section) and the coefficient matrices M, C, and K are defined on the following pages. The M coefficient matrix is defined as

$\delta u_{F''i}$ row:

$\ddot{\breve{u}}_{F''j}$ column: $m\delta_{ij}$

$\ddot{\breve{\theta}}_{F''j}$ column: $\epsilon_{ijk}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk})$

$\ddot{\breve{u}}_j$ column: $m\delta_{ij}$ (8.3.3-6a)

$\ddot{\breve{u}}'_\beta$ column: $\epsilon_{klm} m_m C^{P'F'}_{ki} \dfrac{\partial \kappa_l}{\partial u''_\beta}$

$\ddot{\breve{\theta}}_3$ column: $\epsilon_{k3l} m_l C^{P'F'}_{ki}$

$\delta\psi_{F''i}$ row:

$$\ddot{u}_{F''j} \text{ column:} \quad -\epsilon_{ijk}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk})$$

$$\ddot{\theta}_{F''j} \text{ column:} \quad \delta_{ij}(mR^{P'F'}_{F'k} R^{P'F'}_{F'k} + 2m_l C^{P'F'}_{lk} R^{P'F'}_{F'k}) - mR^{P'F'}_{F'i} R^{P'F'}_{F'j} - m_l(C^{P'F'}_{li} R^{P'F'}_{F'j} + C^{P'F'}_{lj} R^{P'F'}_{F'i})$$

$$\ddot{u}_j \text{ column:} \quad -\epsilon_{ijk}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk}) \tag{8.3.3–6b}$$

$$\ddot{u}'_\beta \text{ column:} \quad m_l(C^{P'F'}_{ki} C^{P'F'}_{lm} - C^{P'F'}_{km} C^{P'F'}_{li}) R^{P'F'}_{F'm} \frac{\partial \kappa_k}{\partial u''_\beta} + i_1 C^{P'F'}_{1i} \frac{\partial \kappa_1}{\partial u''_\beta} + i_2 C^{P'F'}_{2i} \frac{\partial \kappa_2}{\partial u''_\beta} + i_3 C^{P'F'}_{3i} \frac{\partial \kappa_3}{\partial u''_\beta}$$

$$\ddot{\theta}_3 \text{ column:} \quad m_k(C^{P'F'}_{3i} C^{P'F'}_{kl} - C^{P'F'}_{3l} C^{P'F'}_{ki}) R^{P'F'}_{F'l} + i_3 C^{P'F'}_{3i}$$

δu_i row:

$$\ddot{u}_{F''j} \text{ column:} \quad m\delta_{ij}$$

$$\ddot{\theta}_{F''j} \text{ column:} \quad \epsilon_{ijk}(mR^{P'F'}_{F'k} + m_1 C^{P'F'}_{lk})$$

$$\ddot{u}_j \text{ column:} \quad m\delta_{ij} \tag{8.3.3–6c}$$

$$\ddot{u}'_\beta \text{ column:} \quad \epsilon_{klm} m_m C^{P'F'}_{ki} \frac{\partial \kappa_1}{\partial u''_\beta}$$

$$\ddot{\theta}_3 \text{ column:} \quad \epsilon_{k3l} m_1 C^{P'F'}_{ki}$$

$\delta u'_\alpha$ row:

$$\ddot{u}_{F''j} \text{ column: } \epsilon_{klm} m_m C_{kj}^{P'F'} \frac{\partial \kappa_l}{\partial u''_\alpha}$$

$$\ddot{\theta}_{F''j} \text{ column: } m_l (C_{kj}^{P'F'} C_{lm}^{P'F'} - C_{lj}^{P'F'} C_{km}^{P'F'}) R_{F'm}^{P'F'} \frac{\partial \kappa_k}{\partial u''_\alpha} + i_1 \frac{\partial \kappa_1}{\partial u''_\alpha} C_{1j}^{P'F'} + i_2 \frac{\partial \kappa_2}{\partial u''_\alpha} C_{2j}^{P'F'} + i_3 \frac{\partial \kappa_3}{\partial u''_\alpha} C_{3j}^{P'F'}$$

$$\ddot{u}_j \text{ column: } \epsilon_{klm} m_m C_{kj}^{P'F'} \frac{\partial \kappa_l}{\partial u''_\alpha} \tag{8.3.3-6d}$$

$$\ddot{u}'_\beta \text{ column: } i_1 \frac{\partial \kappa_1}{\partial u''_\alpha} \frac{\partial \kappa_1}{\partial u''_\beta} + i_2 \frac{\partial \kappa_2}{\partial u''_\alpha} \frac{\partial \kappa_2}{\partial u''_\beta} + i_3 \frac{\partial \kappa_3}{\partial u''_\alpha} \frac{\partial \kappa_3}{\partial u''_\beta}$$

$$\ddot{\theta}_3 \text{ column: } i_3 \frac{\partial \kappa_3}{\partial u''_\alpha}$$

θ_3 row:

$$\ddot{u}_{F''j} \text{ column: } \epsilon_{k3l} m_l C_{kj}^{P'F'}$$

$$\ddot{\theta}_{F''j} \text{ column: } m_k (C_{3j}^{P'F'} C_{kl}^{P'F'} - C_{3l}^{P'F'} C_{kj}^{P'F'}) R_{F'l}^{P'F'} + i_3 C_{3j}^{P'F'}$$

$$\ddot{u}_j \text{ column: } \epsilon_{k3l} m_l C_{kj}^{P'F'} \tag{8.3.3-6e}$$

$$\ddot{u}'_\beta \text{ column: } i_3 \frac{\partial \kappa_3}{\partial u''_\beta}$$

$$\ddot{\theta}_3 \text{ column: } i_3$$

the C coefficient matrix is defined as

$\delta u_{F''i}$ row:

$\dot{u}_{F''j}$ column: $\quad -2\epsilon_{ijk} m \Omega^{F'I}_{F'k}$

$\dot{\theta}_{F''j}$ column: $\quad 2\delta_{ij}\Omega^{F'I}_{F'k}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk}) - 2\Omega^{F'I}_{F'j}(mR^{P'F'}_{F'i} + m_k C^{P'F'}_{ki})$

$$(8.3.3\text{–}7a)$$

$\dot{u}_j$ column: $\quad -2\epsilon_{ijk} m \Omega^{F'I}_{F'k}$

$\dot{u}'_\beta$ column: $\quad 2m_k\Omega^{F'I}_{F'l}(C^{P'F'}_{mi} C^{P'F'}_{kl} - C^{P'F'}_{ki} C^{P'F'}_{ml})\dfrac{\partial \kappa_m}{\partial u''_\beta}$

$\dot{\theta}_3$ column: $\quad 2m_k\Omega^{F'I}_{F'l}(C^{P'F'}_{3i} C^{P'F'}_{kl} - C^{P'F'}_{3l} C^{P'F'}_{ki})$

$\delta\psi_{F''i}$ row:

$\dot{u}_{F''j}$ column: $\quad -2\delta_{ij}\Omega^{F'I}_{F'k}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk}) + 2\Omega^{F'I}_{F'i}(mR^{P'F'}_{F'j} + m_k C^{P'F'}_{kj})$

$\dot{\theta}_{F''j}$ column: $\quad -2\epsilon_{ijk} m \Omega^{F'I}_{F'l} R^{P'F'}_{F'k} R^{P'F'}_{F'l} - 2\epsilon_{ijk} m_m \Omega^{F'I}_{F'l}(C^{P'F'}_{ml} R^{P'F'}_{F'k} + C^{P'F'}_{mk} R^{P'F'}_{F'l}) - 2\epsilon_{ijk} i_3 \Omega^{F'I}_{F'k}$

$\dot{u}_j$ column: $\quad -2\delta_{ij}\Omega^{F'I}_{F'k}(mR^{P'F'}_{F'k} + m_l C^{P'F'}_{lk}) + 2\Omega^{F'I}_{F'i}(mR^{P'F'}_{F'j} + m_k C^{P'F'}_{kj})$ $\quad (8.3.3\text{–}7b)$

$\dot{u}'_\beta$ column: $\quad 2\epsilon_{klm} m_m(\Omega^{F'I}_{F'i} C^{P'F'}_{kn} - \Omega^{F'I}_{F'n} C^{P'F'}_{ki}) R^{P'F'}_{F'n}\dfrac{\partial \kappa_l}{\partial u''_\beta} - 2\Omega^{F'I}_{F'm}(\epsilon_{kl2} i_1 C^{P'F'}_{2m} + \epsilon_{kl1} i_2 C^{P'F'}_{1m}) C^{P'F'}_{ki}\dfrac{\partial \kappa_l}{\partial u''_\beta}$

$\dot{\theta}_3$ column: $\quad 2\epsilon_{kl3} m_l(\Omega^{F'I}_{F'i} C^{P'F'}_{km} - \Omega^{F'I}_{F'm} C^{P'F'}_{ki}) R^{P'F'}_{F'm} + 2\Omega^{F'I}_{F'k}(i_1 C^{P'F'}_{2k} C^{P'F'}_{1i} - i_2 C^{P'F'}_{1k} C^{P'F'}_{2i})$

δu_i row:

$$\dot{\ddot{u}}_{F''j} \text{ column: } \quad -2\epsilon_{ijk} m \Omega_{F'k}^{F'I}$$

$$\dot{\ddot{\theta}}_{F''j} \text{ column: } \quad 2\delta_{ij}\Omega_{F'k}^{F'I}(mR_{F'k}^{P'F'} + m_l C_{lk}^{P'F'}) - 2\Omega_{F'j}^{F'I}(mR_{F'i}^{P'F'} + m_k C_{ki}^{P'F'}) \tag{8.3.3-7c}$$

$$\dot{\ddot{u}}_j \text{ column: } \quad -2\epsilon_{ijk} m \Omega_{F'k}^{F'I}$$

$$\dot{\ddot{u}}'_\beta \text{ column: } \quad 2m_k\Omega_{F'l}^{F'I}(C_{mi}^{P'F'}C_{kl}^{P'F'} - C_{ki}^{P'F'}C_{ml}^{P'F'})\frac{\partial \kappa_m}{\partial u''_\beta}$$

$$\dot{\ddot{\theta}}_3 \text{ column: } \quad 2m_k\Omega_{F'l}^{F'I}(C_{3i}^{P'F'}C_{kl}^{P'F'} - C_{3l}^{P'F'}C_{ki}^{P'F'})$$

$\delta u'_\alpha$ row:

$$\dot{\ddot{u}}_{F''j} \text{ column: } \quad -2m_k\Omega_{F'l}^{F'I}(C_{mj}^{P'F'}C_{kl}^{P'F'} - C_{kj}^{P'F'}C_{ml}^{P'F'})\frac{\partial \kappa_m}{\partial u''_\alpha}$$

$$\dot{\ddot{\theta}}_{F''j} \text{ column: } \quad -2\epsilon_{klm}m_m(\Omega_{F'j}^{F'I}C_{kn}^{P'F'} - \Omega_{F'n}^{F'I}C_{kj}^{P'F'})R_{F'n}^{P'F'}\frac{\partial \kappa_l}{\partial u''_\alpha} + 2\Omega_{F'm}^{F'I}(\epsilon_{kl2}i_1C_{2m}^{P'F'} + \epsilon_{kl1}i_2C_{1m}^{P'F'})C_{kj}^{P'F'}\frac{\partial \kappa_l}{\partial u''_\alpha} \tag{8.3.3-7d}$$

$$\dot{\ddot{u}}_j \text{ column: } \quad -2m_k\Omega_{F'l}^{F'I}(C_{mj}^{P'F'}C_{kl}^{P'F'} - C_{kj}^{P'F'}C_{ml}^{P'F'})\frac{\partial \kappa_m}{\partial u''_\alpha}$$

$$\dot{\ddot{u}}'_\beta \text{ column: } \quad -2\Omega_{F'm}^{F'I}(\epsilon_{kl2}i_1C_{2m}^{P'F'} + \epsilon_{kl1}i_2C_{1m}^{P'F'})\frac{\partial \kappa_k}{\partial u''_\alpha}\frac{\partial \kappa_l}{\partial u''_\beta}$$

$$\dot{\ddot{\theta}}_3 \text{ column: } \quad 2\Omega_{F'k}^{F'I}(i_1C_{2k}^{P'F'}\frac{\partial \kappa_1}{\partial u''_\alpha} - i_2C_{1k}^{P'F'}\frac{\partial \kappa_2}{\partial u''_\alpha})$$

$\delta\theta_3$ row:

$$\dot{\breve{u}}_{F''j} \text{ column:} \quad -2m_k\Omega_{F'l}^{F'I}(C_{3j}^{P'F'}C_{kl}^{P'F'} - C_{3l}^{P'F'}C_{kj}^{P'F'})$$

$$\dot{\breve{\theta}}_{F''j} \text{ column:} \quad -2\epsilon_{k3l}m_l(\Omega_{F'j}^{F'I}C_{km}^{P'F'} - \Omega_{F'm}^{F'I}C_{kj}^{P'F'})R_{F'm}^{P'F'} - 2\Omega_{F'k}^{F'I}(i_1C_{2k}^{P'F'}C_{1j}^{P'F'} - i_2C_{1k}^{P'F'}C_{2j}^{P'F'})$$

$$\dot{\breve{u}}_j \text{ column:} \quad -2m_k\Omega_{F'l}^{F'I}(C_{3j}^{P'F'}C_{kl}^{P'F'} - C_{3l}^{P'F'}C_{kj}^{P'F'}) \tag{8.3.3–7e}$$

$$\dot{\breve{u}}'_\beta \text{ column:} \quad -2\Omega_{F'k}^{F'I}(i_1C_{2k}^{P'F'}\frac{\partial\kappa_1}{\partial u''_\beta} - i_2C_{1k}^{P'F'}\frac{\partial\kappa_2}{\partial u''_\beta})$$

$$\dot{\breve{\theta}}_3 \text{ column:} \quad 0$$

and the K coefficient matrix is defined as

$\delta u_{F''}$ row:

$$\breve{u}_{F''} \text{ column:} \quad m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$$

$$\breve{\theta}_{F''} \text{ column:} \quad m\left[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + \left(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'}\right)^{\tilde{}} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_{F'}^{P'F'}\right] + \left(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}m_{F'}\right)^{\tilde{}} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'} \tag{8.3.3–8a}$$

$$\breve{u} \text{ column:} \quad m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$$

$$\breve{\theta}_{P'} \text{ column:} \quad -\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'}C^{F'P'}$$

$\delta\psi_{F''}$ row:

$\check{u}_{F''}$ column: $-m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim} - \tilde{R}_{F'}^{P'F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}] - (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}m_{F'})^{\sim} + \tilde{m}_{F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\check{\theta}_{F''}$ column:

$$
\begin{aligned}
&m\{\tilde{R}_{F'}^{P'F'}(\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'})+\\
&\tilde{R}_{F'}^{P'F'}[(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_{F'}^{P'F'}]\}+\\
&\tilde{m}_{F'}[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim} -\\
&\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_{F'}^{P'F'}]+\\
&\tilde{R}_{F'}^{P'F'}[(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}m_{F'})^{\sim} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'}]+\\
&C^{F'P'}(\tilde{H}_{P'}\tilde{\Omega}_{P'}^{F'I} - \tilde{\Omega}_{P'}^{F'I}i_{P'}\tilde{\Omega}_{P'}^{F'I})C^{P'F'}
\end{aligned}
\tag{8.3.3-8b}
$$

$\check{u}$ column: $-m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim} - \tilde{R}_{F'}^{P'F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}] - (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}m_{F'})^{\sim} + \tilde{m}_{F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\check{\theta}_{P'}$ column:

$$
\begin{aligned}
&-\tilde{R}_{F'}^{P'F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'}C^{F'P'} -\\
&(\tilde{m}_{F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim}C^{F'P'} +\\
&\tilde{m}_{F'}(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim}C^{F'P'} -\\
&[\tilde{m}_{F'}(A_{F'}^{F'I} - g_{F'})]^{\sim}C^{F'P'} + \tilde{m}_{F'}(\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'})C^{F'P'} +\\
&C^{F'P'}(\tilde{H}_{P'}\tilde{\Omega}_{P'}^{F'I} - \tilde{\Omega}_{P'}^{F'I}i_{P'}\tilde{\Omega}_{P'}^{F'I})^{T}
\end{aligned}
$$

δu column:

$\check{u}_{F''}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\check{\theta}_{F''}$ column:

$$
\begin{aligned}
&m[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\sim} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_{F'}^{P'F'}]+\\
&(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}m_{F'})^{\sim} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'}
\end{aligned}
\tag{8.3.3-8c}
$$

$\check{u}$ column: $m\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$

$\check{\theta}_{P'}$ column: $-\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{m}_{F'}C^{F'P'}$

$\delta\psi_{P'}$ row:

$$\breve{u}_{F''} \text{ column: } \quad C^{P'F'}\tilde{m}_{F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$$

$$\breve{\theta}_{F''} \text{ column: } \quad C^{P'F'}\tilde{m}_{F'}(\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'}) + C^{P'F'}\tilde{m}_{F'}[(\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\tilde{}} - \tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}\tilde{R}_{F'}^{P'F'}] + (\tilde{H}_{P'}\tilde{\Omega}_{P'}^{F'I} - \tilde{\Omega}_{P'}^{F'I}i_{P'}\tilde{\Omega}_{P'}^{F'I})C^{P'F'} \tag{8.3.3-8d}$$

$$\breve{u} \text{ column: } \quad C^{P'F'}\tilde{m}_{F'}\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}$$

$$\breve{\theta}_{P'} \text{ column: } \quad C^{P'F'}\tilde{m}_{F'}[\tilde{A}_{F'}^{F'I} - \tilde{g}_{F'} + (\tilde{\Omega}_{F'}^{F'I}\tilde{\Omega}_{F'}^{F'I}R_{F'}^{P'F'})^{\tilde{}}]C^{F'P'} + \tilde{H}_{P'}\tilde{\Omega}_{P'}^{F'I} - \tilde{\Omega}_{P'}^{F'I}i_{P'}\tilde{\Omega}_{P'}^{F'I}$$

where

$$H_{P'} = i_{P'}\Omega_{P'}^{F'I}$$

$$i_{P'} = \begin{bmatrix} i_1 & 0 & 0 \\ 0 & i_2 & 0 \\ 0 & 0 & i_3 \end{bmatrix} \tag{8.3.3-9}$$

$$m_{P'} = \begin{Bmatrix} m_1 \\ m_2 \\ 0 \end{Bmatrix}$$

In the foregoing matrices, m is the running mass per unit length, and m_α is the first mass moment about the ξ_α axis. The last block row associated with $\delta\psi_{P'}^{P''P'}$ is used to obtain the terms associated with $\delta u'_\alpha$ and $\delta\theta_3$ by substitution from the equations

$$\delta\psi_{P'i}^{P''P'} = \delta_{3i}\delta\theta_3 + (\frac{\partial\kappa_i}{\partial u''_\alpha} + \frac{\partial^2\kappa_i}{\partial u''_\alpha\partial u'_\beta}\breve{u}'_\beta + \frac{\partial^2\kappa_i}{\partial u''_\alpha\partial\theta_3}\breve{\theta}_3)\delta u'_\alpha \tag{8.3.3-10}$$

and

$$\breve{\theta}_{P'i}^{P''P'} = \delta_{3i}\breve{\theta}_3 + \frac{\partial\kappa_i}{\partial u''_\alpha}\breve{u}'_\alpha \tag{8.3.3-11}$$

The $\delta\psi_{P'}$ row block matrices must then be pre-multiplied by R^T and the $\breve{\theta}_{P'}$ column block matrices must be post-multiplied by R, where

$$R_{i\alpha} = \frac{\partial\kappa_i}{\partial u''_\alpha} \qquad R_{i3} = \delta_{i3} \tag{8.3.3-12}$$

The geometric stiffness matrix K^G is added to the $\delta\psi_{P'}$ rows and $\breve{\theta}_{P'}$ columns where

$$
\begin{aligned}
K^G_{\alpha\beta} &= -\, M^{P'}_{P' i}\frac{\partial^2 \kappa_i}{\partial u''_\alpha \partial u'_\beta} \\
K^G_{\alpha 3} &= -\, M^{P'}_{P' i}\frac{\partial^2 \kappa_i}{\partial u''_\alpha \partial \theta_3} \qquad (8.3.3\text{–}13) \\
K^G_{3i} &= 0
\end{aligned}
$$

The matrix K^G comes from the last two terms in equation (8.3.3–10), which are commonly called the geometric stiffness terms.

8.3.4. Aerodynamic Forces

The aerodynamic forces acting on the aeroelastic beam element are determined from a quasi-steady adaptation of Greenberg's thin-airfoil theory (ref. 29). Before the theory is discussed in detail, two new sets of axes must be introduced for the purposes of defining the directions in which the lift and drag forces and the pitching moment act. In figure 18, the Z axes are associated with the zero-lift line for the airfoil section with the vector $\hat{\underline{b}}_2^Z$ along the zero-lift line toward the trailing edge. The vector $\hat{\underline{b}}_3^Z$ is along the beam axis but in a direction such that a dextral rotation of the airfoil section about this vector results in an increase in the angle of attack. Then, being a dextral system, $\hat{\underline{b}}_1^Z$ turns out to be normal to the zero-lift line (and *nominally* in the direction of positive lift for the section). The other set of axes is the so-called wind axes W. For these axes the base vector $\hat{\underline{b}}_3^W$ is identical to $\hat{\underline{b}}_3^Z$. The base vector $\hat{\underline{b}}_2^Z$ is located along the relative wind vector (in the direction of drag) and $\hat{\underline{b}}_1^W$ is in the direction of lift.

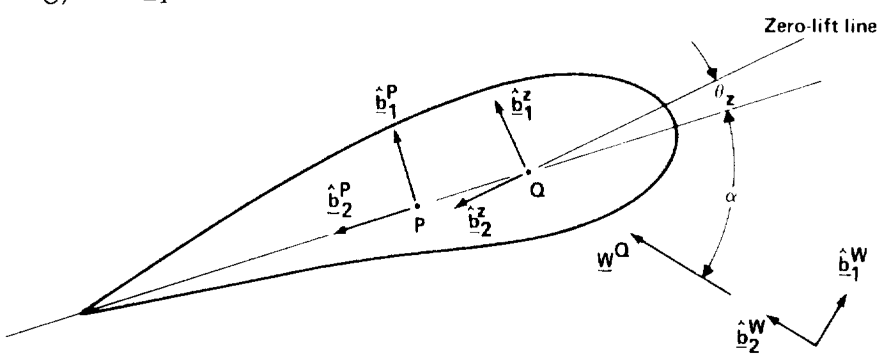


Figure 18. Aeroelastic beam cross section.

The Z basis and the P (principal axes) basis convect with the blade cross section, and are related by the direction cosine array $C^{ZP} = C^{Z'P'} = C^{Z''P''}$.

$$C^{ZP} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sigma\sin\theta_z & \sigma\cos\theta_z & 0 \\ 0 & 0 & \sigma \end{bmatrix} \tag{8.3.4-1}$$

where $\sigma = +1$ if a dextral rotation about $\hat{\underline{b}}_3^P$ results in an increase in the angle of attack and $\sigma = -1$ if a dextral rotation about $\hat{\underline{b}}_3^P$ results in a decrease in the angle of attack.

The wind basis W is related to the Z basis by

$$C^{WZ} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.3.4-2}$$

where α is the angle of attack. Then, $C^{WP} = C^{WZ}C^{ZP}$.

Point Q is the quarter-chord point of the cross section, about which the aerodynamic forces and pitching moment are calculated. The offset position of Q relative to the origin of the local principal axes P is $R^{QP}_{Z2}\hat{\underline{b}}_2^Z$.

Consider the wind velocity vector at the perturbed position of the aerodynamic center Q''. $\underline{W}^{Q''}$ is calculated by subtracting the inertial structural velocity at Q'' ($\underline{V}^{Q''I}$) from the inertial air velocity at Q'' ($\underline{U}^{Q''I}$), where

$$\underline{U}^{Q''I} = -(\overline{U}_1^A + r\overline{\gamma}^A + \dot{\breve{P}}_1^A + R^{QA}_{A2}\dot{\breve{\phi}}_{12}^A + R^{QA}_{A3}\dot{\breve{\phi}}_{13}^A)\hat{\underline{b}}_1^A \tag{8.3.4-3}$$

and

$$\begin{aligned}\underline{V}^{Q''I} =& (\underline{\tilde{\Omega}}^{F'I} + \underline{\tilde{\Omega}}^{F''F'} + \underline{\tilde{\Omega}}^{PF})\underline{R}^{QP} + (\underline{\tilde{\Omega}}^{F'I} + \underline{\tilde{\Omega}}^{F''F'})\underline{R}^{PF''} + \\ & {}^{F''}\dot{\underline{R}}^{PF''} + \underline{\tilde{\Omega}}^{F'I}R^{F''F'} + {}^{F'}\dot{\underline{R}}^{F''F'} + {}^{I}\dot{\underline{R}}^{F'I}\end{aligned}$$

The relative wind velocity components in the Z'' basis are then

$$\begin{aligned} W^{Q''}_{Z''i} =& -C^{Z''A}_{i1}(\bar{U}_1^A + r\bar{\gamma}^A + \dot{\breve{P}}_1^A + R^{Q''A}_{A2}\dot{\breve{\phi}}_{12}^A + R^{Q''A}_{A3}\dot{\breve{\phi}}_{13}^A) - \\ & [(C^{Z''F''}\Omega^{F'I}_{F'})^{\sim} + (C^{Z''F''}\dot{\breve{\theta}}_{F''})^{\sim} + (C^{Z''P''}\dot{\breve{\theta}}_{P'}^{P''P'})^{\sim}]R^{QP}_{Z} - \\ & [(C^{Z''F''}\Omega^{F'I}_{F'})^{\sim} + (C^{Z''F''}\dot{\breve{\theta}}_{F''})^{\sim}]C^{Z''F''}R^{P''F''}_{F''} - \\ & C^{Z''F''}\dot{\breve{u}} - [(C^{Z''F''}\Omega^{F'I}_{F'})^{\sim}]C^{Z''F''}\breve{u}_{F''} - \\ & (C^{Z''F''}\dot{\breve{\theta}}_{F''})^{\sim}C^{Z''F''}\breve{u}_{F''} - C^{Z''F''}\breve{u}_{F''} - C^{Z''F''}V^{F'I}_{F'} \end{aligned} \tag{8.3.4-4}$$

and the local air flow velocity gradient is

$$G^{Q''}_{Z''12} = \frac{\partial W^{Q''}_{Z''1}}{\partial R^{QP}_{Z2}}$$

$$= - C^{ZA}_{11}\left(\frac{C^{AZ}_{22} R^{QA}_{A2} + C^{AZ}_{32} R^{QA}_{A3}}{r} \bar{\gamma}^{A} + C^{AZ}_{22} \dot{\phi}^{A}_{12} + C^{AZ}_{32} \dot{\phi}^{A}_{13}\right) + \tag{8.3.4-5}$$

$$C^{IF'}_{3j} \Omega^{ZF'}_{F'j} + C^{ZF''}_{3j} \dot{\theta}^{F}_{j} + C^{ZP}_{3j} \dot{\theta}^{P}_{j}$$

where

$$r = \sqrt{{R^{QA}_{A2}}^2 + {R^{QA}_{A3}}^2}$$

If the time derivatives in equations (8.3.4–4) and (8.3.4–5) are replaced with variations, the relative virtual displacements and rotations of an element of air with respect to the structure are obtained.

$$\delta S^{Q''}_{Z''i} = - C^{Z''A}_{i1} (\delta P^{A}_{1} + R^{Q''A}_{A2} \delta\phi^{A}_{12} + R^{Q''A}_{A3} \delta\phi^{A}_{13}) -$$

$$[(C^{Z''F''} \delta\psi_{F''})^{\tilde{}} + (C^{Z''P''} \delta\psi^{P''P'}_{P''})^{\tilde{}}]_{i2} R^{QP}_{Z2} -$$

$$(C^{Z''F''} \delta\psi_{F''})^{\tilde{}}_{ij} C^{Z''F''}_{jk} R^{P''F''}_{F'k} - C^{Z''F''}_{ij} \delta u_{j} -$$

$$C^{Z''F''}_{ij} \delta u_{F''j} - (C^{Z''F''} \delta\psi_{F''})^{\tilde{}} C^{Z''F''} R^{F''F'}_{F''}$$

$$\delta\Upsilon^{Q''}_{Z''3} = - C^{Z''A}_{11} C^{Z''A}_{22} \delta\phi^{A}_{12} - C^{Z''A}_{11} C^{Z''A}_{23} \delta\phi^{A}_{13} + C^{Z''F''}_{3j} \delta\psi_{F''j} + C^{Z''P''}_{3j} \delta\psi^{P''P'}_{P'} \tag{8.3.4-6}$$

The relative wind velocity magnitude and components are time-dependent quantities. For the magnitude note that

$$W^2 = (W^{Q''}_{Z''1})^2 + (W^{Q''}_{Z''2})^2 \tag{8.3.4-7}$$

for which the static part is

$$\bar{W}^2 = (\bar{W}^{Q'}_{Z'1})^2 + (\bar{W}^{Q'}_{Z'2})^2 \tag{8.3.4-8}$$

and the dynamic part is

$$\check{W} = \frac{\bar{W}^{Q'}_{Z'1} \check{W}^{Q''}_{Z''1} + \bar{W}^{Q'}_{Z'2} \check{W}^{Q''}_{Z''2}}{\bar{W}} \tag{8.3.4-9}$$

Likewise, the angle of attack is a time-dependent quantity. In the equations written below, it is necessary only to develop the static part and the linearized dynamic perturbation part. These quantities are easily determined from the definition of α.

$$\tan\alpha = \frac{W^{Q''}_{Z''1}}{W^{Q''}_{Z''2}} \tag{8.3.4-10}$$

The static part is simply

$$\tan \bar{\alpha} = \frac{\bar{W}^{Q'}_{Z'1}}{\bar{W}^{Q'}_{Z'2}} \tag{8.3.4–11}$$

while the dynamic part is

$$\breve{\alpha} = \frac{\bar{W}^{Q'}_{Z'2}\breve{W}^{Q''}_{Z''1} - \bar{W}^{Q'}_{Z'1}\breve{W}^{Q''}_{Z''1}}{\bar{W}^2} \tag{8.3.4–12}$$

The applied force is assumed to be

$$\underline{F} = \mathcal{L}_c \hat{\underline{b}}_1^{W''} + \mathcal{D}\hat{\underline{b}}_2^{W''} + \mathcal{L}_{nc}\underline{b}_1^{Z''} \tag{8.3.4–13}$$

and the applied moment is

$$\underline{M} = \mathcal{M}\hat{\underline{b}}_3^{Z''} \tag{8.3.4–14}$$

The equations governing the aerodynamic force components are

$$\begin{aligned}
\mathcal{L}_c &= \frac{1}{2}\rho_a W^2 c c_l + \frac{\pi}{2}\rho_a c^2 W G^{Q''}_{Z''12} \\
\mathcal{D} &= \frac{1}{2}\rho_a W^2 c c_d \\
\mathcal{M} &= \frac{1}{2}\rho_a W^2 c^2 c_m - \frac{\pi}{16}\rho_a c^3 (W G^{Q''}_{Z''12} + \dot{W}^{Q''}_{Z''1} + \frac{3c}{8}\dot{G}^{Q''}_{Z''12}) \\
\mathcal{L}_{nc} &= \frac{\pi}{4}\rho_a c^2 (\dot{W}^{Q''}_{Z''1} + \frac{c}{4}\dot{G}^{Q''}_{Z''12})
\end{aligned} \tag{8.3.4–15}$$

Now, all of the quantities that are needed to define the virtual work are available.

$$\delta\mathcal{W} = \int_0^{\ell} (-\delta S^{Q''}_{Z''i} F_{Z''i} + \delta\Upsilon^{Q''}_{Z''3}\mathcal{M}) dx_3 \tag{8.3.4–16}$$

Steady-State. The static generalized forces can be removed from the expression for the virtual work and written in the form $\delta\mathcal{W} = \int_0^{\ell} \delta q^T Q dx_3$, where δq is

$$\delta q = \begin{Bmatrix} \delta P_1^A \\ \delta\phi_{1r}^A \\ \delta\phi_{12}^A \\ \delta\phi_{13}^A \\ \delta u_{F''i} \\ \delta\psi_{F''i} \\ \delta u_i \\ \delta u'_\alpha \\ \delta\theta_3 \end{Bmatrix} \tag{8.3.4–17}$$

and the elements of Q are

$$
\begin{aligned}
\delta P_1^A : \quad & \mathcal{L}_c C_{11}^{AW'} + \mathcal{D} C_{12}^{AW'} \\
\delta \phi_{1r}^A : \quad & r(\mathcal{L}_c C_{11}^{AW'} + \mathcal{D} C_{12}^{AW'}) - \frac{\mathcal{M} C_{11}^{Z'A}}{r}(R_{A2}^{Q'A} C_{22}^{Z'A} + R_{A3}^{Q'A} C_{23}^{Z'A}) \\
\delta \phi_{12}^A : \quad & 0 \\
\delta \phi_{13}^A : \quad & 0 \\
\delta u_{F'i} : \quad & \mathcal{L}_c C_{1i}^{W'F'} + \mathcal{D} C_{2i}^{W'F'} = F_i^A \\
\delta \psi_{F'i} : \quad & \mathcal{M} C_{3i}^{Z'F'} + (\tilde{R}_{F'}^{Q'F'} F^A)_i \\
\delta u_i : \quad & F_i^A \\
\delta u'_\alpha : \quad & C_{33}^{ZP}(\mathcal{M} - R_{Z2}^{QP} F_1^A) \frac{\partial \kappa_3}{\partial u''_\alpha} \\
\delta \theta_3 : \quad & C_{33}^{ZP}(\mathcal{M} - R_{Z2}^{QP} F_1^A)
\end{aligned}
\tag{8.3.4–18}
$$

Dynamic. After removing the steady-state contribution to the virtual work, the virtual work per unit of beam element length done by the aerodynamic forces and pitching moment can be put into the following form:

$$-\delta\mathcal{W} = \begin{Bmatrix} \delta P_1^A \\ \delta\phi_{1r}^A \\ \delta\phi_{12}^A \\ \delta\phi_{13}^A \\ \delta u_{F''i} \\ \delta\psi_{F''i} \\ \delta u_i \\ \delta u'_\alpha \\ \delta\theta_3 \end{Bmatrix}^T \left\{ [A] \begin{Bmatrix} \check{\mathcal{L}}_c \\ \check{\mathcal{D}} \\ \check{\mathcal{L}}_{nc} \\ \check{\mathcal{M}} \end{Bmatrix} + [B] \begin{Bmatrix} \check{W}_{Z\alpha}^Q \\ \check{G}_{Z12}^Q \end{Bmatrix} + [D] \begin{Bmatrix} \check{P}_1^A \\ \check{\phi}_{1r}^A \\ \check{\phi}_{12}^A \\ \check{\phi}_{13}^A \\ \check{u}_{F''j} \\ \check{\theta}_{F''j} \\ \check{u}_j \\ \check{u}'_\beta \\ \check{\theta}_3 \end{Bmatrix} \right\}$$

$$\begin{Bmatrix} \check{\mathcal{L}}_c \\ \check{\mathcal{D}} \\ \check{\mathcal{L}}_{nc} \\ \check{\mathcal{M}} \end{Bmatrix} = [E] \begin{Bmatrix} \check{W}_{Z\alpha}^Q \\ \check{G}_{Z12}^Q \end{Bmatrix} + [F] \begin{Bmatrix} \dot{\check{W}}_{Z\alpha}^Q \\ \dot{\check{G}}_{Z12}^Q \end{Bmatrix} \tag{8.3.4-19}$$

$$\begin{Bmatrix} \check{W}_{Z\alpha}^Q \\ \check{G}_{Z12}^Q \end{Bmatrix} = [G] \begin{Bmatrix} \check{P}_1^A \\ \check{\phi}_{1r}^A \\ \check{\phi}_{12}^A \\ \check{\phi}_{13}^A \\ \check{u}_{F''j} \\ \check{\theta}_{F''j} \\ \check{u}_j \\ \check{u}'_\beta \\ \check{\theta}_3 \end{Bmatrix} + [H] \begin{Bmatrix} \dot{\check{P}}_1^A \\ \dot{\check{\phi}}_{1r}^A \\ \dot{\check{\phi}}_{12}^A \\ \dot{\check{\phi}}_{13}^A \\ \dot{\check{u}}_{F''j} \\ \dot{\check{\theta}}_{F''j} \\ \dot{\check{u}}_j \\ \dot{\check{u}}'_\beta \\ \dot{\check{\theta}}_3 \end{Bmatrix}$$

This equation can then be rewritten in terms of aerodynamic M, C, and K matrices, where

$$\begin{aligned} M &= AFH \\ C &= AEH + AFG + BH \\ K &= AEG + BG + D \end{aligned} \tag{8.3.4-20}$$

The elements of A are

δP_1^A row:

$\check{\mathcal{L}}_c$ column: $-C_{11}^{W'A}$

$\check{\mathcal{D}}$ column: $-C_{21}^{W'A}$ (8.3.4–21a)

$\check{\mathcal{L}}_{nc}$ column: $-C_{11}^{Z'A}$

$\check{\mathcal{M}}$ column: 0

$\delta\phi_{1r}^A$ row:

$\check{\mathcal{L}}_c$ column: 0

$\check{\mathcal{D}}$ column: 0 (8.3.4–21b)

$\check{\mathcal{L}}_{nc}$ column: 0

$\check{\mathcal{M}}$ column: 0

$\delta\phi_{12}^A$ row:

$\check{\mathcal{L}}_c$ column: $-C_{11}^{W'A}R_{A2}^{Q'A}$

$\check{\mathcal{D}}$ column: $-C_{21}^{W'A}R_{A2}^{Q'A}$ (8.3.4–21c)

$\check{\mathcal{L}}_{nc}$ column: $-C_{11}^{Z'A}R_{A2}^{Q'A}$

$\check{\mathcal{M}}$ column: $C_{11}^{Z'A}C_{22}^{Z'A}$

$\delta\phi_{13}^A$ row:

$\check{\mathcal{L}}_c$ column: $-C_{11}^{W'A}R_{A3}^{Q'A}$

$\check{\mathcal{D}}$ column: $-C_{21}^{W'A}R_{A3}^{Q'A}$ (8.3.4–21d)

$\check{\mathcal{L}}_{nc}$ column: $-C_{11}^{Z'A}R_{A3}^{Q'A}$

$\check{\mathcal{M}}$ column: $C_{11}^{Z'A}C_{23}^{Z'A}$

$\delta u_{F''i}$ row:

$$\check{\mathcal{L}}_c \text{ column: } \quad -C_{1i}^{W'F'}$$

$$\check{\mathcal{D}} \text{ column: } \quad -C_{2i}^{W'F'} \tag{8.3.4–21e}$$

$$\check{\mathcal{L}}_{nc} \text{ column: } \quad -C_{1i}^{Z'F'}$$

$$\check{\mathcal{M}} \text{ column: } \quad 0$$

$\delta \psi_{F''i}$ row:

$$\check{\mathcal{L}}_c \text{ column: } \quad \epsilon_{ijk} C_{1j}^{W'F'} R_{F'k}^{Q'F'}$$

$$\check{\mathcal{D}} \text{ column: } \quad \epsilon_{ijk} C_{2j}^{W'F'} R_{F'k}^{Q'F'} \tag{8.3.4–21f}$$

$$\check{\mathcal{L}}_{nc} \text{ column: } \quad \epsilon_{ijk} C_{1j}^{Z'F'} R_{F'k}^{Q'F'}$$

$$\check{\mathcal{M}} \text{ column: } \quad -C_{3i}^{Z'F'}$$

δu_i row:

$$\check{\mathcal{L}}_c \text{ column: } \quad -C_{1i}^{W'F'}$$

$$\check{\mathcal{D}} \text{ column: } \quad -C_{2i}^{W'F'} \tag{8.3.4–21g}$$

$$\check{\mathcal{L}}_{nc} \text{ column: } \quad -C_{1i}^{Z'F'}$$

$$\check{\mathcal{M}} \text{ column: } \quad 0$$

$\delta u'_\alpha$ row:

$$\check{\mathcal{L}}_c \text{ column: } \quad C_{11}^{W'Z'} C_{33}^{ZP} R_{Z2}^{QP} \frac{\partial \kappa_3}{\partial u''_\alpha}$$

$$\check{\mathcal{D}} \text{ column: } \quad C_{21}^{W'Z'} C_{33}^{ZP} R_{Z2}^{QP} \frac{\partial \kappa_3}{\partial u''_\alpha} \tag{8.3.4–21h}$$

$$\check{\mathcal{L}}_{nc} \text{ column: } \quad C_{33}^{ZP} R_{Z2}^{QP} \frac{\partial \kappa_3}{\partial u''_\alpha}$$

$$\check{\mathcal{M}} \text{ column: } \quad -\frac{\partial \kappa_3}{\partial u''_\alpha} C_{33}^{ZP}$$

$\delta\theta_3$ row:

$$\check{\mathcal{L}}_c \text{ column: } C_{11}^{W'Z'} C_{33}^{ZP} R_{Z2}^{QP}$$

$$\check{\mathcal{D}} \text{ column: } C_{21}^{W'Z'} C_{33}^{ZP} R_{Z2}^{QP} \tag{8.3.4-21i}$$

$$\check{\mathcal{L}}_{nc} \text{ column: } C_{33}^{ZP} R_{Z2}^{QP}$$

$$\check{\mathcal{M}} \text{ column: } -C_{33}^{ZP}$$

The elements of B are

δP_1^A row:

$$\check{W}_{Z1}^{Q} \text{ column: } \frac{-\bar{W}_{Z2}^{Q} C_{i1}^{Z'A}(\mathcal{L}_c \frac{\partial C_{1i}^{W'z'}}{\partial \bar{\alpha}} + \mathcal{D} \frac{\partial C_{2i}^{W'z'}}{\partial \bar{\alpha}})}{|W_Z^Q|^2}$$

$$\check{W}_{Z2}^{Q} \text{ column: } \frac{\bar{W}_{Z1}^{Q} C_{i1}^{Z'A}(\mathcal{L}_c \frac{\partial C_{1i}^{W'z'}}{\partial \bar{\alpha}} + \mathcal{D} \frac{\partial C_{2i}^{W'z'}}{\partial \bar{\alpha}})}{|W_Z^Q|^2} \tag{8.3.4-22a}$$

$$\check{G}_{Z12}^{Q} \text{ column: } 0$$

$\delta\phi_{1r}^A$ row:

$$\check{W}_{Z1}^{Q} \text{ column: } 0$$

$$\check{W}_{Z2}^{Q} \text{ column: } 0 \tag{8.3.4-22b}$$

$$\check{G}_{Z12}^{Q} \text{ column: } 0$$

$\delta\phi_{12}^A$ row:

$$\check{W}_{Z1}^{Q} \text{ column: } \frac{-\bar{W}_{Z2}^{Q} C_{i1}^{Z'A} R_{A2}^{Q'A}(\mathcal{L}_c \frac{\partial C_{1i}^{W'z'}}{\partial \bar{\alpha}} + \mathcal{D} \frac{\partial C_{2i}^{W'z'}}{\partial \bar{\alpha}})}{|W_Z^Q|^2}$$

$$\check{W}_{Z2}^{Q} \text{ column: } \frac{\bar{W}_{Z1}^{Q} C_{i1}^{Z'A} R_{A2}^{Q'A}(\mathcal{L}_c \frac{\partial C_{1i}^{W'z'}}{\partial \bar{\alpha}} + \mathcal{D} \frac{\partial C_{2i}^{W'z'}}{\partial \bar{\alpha}})}{|W_Z^Q|^2} \tag{8.3.4-22c}$$

$$\check{G}_{Z12}^{Q} \text{ column: } 0$$

$\delta\phi^A_{13}$ row:

$$\check{W}^Q_{Z1}\ \text{column:}\quad \frac{-\bar{W}^Q_{Z2} C^{Z'A}_{i1} R^{Q'A}_{A3}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1i}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2i}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2}\ \text{column:}\quad \frac{\bar{W}^Q_{Z1} C^{Z'A}_{i1} R^{Q'A}_{A3}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1i}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2i}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2} \tag{8.3.4–22d}$$

$$\check{G}^Q_{Z12}\ \text{column:}\quad 0$$

$\delta u_{F''i}$ row:

$$\check{W}^Q_{Z1}\ \text{column:}\quad \frac{-\bar{W}^Q_{Z2} C^{Z'A}_{ji}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1j}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2j}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2}\ \text{column:}\quad \frac{\bar{W}^Q_{Z1} C^{Z'A}_{ji}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1j}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2j}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2} \tag{8.3.4–22e}$$

$$\check{G}^Q_{Z12}\ \text{column:}\quad 0$$

$\delta\psi_{F''i}$ row:

$$\check{W}^Q_{Z1}\ \text{column:}\quad \frac{\epsilon_{ijk}\bar{W}^Q_{Z2} C^{Z'F'}_{lj} R^{QF'}_{F'k}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1l}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2l}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2}\ \text{column:}\quad \frac{-\epsilon_{ijk}\bar{W}^Q_{Z1} C^{Z'F'}_{lj} R^{QF'}_{F'k}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1l}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2l}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2} \tag{8.3.4–22f}$$

$$\check{G}^Q_{Z12}\ \text{column:}\quad 0$$

δu_i row:

$$\check{W}^Q_{Z1}\ \text{column:}\quad \frac{-\bar{W}^Q_{Z2} C^{Z'F'}_{ji}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1j}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2j}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2}\ \text{column:}\quad \frac{\bar{W}^Q_{Z1} C^{Z'F'}_{ji}\left(\mathcal{L}_c \frac{\partial C^{W'z'}_{1j}}{\partial\tilde{\alpha}} + \mathcal{D}\frac{\partial C^{W'z'}_{2j}}{\partial\tilde{\alpha}}\right)}{|W^Q_Z|^2} \tag{8.3.4–22g}$$

$$\check{G}^Q_{Z12}\ \text{column:}\quad 0$$

$\delta u'_\alpha$ row:

$$\check{W}^Q_{Z1} \text{ column: } \quad \frac{\bar{W}^Q_{Z2} C^{ZP}_{33} R^{QP}_{Z2} \frac{\partial \kappa_3}{\partial u''_\alpha}(\mathcal{L}_c \frac{\partial C^{w'z'}_{11}}{\partial \bar{\alpha}} + \mathcal{D}\frac{\partial C^{w'z'}_{21}}{\partial \bar{\alpha}})}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2} \text{ column: } \quad \frac{-\bar{W}^Q_{Z1} C^{ZP}_{33} R^{QP}_{Z2} \frac{\partial \kappa_3}{\partial u''_\alpha}(\mathcal{L}_c \frac{\partial C^{w'z'}_{11}}{\partial \bar{\alpha}} + \mathcal{D}\frac{\partial C^{w'z'}_{21}}{\partial \bar{\alpha}})}{|W^Q_Z|^2} \tag{8.3.4-22h}$$

$\check{G}^Q_{Z12}$ column: 0

$\delta\theta_3$ row:

$$\check{W}^Q_{Z1} \text{ column: } \quad \frac{\bar{W}^Q_{Z2} C^{ZP}_{33} R^{QP}_{Z2}(\mathcal{L}_c \frac{\partial C^{w'z'}_{11}}{\partial \bar{\alpha}} + \mathcal{D}\frac{\partial C^{w'z'}_{21}}{\partial \bar{\alpha}})}{|W^Q_Z|^2}$$

$$\check{W}^Q_{Z2} \text{ column: } \quad \frac{-\bar{W}^Q_{Z1} C^{ZP}_{33} R^{QP}_{Z2}(\mathcal{L}_c \frac{\partial C^{w'z'}_{11}}{\partial \bar{\alpha}} + \mathcal{D}\frac{\partial C^{w'z'}_{21}}{\partial \bar{\alpha}})}{|W^Q_Z|^2} \tag{8.3.4-22i}$$

$\check{G}^Q_{Z12}$ column: 0

The elements of D are

δP_1^A row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

$\check{u}_{F''j}$ column: 0 (8.3.4–23a)

$\check{\theta}_{F''j}$ column: $\epsilon_{klj}(\mathcal{L}_c C_{1l}^{W'F'} + \mathcal{D}C_{2l}^{W'F'})C_{k1}^{F'A}$

$\check{u}_j$ column: 0

$\check{u}'_\beta$ column: $-\epsilon_{klm}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D}C_{2k}^{W'P'})C_{k1}^{F'A}\dfrac{\partial \kappa_m}{\partial u''_\beta}$

$\check{\theta}_3$ column: $-\epsilon_{kl3}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D}C_{2k}^{W'P'})\bar{C}_{l1}^{P'A}$

$\delta\phi_{1r}^{A}$ row:

$$
\begin{aligned}
&\check{P}_1^A \text{ column: } 0\\
&\check{\phi}_{1r}^A \text{ column: } 0\\
&\check{\phi}_{12}^A \text{ column: } 0\\
&\check{\phi}_{13}^A \text{ column: } 0\\
&\check{u}_{F''} \text{ column: } 0\\
&\check{\theta}_{F''j} \text{ column: } 0\\
&\check{u}_j \text{ column: } 0\\
&\check{u}'_\beta \text{ column: } 0\\
&\check{\theta}_3 \text{ column: } 0
\end{aligned}
\tag{8.3.4–23b}
$$

$\delta\phi^A_{12}$ row:

$\check{P}^A_1$ column: 0

$\check{\phi}^A_{1r}$ column: 0

$\check{\phi}^A_{12}$ column: 0

$\check{\phi}^A_{13}$ column: 0

$\check{u}_{F''j}$ column: $-(\mathcal{L}_c C^{W'A}_{11} + \mathcal{D}C^{W'A}_{21})C^{F'A}_{j2}$

$$\begin{aligned}\check{\theta}_{F''j} \text{ column: } & -\epsilon_{jkl}[(\mathcal{L}_c C^{W'Z'}_{1m} + \mathcal{D}C^{W'Z'}_{2m})(C^{Z'A}_{m1}C^{F'A}_{l2}\bar{R}^{Q'F'}_{F'k} - \\ & C^{Z'F'}_{ml}C^{F'A}_{k1}\bar{R}^{Q'A}_{A2}) + \\ & \mathcal{M}(C^{Z'F'}_{1l}C^{F'A}_{k1}C^{Z'A}_{22} - C^{Z'A}_{11}C^{F'A}_{l2}C^{Z'F'}_{2k})]\end{aligned} \tag{8.3.4-23c}$$

$\check{u}_j$ column: $-(\mathcal{L}_c C^{W'A}_{11} + \mathcal{D}C^{W'A}_{21})C^{F'A}_{j2}$

$$\begin{aligned}\check{u}'_\beta \text{ column: } & \epsilon_{klm}[(\mathcal{L}_c C^{W'Z'}_{1i} + \mathcal{D}C^{W'Z'}_{2i})(C^{Z'A}_{i1}C^{P'A}_{k2}C^{ZP}_{2l}R^{QP}_{Z2} - \\ & C^{ZP}_{ik}C^{P'A}_{l1}R^{Q'A}_{A2}) + \\ & \mathcal{M}(C^{ZP}_{1k}C^{P'A}_{l1}C^{Z'A}_{22} - C^{Z'A}_{11}C^{P'A}_{k2}C^{ZP}_{2l})]\frac{\partial\kappa_m}{\partial u''_\beta}\end{aligned}$$

$$\begin{aligned}\check{\theta}_3 \text{ column: } & \epsilon_{kl3}[(\mathcal{L}_c C^{W'Z'}_{1i} + \mathcal{D}C^{W'Z'}_{2i})(C^{Z'A}_{i1}C^{P'A}_{k2}C^{ZP}_{2l}R^{QP}_{Z2} - \\ & C^{ZP}_{ik}C^{P'A}_{l1}R^{Q'A}_{A2}) + \\ & \mathcal{M}(C^{ZP}_{1k}C^{P'A}_{l1}C^{Z'A}_{22} - C^{Z'A}_{11}C^{P'A}_{k2}C^{ZP}_{2l})]\end{aligned}$$

$\delta\phi^A_{13}$ row:

$\check{P}^A_1$ column: 0

$\check{\phi}^A_{1r}$ column: 0

$\check{\phi}^A_{12}$ column: 0

$\check{\phi}^A_{13}$ column: 0

$\check{u}_{F''j}$ column: $-(\mathcal{L}_c C^{W'A}_{11} + \mathcal{D} C^{W'A}_{21}) C^{F'A}_{j3}$

$$\check{\theta}_{F''j} \text{ column: } -\epsilon_{jkl}[(\mathcal{L}_c C^{W'Z'}_{1m} + \mathcal{D} C^{W'Z'}_{2m})(C^{Z'A}_{m1} C^{F'A}_{l3} \bar{R}^{Q'F'}_{F'k} - C^{Z'F'}_{ml} C^{F'A}_{k1} R^{Q'A}_{A3}) + \mathcal{M}(C^{Z'F'}_{1l} C^{F'A}_{k1} C^{Z'A}_{23} - C^{Z'A}_{11} C^{F'A}_{l3} C^{Z'F'}_{2k})] \tag{8.3.4-23d}$$

$\check{u}_j$ column: $-(\mathcal{L}_c C^{W'A}_{11} + \mathcal{D} C^{W'A}_{21}) C^{F'A}_{j3}$

$$\check{u}'_\beta \text{ column: } \epsilon_{klm}[(\mathcal{L}_c C^{W'Z'}_{1i} + \mathcal{D} C^{W'Z'}_{2i})(C^{Z'A}_{i1} C^{P'A}_{k3} C^{ZP}_{2l} R^{QP}_{Z2} - C^{ZP}_{ik} C^{P'A}_{l1} R^{Q'A}_{A3}) + \mathcal{M}(C^{ZP}_{1k} C^{P'A}_{l1} C^{Z'A}_{23} - C^{Z'A}_{11} C^{P'A}_{k3} C^{ZP}_{2l})] \frac{\partial \kappa_m}{\partial u''_\beta}$$

$$\check{\theta}_3 \text{ column: } \epsilon_{kl3}[(\mathcal{L}_c C^{W'Z'}_{1i} + \mathcal{D} C^{W'Z'}_{2i})(C^{Z'A}_{i1} C^{P'A}_{k3} C^{ZP}_{2l} R^{QP}_{Z2} - C^{ZP}_{ik} C^{P'A}_{l1} R^{Q'A}_{A3}) + \mathcal{M}(C^{ZP}_{1k} C^{P'A}_{l1} C^{Z'A}_{23} - C^{Z'A}_{11} C^{P'A}_{k3} C^{ZP}_{2l})]$$

$\delta u_{F''i}$ row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

$\check{u}_{F''j}$ column: 0 **(8.3.4–23e)**

$\check{\theta}_{F''j}$ column: 0

$\check{u}_j$ column: 0

$\check{u}'_\beta$ column: $\quad -\epsilon_{klm}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D} C_{2k}^{W'P'}) C_{li}^{P'F'} \dfrac{\partial \kappa_m}{\partial u''_\beta}$

$\check{\theta}_3$ column: $\quad -\epsilon_{kl3}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D} C_{2k}^{W'P'}) C_{li}^{P'F'}$

$\delta\psi_{F''i}$ row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

$\check{u}_{F''j}$ column: $\epsilon_{mkl}(\mathcal{L}_c C_{1m}^{W'Z'} + \mathcal{D}C_{2m}^{W'Z'})C_{li}^{Z'F'}C_{kj}^{Z'F'}$

$\check{\theta}_{F''j}$ column: 0 (8.3.4–23f)

$\check{u}_j$ column: $\epsilon_{mkl}(\mathcal{L}_c C_{1m}^{W'Z'} + \mathcal{D}C_{2m}^{W'Z'})C_{li}^{Z'F'}C_{kj}^{Z'F'}$

$\check{u}'_\beta$ column: $\epsilon_{3lk}[(\mathcal{L}_c C_{11}^{W'Z'} + \mathcal{D}C_{21}^{W'Z'})R_{Z2}^{QP} - \mathcal{M}]C_{33}^{ZP}C_{li}^{P'F'}\dfrac{\partial\kappa_k}{\partial u''_\beta} +$

$\epsilon_{mkp}\epsilon_{lnq}(\mathcal{L}_c C_{1m}^{W'Z'} + \mathcal{D}C_{2m}^{W'Z'})(C_{kl}^{ZP}C_{pi}^{ZF}C_{no}^{P'F'} +$

$C_{pl}^{ZP}C_{ko}^{ZF}C_{ni}^{P'F'})R_{F'o}^{P'F'}\dfrac{\partial\kappa_q}{\partial u''_\beta}$

$\check{\theta}_3$ column: $\epsilon_{mkp}\epsilon_{ln3}(\mathcal{L}_c C_{1m}^{W'Z'} + \mathcal{D}C_{2m}^{W'Z'})(C_{kl}^{ZP}C_{pi}^{ZF}C_{no}^{P'F'} +$

$C_{pl}^{ZP}C_{ko}^{ZF}C_{ni}^{P'F'})R_{F'o}^{P'F'}$

δu_i row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

$\check{u}_{F''j}$ column: 0 (8.3.4-23g)

$\check{\theta}_{F''j}$ column: 0

$\check{u}_j$ column: 0

$\check{u}'_\beta$ column: $-\epsilon_{klm}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D} C_{2k}^{W'P'}) C_{li}^{P'F'} \dfrac{\partial \kappa_m}{\partial u''_\beta}$

$\check{\theta}_3$ column: $-\epsilon_{kl3}(\mathcal{L}_c C_{1k}^{W'P'} + \mathcal{D} C_{2k}^{W'P'}) C_{li}^{P'F'}$

$\delta u'_\alpha$ row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

$\check{u}_{F''j}$ column: 0 (8.3.4-23h)

$\check{\theta}_{F''j}$ column: 0

$\check{u}_j$ column: 0

$\check{u}'_\beta$ column: $[(\mathcal{L}_c C_{11}^{W'Z'} + \mathcal{D} C_{21}^{W'Z'}) R_{Z2}^{QP} - \mathcal{M}] C_{33}^{ZP} \dfrac{\partial^2 \kappa_3}{\partial u''_\alpha \partial u'_\beta}$

$\check{\theta}_3$ column: 0

$\delta\theta_3$ row:

$\check{P}_1^A$ column: 0

$\check{\phi}_{1r}^A$ column: 0

$\check{\phi}_{12}^A$ column: 0

$\check{\phi}_{13}^A$ column: 0

(8.3.4-23i)

$\check{u}_{F''j}$ column: 0

$\check{\theta}_{F''j}$ column: 0

$\check{u}_j$ column: 0

$\check{u}'_\beta$ column: 0

$\check{\theta}_3$ column: 0

The elements of E and F are determined from perturbations of Eqs. (8.3.4–15) which govern the lift, drag, and pitching moment. Thus the E matrix may be defined as:

$\check{\mathcal{L}}_c$ row:

$$\check{W}^Q_{Z1} \text{ column: } \rho_a c c_l \bar{W}^Q_{Z1} + \frac{1}{2}\rho_a c \frac{dc_l}{d\alpha}\bar{W}^Q_{Z2} + \frac{1}{2}\pi\rho_a c^2 \frac{\bar{W}^Q_{Z1}}{|\bar{W}|}\bar{G}^Q_{Z12}$$

$$\check{W}^Q_{Z2} \text{ column: } \rho_a c c_l \bar{W}^Q_{Z2} - \frac{1}{2}\rho_a c \frac{dc_l}{d\alpha}\bar{W}^Q_{Z1} + \frac{1}{2}\pi\rho_a c^2 \frac{\bar{W}^Q_{Z2}}{|\bar{W}|}\bar{G}^Q_{Z12} \tag{8.3.4–24a}$$

$$\check{G}^Q_{Z12} \text{ column: } \frac{1}{2}\pi\rho_a c^2 |\bar{W}|$$

$\check{\mathcal{D}}$ row:

$$\check{W}^Q_{Z1} \text{ column: } \rho_a c c_d \bar{W}^Q_{Z1} + \frac{1}{2}\rho_a c \frac{dc_d}{d\alpha}\bar{W}^Q_{Z2}$$

$$\check{W}^Q_{Z2} \text{ column: } \rho_a c c_d \bar{W}^Q_{Z2} - \frac{1}{2}\rho_a c \frac{dc_d}{d\alpha}\bar{W}^Q_{Z1} \tag{8.3.4–24b}$$

$$\check{G}^Q_{Z12} \text{ column: } 0$$

$\check{\mathcal{L}}_{nc}$ row:

$$\check{W}^Q_{Z1} \text{ column: } 0$$

$$\check{W}^Q_{Z2} \text{ column: } 0 \tag{8.3.4–24c}$$

$$\check{G}^Q_{Z12} \text{ column: } 0$$

$\check{\mathcal{M}}$ row:

$$\check{W}^Q_{Z1} \text{ column: } \quad \rho_a c^2 c_m \bar{W}^Q_{Z1} + \frac{1}{2}\rho_a c^2 \frac{dc_m}{d\alpha}\bar{W}^Q_{Z2} - \frac{1}{16}\pi\rho_a c^3 \frac{\bar{W}^Q_{Z1}}{|\bar{W}|}\bar{G}^Q_{Z12}$$

$$\check{W}^Q_{Z2} \text{ column: } \quad \rho_a c^2 c_m \bar{W}^Q_{Z2} - \frac{1}{2}\rho_a c^2 \frac{dc_m}{d\alpha}\bar{W}^Q_{Z1} - \frac{1}{16}\pi\rho_a c^3 \frac{\bar{W}^Q_{Z2}}{|\bar{W}|}\bar{G}^Q_{Z12} \tag{8.3.4-24d}$$

$$\check{G}^Q_{Z12} \text{ column: } \quad -\frac{1}{16}\pi\rho_a c^3 |\bar{W}|$$

and for the elements of F

$\check{\mathcal{L}}_c$ row:

$$\dot{\check{W}}^Q_{Z1} \text{ column: } \quad 0$$

$$\dot{\check{W}}^Q_{Z2} \text{ column: } \quad 0 \tag{8.3.4-25a}$$

$$\dot{\check{G}}^Q_{Z12} \text{ column: } \quad 0$$

$\check{\mathcal{D}}$ row:

$$\dot{\check{W}}^Q_{Z1} \text{ column: } \quad 0$$

$$\dot{\check{W}}^Q_{Z2} \text{ column: } \quad 0 \tag{8.3.4-25b}$$

$$\dot{\check{G}}^Q_{Z12} \text{ column: } \quad 0$$

$\check{\mathcal{L}}_{nc}$ row:

$$\dot{\check{W}}^Q_{Z1} \text{ column: } \quad \frac{\pi\rho_a c^2}{4}$$

$$\dot{\check{W}}^Q_{Z2} \text{ column: } \quad 0 \tag{8.3.4-25c}$$

$$\dot{\check{G}}^Q_{Z12} \text{ column: } \quad \frac{\pi\rho_a c^3}{16}$$

$\dot{\mathcal{M}}$ row:

$$\dot{\breve{W}}^{Q}_{Z1} \text{ column: } \quad -\frac{\pi\rho_a c^3}{16}$$

$$\dot{\breve{W}}^{Q}_{Z2} \text{ column: } \quad 0 \tag{8.3.4-25d}$$

$$\dot{\breve{G}}^{Q}_{Z12} \text{ column: } \quad -\frac{3\pi\rho_a c^4}{128}$$

From the relations defining the relative velocity components and gradient, the elements of G and H can be determined. The elements of G are defined as

$\breve{W}^{Q}_{Z\alpha}$ row:

$$\breve{P}^{A}_{1} \text{ column: } \quad 0$$

$$\breve{\phi}^{A}_{1r} \text{ column: } \quad 0$$

$$\breve{\phi}^{A}_{12} \text{ column: } \quad 0$$

$$\breve{\phi}^{A}_{13} \text{ column: } \quad 0$$

$$\breve{u}_{F''j} \text{ column: } \quad \epsilon_{\alpha kl} C^{Z'F'}_{lm} \Omega^{F'I}_{F'm} C^{Z'F'}_{kj} - \frac{\bar{\gamma}^{A}}{\bar{r}} C^{Z'A}_{\alpha 1} (C^{F'A}_{j2} \bar{R}^{Q'A}_{A2} + C^{F'A}_{j3} \bar{R}^{Q'A}_{A3}) \tag{8.3.4-26a}$$

$$\breve{\theta}_{F''j} \text{ column: } \quad -\epsilon_{jkl} \bar{C}^{Z'F'}_{\alpha k} [C^{F'A}_{l1} (\bar{U}^{A}_{1} + \bar{r}\bar{\gamma}^{A}) + V^{F'I}_{F'l}] + \delta_{\alpha 1} \epsilon_{jkl} \bar{C}^{Z'F'}_{3k} \Omega^{F'I}_{F'l} R^{QP}_{Z2} + \epsilon_{jkl} \frac{\bar{\gamma}^{A}}{\bar{r}} \bar{C}^{Z'A}_{\alpha 1} \bar{R}^{Q'F'}_{F'l} (C^{F'A}_{k2} \bar{R}^{Q'A}_{A2} + C^{F'A}_{k3} \bar{R}^{Q'A}_{A3}) + \epsilon_{\alpha km} \epsilon_{noj} C^{Z'F'}_{kl} C^{Z'F'}_{mn} \Omega^{F'I}_{F'o} R^{P'F'}_{F'l}$$

$$\breve{u}_{j} \text{ column: } \quad \epsilon_{\alpha kl} C^{Z'F'}_{lm} \Omega^{F'I}_{F'm} C^{Z'F'}_{kj} - \frac{\bar{\gamma}^{A}}{\bar{r}} C^{Z'A}_{\alpha 1} (C^{F'A}_{j2} \bar{R}^{Q'A}_{A2} + C^{F'A}_{j3} \bar{R}^{Q'A}_{A3})$$

$\breve{u}'_\beta$ column: $\quad - \epsilon_{klm} C^{ZP}_{\alpha k} [C^{P'A}_{l1} (\bar{U}^A_1 + \bar{r}\bar{\gamma}^A) + C^{P'F'}_{ln} V^{F'I}_{F'n}] \dfrac{\partial \kappa_m}{\partial u''_\beta} +$

$$\delta_{\alpha 1} \epsilon_{3lm} C^{P'F'}_{lk} C^{ZP}_{33} \Omega^{F'I}_{F'k} R^{QP}_{Z2} \frac{\partial \kappa_m}{\partial u''_\beta} +$$

$$\epsilon_{klm} \frac{\bar{\gamma}^A}{\bar{r}} C^{ZP}_{2l} C^{Z'A}_{\alpha 1} R^{QP}_{Z2} (C^{P'A}_{k2} \bar{R}^{Q'A}_{A2} + C^{P'A}_{k3} \bar{R}^{Q'A}_{A3}) \frac{\partial \kappa_m}{\partial u''_\beta} +$$

$$\epsilon_{klm} \epsilon_{\alpha no} \Omega^{F'I}_{F'p} \bar{R}^{P'F'}_{F'q} (C^{P'F'}_{lp} C^{ZP}_{ok} C^{Z'F'}_{nq} +$$

$$C^{P'F'}_{lq} C^{ZP}_{nk} C^{Z'F'}_{op}) \frac{\partial \kappa_m}{\partial u''_\beta}$$

$\breve{\theta}_3$ column: $\quad - \epsilon_{kl3} C^{ZP}_{\alpha k} [C^{P'A}_{l1} (\bar{U}^A_1 + \bar{r}\bar{\gamma}^A) + C^{P'F'}_{ln} V^{F'I}_{F'n}] +$

$$\epsilon_{kl3} \epsilon_{\alpha no} (C^{P'F'}_{lp} C^{ZP}_{ok} C^{Z'F'}_{nq} + C^{P'F'}_{lq} C^{ZP}_{nk} C^{Z'F'}_{op}) \Omega^{F'I}_{F'p} \bar{R}^{P'F'}_{F'q} +$$

$$\epsilon_{kl3} \frac{\bar{\gamma}^A}{\bar{r}} C^{ZP}_{2l} C^{Z'A}_{\alpha 1} R^{QP}_{Z2} (C^{P'A}_{k2} \bar{R}^{Q'A}_{A2} + C^{P'A}_{k3} \bar{R}^{Q'A}_{A3})$$

$\breve{G}^Q_{Z12}$ row:

$\breve{P}^A_1$ column: 0

$\breve{\phi}^A_{1r}$ column: 0

$\breve{\phi}^A_{12}$ column: 0

$\breve{\phi}^A_{13}$ column: 0

$\breve{u}_{F''j}$ column: $\quad - \dfrac{\bar{\gamma}^A}{\bar{r}} C^{Z'A}_{11} \Big[(C^{Z'A}_{22} C^{F'A}_{j2} + C^{Z'A}_{23} C^{F'A}_{j3}) -$

$$\frac{1}{\bar{r}^2} (C^{Z'A}_{22} \bar{R}^{Q'A}_{A2} + C^{Z'A}_{23} \bar{R}^{Q'A}_{A3})(C^{F'A}_{j2} \bar{R}^{Q'A}_{A2} + C^{F'A}_{j2} \bar{R}^{Q'A}_{A3}) \Big] \tag{8.3.4-26b}$$

$$
\begin{aligned}
\breve{\theta}_{F''j}\ \text{column:}\quad & \epsilon_{jkl}\bigg\{C_{3k}^{Z'F'}\Omega_{F'l}^{F'I}+ \\
& \frac{\bar{\gamma}^A}{\bar{r}}C_{11}^{Z'A}R_{F'l}^{QF'}\bigg[(C_{22}^{Z'A}C_{k2}^{F'A}+C_{23}^{Z'A}C_{k3}^{F'A})- \\
& \frac{1}{\bar{r}^2}(C_{22}^{Z'A}\bar{R}_{A2}^{Q'A}+C_{23}^{Z'A}\bar{R}_{A3}^{Q'A})(C_{k2}^{F'A}\bar{R}_{A2}^{Q'A}+C_{k2}^{F'A}\bar{R}_{A3}^{Q'A})\bigg]- \\
& \frac{\bar{\gamma}^A}{\bar{r}}\bigg[C_{1k}^{Z'F'}C_{l1}^{F'A}(C_{22}^{Z'A}R_{A2}^{Q'A}+C_{23}^{Z'A}R_{A3}^{Q'A})+ \\
& C_{11}^{Z'A}C_{2k}^{Z'F'}(C_{l2}^{F'A}R_{A2}^{Q'A}+C_{l3}^{F'A}R_{A3}^{Q'A})\bigg]\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\breve{u}_j\ \text{column:}\quad & -\frac{\bar{\gamma}^A}{\bar{r}}C_{11}^{Z'A}\bigg[(C_{22}^{Z'A}C_{j2}^{F'A}+C_{23}^{Z'A}C_{j3}^{F'A})- \\
& \frac{1}{\bar{r}^2}(C_{22}^{Z'A}\bar{R}_{A2}^{Q'A}+C_{23}^{Z'A}\bar{R}_{A3}^{Q'A})(C_{j2}^{F'A}\bar{R}_{A2}^{Q'A}+C_{j2}^{F'A}\bar{R}_{A3}^{Q'A})\bigg]
\end{aligned}
$$

$$
\begin{aligned}
\breve{u}'_\beta\ \text{column:}\quad & \epsilon_{3lm}C_{33}^{ZP}C_{lk}^{P'F'}\Omega_{F'k}^{F'I}\frac{\partial\kappa_m}{\partial u''_\beta}+ \\
& \epsilon_{klm}\bigg\{\frac{\bar{\gamma}^A}{\bar{r}}C_{11}^{Z'A}C_{l2}^{PZ}R_{Z2}^{QP}\bigg[(C_{22}^{Z'A}C_{k2}^{P'A}+C_{23}^{Z'A}C_{k3}^{P'A})- \\
& \frac{1}{\bar{r}^2}(C_{22}^{Z'A}R_{A2}^{Q'A}+C_{23}^{Z'A}R_{A3}^{Q'A})(C_{k2}^{P'A}R_{A2}^{Q'A}+C_{k3}^{P'A}R_{A3}^{Q'A})\bigg]- \\
& \frac{\bar{\gamma}^A}{\bar{r}}\bigg[C_{1k}^{ZP}C_{l1}^{P'A}(C_{22}^{Z'A}R_{A2}^{Q'A}+C_{23}^{Z'A}R_{A3}^{Q'A})+ \\
& C_{11}^{Z'A}C_{2k}^{ZP}(C_{l2}^{P'A}R_{A2}^{Q'A}+C_{l3}^{P'A}R_{A3}^{Q'A})\bigg]\bigg\}\frac{\partial\kappa_m}{\partial u''_\beta}
\end{aligned}
$$

$$
\begin{aligned}
\breve{\theta}_3\ \text{column:}\quad & \epsilon_{kl3}\frac{\bar{\gamma}^A}{\bar{r}}\bigg\{C_{11}^{Z'A}C_{l2}^{PZ}R_{Z2}^{QP}\bigg[(C_{22}^{Z'A}C_{k2}^{P'A}+C_{23}^{Z'A}C_{k3}^{P'A})- \\
& \frac{1}{\bar{r}^2}(C_{22}^{Z'A}R_{A2}^{Q'A}+C_{23}^{Z'A}R_{A3}^{Q'A})(C_{k2}^{P'A}R_{A2}^{Q'A}+C_{k3}^{P'A}R_{A3}^{Q'A})\bigg]- \\
& C_{1k}^{ZP}C_{l1}^{P'A}(C_{22}^{Z'A}R_{A2}^{Q'A}+C_{23}^{Z'A}R_{A3}^{Q'A})- \\
& C_{11}^{Z'A}C_{2k}^{ZP}(C_{l2}^{P'A}R_{A2}^{Q'A}+C_{l3}^{P'A}R_{A3}^{Q'A})\bigg\}
\end{aligned}
\tag{8.3.4–26c}
$$

The elements of H are defined as:

$\check{W}^{Q}_{Z\alpha}$ row:

$$
\begin{aligned}
&\dot{\check{P}}_1^A \text{ column:} \quad -C_{\alpha 1}^{Z'A} \\
&\dot{\check{\phi}}_{1r}^A \text{ column:} \quad 0 \\
&\dot{\check{\phi}}_{12}^A \text{ column:} \quad -C_{\alpha 1}^{Z'A}\bar{R}_{A2}^{Q'A} \\
&\dot{\check{\phi}}_{13}^A \text{ column:} \quad -C_{\alpha 1}^{Z'A}\bar{R}_{A3}^{Q'A} \\
&\dot{\check{u}}_{F''j} \text{ column:} \quad -C_{\alpha j}^{Z'F'} \\
&\dot{\check{\theta}}_{F''j} \text{ column:} \quad \delta_{\alpha 1}C_{3j}^{Z'F'}R_{Z2}^{QP} + \epsilon_{\alpha kl}C_{lj}^{Z'F'}C_{km}^{Z'F'}\bar{R}_{F'm}^{P'F'} \\
&\dot{\check{u}}_j \text{ column:} \quad -C_{\alpha j}^{Z'F'} \\
&\dot{\check{u}}'_\beta \text{ column:} \quad \delta_{\alpha 1}C_{33}^{ZP}R_{Z2}^{QP}\frac{\partial \kappa_3}{\partial u''_\beta} \\
&\dot{\check{\theta}}_3 \text{ column:} \quad \delta_{\alpha 1}C_{33}^{ZP}R_{Z2}^{QP}
\end{aligned}
\tag{8.3.4–27a}
$$

$\breve{G}^{Q}_{Z12}$ row:

$$
\begin{aligned}
&\dot{\breve{P}}^{A}_{1} \text{ column: } 0 \\
&\dot{\breve{\phi}}^{A}_{1r} \text{ column: } 0 \\
&\dot{\breve{\phi}}^{A}_{12} \text{ column: } -C^{Z'A}_{11}C^{Z'A}_{22} \\
&\dot{\breve{\phi}}^{A}_{13} \text{ column: } -C^{Z'A}_{11}C^{Z'A}_{23} \\
&\dot{\breve{u}}_{F''j} \text{ column: } 0 \\
&\dot{\breve{\theta}}_{F''j} \text{ column: } C^{Z'F'}_{3j} \\
&\dot{\breve{u}}_{j} \text{ column: } 0 \\
&\dot{\breve{u}}'_{\beta} \text{ column: } C^{ZP}_{33}\frac{\partial\kappa_3}{\partial u''_{\beta}} \\
&\dot{\breve{\theta}}_{3} \text{ column: } C^{ZP}_{33}
\end{aligned}
\qquad (8.3.4\text{–}27b)
$$

8.3.5. Spatial Discretization

The variables u_i and θ_3 are expanded in a set of polynomials based on reference 30. The "C0" functions (u_3 and θ_3) are expanded in terms of $\psi_i(x)$ where $x = x_3/\ell$. The functions used beyond the first two standard linear functions are orthonormalized. The C1 functions (u_α) are expanded in terms of $\beta_i(x)$. The functions used beyond the first four standard cubic functions are orthonormalized. The details of the orthonormalization procedure are specified below.

The expansions are given by

$$
\begin{aligned}
u_\alpha &= \sum_{i=1}^{N_\alpha} q_{\alpha i}\beta_i(x) \\
u_3 &= \sum_{i=1}^{N_3} q_{3i}\psi_i(x) \\
\theta_3 &= \sum_{i=1}^{N_4} q_{4i}\psi_i(x)
\end{aligned}
\qquad (8.3.5\text{–}1)
$$

The functions ψ_i for $i > 2$ and β_i for $i > 4$ are constructed from the Jacobi polynomials $g_n(x) = G_{n-1}(p, q; x)$ where $p = 5$ and $q = 3$ for the C0 functions and where $p = 9$ and $q = 5$ for the C1 functions.

Letting $x = \frac{x_s}{\ell}$, the C0 shape functions are

$$
\begin{aligned}
\psi_1 &= 1 - x \\
\psi_2 &= x \\
\psi_i &= x(1 - x) g_{i-2}(x) f_{i-2}
\end{aligned}
\tag{8.3.5-2}
$$

where $3 \leq i \leq N + 1$ and $N = N_3$ or N_4. The recursion relations used to compute the polynomials are

$$
\begin{aligned}
g_1(x) &= 1 \\
g_2(x) &= x - \frac{1}{2} \\
g_j(x) &= g_2(x) g_{j-1}(x) - g_{j-2}(x) A_{j-2} \\
g_1'(x) &= 0 \\
g_2'(x) &= 1 \\
g_j'(x) &= g_{j-1}(x) + g_2(x) g_{j-1}'(x) - g_{j-2}'(x) A_{j-2} \\
g_1''(x) &= 0 \\
g_2''(x) &= 0 \\
g_j''(x) &= 2 g_{j-1}'(x) + g_2(x) g_{j-1}''(x) - g_{j-2}'' A_{j-2}
\end{aligned}
\tag{8.3.5-3}
$$

where

$$
A_i = \frac{i(i+q-1)(i+p-1)(i+p-q)}{(2i+p-2)(2i+p-1)^2(2i+p)} \; ; \qquad p = 5, \, q = 3
$$

$$
\begin{aligned}
{f_1}^2 &= 30 \\
{f_{i+1}}^2 &= \frac{{f_i}^2}{A_i}
\end{aligned}
\tag{8.3.5-4}
$$

The derivatives of the shape functions are then

$$
\begin{aligned}
\psi_1' &= -1 \\
\psi_1'' &= 0 \\
\psi_2' &= 1 \\
\psi_2'' &= 0 \\
\psi_i' &= [(1-2x) g_{i-2} + x(1-x) g_{i-2}'] f_{i-2} \\
\psi_i'' &= [-2 g_{i-2}(x) - 4(x - \frac{1}{2}) g_{i-2}'(x) + x(1-x) g_{i-2}''(x)] f_{i-2}
\end{aligned}
\tag{8.3.5-5}
$$

Similarly, the C1 shape functions are

$$\begin{aligned}
\beta_1 &= 1 - 3x^2 + 2x^3 \\
\beta_2 &= x - 2x^2 + x^3 \\
\beta_3 &= 3x^2 - 2x^3 \\
\beta_4 &= x^3 - x^2 \\
\beta_j &= x^2(1-x)^2 g_{i-4}(x) f_{i-4}
\end{aligned} \tag{8.3.5-6}$$

where $5 \le i \le N+1$ and $N = N_1$ or N_2. The g's are the same as above for the C0 shape functions, A_i is evaluated for $p = 9$, $q = 5$, and $f_1^2 = 630$. The higher derivatives are

$$\begin{aligned}
g_1'''(x) &= g_2'''(x) = 0 \\
g_j'''(x) &= 3g_{j-1}''(x) + g_2(x)g_{j-1}''' - g_{j-2}'''(x)A_{j-2} \\
g_1''''(x) &= g_2''''(x) = 0 \\
g_j''''(x) &= 4g_{j-1}'''(x) + g_2(x)g_{j-1}'''' - g_{j-2}''''(x)A_{j-2}
\end{aligned} \tag{8.3.5-7}$$

and derivatives of the shape functions are

$$\begin{array}{llll}
\beta_1' = -6x(1-x) & \beta_1'' = 12(x - \frac{1}{2}) & \beta_1''' = 12 & \beta_1'''' = 0 \\
\beta_2' = 3x^2 - 4x + 1 & \beta_2'' = 6(x - \frac{2}{3}) & \beta_2''' = 6 & \beta_2'''' = 0 \\
\beta_3' = 6x(1-x) & \beta_3'' = -12(x - \frac{1}{2}) & \beta_3''' = -12 & \beta_3'''' = 0 \\
\beta_4' = 3x^2 - 2x & \beta_4'' = 6(x - \frac{1}{3}) & \beta_4''' = 6 & \beta_4'''' = 0
\end{array} \tag{8.3.5-8}$$

$$\begin{aligned}
\beta_i' &= [-4x(1-x)(x - \tfrac{1}{2})g_{i-4}(x) + x^2(1-x)^2 g_{i-4}'(x)] f_{i-4} \\
\beta_i'' &= [12(x^2 - x + \tfrac{1}{6})g_{i-4}(x) - 8x(1-x)(x - \tfrac{1}{2})g_{i-4}' + x^2(1-x)^2 g_{i-4}'''(x)] f_{i-4} \\
\beta_i''' &= [24(x - \tfrac{1}{2})g_{i-4}(x) + 36(x^2 - x + \tfrac{1}{6})g_{i-4}'(x) - 12x(1-x)(x - \tfrac{1}{2})g_{i-4}''(x) + \\
&\quad x^2(1-x)^2 g_{i-4}'''(x)] f_{i-4} \\
\beta_i'''' &= [24g_{i-4}(x) + 96(x - \tfrac{1}{2})g_{i-4}'(x) + 72(x^2 - x + \tfrac{1}{6})g_{i-4}''(x) - \\
&\quad 16x(1-x)(x - \tfrac{1}{2})g_{i-4}'''(x) + x^2(1-x)^2 g_{i-4}''''(x)] f_{i-4}
\end{aligned} \tag{8.3.5-9}$$

These formulas for shape functions, when substituted into expressions for virtual work of either internal, inertial, or applied loads, produce integrands that depend only on x_3. These integrals can be evaluated to any accuracy desired by use of Gaussian quadrature.

8.3.6. Transformation from Nodal Coordinates

In GRASP, a different set of generalized coordinates are used for the beam element than those for the nodes. It is therefore necessary to calculate the beam generalized coordinates in terms of the nodal displacements and rotational variables at both the root and tip of the beam, so that the beam equations can be written using a convenient set of generalized coordinates.

The beam generalized coordinates $q_{\alpha i}$ for $i = 1,2,3,4$ determine the u_α displacements at the beam root and tip. Similarly, $q_{3\alpha}$ determines the u_3 displacement at the root and tip, and $q_{4\alpha}$ determines the θ_3 rotation at the root and tip. The exact relations are

$$
\begin{aligned}
q_{i1} &= u_{Ri} \\
q_{12} &= C_{31}^{R'R} \\
q_{14} &= C_{31}^{T'R} \\
q_{22} &= C_{32}^{R'R} \\
q_{24} &= C_{32}^{T'R} \\
\begin{Bmatrix} q_{13} \\ q_{23} \\ q_{32} \end{Bmatrix} &= C^{RT} u_T \qquad\qquad (8.3.6\text{–}1) \\
q_{41} &= \sin^{-1}\left[\frac{-C_{21}^{R'R}}{\sqrt{1-(C_{31}^{R'R})^2}}\right] \\
q_{42} &= \sin^{-1}\left[\frac{-C_{21}^{T'R}}{\sqrt{1-(C_{31}^{T'R})^2}}\right] \\
C^{T'R} &= C^{T'T} C^{TR}
\end{aligned}
$$

The rotation expressions are derived from expressions for $C^{P''F''}$ written in terms of u'_α and θ_3 (see ref. 24, equations 4, 17, and 60-62) for which

$$
\begin{aligned}
C_{31}^{P''F''} &= \sin\theta_2 = u'_1 \\
C_{32}^{P''F''} &= -\cos\theta_2 \sin\theta_1 = u'_2 \qquad\qquad (8.3.6\text{–}2) \\
C_{21}^{P''F''} &= -\cos\theta_2 \sin\theta_3 = -\sqrt{1-{u'_1}^2}\,\sin\theta_3
\end{aligned}
$$

8.3.7. Transformation to Forces and Moments

The generalized forces calculated for the beam element root and tip correspond to the beam generalized coordinates. These forces must now be transformed into forces and moments at the root and tip nodes. The virtual work at the root can be written in terms of the static residuals Q_R and the linear coefficient matrices. In terms of the beam generalized coordinates, this relation is

$$-\delta W_R = \delta q_R{}^T(-Q_R + L_R\check{q}_R) \tag{8.3.7-1}$$

where $\delta q_R{}^T = \lfloor \delta q_{11}\ \delta q_{21}\ \delta q_{31}\ \delta q_{12}\ \delta q_{22}\ \delta q_{41} \rfloor$ and $\check{q}_R^T = \lfloor \check{q}_{11}\ \check{q}_{21}\ \check{q}_{31}\ \check{q}_{12}\ \check{q}_{22}\ \check{q}_{41} \rfloor$, and L_R is a linear operator representing $(M_R\frac{d^2}{dt^2} + C_R\frac{d}{dt} + K_R)$. Note that this equation defines the negative of the virtual work. The explanation for treating the virtual work in this manner is that it is conventional for L_R to be positive, and L_R is normally considered to be positive on the left-hand side of the equations of motion, while Q_R is positive on the right-hand side.

The root node virtual displacements and rotations may be related to the beam virtual generalized coordinates by the expression

$$\begin{Bmatrix} \delta q_{11} \\ \delta q_{21} \\ \delta q_{31} \\ \delta q_{12} \\ \delta q_{22} \\ \delta q_{41} \end{Bmatrix} = \begin{bmatrix} \underset{3\times3}{\Delta} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{R_R} \end{bmatrix} \begin{Bmatrix} \underset{3\times1}{\delta u_R} \\ \underset{3\times1}{\delta\psi_R} \end{Bmatrix} \tag{8.3.7-2}$$

where the root node virtual displacements are $\delta u_R = \delta R_R^{R'R}$, and the root node virtual rotations are $\delta\psi_R = \delta\psi_R^{R'R}$. The 6×6 coefficient matrix that premultiplies the root node virtual displacement and rotation vector is called T_R and matrix R_R (ref. 24, eq. 67) is

$$R_R = \begin{bmatrix} 0 & \ell C_{33}^{R'R} & -\ell C_{32}^{R'R} \\ -\ell C_{33}^{R'R} & 0 & \ell C_{31}^{R'R} \\ 0 & \frac{C_{32}^{R'R}}{1-(C_{31}^{R'R})^2} & \frac{C_{33}^{R'R}}{1-(C_{31}^{R'R})^2} \end{bmatrix} \tag{8.3.7-3}$$

Similarly, the perturbed root node displacements $\check{u}_R$ and rotations $\check{\theta}_R$ are related to the perturbed element generalized coordinates $\check{q}_R$ through the expression

$$\check{q}_R = T_R \begin{Bmatrix} \check{u}_R \\ \check{\theta}_R \end{Bmatrix} \tag{8.3.7-4}$$

When the virtual work at the beam root is transformed into nodal coordinates by the substitution of equations (8.3.7–2) and (8.3.7–4) into equation (8.3.7–1), the following expression is obtained:

$$
\begin{aligned}
-\delta W_R &= \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor T_R{}^T \left\{ \{-Q_R\} + L_R T_R \begin{Bmatrix} \breve{u}_R \\ \breve{\theta}_R \end{Bmatrix} \right\} \\
&= \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor \left\{ \{-T_R{}^T Q_R\} + T_R{}^T L_R T_R \begin{Bmatrix} \breve{u}_R \\ \breve{\theta}_R \end{Bmatrix} \right\}
\end{aligned}
\tag{8.3.7–5}
$$

First consider the transformation of L_R, which contains the dynamic matrices M_R, C_R, and K_R. The transformation of the element generalized coordinates into the nodal generalized coordinates introduces the transformation matrix T_R into the expression for the virtual work. Since T_R is a function of $C^{R'R}$, which is a function of the nodal rotations, it must also be perturbed to recover any additional perturbation contributions. In the case of the linearized dynamic matrices M_R, C_R, and K_R, no new perturbation terms are introduced by the transformation, since any such contributions would be nonlinear.

The transformation of the static generalized force Q_R is, however, another matter. In this case, transformation does contribute an additional term, called the geometric stiffness term K_R^G, to the linearized dynamic equations. Geometric stiffness originates from the perturbation of T_R.

$$
-\delta q_R Q_R = -\lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor T_R{}^T \{Q_R\} \tag{8.3.7–6}
$$

When T_R is perturbed,

$$
\begin{aligned}
-\delta q_R Q_R &= -\ \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor \frac{\partial T_R{}^T}{\partial q_R} q_R Q_R \\
&= -\ \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor \frac{\partial T_R{}^T}{\partial q_R} T_R \begin{Bmatrix} \breve{u}_R \\ \breve{\theta}_R \end{Bmatrix} Q_R
\end{aligned}
\tag{8.3.7–7}
$$

where

$$
\frac{\partial T_R}{\partial q_R} = \begin{bmatrix} \underset{3\times 3}{0} & \underset{3\times 3}{0} \\ \underset{3\times 3}{0} & \underset{3\times 3}{\frac{\partial R_R}{\partial q_R}} \end{bmatrix} \tag{8.3.7–8}
$$

When equation (8.3.7–7) is multiplied out, only one of the 3×3 submatrices is nonzero. This submatrix is called the root geometric stiffness matrix k_{GR}, and it contains only terms that are related to the nodal rotations.

$$
K_R^G \breve{\theta}_R = -\frac{\partial R_R{}^T}{\partial q_R} R_R \breve{\theta}_R Q_{R\theta} \tag{8.3.7–9}
$$

where

$$Q_R = \begin{Bmatrix} Q_{Ru} \\ Q_{R\theta} \end{Bmatrix} \tag{8.3.7-10}$$

Written in index notation to allow the isolation of k_{GR}, the root geometric stiffness is

$$K_{GR_{ij}} \check{\theta}_{R_j} = -\frac{\partial R_{R_{ki}}}{\partial q_{R_l}} R_{R_{ij}} \check{\theta}_{R_j} Q_{R\theta_k} \tag{8.3.7-11}$$

and

$$k_{GR_{ij}} = -\frac{\partial R_{R_{ki}}}{\partial q_{R_l}} R_{R_{ij}} Q_{R\theta_k} \tag{8.3.7-12}$$

The geometric stiffness matrix used to transform all of the root nodal degrees of freedom is then

$$K_R^G = \begin{bmatrix} \underset{3\times 3}{0} & \underset{3\times 3}{0} \\ \underset{3\times 3}{0} & \underset{3\times 3}{k_{GR}} \end{bmatrix} \tag{8.3.7-13}$$

The virtual work at the root can now be written in the form

$$\begin{aligned} -\delta W_R &= \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor \left\{ -T_R{}^T Q_R + T_R{}^T L_R T_R \begin{Bmatrix} \check{u}_R \\ \check{\theta}_R \end{Bmatrix} + K_R^G \begin{Bmatrix} \check{u}_R \\ \check{\theta}_R \end{Bmatrix} \right\} \\ &= \lfloor \delta u_R{}^T \ \delta\psi_R{}^T \rfloor \left\{ -Q_R^* + L_R^* \begin{Bmatrix} \check{u}_R \\ \check{\theta}_R \end{Bmatrix} \right\} \end{aligned} \tag{8.3.7-14}$$

where

$$\begin{aligned} Q_R^* &= T_R{}^T Q_R \\ L_R^* &= T_R{}^T L_R T_R + T_R{}^T K_R^G \end{aligned} \tag{8.3.7-15}$$

The transformation of the generalized forces and moments at the tip of the element into nodal forces and moments is similar to that for the root. In beam element generalized coordinates, the virtual work at the tip is

$$-\delta W_T = \delta q_T{}^T \left(-Q_T + L_T \check{q}_T \right) \tag{8.3.7-16}$$

where $\delta q_T{}^T = \lfloor \delta q_{13} \ \delta q_{23} \ \delta q_{32} \ \delta q_{14} \ \delta q_{24} \ \delta q_{42} \rfloor$ and $\check{q}_T^T = \lfloor \check{q}_{13} \ \check{q}_{23} \ \check{q}_{32} \ \check{q}_{14} \ \check{q}_{24} \ \check{q}_{42} \rfloor$, and L_T is a linear operator representing $(M_T \frac{d^2}{dt^2} + C_T \frac{d}{dt} + K^T)$. Note the similarity with equation (8.3.7-1).

The equation that relates the tip node virtual displacements to the element virtual generalized coordinates is

$$\begin{Bmatrix} \delta q_{13} \\ \delta q_{23} \\ \delta q_{32} \\ \delta q_{14} \\ \delta q_{24} \\ \delta q_{42} \end{Bmatrix} = \begin{bmatrix} \underset{3\times3}{C^{RT}} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{R_T C^{RT}} \end{bmatrix} \begin{Bmatrix} \underset{3\times1}{\delta u_T} \\ \underset{3\times1}{\delta\psi_T} \end{Bmatrix} \tag{8.3.7-17}$$

where the tip node virtual displacements are $\delta u_T = \delta R_R^{T'R}$, and the tip node virtual rotations are $\delta\psi_T = \delta\psi_R^{T'R}$. The 6×6 coefficient matrix that premultiplies the tip node virtual displacement and rotation vector is called T_T and matrix R_T is

$$R_T = \begin{bmatrix} 0 & \ell C_{33}^{T'R} & -\ell C_{32}^{T'R} \\ -\ell C_{33}^{T'R} & 0 & \ell C_{31}^{T'R} \\ 0 & \frac{C_{32}^{T'R}}{1-(C_{31}^{T'R})^2} & \frac{C_{33}^{T'R}}{1-(C_{31}^{T'R})^2} \end{bmatrix} \tag{8.3.7-18}$$

Similarly, the perturbed tip node displacements $\check{u}_T$ and rotations $\check{\theta}_T$ are related to the perturbed element generalized coordinates $\check{q}_T$ through the expression

$$\check{q}_T = T_T \begin{Bmatrix} \check{u}_T \\ \check{\theta}_T \end{Bmatrix} \tag{8.3.7-19}$$

The expression for the virtual work at the tip is similar to the expression for the virtual work at the root.

$$-\delta W_T = \lfloor \delta u_T{}^T \;\; \delta\psi_T{}^T \rfloor \left\{ \{-T_T{}^T Q_T\} + T_T{}^T L_T T_T \begin{Bmatrix} \check{u}_T \\ \check{\theta}_T \end{Bmatrix} \right\} \tag{8.3.7-20}$$

As in the derivation of the transformation of L_R, no additional terms result from the transformation of L_T. There are, however, geometric stiffness terms that result from the transformation of Q_T. Following the derivation of the root geometric stiffness,

$$-\delta q_T Q_T = -\lfloor \delta u_T{}^T \;\; \delta\psi_T{}^T \rfloor \frac{\partial T_T{}^T}{\partial q_T} T_T \begin{Bmatrix} \check{u}_T \\ \check{\theta}_T \end{Bmatrix} Q_T \tag{8.3.7-21}$$

where

$$\frac{\partial T_T}{\partial q_T} = \begin{bmatrix} \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{\frac{\partial R_T}{\partial q_T} C^{RT}} \end{bmatrix} \tag{8.3.7-22}$$

When equation (8.3.7–21) is expanded, only one, nonzero 3×3 submatrix remains. It is called the tip geometric stiffness matrix k_{GT}, and

$$k_{GT}\breve{\theta}_T = -C^{TR}\frac{\partial R_T{}^T}{\partial q_T}R_T C^{TR}\breve{\theta}_T Q_{T\theta} \tag{8.3.7–23}$$

where

$$Q_T = \begin{Bmatrix} Q_{Tu} \\ Q_{T\theta} \end{Bmatrix} \tag{8.3.7–24}$$

Written in index notation to allow the isolation of k_{GT}, the tip geometric stiffness is

$$k_{GT_{ij}} = -C^{TR}_{ik}\frac{\partial R_{T_{lk}}}{\partial q_{T_m}}R_{T_{mn}}C^{RT}_{nj}Q_{T\theta_l} \tag{8.3.7–25}$$

or

$$k_{GT_{ij}} = (C^{TR}\bar{k}_{GT}C^{RT})_{ij} \tag{8.3.7–26}$$

where

$$\bar{k}_{GT_{ij}} = -\frac{\partial R_{T_{ki}}}{\partial q_{R_l}}R_{T_{lj}}Q_{T\theta_k} \tag{8.3.7–27}$$

Therefore,

$$K^G_T = \begin{bmatrix} \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{k_{GT}} \end{bmatrix} \tag{8.3.7–28}$$

The virtual work at the tip can now be written in the form

$$\begin{aligned} -\delta\mathcal{W}_T &= \lfloor \delta u_T{}^T \; \delta\psi_T{}^T \rfloor \left\{ -T_T{}^T Q_T + T_T{}^T L_T T_T \begin{Bmatrix} \breve{u}_T \\ \breve{\theta}_T \end{Bmatrix} + K^G_T \begin{Bmatrix} \breve{u}_T \\ \breve{\theta}_T \end{Bmatrix} \right\} \\ &= \lfloor \delta u_T{}^T \; \delta\psi_T{}^T \rfloor \left\{ -Q^*_T + L^*_T \begin{Bmatrix} \breve{u}_T \\ \breve{\theta}_T \end{Bmatrix} \right\} \end{aligned} \tag{8.3.7–29}$$

where

$$\begin{aligned} Q^*_T &= T_T{}^T Q_T \\ L^*_T &= T_T{}^T L_T T_T + T_T{}^T K^G_T \end{aligned} \tag{8.3.7–30}$$

For both the root and tip, derivatives of R_R and R_T with respect to the q_{ij} are needed. The only nonzero elements of these arrays may determined from

$$\frac{\partial R}{\partial u'_1} = \begin{bmatrix} 0 & -\ell\frac{C_{31}}{C_{33}} & 0 \\ \ell\frac{C_{31}}{C_{33}} & 0 & \ell \\ 0 & \frac{2C_{31}C_{32}}{(1-C^2_{31})^2} & \frac{C_{31}(1-C^2_{31}-2C^2_{32})}{C_{33}(1-C^2_{31})^2} \end{bmatrix} \tag{8.3.7–31}$$

$$\frac{\partial R}{\partial u_2'} = \begin{bmatrix} 0 & -\ell\frac{C_{32}}{C_{33}} & -\ell \\ \ell\frac{C_{32}}{C_{33}} & 0 & \ell \\ 0 & \frac{1}{1-C_{31}^2} & \frac{-C_{32}}{C_{33}(1-C_{31}^2)} \end{bmatrix} \tag{8.3.7-32}$$

where

$$\begin{aligned} \frac{\partial R}{\partial q_{12}} &= \frac{1}{\ell}\frac{\partial R}{\partial u_1'}\Big|_{x_3=0} \\ \frac{\partial R}{\partial q_{14}} &= \frac{1}{\ell}\frac{\partial R}{\partial u_1'}\Big|_{x_3=\ell} \\ \frac{\partial R}{\partial q_{22}} &= \frac{1}{\ell}\frac{\partial R}{\partial u_2'}\Big|_{x_3=0} \\ \frac{\partial R}{\partial q_{24}} &= \frac{1}{\ell}\frac{\partial R}{\partial u_2'}\Big|_{x_3=\ell} \end{aligned} \tag{8.3.7-33}$$

where C is $C^{R'R}$ at the root and $C^{T'R}$ at the tip and R is R_R at the root and R_T at the tip.

9. CONCLUDING REMARKS

In response to the limitations of previous methods for analyzing rotorcraft, GRASP has been developed. GRASP is a general-purpose program which treats the nonlinear static and linearized dynamic behavior of rotorcraft represented by arbitrarily connected rigid-body and beam elements. Large relative motions and deformation-induced displacements and rotations are permitted (as long as the strains in the beam element are small). Periodic coefficients are not treated, restricting the solutions to rotorcraft in axial flight and on the ground.

GRASP uses a modern approach for modeling structures, incorporating the features of several traditional methods. The basic approach which provides the foundation for large relative motion kinematics is derived from "multibody" research with an expanded emphasis on multiple levels of substructures. This is combined with the finite element approach which provides flexible modeling through the use of libraries of elements, constraints, and nodes. The use of a variable-order polynomial beam element makes the finite element approach more effective. The incorporation of aeroelastic effects, including inflow dynamics and nonlinear aerodynamic coefficients for the beam element, further extends the capabilities.

Due to the fact that GRASP was developed using structured, modular, software methods, changes to the code are relatively easy to perform. This makes it practical to modify the code in order to enhance its functionality. Some of the many areas where possibilities for enhancements exist are expanded solution procedures, improved aerodynamic models, expanded modeling capabilities, new elements, and new constraints.

Existing solution procedures (steady-state and asymmetric eigenproblem) could easily be expanded to include a symmetric eigensolution. This solution procedure would take the symmetric part of the linearized, perturbed equations of motion, then calculate the eigenvalues and eigenvectors. The symmetric eigensolution would be to generate the modes for another new solution procedure, the subspace reduction. The subspace reduction would allow the user to solve for the asymmetric eigensolution using a reduced set of admissible functions. A reference deformations solution procedure would allow a user to take any steady-state solution and use it either as an initial guess for another steady-state problem, or as the state about which the linearization is performed for an eigensolution. The reference deformations solution would lift the restriction that the same model must be used in the the steady-state solution and the eigensolution. Another valuable enhancement would be to extend GRASP to forward flight using either a time-domain solution, a periodic solution, or both.

Enhancements to the aerodynamics could include adding the capability for table-lookup for the aerodynamic coefficients, and perhaps making those coefficients functions of Mach number. Another possibility would be to incorporate a lifting-line or lifting surface theory to calculate the aerodynamic forces. Wake geometry could also be included. Other valuable enhancements to the aerodynamic model would be the inclusion of transonic and dynamic stall effects.

The modeling capabilities could also be improved with the addition of the ability to model applied loads. It might be advantageous to include simple, dead loads (forces), and geometrically nonlinear loads such as applied moments and nonconservative forces. With the rapid growth of control theory, some sort of control representation should be included in GRASP. This could be as simple as specifying the thrust level of the rotor, or as complex as a complete control representation including sensors, actuators, and control laws. In addition, it would be convenient to implement a "generic" node. Such a node would be used to allow the user to define generalized coordinates not associated with any of the predefined nodes.

GRASP would greatly benefit from the addition of a composite beam element and a direct-input element. The composite beam element would be able to rigorously treat the structural couplings introduced by composite layups. This element might also include the effects of shear deformation, initial curvature, and warping rigidity. The direct-input element would be used in conjunction with the generic node to allow the user to define the properties of elements that are not included in GRASP. An example of such a use would be taking a set of modes from a NASTRAN analysis to represent the fuselage of a helicopter.

New constraints that would enhance the capabilities of GRASP include a moving-frame constraint, a pin constraint, and a clamp constraint. The moving-frame constraint would allow a frame to deform with the structure. Currently, frame motion is independent of the structure. The pin constraint would allow a node to rotate arbitrarily about either a frame or another node. Eliminating all motion of a node would be accomplished using the clamp constraint.

From this description of possible enhancements, it should be obvious that GRASP has a great potential for growth. Because of its modular construction, GRASP has the capability to handle expansion without requiring massive rewriting of the existing equations and code. This framework makes GRASP a desirable platform for future development.

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16. Abstract

The Rotorcraft Dynamics Division, Aeroflightdynamics Directorate, U.S. Army Aviation Research and Technology Activity (AVSCOM) has developed the General Rotorcraft Aeromechanical Stability Program (GRASP) to calculate aeroelastic stability for rotorcraft in hovering flight, vertical flight, and ground contact conditions. In this report, GRASP is described in terms of its capabilities and its philosophy of modeling. The equations of motion that govern the physical system are described, as well as the analytical approximations used to derive the equations. These equations include the kinematical equation, the element equations, and the constraint equations. In addition, the solution procedures used by GRASP are described.

GRASP is capable of treating the nonlinear static and linearized dynamic behavior of structures represented by arbitrary collections of rigid-body and beam elements. These elements may be connected in an arbitrary fashion, and are permitted to have large relative motions. The main limitation of this analysis is that periodic coefficient effects are not treated, restricting rotorcraft flight conditions to hover, axial flight, and ground contact. Instead of following the methods employed in other rotorcraft programs, GRASP is designed to be a hybrid of the finite-element method and the multibody methods used in spacecraft analyses. GRASP differs from traditional finite-element programs by allowing multiple levels of substructures in which the substructures can move and/or rotate relative to others with no small-angle approximations. This capability facilitates the modeling of rotorcraft structures, including the rotating/nonrotating interface and the details of the blade/root kinematics for various rotor types. GRASP differs from traditional multibody programs by considering aeroelastic effects, including inflow dynamics (simple unsteady aerodynamics) and nonlinear aerodynamic coefficients.

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