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ON A CLASS OF UNSTEADY THREE-DIMENSIONAL NAVIER STOKES SOLUTIONS RELEVANT TO ROTATING DISC FLOWS: THRESHOLD AMPLITUDES AND FINITE TIME SINGULARITIES [1]

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Abstract

A class of exact steady and unsteady solutions of the Navier Stokes equations in cylindrical polar coordinates is given. The flows correspond to the motion induced by an infinite disc rotating with constant angular velocity about the z-axis in a fluid occupying a semi-infinite region which, at large distances from the disc, has velocity field proportional to $(x, -y, 0)$ with respect to a Cartesian coordinate system. It is shown that when the rate of rotation is large Karman's exact solution for a disc rotating in an otherwise motionless fluid is recovered. In the limit of zero rotation rate a particular form of Howarth's exact solution for three-dimensional stagnation point flow is obtained. The unsteady form of the partial differential system describing this class of flow may be generalized to time-periodic equilibrium flows. In addition the unsteady equations are shown to describe a strongly nonlinear instability of Karman's rotating disc flow. It is shown that sufficiently large perturbations lead to a finite time breakdown of that flow whilst smaller disturbances decay to zero. If the stagnation point flow at infinity is sufficiently strong the steady basic states become linearly unstable. In fact there is then a continuous spectrum of unstable eigenvalues of the stability equations but, if the initial value problem is considered, it is found that, at large values of time, the continuous spectrum leads to a velocity field growing exponentially in time with an amplitude decaying algebraically in time.

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1. Introduction.

Our concern is with the strongly nonlinear instability of the boundary layer on a rotating disc, the important feature of our investigation is that the stability problem which we discuss corresponds to an exact solution of the Navier Stokes equations. In addition our investigation uncovers a class of exact steady and unsteady Navier Stokes solutions relevant to the flow over a rotating disc immersed in a three-dimensional stagnation point flowfield.

The importance of the stability problem for the boundary layer on a rotating disc is due to the fact that the flow can be thought of as a prototype stability problem for boundary layer flows over swept wings. Some years ago Gregory, Stuart and Walker (1955) showed that the flow over a rotating disc is highly unstable to an inviscid 'crossflow' instability associated with the highly inflectional velocity profiles which occur in certain directions. Their calculations pointed to the particular importance of a stationary mode of instability associated with an effective velocity profile having an inflection point at a position of zero flow velocity. The structure of the Gregory, Stuart and Walker mode in the nonlinear regime was later discussed by Bassom and Gajjar (1988), the latter authors show that in that regime a nonlinear critical layer structure develops. This mode is apparently the one most preferred in an experimental situation if the background disturbance level is sufficiently small. However, though most experimental investigations of the rotating disc problem have clearly identified the stationary crossflow structure described by Gregory, Stuart and Walker (1955), some experiments have pointed to the existence of a second type of vortex structure associated with some type of subcritical response caused by another type of instability, see for example Federov et al (1976).

A possible cause for this second type of stationary vortex structure is the viscous stationary crossflow vortex identified numerically by Malik (1986) and described using essentially triple-deck theory by Hall (1986). Later MacKerrell (1987) was able to show that this mode is destabilized by nonlinear effects and therefore might cause the subcritical instability observed experimentally. However a key feature of the mechanism described by Hall (1986) is that the crucial balance of forces leading to instability is one between Coriolis and viscous forces, thus in swept wing flows this mechanism is possibly not operational. An alternative source of finite amplitude instability in more general three-dimensional boundary layers is the one discussed in this paper. Though we shall formulate and solve the resulting nonlinear interaction equations in cylindrical polar coordinates it is easy to see the relevance of the structure we find to flows more naturally described in Cartesian coordinates.

In Section 2 we shall formulate the nonlinear interaction equations describing the flow over a rotating disc immersed in a three-dimensional stagnation point flow. In Section 3 we shall discuss some steady equilibrium states of these equations, in particular we describe the nonunique nature of the solutions of these equations. In Section 4 we concentrate on the unsteady forms of the interaction equations and discuss the linear and nonlinear instability of the flow over a rotating disc in an otherwise still fluid. Our calculations in that Section point clearly to a threshold amplitude response of the flow; thus a sufficiently large initial disturbance causes an unbounded velocity field to develop after a finite time. The singularity structure associated with this 'blow-up' is discussed in Section 5. Since the interaction equations which we derive in Section 3 are obtained without neglecting any terms in the Navier Stokes equations the singularity discussed in Section 5 is a singularity of the full Navier Stokes equations in three dimensions. Finally in Section 6 we draw some conclusions.

2. The equations for combined disc-stagnation point flows.

With respect to cylindrical polar coordinates (r, θ, z) the Navier Stokes equations may be written

$$\begin{gathered}\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \bullet \nabla)\underset{\sim}{u} + \begin{pmatrix} -\frac{v^2}{r} \\ \frac{uv}{r} \\ 0 \end{pmatrix} = \frac{-1}{\rho}\nabla p + \nu \Delta \underset{\sim}{u} + \nu \begin{pmatrix} -\frac{u}{r^2} - \frac{2}{r^2} v_\theta \\ -\frac{v}{r^2} + \frac{2u_\theta}{r^2} \\ 0 \end{pmatrix}, \\ div\, \underset{\sim}{u} = 0, \end{gathered} \qquad (2.1)$$

where (u, v, w) is the velocity field corresponding to (r, θ, z) and p, ρ and ν are the fluid pressure, density and kinematic viscosity respectively. The operators ∇ and Δ appearing in (2.1) are the gradient and Laplacian operators in cylindrical polar coordinates. We define dimensionless time and axial variables T and ζ by

$$T = Kt, \quad \zeta = \left(\frac{K}{\nu}\right)^{\frac{1}{2}} z \qquad (2.2a, b)$$

where K is a constant. We seek a solution of (2.1) in the form

$$u = Kr\bar{u}(\zeta,T) + \frac{1}{2}\{K\gamma U(\zeta,T)e^{2i\theta} + c.c.\},$$

$$v = \Omega r\bar{v}(\zeta,T) + \frac{1}{2}\{iK\gamma U(\zeta,T)e^{2i\theta} + c.c\}, \tag{2.3a, b, c, d}$$

$$w = (K\nu)^{\frac{1}{2}}\bar{w}(\zeta,T),$$

$$\frac{p}{\rho} = \frac{1}{2}\lambda K^2r^2 + \nu K\bar{p}(\zeta,T) + \frac{1}{2}\gamma^2K^2r^2\{J(T)e^{2i\theta} + c.c.\}.$$

Here γ, λ are constants whilst $c.c.$ denotes 'complex conjugate'. We note that (2.3) reduces to Karman's solution for the flow over a rotating disc if we choose $\Omega = K$ and set $\gamma = 0$. If we substitute (2.3) into (2.1) we find the crucial result that the nonlinear terms in the radial and azimuthal momentum equations generate no terms proportional to $e^{\pm 4i\theta}$. This means that (2.3) is an exact Navier Stokes solution and we find that the equations to determine the functions appearing in (2.3) are

$$\left.\begin{aligned} &\bar{u}_T + \bar{u}^2 + \gamma^2|U|^2 - \left(\frac{\Omega}{K}\right)^2\bar{v}^2 + \bar{w}\bar{u}_\zeta = -\lambda + \bar{u}_{\zeta\zeta},\\ &\bar{v}_T + 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta},\\ &\bar{w}_T + \bar{w}\bar{w}_\zeta = -\bar{p}_\zeta + \bar{w}_{\zeta\zeta},\\ &2\bar{u} + \bar{w}_\zeta = 0, \end{aligned}\right\} \tag{2.4}$$

and

$$U_T + 2\bar{u}U + \bar{w}U_\zeta = -J + U_{\zeta\zeta}. \tag{2.5}$$

From now on we shall restrict our attention to the situation when $\Omega = K$; note that Karman's solution is retrieved by letting $\gamma \to 0$. Before writing down boundary conditions appropriate to (2.4), (2.5) it is perhaps worthwhile commenting on the motivation for the choice of the special form (2.1). In a recent paper, Balakumar, Hall and Malik (1990), investigated the instability of Karman's solution to nonparallel Rayleigh modes of wavenumber n in the azimuthal direction. These high Reynolds number modes can be made nonlinear in the manner suggested by the vortex-wave interaction structure of Hall and Smith (1990).

In that structure the amplitude of the nonparallel mode with azimuthal wavenumber n is adjusted until it drives a mean flow correction comparable with the unperturbed state. For 0(1) values of n it turns out that, because of the comparable size of the three disturbance velocity components in the analysis of Balakumar, Hall and Malik (1990) , only the $n = \pm 2$ modes can be made strongly nonlinear in the manner described by Hall and Smith (1990) and then the appropriate form of the disturbed flow is (2.1) . However the structure (2.1), suggested by the interaction described by Hall and Smith, is applicable at all Reynolds numbers rather than just at high Reynolds numbers where the work of Balakumar, Hall and Malik (1990) applies. Thus we can interpret (2.1) as a 'mean field' type of disturbed flow with the θ-dependent part representing a wave superimposed on Karman's solution which now evolves in time as the disturbance develops. We note that in this evolution procedure no terms in the Navier Stokes equations have been neglected.

We close this section with a discussion of the boundary conditions required to enable a solution of the system (2,4), (2.5). We assume that as the flow evolves the mean (with respect to θ) part of the velocity field, i.e. $(\bar{u}, \bar{v}, \bar{w})$, satisfies

$$\bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad \zeta = 0, \qquad \bar{u} = 0, \quad \bar{v} = 0, \quad \zeta \to \infty. \tag{2.6a, b}$$

Next we assume that the wavelike part of the flow satisfies

$$U = 0, \quad \zeta = 0, \quad U = e^{iNT}, \quad \zeta \to \infty \tag{2.7a, b}$$

where N is a constant dimensionless frequency. In order that the $\bar{u}$, and U equations are consistent with the above conditions λ and J must be choosen such that

$$\lambda = -\gamma^2, \quad J = -iNe^{iNT}. \tag{2.8a, b}$$

Having made the above choice of boundary conditions we can seek solutions of (2.4), (2.5) which are periodic in time with period $\frac{2\pi}{N}$, the steady states of (2.4),(2.5) are then found by setting $N = 0$. It follows from the form of the nonlinear terms in (2.4), (2.5) that periodic solutions have $(\bar{u}, \bar{v}, \bar{w})$ independent of time so that

$(\bar{u}, \bar{v}, \bar{w})$ and U satisfy

$$
\left.\begin{aligned}
& iNU + 2\bar{u}U + \bar{w}U_\zeta = U_{\zeta\zeta} + iN \\
& \bar{u}^2 + \gamma^2 \, |U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta = \gamma^2 + \bar{u}_{\zeta\zeta} \\
& 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta} \\
& 2\bar{u} + \bar{v}_\zeta = 0 \\
& \bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad U = 0, \quad \zeta = 0 \\
& \bar{u} = \bar{v} = 0, \quad \zeta \to \infty, \quad U = 1, \quad \zeta = \infty
\end{aligned}\right\} \tag{2.9}
$$

where we have replaced $U(\zeta, T)$ by $e^{iNT}U(\zeta)$. The nonperiodic solutions satisfy

$$
\left.\begin{aligned}
& U_T + 2\bar{u}U + \bar{w}U_\zeta = U_{\zeta\zeta} + iNe^{iNT} \\
& \bar{u}_T + \bar{u}^2 + \gamma^2|U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta = \gamma^2 + \bar{u}_{\zeta\zeta}, \\
& \bar{v}_t + 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta = \bar{v}_{\zeta\zeta}, \\
& 2\bar{u} + \bar{w}_\zeta = 0 \\
& \\
& \bar{u} = 0, \quad \bar{v} = 1, \quad \bar{w} = 0, \quad U = 0, \qquad \zeta = 0 \\
& \bar{u} = \bar{v} = 0, \quad \zeta = 0, \quad U = e^{iNT}, \quad \zeta = \infty \\
& \\
& \bar{u} = \hat{u}(\zeta), \quad \bar{v} = \hat{v}(\zeta), \quad U = \hat{U}(\zeta), \quad T = 0
\end{aligned}\right\} \tag{2.10}
$$

Thus the periodic solutions can be found by integrating an ordinary differential system, (2.9), wheras the unsteady modes satisfy a parabolic partial differential system. For that reason we have been required in (2.10) to give initial conditions to completely specify the problem for $\bar{u}, \bar{v}, \bar{w}$ and U. Furthermore we note that (2.10) can be regarded as the appropriate nonlinear initial value instability problem for the periodic problem (2.9). In the next section we shall discuss the solutions of (2.9), the solution of (2.10) will be discussed in §4.

3 Equilibrium solutions of the interaction equations.

In order to begin the solution of (2.9) by some appropriate numerical method it is convenient to discuss a limiting form of that system which can then be used to begin the calculation. The first limit we consider is $\gamma \to 0$ in which case the problem for $\bar{u}, \bar{v}$ and $\bar{w}$ becomes uncoupled from that for U and is in fact simply Karman's solution. Thus we know that in the limit $\gamma \to 0, \bar{u}'(0) \simeq .50, \bar{v}'(0) \simeq -.61$. Another known flow is found in the limit $\gamma \to \infty$. In that limit we write

$$\begin{aligned} \bar{u} &= \gamma \bar{u}_0 + \cdots, \\ \bar{v} &= \bar{v}_0 + \cdots, \\ \bar{w} &= \gamma^{\frac{1}{2}} \bar{w}_0 + \cdots, \\ U &= U_0 + \cdots \end{aligned} \tag{3.1a, b, c, d}$$

where the functions appearing in the above expansions are functions of the stretched variable $\xi = \gamma^{\frac{1}{2}}\zeta$. In this limit the problem for $\bar{u}_0, U_0, \bar{w}_0$ decouples from $\bar{v}_0$ and we obtain the system

$$\left.\begin{aligned} & iNU_0 + 2\bar{u}_0 U_0 + \bar{w}_0 U_0' = U_0'' + iN, \\ & \bar{u}_0^2 + |U_0|^2 + \bar{w}_0 \bar{u}_0' = 1 + \bar{u}_0'', \\ & 2\bar{u}_0 + \bar{w}_0' = 0, \\ & \bar{u}_0 = 0, \quad \bar{w}_0 = 0, \quad U_0 = 0, \quad \xi = 0, \\ & \bar{u}_0 = 0, \quad U_0 = 1, \quad \xi \to \infty. \end{aligned}\right\} \tag{3.2}$$

In the special case $N = 0$ we can relate (3.2) to a special case of the three-dimensional stagnation point flows considered by Howarth (1951), Davey (1961), Banks and Zaturska (1988). We recall that, with respect to Cartesian coordinates x, y, z Howarth identified the class of exact Navier-Stokes solutions given by

$$u = \frac{Ux}{\ell} f'(\eta), \quad v = \frac{Vy}{\ell} g'(\eta), \quad w = -\left(\frac{\nu}{\ell U}\right)^{\frac{1}{2}} (Uf(\eta) + Vg(\eta)), \tag{3.3}$$

where ℓ is a length and

$$\eta = \left(\frac{U}{\gamma \ell}\right)^{\frac{1}{2}} z. \tag{3.4}$$

Here f', g' satisfy

$$\left.\begin{aligned} & f'^2 - (f + \alpha g) f'' = 1 + f''' \\ & g'^2 - (g + \frac{1}{\alpha} f) g'' = 1 + \frac{1}{\alpha} g''' \end{aligned}\right\} \tag{3.5}$$

with $\alpha = V/U$ and subject to

$$
\begin{aligned}
f = g = f' = g' = 0, &\quad \eta = 0 \\
f' = g' = 1, &\quad \eta \to \infty.
\end{aligned}
\tag{3.6}
$$

These solutions correspond to a three-dimensional stagnation point flow above the plane $z = 0$. In the special case $\alpha = -1$ we can relate f' and g' to the functions $\bar{u}_0, U_0$ with $N = 0$ by writing

$$2\bar{u}_0 = f' - g', \quad -\bar{w}_0 = f - g, \quad 2U_0 = f' + g'.$$

Davey (1961) gives $f'' = 1.2729, g'' = -.8112$ which suggests that for large γ the solutions of (2.9) with $N = 0$ are such that

$$
\begin{aligned}
\bar{u}'(0) &= 1.042\gamma^{\frac{3}{2}} + \cdots, \\
U'(0) &= .231\gamma^{\frac{1}{2}} + \cdots.
\end{aligned}
\tag{3.7a, b}
$$

However we shall see below that the solution of (2.9) is not unique so Davey's solution correspond to only one of our solutions at large values of γ. For nonzero values of N a similar asymptotic structure can be obtained but the coefficients in the expansions (3.7a,b) will, of course, be functions of the frequency. Before giving the results of our numerical investigation of (2.9) we note that, for $\gamma >> 1$, the dominant terms in the steady state solution of (2.3a,b,c) are such that

$$(u, v, w) \sim rK\gamma(\bar{u}_0 + U_0 \cos 2\theta, \quad -U_0 \sin 2\theta, \quad \left(\frac{\nu}{K\gamma}\right)^{\frac{1}{2}} w_0) \tag{3.8}$$

where without any loss of generality we have taken U_0 to be real. If we transform (3.8) to Cartesian coordinates we obtain a velocity field

$$\underline{u} \sim K\gamma[x\bar{u}_0 + xU_0, \quad y\bar{u}_0 - yU_0, \quad \left(\frac{\nu}{K\gamma}\right)^{\frac{1}{2}} \bar{w}_0], \tag{3.9}$$

and comparison of (3.3), (3.9) then confirms our previous result (3.6).

In practice the numerical solution of (2.9) and indeed the reduced large γ problem, (3.2), is not straightforward. The reason why there is a difficulty with the numerical solution of (3.2) was first discussed by Davey (1961) and later in more detail by Schofield and Davey (1967). In order to see what this difficulty is we consider the large ζ limit of the equations to determine $(\bar{u}, \bar{v}, \bar{w}, U)$ in (2.9) with $N = 0$. Suppose that for $\zeta >> 1$ we write

$$(\bar{u}, \bar{v}, \bar{w}, U) = (u^+, v^+, w^+ + w_\infty, \quad 1 + U^+)$$

where u^+, v^+, w^+ and U^+ are small and for simplicity we assume that U^+ is real. It is an easy matter to show that the linearised equations for (u^+, v^+, w^+, U^+) can be reduced to

$$\left(\frac{\partial^2}{\partial\zeta^2} - w_\infty \frac{\partial}{\partial\zeta}\right)^2 u^+ = 4\gamma^2 u^+,$$

$$\left(\frac{\partial^2}{\partial\zeta^2} - w_\infty \frac{\partial}{\partial\zeta}\right) v^+ = 0, \tag{3.10a, b}$$

and U^+, w^+ can be found in terms of u^+, v^+. Thus for large ζ we can write

$$u^+ = A_1 e^{m_1\zeta} + A_2 e^{m_2\zeta} + A_3 e^{m_3\zeta},$$

$$v^+ = B_1 e^{w_\infty \zeta}.$$

Here m_1, m_2, m_3 are given by the values of m which satisfy

$$2m = w_\infty \pm \sqrt{w_\infty^2 \pm 8\gamma}, \quad m_r < 0, \tag{3.11}$$

and we note that two of the required values will be complex when $\gamma > \dfrac{w_\infty^2}{8}$. Thus we have five independent constants w_∞, A_1, A_2, A_3 and B_1 which can be iterated upon in order to satisfy the conditions at $\zeta = 0$. It follows that there will be a continuum of solutions of (2.9) since there is no reason to ignore any of the decaying exponential solutions. Clearly it is to be expected that w_∞ will vary in this continuum so the solutions can be labeled by the size of the inflow towards the disc at infinity. This situation persists in the large γ limit where $w_\infty^2 \sim \gamma$; Schofield and Davey (1967) argued that the solutions should in this case be fixed by discarding the slowest decaying exponential. Whilst it is certainly true that this fixes the solution, there is no basis for making such an assumption. Having made that assumption Schofield and Davey concluded that, in our notation, $w_\infty^2 = 8\gamma$. Interestingly enough we shall see later that this choice of w_∞ fixes the boundary between linearly stable and unstable solutions of (2.9). We now present results obtained for the system (2.9) in the steady case $N = 0$.

As mentioned above, at large values of ζ we have five constants, $A_1, B_1, A_2, A_3, w_\infty$ at our disposal once the constant γ has been fixed. In our calculations we fixed γ, w_∞ and iterated on the remaining four constants until the required boundary conditions at $\zeta = 0$ were satisfied after integrating the differential equations in (2.9) from a suitably large value of ζ to the origin. This integration was carried out using a fourth order Runge-Kutta scheme or a compact finite difference scheme We restricted our attention to the cases $\gamma = .02, .1, .5$ and in Table I we show the values of $\bar{u}'(0), \bar{v}'(0), \gamma U'(0)$ obtained for the different values of γ, w_∞ shown. We see that for each of the values of γ there are values of w_∞^2 greater and less than 8γ. For each of the values of γ used we were able to find solutions of (2.9) only for w_∞ less than some critical value. Thus for example when $\gamma = .02$ we were unable to find solutions of (2.9) for w_∞ greater than$-.355$. Our calculations suggested that this minimum value decreases when γ increases. In Figures (3.1-3.4) we show the functions $\bar{u}, \bar{v}, \bar{w}, U$ for different values of γ, w_∞ Further solutions of (2.9) were obtained by fixing w_∞ and varying γ. In particular we obtained solutions for the case $w_\infty^2 = 8\gamma$ and our results are shown in Figures (3.5-3.6). We recall that Davey obtained solutions of the large γ problem in a different context and that his results were obtained by neglecting the slowest decaying exponential solution at large ζ. Having made that approximation Davey found numerically that $w_\infty^2 = 8\gamma$ to the numerical accuracy of his calculations. Thus we expect that our results in Figures (3.5-3.6) should reduce to those of Davey at sufficiently large values of ζ. In fact we have in these figures shown Davey's results expressed in our notation and we see that our results approach (3.7) for large γ. However we stress at this point that there is no reason why the solutions of (2.9) obtained by rejecting a particular decaying exponential solution of that system at large ζ should be preferred, we hope to shed some light on the selection mechanism for the different solutions later in this paper.

4 Time-dependent solutions of the interaction equations

We shall now discuss the solutions we have obtained for the unsteady form of the interaction equations. We restrict our attention to the case when the stagnation point flow vanishes at infinity. This means that we are in effect discussing the finite amplitude instability of Karman's solution. In the Appendix we shall give a limited discussion of the more general problem with $\gamma \neq 0$. The appropriate simplified form of (2.10) is found by setting $\gamma = 1$ and applying the conditions $U = 0, \zeta = 0, \infty$; we obtain

$$
\left.\begin{aligned}
U_{\zeta\zeta} - U_T &= 2\bar{u}U + \bar{w}U_\zeta, \\
\bar{u}_{\zeta\zeta} - \bar{u}_T &= \bar{u}^2 + |U|^2 - \bar{v}^2 + \bar{w}\bar{u}_\zeta, \\
\bar{v}_{\zeta\zeta} - \bar{v}_T &= 2\bar{u}\bar{v} + \bar{w}\bar{v}_\zeta, \\
\bar{u} = \bar{w} = U &= 0, \bar{v} = 1, \zeta = 0, \\
\bar{u} = \bar{v} = U &= 0, \zeta = \infty, \\
\bar{u} = \hat{u}(\zeta), v = \hat{v}(\zeta), U &= \hat{U}(\zeta), T = 0.
\end{aligned}\right\} \tag{4.1}
$$

The above system is parabolic in T and can be solved by marching forward in time from $T = 0$; we note here in passing that $\bar{w}$ cannot be specified arbitrarily at $T = 0$ and must be deduced from $\hat{u}$ via the equation of continuity. For large values of T the solution of (4.1) will approach Karman's solution if that flow is stable. We can therefore regard (4.1) as the nonlinear initial value instability problem for Karman's rotating disc flow. However, we should bear in mind that (4.1) describes only finite amplitude disturbances with azimuthal wavenumbers ± 2.

In the first instance we restrict our attention to small initial perturbations from Karman's solution, we therefore write

$$
(\hat{u}, \hat{v}, \hat{U}) = (\bar{\bar{u}}, \bar{\bar{v}}, 0) + (u^*, v^*, U^*)
$$

where $(\bar{\bar{u}}, \bar{\bar{v}})$ is Karman's solution and u^* etc are small. We now substitute

$$
(\bar{u}, \bar{v}, \bar{w}, U) = (\bar{\bar{u}} + \tilde{u}, \bar{\bar{v}} + \tilde{v}, \bar{\bar{w}} + \tilde{w}, \tilde{U})
$$

in (4.1) and linearize to obtain the following decoupled problems:

$$
\left.\begin{aligned}
\tilde{U}_{\zeta\zeta} - \tilde{U}_T &= 2\bar{\bar{u}}\tilde{U} + \bar{\bar{w}}\tilde{U}_\zeta, \\
\tilde{U} &= 0, \zeta = 0, \infty, \\
\tilde{U} &= U^*, T = 0,
\end{aligned}\right\} \tag{4.2}
$$

and

$$
\left.\begin{aligned}
&\tilde{u}_{\zeta\zeta} - \tilde{u}_T = 2\bar{u}\tilde{u} - 2\bar{v}\tilde{v} + \bar{w}\tilde{u}_\zeta + \tilde{w}\bar{u}_\zeta, \\
&\tilde{v}_{\zeta\zeta} - \tilde{v}_T = 2\bar{u}\tilde{v} + 2\bar{v}\tilde{u} + \bar{w}\tilde{v}_\zeta + \tilde{w}\bar{v}_\zeta, \\
&2\tilde{u} + \tilde{w}_\zeta = 0, \\
&\quad \tilde{u} = \tilde{v} = \tilde{w} = 0, \zeta = 0, \\
&\quad \tilde{u} = \tilde{v} = 0, \zeta = \infty, \\
&\quad (\tilde{u}, \tilde{v}) = (u^*, v^*), \quad T = 0.
\end{aligned}\right\} \tag{4.3}
$$

It should be pointed out that in the above equations we have in effect assumed that $O(U^*) \sim O(u^*) \sim O(v^*)$; a modified form of the equations can be derived when $O(U^*) \sim O(u^*)^{\frac{1}{2}}$. In that case a nonlinear term $|\tilde{U}|^2$ must be inserted into the right hand side of the $\tilde{u}$ equation. This particular case would be important only if the $\tilde{U}$ problem were unstable. We can integrate (4.2),(4.3) formally by taking a Laplace transform in time, when the transform is inverted the nature of the solution will depend crucially on whether either of the eigenvalue problems

$$
\left.\begin{aligned}
&y'' - \sigma y = 2\bar{u}y + \bar{w}y', \\
&y(0) = y(\infty) = 0,
\end{aligned}\right\} \tag{4.4}
$$

or

$$
\left.\begin{aligned}
&y'' - \sigma y = 2\bar{u}y - 2\bar{v}z + \bar{w}y' + x\bar{u}', \\
&z'' - \sigma z = 2\bar{u}z + 2y\bar{v} + \bar{w}z' + x\bar{v}', \\
&2y + x' = 0, \\
&x(0) = y(0) = z(0) = y(\infty) = z(\infty) = 0,
\end{aligned}\right\} \tag{4.5}
$$

has an eigenvalue σ with positive real part. Thus we shall now discuss these eigenvalue problems.

The above eigenvalue problems were solved numerically; no unstable eigenvalues of either system were found so we conclude that the Karman's solution is stable to small amplitude perturbations of the type discussed here. In fact no discrete stable eigenvalues were obtained either, this is because both eigenvalue problems have a continuous spectrum over part of the plane $\sigma_r < 0$. The origin of this continuous spectrum can be seen from (4.4) by taking $\zeta >> 1$. We see then that the two exponential solutions of the equation for y both decay if σ is within the parabola $\sigma_r = -\sigma_i^2/w_\infty^2$ where $w_\infty = \bar{w}(\infty)$. Thus in this region we can always find a solution of (4.4) by combining the two independent solutions for y to satisfy the required condition at the wall. A similar continuous spectrum can be seen to exist for the system (4.5); we expect the continuous spectra play an important role in the initial value problems (4.2),(4.3). Indeed since there is apparently no discrete spectrum associated with (4.4),(4.5) it is clear that the initial value problem must in some sense be described completely by the continuous spectrum.

The initial value problem can be solved by taking Laplace transforms and inverting for particular forms of the initial perturbation, these inversions cannot in general be carried out analytically but their large time behaviour can be approximated asymptotically in a routine manner. Rather than use the Laplace transform method we shall instead look directly for the large time behaviour of (4.2),(4.3). We shall in fact restrict our attention to (4.2), but a similar approach can be used for (4.3).

Suppose then that w_∞ denotes the limiting value of $\bar{w}$ at large values of ζ, we choose to express $\tilde{U}$ in the form

$$\tilde{U} = N(\zeta, T)e^{\frac{w_\infty \zeta}{2} - \frac{w_\infty^2 T}{4}}, \tag{4.6}$$

where the function N then satisfies the modified equation

$$N_{\zeta\zeta} + (w_\infty - \bar{w})N_\zeta + \frac{(-\bar{w}w_\infty + w_\infty^2)}{2}N = N_T + 2\bar{u}N. \tag{4.7}$$

At this stage we assume that all the exponential time dependence of the disturbance has been taken out by the substitution (4.6) so that N has only an algebraic dependence on T. It is well-known for the heat equation that the similarity variable $\frac{\zeta}{T^{\frac{1}{2}}}$ essentially replaces ζ when the the initial value problem is solved. Here the situation is slightly more complicated and we must seek a solution of (4.7) for $\zeta = O(1)$ and $\zeta = O(T^{\frac{1}{2}})$ for large values of T. Thus for $\zeta = O(1)$ we write

$$N = T^j N_0(\zeta) + \cdots,$$

where j is to be determined and N_0 satisfies the ordinary differential equation

$$N_0'' + (w_\infty - \bar{w})N_0' + \frac{(-\bar{w}w_\infty + w_\infty^2)}{2}N_0 = 2\bar{u}N_0. \tag{4.8}$$

This equation must be solved subject to $N_0 = 0, \zeta = 0$ and the resulting solution will then have $N_0 \sim c\zeta$ for large ζ, here c is an arbitrary constant which can be set equal to unity but whose actual value depends on the form of the initial disturbance. Now let us find the required form for N in the upper region, before doing so we note that the constant j is left unknown at this stage since it plays no role in the zeroth order solution in the lower region. In the upper layer we write

$$N = T^{j+\frac{1}{2}}\tilde{N}_0(\chi) + \cdots, \tag{4.9}$$

where χ is the similarity variable $\frac{\zeta}{T^{\frac{1}{2}}}$. Note here that the matching condition with the lower layer now requires that for small χ, $\tilde{N}_0 \sim \chi$.

The equation to determine $\tilde{N}_0$ is found to be

$$\tilde{N}_0'' + \frac{\chi}{2}N_0' - (j + \frac{1}{2})\tilde{N}_0 = 0. \tag{4.10}$$

In order that the disturbance decays to zero at large values of χ we must insist that $\tilde{N}_0$ behaves like the exponentially decaying solution of the above equation; since it must also go to zero like χ for small χ we can show that the required solution is

$$\tilde{N}_0 = \frac{2^{\frac{1}{2}}e^{\frac{-\chi^2}{8}}U(-n-\frac{1}{2}, 2^{-\frac{1}{2}}\chi)}{U'(-n-\frac{1}{2}, 0)}, \tag{4.11}$$

Here n is an odd integer, $U(a, x)$ is the parabolic cylinder function whilst the constant j has been choosen to satisfy the matching condition at $\chi = 0$, this gives

$$j = -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2} + \cdots. \tag{4.12}$$

It follows that the solution will be dominated by the $j = -\frac{3}{2}$ eigensolution for large enough values of the time, the overall amplitude of this and the other decaying modes can only be determined by solving the initial value problem. In order to verify the above large time behaviour of the solution of the linearized perturbation equation for $\tilde{U}$ we integrated (4.2) forward in time from $T = 0$ for the three cases:

CASE a $\quad U^* = \zeta e^{-\zeta^2}$,

CASE b $\quad U^* = \zeta cos\zeta e^{-\zeta}$,

CASE c $\quad U^* = \zeta e^{-(\zeta-2)^2}$.

We note that it is sufficient for us to consider real initial data for $\tilde{U}$ if we are solving (4.2). The results we obtained are shown in Figure (4.1); in order to pick out the dominant exponential decay factor of $\tilde{U}$

we have plotted $(log(T^{\frac{3}{2}}\tilde{U}'(0,T))_T$. On the basis of our discussion above we see that this quantity should tend to $-\frac{w_\infty{}^2}{4} \sim -.2$ for large T. We see that each of the above cases leads to results consistent with our predictions. A similar analysis to that given above for (4.2) can be given for (4.3), again the outcome is that a two-layer structure is required to describe the large time behaviour of the disturbance, furthermore the functions $\tilde{u}, \tilde{v}$ are found to decay exponentially for large T with the same decay rate as that found above.

We shall now report on some calculations carried out for the full nonlinear problem (4.1) with initial conditions

$$\hat{u} = \hat{v} = 0, \quad \hat{U} = \delta\zeta e^{-(\zeta-2)^2}, \tag{4.13}$$

for $\delta = .35, .45, .55, .65$. The results obtained for the initial conditions given above are typical of those we have found for a wide range of disturbances. The results we found are shown in Figure (4.2a,b,c,d,) where we have shown the growth rates $\left(log\int_0^\infty u^2 d\zeta\right)_T, \left(log\int_0^\infty U^2 d\zeta\right)_T, \left(log\int_0^\infty V^2 d\zeta\right)_T, (log(U_\zeta(0,T).T^{\frac{3}{2}})_T$. For the two smaller values of the amplitude constant δ we see that the disturbance decays to zero so that Karman's solution is stable, note that Figure (4.2a) for $\delta = .35$ confirms the linear decay rate, -.2, at sufficiently large values of T when the disturbance field is now small. The calculations for the three larger values of δ demonstrate that Karman's solution is subcritically unstable. At a finite value of T our calculations encountered a singularity and could not be continued further. We did, of course, check that the singularity remains when the ζ and T step lengths were decreased. In Figures (4.3a,b,c,d) we show the velocity field for the case $T = 3.3, \delta = .55$. Further calculations for other initial disturbances confirmed the threshold type of response described above.

Thus we have found that Karman's rotating disc solution is unstable to finite amplitude $n = \pm 2$ modes wheras in the linear regime we have stability to this type of disturbance. We have made no attempt to investigate the dependence of the required threshold amplitude of the instability for different initial disturbance profiles. For the case discussed above instability occurs when the disturbance velocity field is of size comparable to the unperturbed state. Thus this particular type of disturbance is unlikely to be present in an experimental investigation so that it is unlikely that the this disturbance would cause transition. However it is not unreasonable to expect that a detailed investigation of a more general class of initial conditions would isolate much more dangerous disturbances likely to cause instability in an experimental facility with moderate background disturbances.

5. The structure of the Singular solutions

Suppose that the singularity encountered in the numerical calculations occurs at time $T = \bar{T}$. For 0(1) values of ζ then we expect that the inviscid terms in (2.9) will dominate the flow; thus in the neighbourhood of $\bar{T}$ we expect that $\bar{u} \sim \bar{v} \sim U \sim \bar{w} \sim 0(T-\bar{T})^{-1}$. We therefore expand the velocity components as

$$\begin{aligned}
\bar{u} &= \frac{u_s(\zeta)}{T-\bar{T}} + \dots, \\
\bar{v} &= \frac{v_s(\zeta)}{T-\bar{T}} + \dots, \\
\bar{w} &= \frac{w_s(\zeta)}{T-\bar{T}} + \dots, \\
U &= \frac{U_s(\zeta)}{T-\bar{T}} + \dots \quad .
\end{aligned} \tag{5.1a, b, c, d}$$

We can then show from (2.9) that the system to determine (u_s, v_s, w_s, U_s) is

$$\begin{aligned}
u_s &= u_s{}^2 - v_s{}^2 + \gamma^2 \mathrm{U}_s^2 + w_s u_{s\zeta}\,, \\
v_s &= 2u_s v_s + w_s v_{s\zeta}\,, \\
U_s &= 2U_s u_s + w_s U_{s\zeta}\,, \\
w_{s\zeta} &+ 2u_s = 0
\end{aligned} \tag{5.2a, b, c, d}$$

$$\begin{aligned}
w_s &= 0, \ \ \zeta = 0\,, \\
u_s &= v_s = U_s = 0, \ \ \zeta = \infty\,.
\end{aligned} \tag{5.3a, b}$$

Since the boundary conditions and differential equations for U_s and v_s are the same we can write

$$U_s = \Delta v_s$$

where Δ is a constant. In fact the equation for v_s is easily integrated to give

$$v_s = Q w_s e^{\int_\infty^\zeta \{\frac{1}{w_s} - \frac{1}{w_{s\infty}}\} d\zeta + \frac{\zeta}{w_{s\infty}}}\,,$$

where Q is a constant and $w_{s\infty}$ is $w_s(\infty)$. It is sufficient for us to choose $Q = 1$ and write the u_s equation in the form

$$u_s = u_s{}^2 + \delta v_s{}^2 + w_s v_{s\zeta} \tag{5.4}$$

when δ is a constant to be specified and it lies in the range $(-\infty, \infty)$. The case $\delta = 0$ corresponds to $U_s = v_s$ and in that case the equations for u_s, w_s can be integrated directly to give

$$w_s = w_{s\infty}(u-1)^2 \ ,$$
$$u_s - \ell n u_s = 1 - \frac{\zeta}{w_{s\infty}} \ , \tag{5,5a,b}$$

It is easy to show that (5,5b) then defines a single-valued function $u_s = u_s(\zeta)$ satisfying

$$u_s \sim 1, \quad \zeta \to 0$$
$$u_s \sim e^{\frac{\zeta}{w_{s\infty}}}, \quad \zeta \to \infty . \tag{5.6a,b}$$

For small values of ζ the above solution must be matched onto a wall layer solution valid for $\zeta = 0(T-\bar{T})^{-\frac{1}{2}}$. However we will now return to the more general case $\delta \neq 0$; it is easy to show that w_s satisfies

$$w_s{}' = \frac{w_s{}'^2}{2} - \delta w_s e^{\int_\infty^\zeta \left(\frac{1}{w_s} - \frac{1}{w_{s\infty}}\right)d\zeta + \frac{\zeta}{w_{s\infty}}} + w_s w_s'' \tag{5.7}$$

and for large values of ζ we then find that

$$w_s \sim -w_{s\infty} + Be^{\frac{\zeta}{w_{s\infty}}}, \tag{5.8}$$

where B is an arbitrary constant which is fixed by the condition $w_s(0) = 0$. For small values of ζ we can show that w_s, v_s have the forms

$$v_s \sim -b\zeta^{\frac{1}{2}} + \cdots$$
$$w_s \sim 2\zeta + a\zeta^{\frac{3}{2}} + \cdots \tag{5.9a,b}$$

where a, b are constants fixed by the choice of the arbitrary constant B appearing in the large ζ form of w_s. In Figure (5.1) we show the function w_s obtained by integrating (5.7) for three values of δ. In these calculations we have fixed $w_{s\infty} = -1$. It remains for us to show how the inviscid forms (5.9) for $\zeta << 1$ can be matched onto a wall layer solution valid for $\zeta = 0(T - \bar{T})^{-\frac{1}{2}}$. If we define the similarity variable $\eta = \dfrac{\zeta}{(T-\bar{T})^{\frac{1}{2}}}$ then, in view of (5.9), we see that the appropriate expansions are

$$u = \frac{u^0(\eta)}{(T-\bar{T})} + \cdots,$$
$$v = \frac{v^0(\eta)}{(T-\bar{T})^{\frac{3}{4}}} + \cdots, \tag{5.10a,b,c}$$
$$w = \frac{w^0(\eta)}{(T-\bar{T})^{\frac{1}{2}}} + \cdots .$$

The equations to determine u^0, v^0, w^0 are found to be

$$\left.\begin{aligned} &u^{0''} + \eta/2u^{0'} + \frac{1}{2}u^0 = u^{0^2} + w^0 u^{0'}, \\ &v^{0''} + \eta/2v^{0'} + \frac{3}{4}v^0 = w^0 v^{0'} + 2u^0 v^0, \\ &w^{0'} + 2u^0 = 0, \\ &u^0 = v^0 = w^0 = 0, \quad \eta = 0 \\ &u^0 \sim 1, \quad v^0 \sim \eta^{\frac{1}{2}}, \quad w^0 \sim \eta, \quad \eta \to \infty, \end{aligned}\right\} \tag{5.11}$$

From the asymptotic forms of the equations for u^0, v^0, w^0 at large η we find that there are three free constants which can be chosen so that the required conditions at $\eta = 0$ are satisfied. Thus we conclude that the viscous wall layer at $\zeta = 0(T - \bar{T})^{-\frac{1}{2}}$ smoothes out the singular inviscid solutions so as to satisfy the boundary conditions. We conclude then by noting that the singularity structure given above is of course a singularity of the full Navier Stokes equations; in the physical situation we would anticipate that as the singularity develops the flow becomes unstable to disturbances with a more general θ dependence. Presumably the growth of these other modes destroys the singularity structure and the Navier Stokes equations remain valid.

6. Conclusion

We have seen in the previous section that sufficiently large initial perturbations to Karman's rotating disc flow lead to the development of a singularity of the Navier Stokes equations. We note that the structure of the singularity we have described does not depend at zeroth order on the nature of the underlying basic state. Therefore it is likely that the singularity structure we have found could be set up by sufficiently large amplitude perturbations to other basic states.

If we assume that the structure we have found can be induced experimentally then a question of some importance is that of how the singularity can be controlled through the Navier Stokes equations once it has begun. Since no terms in the Navier Stokes equations have been neglected in the analysis leading to the singularity it might appear that the singularity must be controlled by an alternative set of field equations. However we do not believe that is the case, more precisely we believe that once the singularity has begun to develop the velocity profiles associated with it will be massively unstable to inviscid modes with azimuthal wavenumbers $\neq \pm 2$; these modes will then grow and prevent the further development of the singularity.

The latter conjecture could of course be verified by Navier Stokes simulations, we do not attempt such an investigation here.

The typical size of initial amplitude associated with (4.13) required to cause the development of a singularity was found to be comparable with a typical basic state velocity. Thus experimentally the breakdown we have described could only be provoked by a disturbance amplitude unlikely to be present in an experimental situation. However it would be very surprising if the initial amplitude required to induce the singularity could not be significantly reduced by allowing a much more general initial perturbation. A possible way of isolating the most dangerous type of initial perturbation would be to formulate the Energy Stability problem associated with the interaction equations.

Another open question following our discussion in Section 3 is that of which, if any, of the different equilibrium states for non zero γ are most physically relevant. Our linear instability analysis in the Appendix strongly suggests that all those equilibrium states with $w_\infty^2 < 8\gamma$ are not relevant physically since they are linearly unstable. It remains an open question as to whether nonlinear effects are able to further reduce the class of physically realizable flows.

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Appendix. The linear instability problem for the steady states with $\gamma \neq 0$.

Here we shall discuss the instability of the steady equilibrium solutions of (2.9) with $N = 0$. We denote this steady state by $(\bar{u}, \bar{v}, \bar{w}, \bar{U})$ and suppose that we consider the instability of this flow to a time-dependent small perturbation $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U})$ such that

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U}) = (u^*, v^*, w^*, U^*), \quad T = 0. \tag{A1}$$

From (2.10) the linearised perturbation equations for $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{U})$ are

$$\left.\begin{aligned}
&\tilde{U}_T + 2\bar{u}\tilde{U} + 2\tilde{u}\bar{U} + \bar{w}\tilde{U}_\zeta + \tilde{w}\bar{U}_\zeta = \tilde{U}_{\zeta\zeta},\\
&\tilde{u}_T + 2\bar{u}\tilde{u} + 2\gamma^2\bar{U}\tilde{U} - 2\bar{v}\tilde{v} + \bar{w}\tilde{u}_\zeta + \tilde{w}\bar{u}_\zeta = \tilde{u}_{\zeta\zeta},\\
&\tilde{v}_T + 2\bar{u}\tilde{v} + 2\tilde{u}\bar{v} + \bar{w}\tilde{v}_\zeta + \tilde{w}\bar{v}_\zeta = \tilde{v}_{\zeta\zeta},\\
&2\tilde{u} + \tilde{w}_\zeta = 0,
\end{aligned}\right\} \tag{A2}$$

subject to

$$\begin{aligned}
&\tilde{u} = \tilde{v} = \tilde{w} = \tilde{U} = 0, \quad \zeta = 0,\\
&\tilde{u} = \tilde{v} = \tilde{U} = 0, \quad \zeta = \infty.
\end{aligned} \tag{A3}$$

and (A1). Again the initial value problem can be solved by a Laplace transform in T but the form of (A2) means that the resulting ordinary differential equations in ζ must be solved numerically. Following our approach in §4 we consider the eigenvalue problem $\sigma = \sigma(\gamma)$ obtained by replacing ∂_T by σ in (A2) and applying (A3). The structure of the disturbed flow at $\zeta = \infty$ is then found by letting $\zeta \to \infty$ in (A2) after replacing ∂_T by σ and ∂_ζ by m; the appropriate equations to determine m are

$$\begin{aligned}
&(m^2 - w_\infty m - \sigma)\tilde{U} = 2\tilde{u},\\
&(m^2 - w_\infty m - \sigma)\tilde{u} = 2\gamma\tilde{U},\\
&(m^2 - w_\infty m - \sigma)\tilde{v} = 0.
\end{aligned} \tag{A4a, b, c}$$

Thus m is given by

$$\begin{aligned}
&2m = w_\infty \pm \sqrt{w_\infty^2 + 4\sigma}\\
&2m = w_\infty \pm \sqrt{w_\infty^2 + 4[\sigma \pm 2\gamma]}.
\end{aligned} \tag{A5a, b}$$

The first of these equations corresponds to (A4c) and leads to the continuous spectrum $\sigma_r < -\dfrac{\sigma_i^2}{w_\infty^2}$ found in §4. However (A5a,b) lead to the continuous spectra,

$$\sigma_r \leq \pm 2\gamma - \frac{\sigma_i^2}{w_\infty^2}. \tag{A6a, b}$$

The positive root corresponds to the case when $\tilde{u} = -\gamma\tilde{U}$ in (A4a,b) and, surprisingly, we obtain an unstable continuous spectrum for $\gamma > 0$. However, it remains to be seen whether this unstable continuous spectrum can induce a physically relevant exponentially growing solution. In order to answer this question we shall now seek a large time solution of (A2); the structure we choose is based on the assumption that at large times the unstable spectrum (A6a) effectively dominates the flow. The first step in our solution procedure is the substitution

$$(\tilde{u},\tilde{v},\tilde{w},\tilde{U}) = (u^+,v^+,w^+,U^+)\exp\left\{\frac{w_\infty}{2}\zeta - \frac{w_\infty^2}{4}T + 2\gamma T\right\}. \tag{A7}$$

The functions u^+, v^+ etc. are then found in the regions where $\zeta = 0(1), \zeta = 0(T^{\frac{1}{2}})$. In the region where $\zeta = 0(1)$ we write

$$(u^+,v^+,w^+,U^+) = T^j(u_0^+ v_0^+, w_0^+, U_0^+) + \cdots$$

where u_0^+ etc. depend only on ζ. The problem for these functions is obtained from (A2) and solved subject to

$$u_0^+ = v_0^+ = w_0^+ = U_0^+ = 0 \quad \zeta = 0$$

and the unknown quantities $u_0^{+'}(0), v_0^{+'}(0), w_0^{+'}$ are chosen such that

$$u_0^+ \sim \gamma\zeta, \quad U_0^+ \sim -\zeta, \quad v_0 \to 0 \text{ exponentially}\, \zeta \to \infty$$

The constant j again remains unknown in the solution of the $\zeta = O(1)$ problem; however an investigation of the problem with $\zeta = O(T^{\frac{1}{2}})$ shows that matching with the above solution can only be achieved if j is again determined by (4.12). Thus we conclude that the steady state solutions with $w_\infty > 8\gamma$ are linearly unstable and so not likely to be observed.

TABLE I

γ	W_∞	$\bar{u}'(0)$	$\bar{v}'(0)$	$\gamma U'(0)$
.02	-0.35500000	0.51152874	-0.60931513	0.01281756
.02	-0.39000000	0.51089963	-0.61151223	0.01015261
.02	-0.41000000	0.51080189	-0.61208108	0.00945387
.02	-0.45000000	0.51069253	-0.61291648	0.00842226
.02	-0.50000000	0.51062290	-0.61366951	0.00748787
.02	-0.55000000	0.51058552	-0.61424223	0.00677497
.02	-0.60000000	0.51056418	-0.61470076	0.00620302
.02	-0.65000000	0.51055185	-0.61507974	0.00572962
.02	-0.70000000	0.51054493	-0.61540001	0.00532912
.02	-0.75000000	0.51054144	-0.61567522	0.00498469
.02	-0.80000000	0.51054015	-0.61591483	0.00468459
.02	-0.85000000	0.51054034	-0.61612571	0.00442034
.02	-0.90000000	0.51054151	-0.61631296	0.00418557
.02	-0.95000000	0.51054334	-0.61648051	0.00397542
.02	-1.00000000	0.51054560	-0.61663143	0.00378607
.02	-1.05000000	0.51054815	-0.61676816	0.00361447
.02	-1.10000000	0.51055088	-0.61689265	0.00345817
.02	-1.15000000	0.51055370	-0.61700654	0.00331516
.02	-1.20000000	0.51055657	-0.61711115	0.00318377
.02	-1.25000000	0.51055945	-0.61720760	0.00306261
.02	-1.30000000	0.51056231	-0.61729682	0.00295050
.02	-1.35000000	0.51056513	-0.61737962	0.00284646
.02	-1.40000000	0.51056790	-0.61745666	0.00274962
.02	-1.50000000	0.51057324	-0.61759580	0.00257472

.02	-1.60000000	0.51057830	-0.61771805	0.00242100
.02	-1.70000000	0.51058306	-0.61782635	0.00228479
.02	-1.80000000	0.51058753	-0.61792299	0.00216323
.1	-0.51000000	0.52025270	-0.60663520	0.03934270
.1	-0.55000000	0.51846172	-0.61209594	0.03382640
.1	-0.60000000	0.51796137	-0.61511910	0.03074344
.1	-0.65000000	0.51779186	-0.61725674	0.02852767
.1	-0.70000000	0.51775162	-0.61895371	0.02674543
.1	-0.75000000	0.51777457	-0.62037081	0.02524172
.1	-0.80000000	0.51783140	-0.62158902	0.02393845
.1	-0.85000000	0.51790706	-0.62265639	0.02278891
.1	-0.90000000	0.51799309	-0.62360449	0.02176219
.1	-0.95000000	0.51808449	-0.62445542	0.02083638
.1	-1.00000000	0.51817818	-0.62522548	0.01999520
.1	-1.10000000	0.51836538	-0.62656993	0.01851929
.1	-1.20000000	0.51854617	-0.62770915	0.01726178
.1	-1.30000000	0.51871723	-0.62869011	0.01617416
.1	-1.40000000	0.51887748	-0.62954560	0.01522217
.1	-1.50000000	0.51902693	-0.63029943	0.01438070
.1	-1.60000000	0.51916605	-0.63096949	0.01363075
.1	-1.70000000	0.51929552	-0.63156952	0.01295762
.1	-1.80000000	0.51941606	-0.63211033	0.01234970
.1	-1.90000000	0.51952843	-0.63260051	0.01179769
.5	-0.60000000	0.68197809	-0.66231775	0.20129254
.5	-0.80000000	0.68831733	-0.71390532	0.16251868
.5	-1.00000000	0.69755716	-0.73310149	0.14469129
.5	-1.50000000	0.71653343	-0.75988876	0.11597686

.5	-2.00000000	0.73026710	-0.77552448	0.09744803
.5	-2.50000000	0.74055061	-0.78614433	0.08420684
.5	-3.00000000	0.74852620	-0.79390866	0.07420619
.5	-3.50000000	0.75489053	-0.79985908	0.06636331
.5	-4.00000000	0.76008693	-0.80457555	0.06003820
.5	-4.50000000	0.76441004	-0.80841092	0.05482446
.5	-5.00000000	0.76806312	-0.81159360	0.05045039

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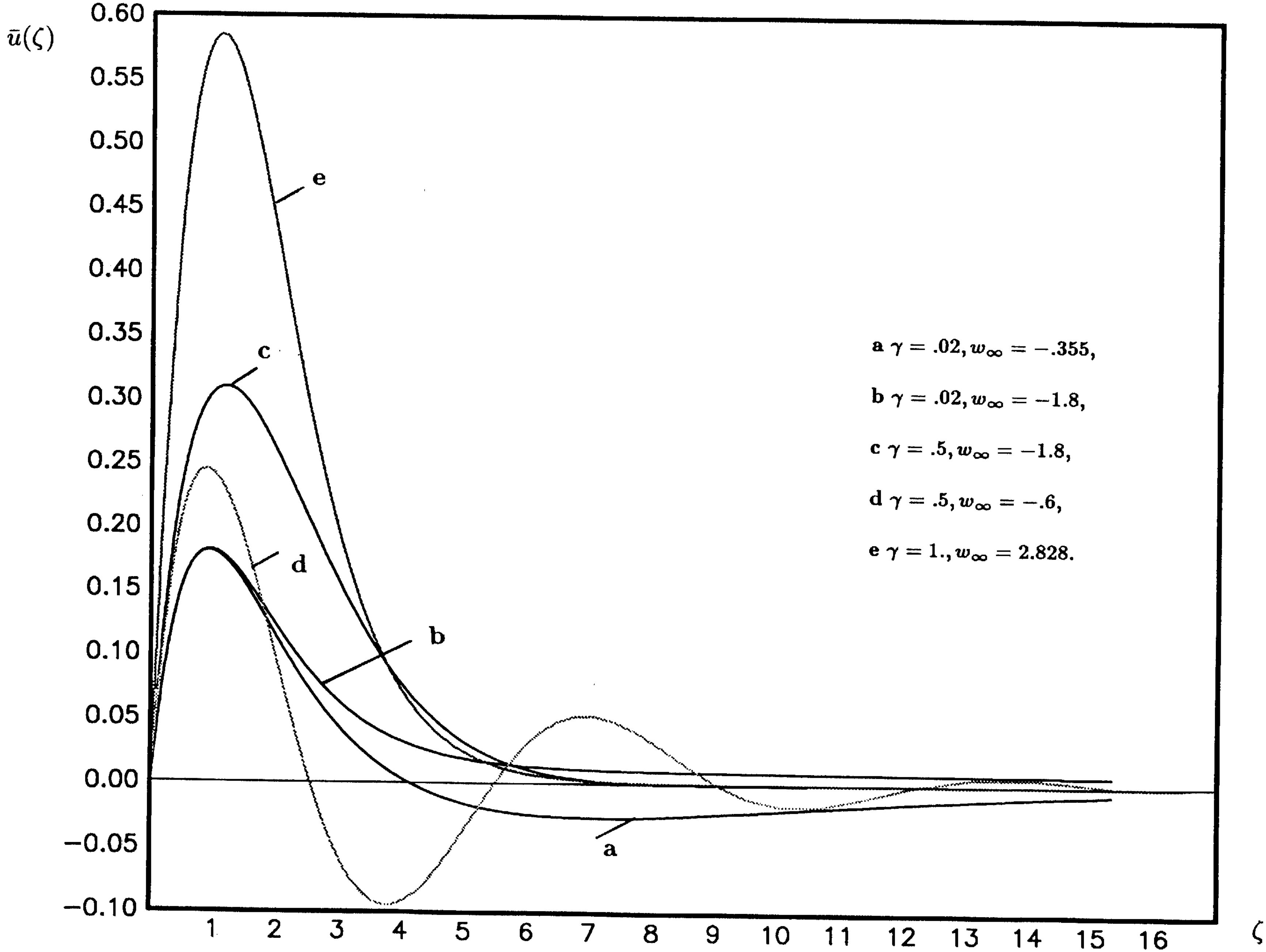


Figure (3.1) The function $\bar{u}(\zeta)$ for different values of γ, w_∞.

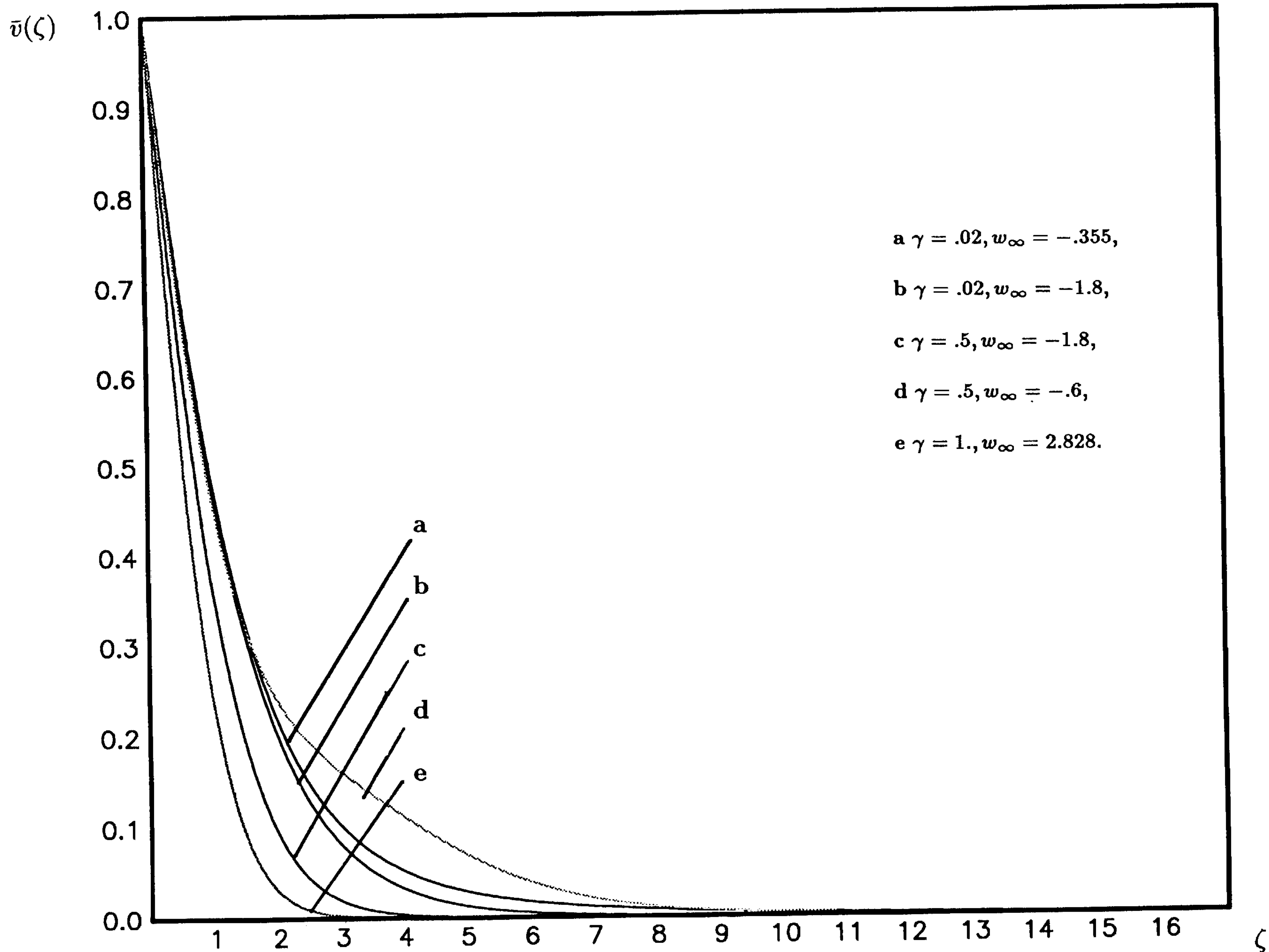


Figure (3.2) The function $\bar{v}(\zeta)$ for different values of γ, w_∞.

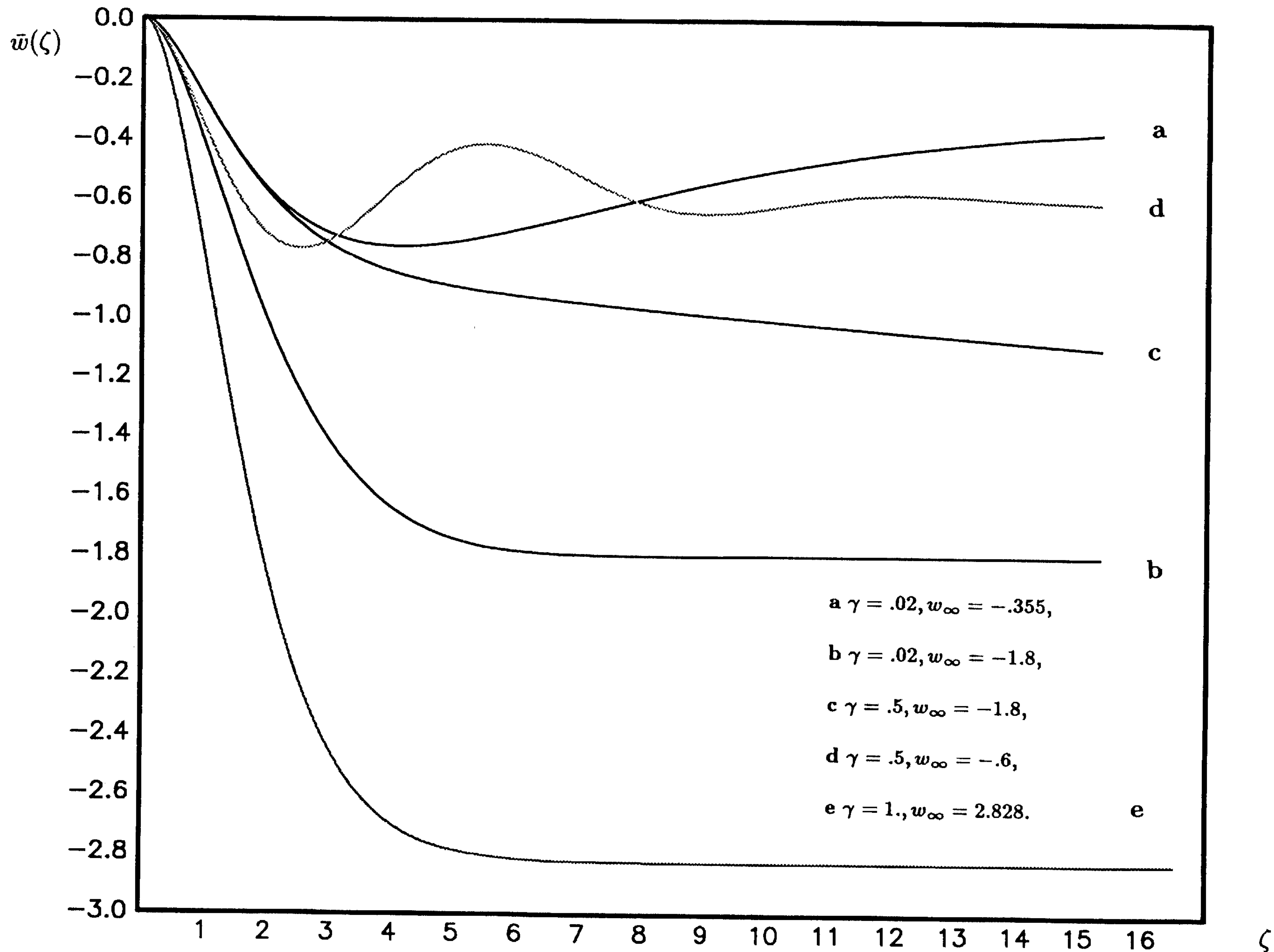


Figure (3.3) The function $\bar{w}(\zeta)$for different values of γ, w_∞.

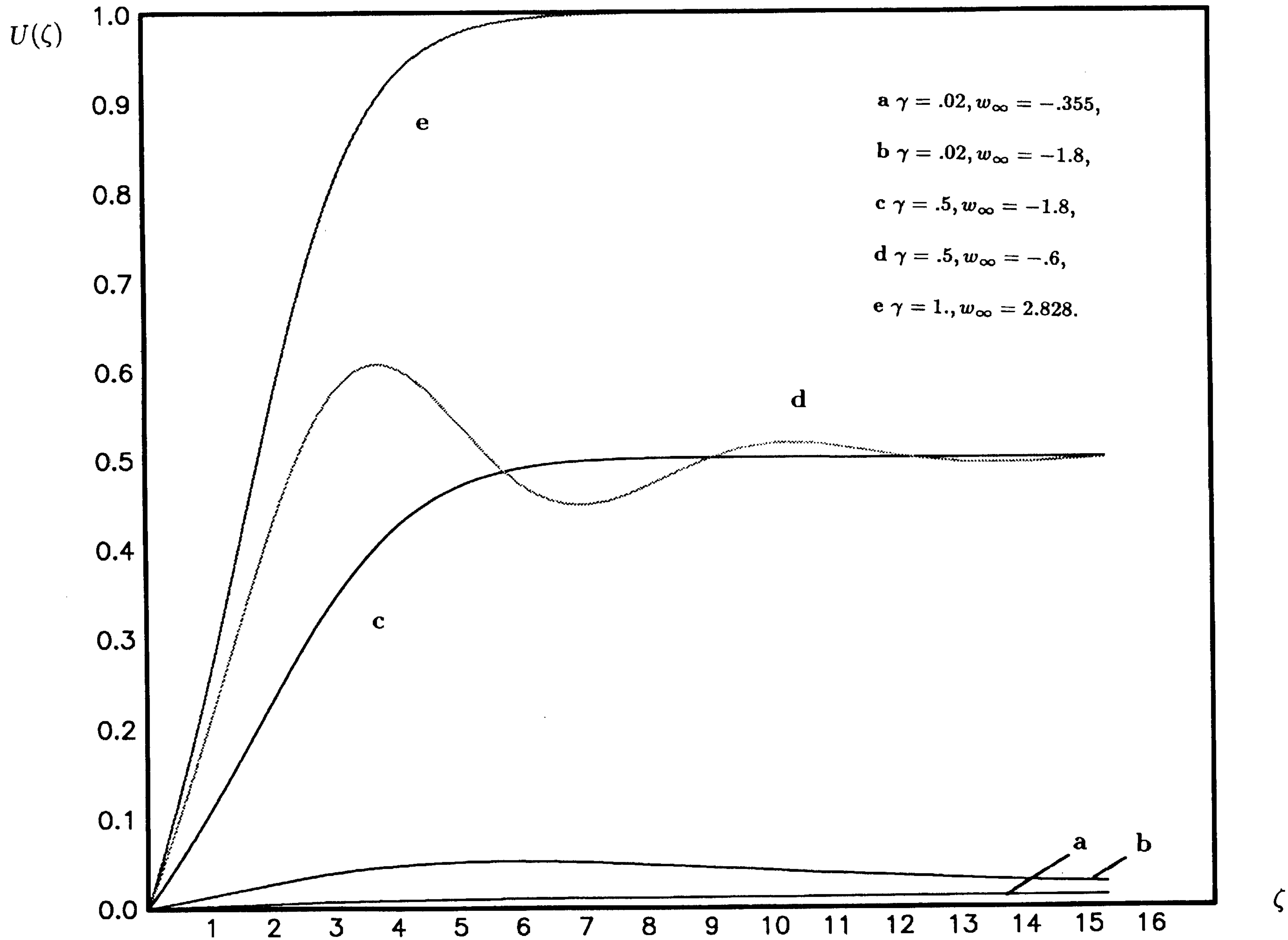


Figure (3.4) The function $\gamma U(\zeta)$ for different values of γ, w_∞.

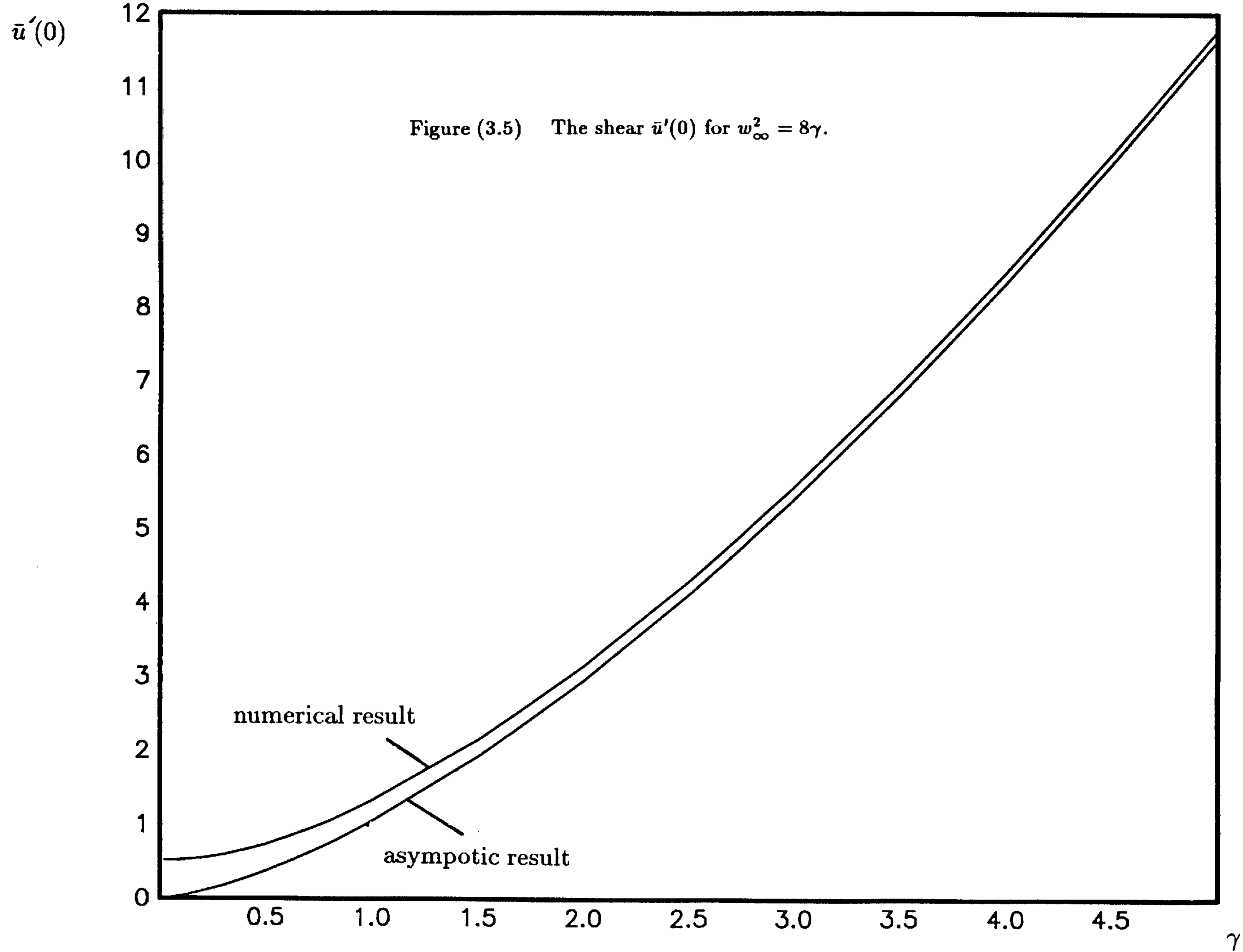


Figure (3.5) The shear $\bar{u}'(0)$ for $w_\infty^2 = 8\gamma$.

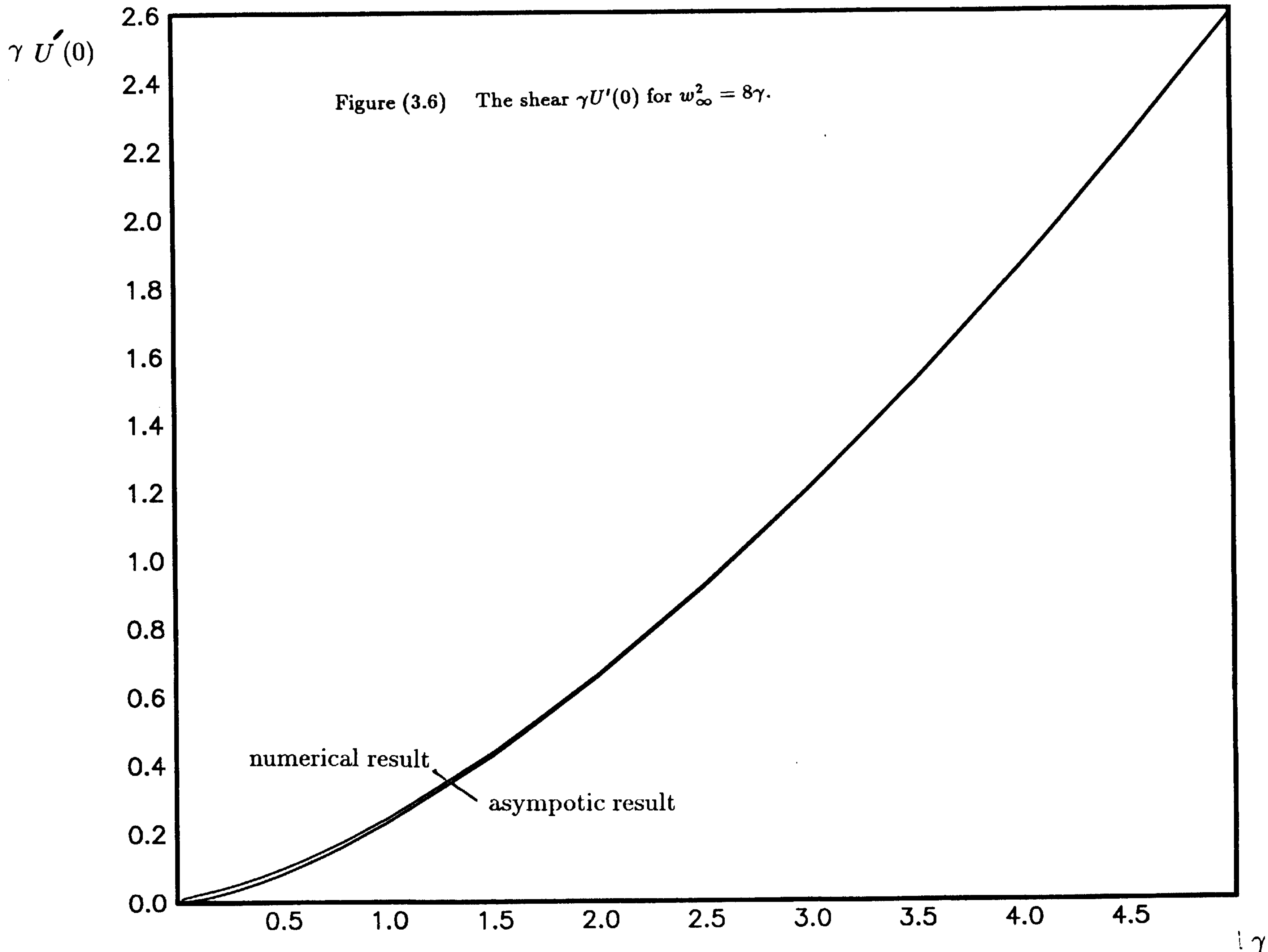
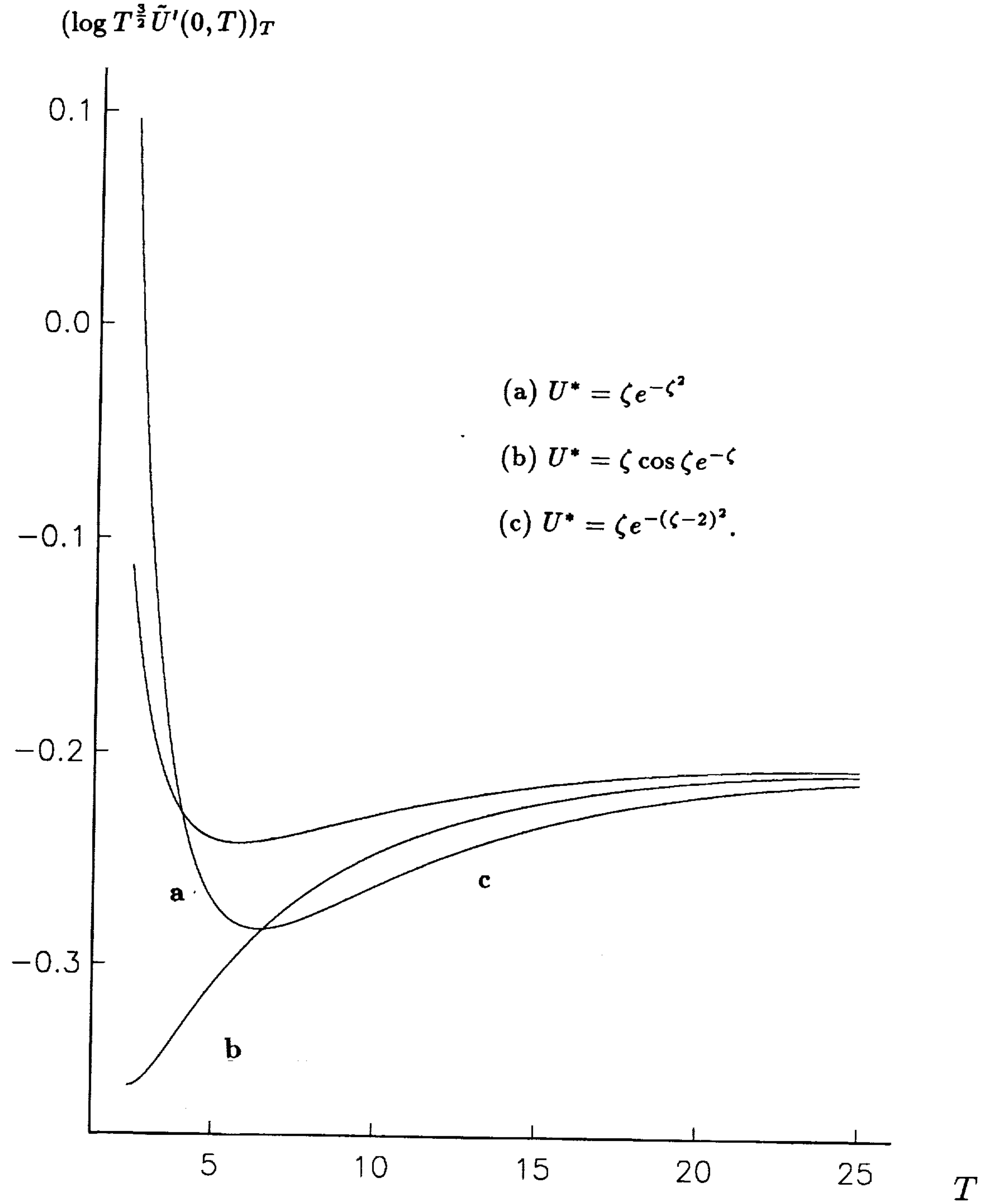


Figure (3.6) The shear $\gamma U'(0)$ for $w_\infty^2 = 8\gamma$.

Figure (4.1) The growth rate $(\log T^{\frac{3}{2}}\tilde{U}'(0,T))_T$ for

(a) $U^* = \zeta e^{-\zeta^2}$, (b) $U^* = \zeta\cos\zeta e^{-\zeta}$, (c) $U^* = \zeta e^{-(\zeta-2)^2}$.

The predicted growth rate at large T is $-\frac{w_\infty^2}{4} \simeq -.2$.



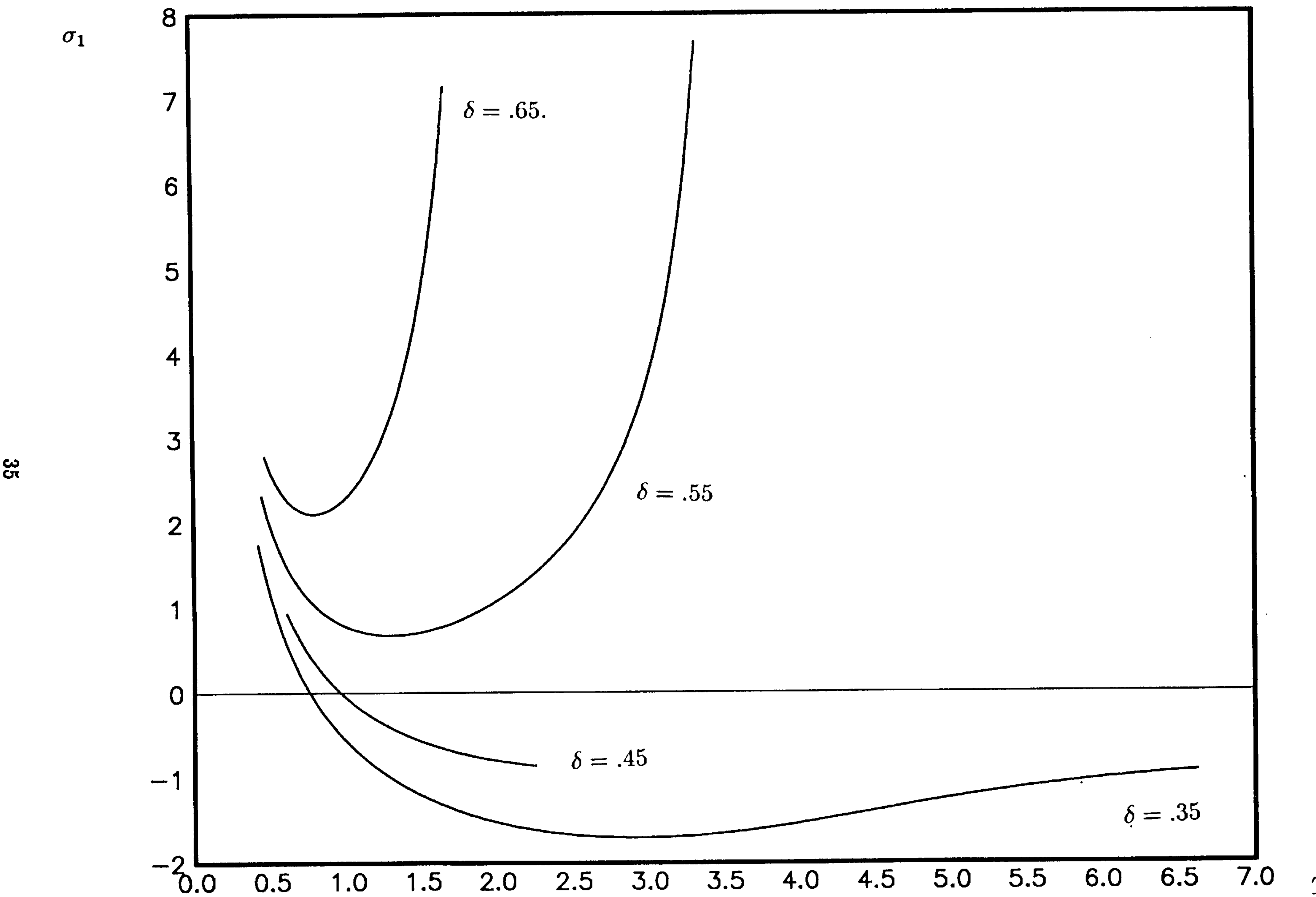


Figure (4.2a) **The growth rate $\sigma_1 = (\log \int_0^\infty u^2 d\zeta)_T$ for $\delta = .35, .45, .55, .65$.**

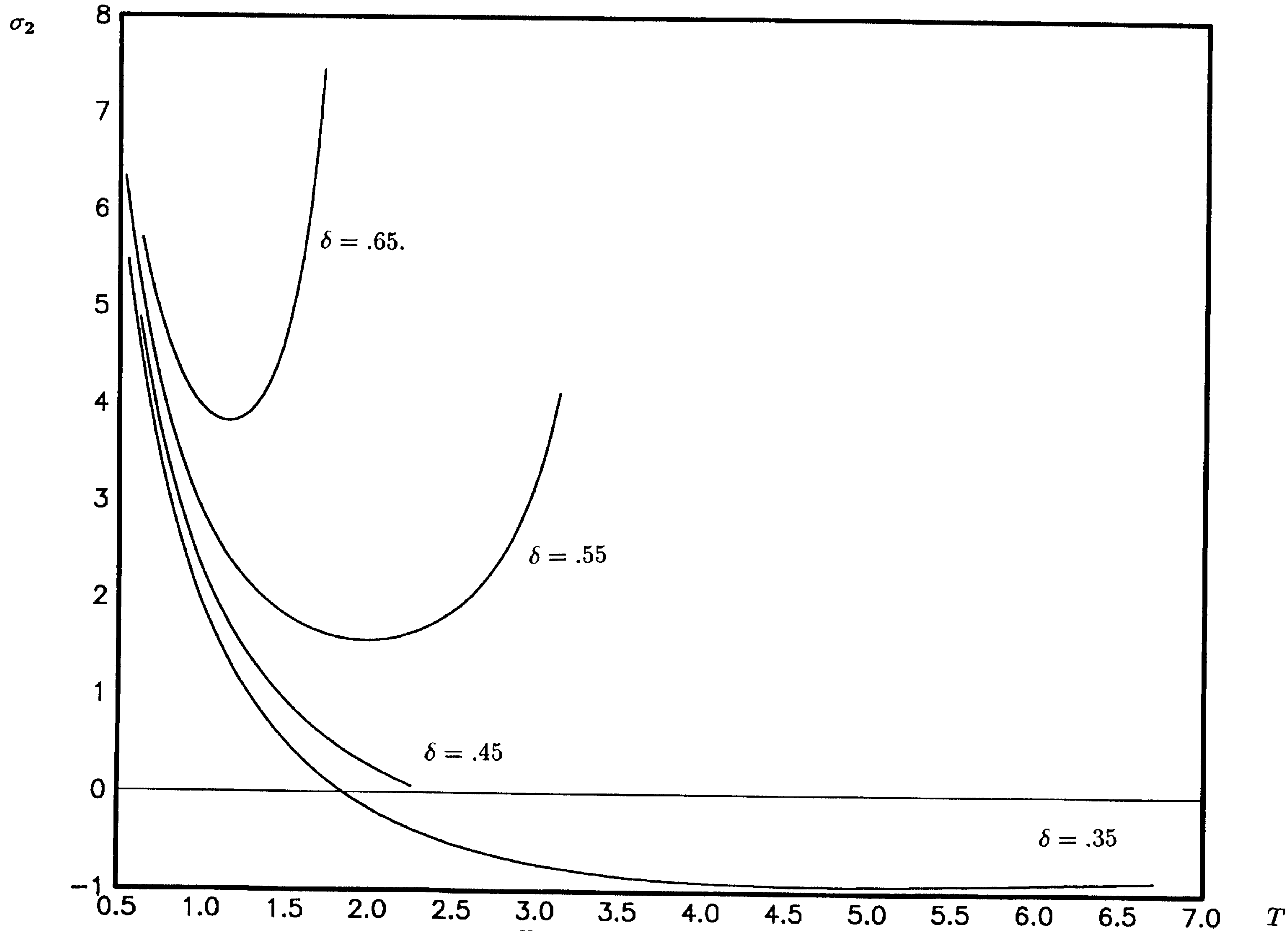


Figure (4.2b) **The growth rate $\sigma_2 = (\log \int_0^\infty v^2 d\zeta)_T$ for $\delta = .35, .45, .55, .65.$**

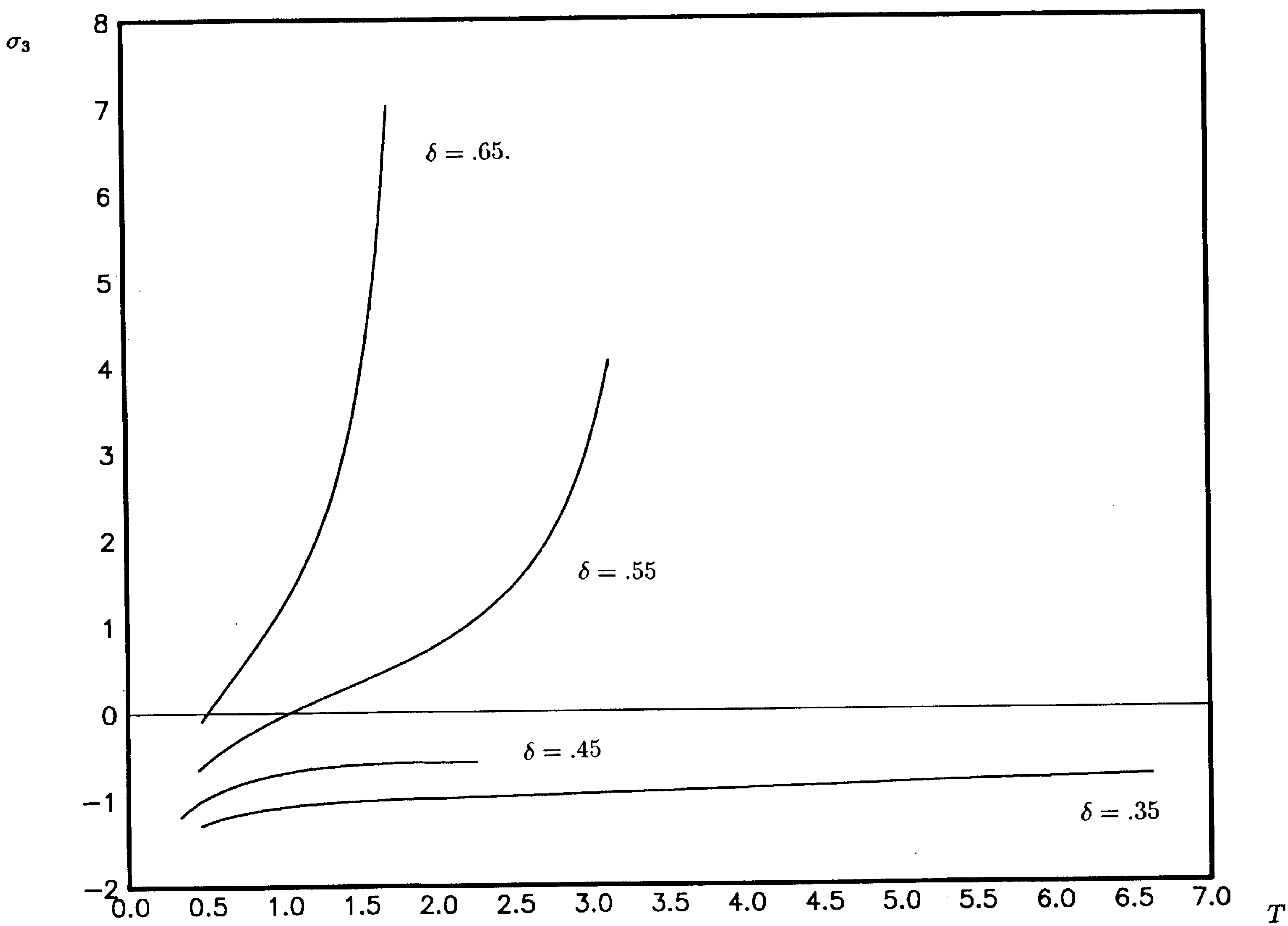


Figure (4.2c) The growth rate $\sigma_3 = (\log \int_0^\infty w^2 d\zeta)_T$ for $\delta = .35, .45, .55, .65$.

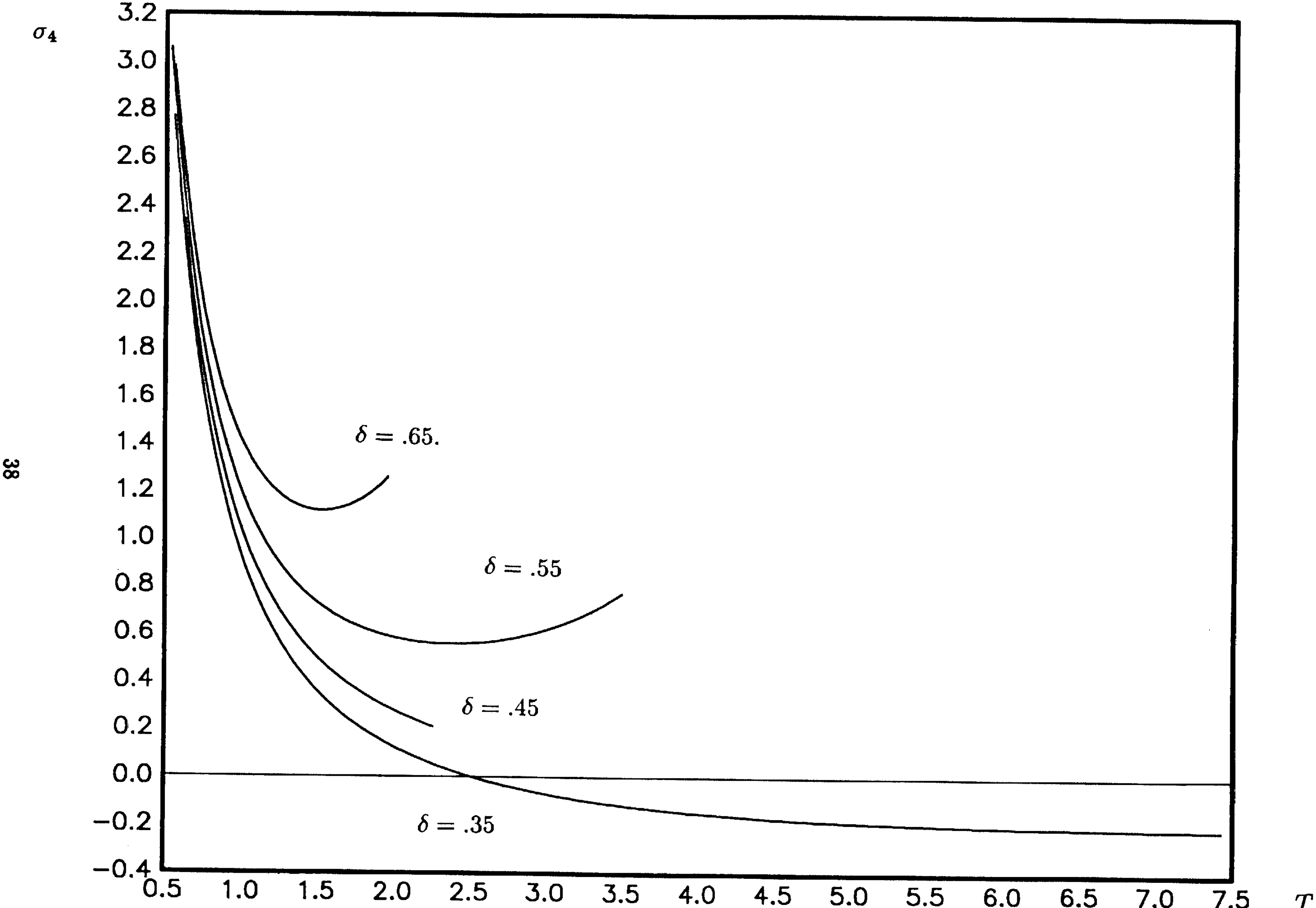


Figure (4.2d) The growth rate $\sigma_4 = (\log T^{\frac{3}{2}} U'(0,T))_T$ for $\delta = .35, .45, .55, .65.$

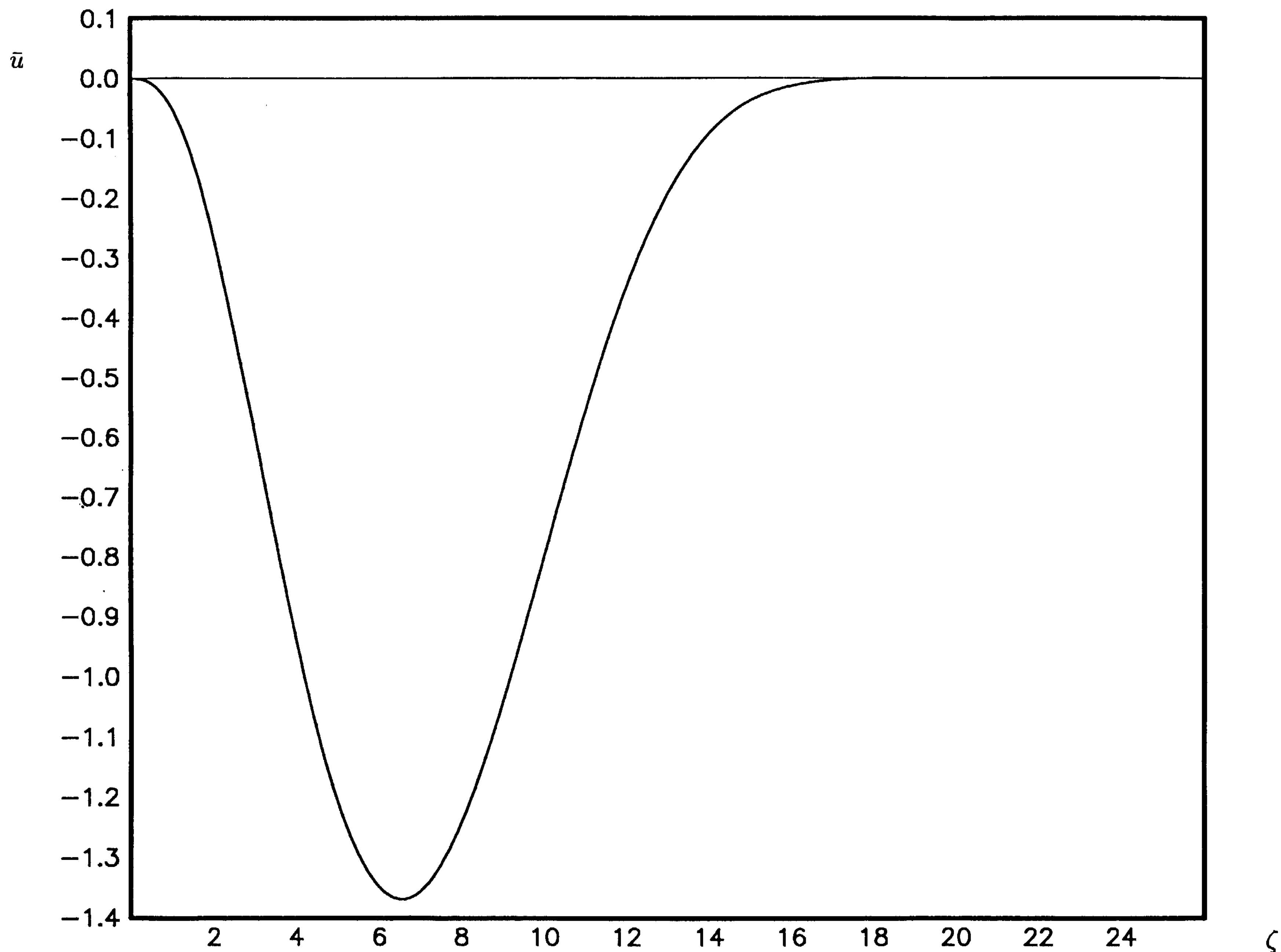


Figure (4.3a) The velocity field $\bar{u}(\zeta)$ for $\delta = .55, T = 3.3$.

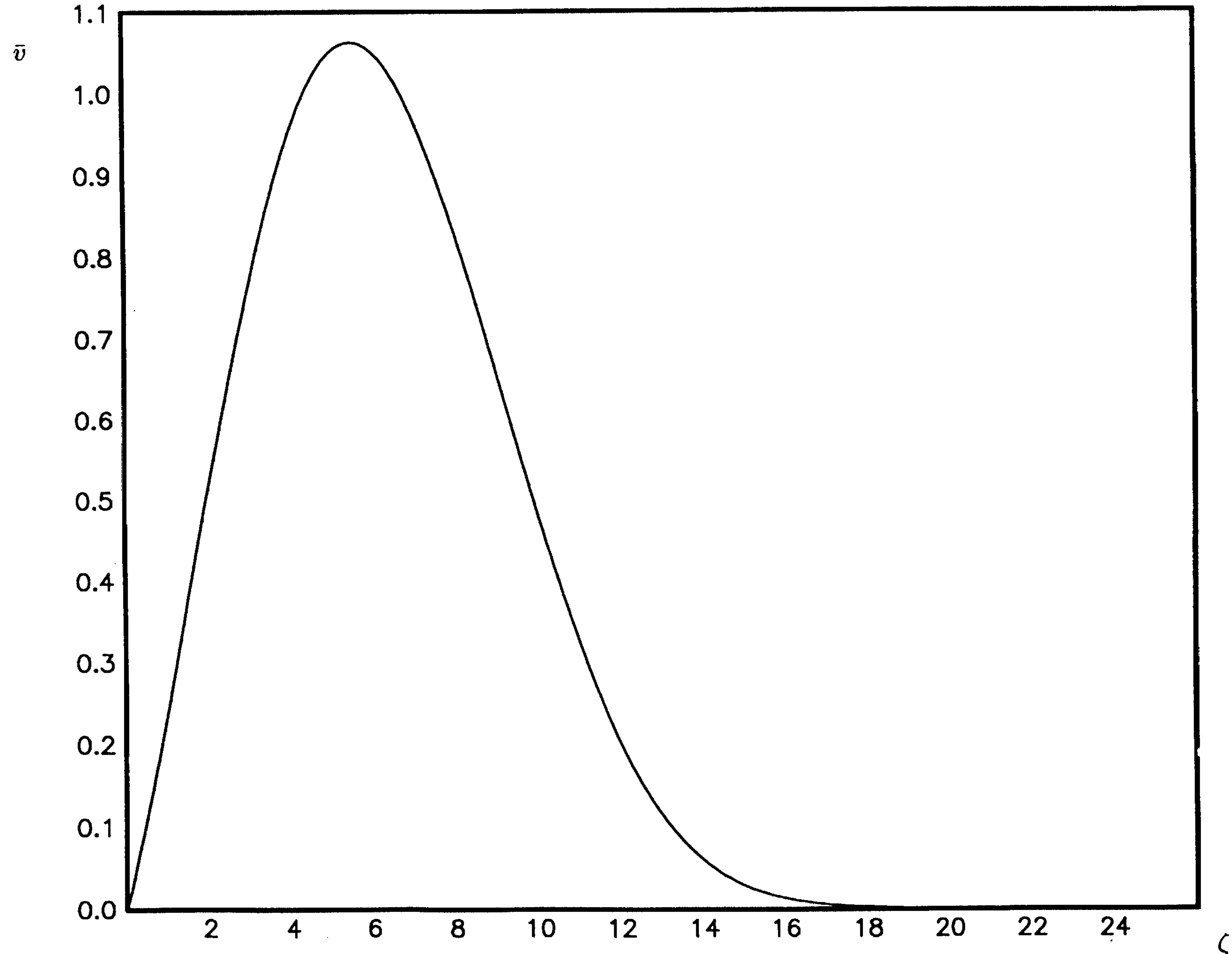


Figure (4.3b) The velocity field $\bar{v}(\zeta)$ for $\delta = .55, T = 3.3$.

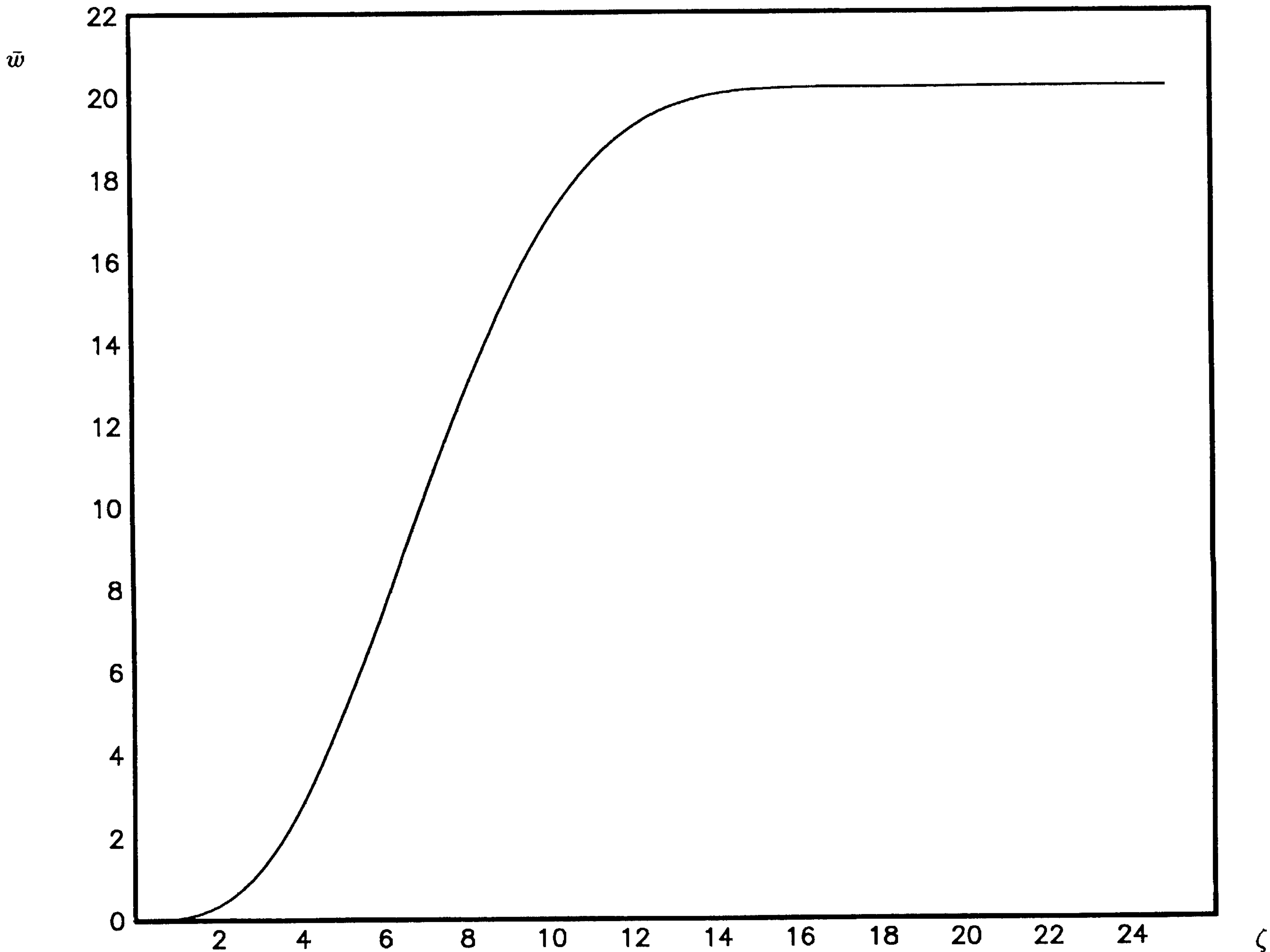


Figure (4.3c) The velocity field $\bar{w}(\zeta)$ for $\delta = .55, T = 3.3$.

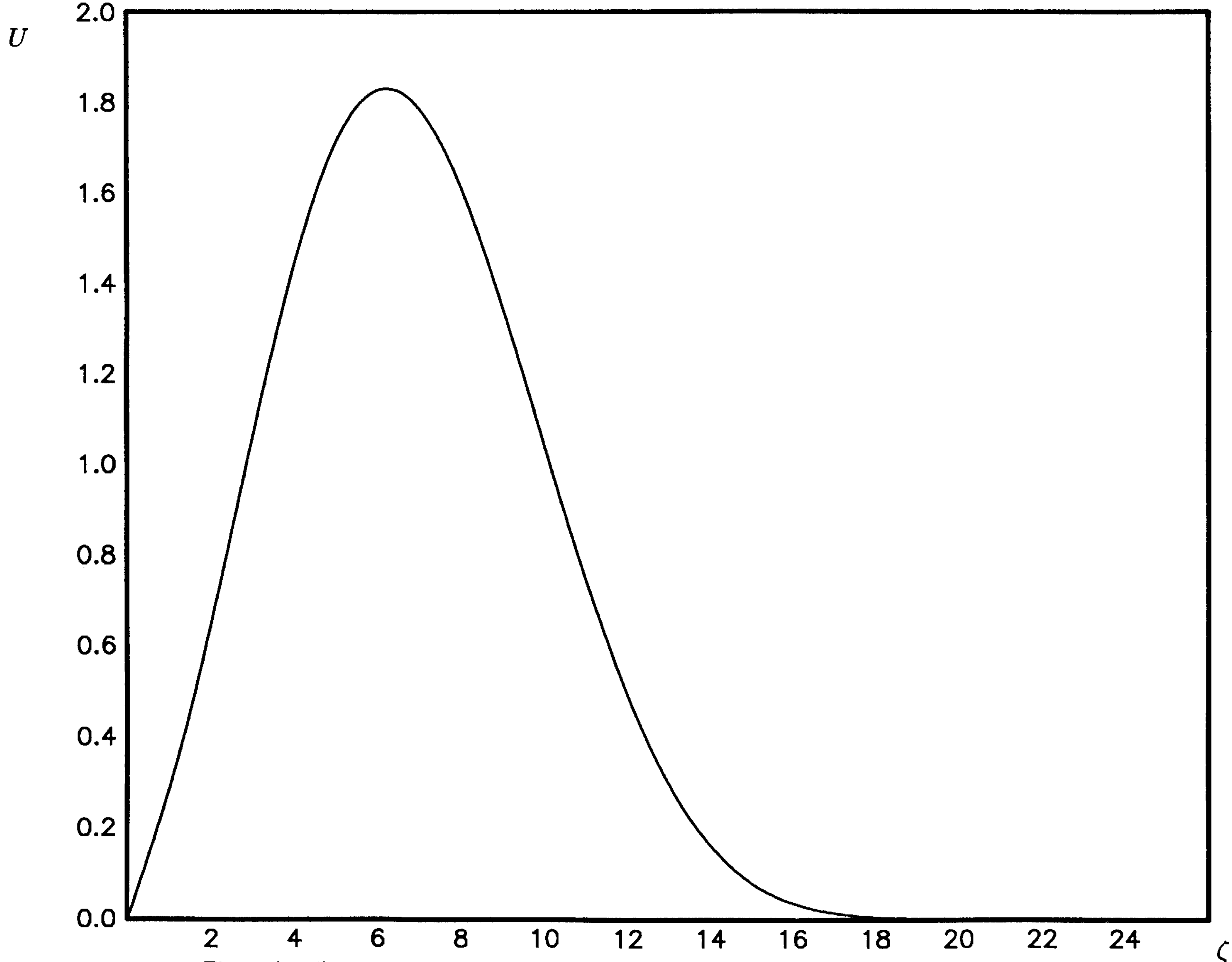


Figure (4.3d) The velocity field $U(\zeta)$ for $\delta = .55, T = 3.3$.

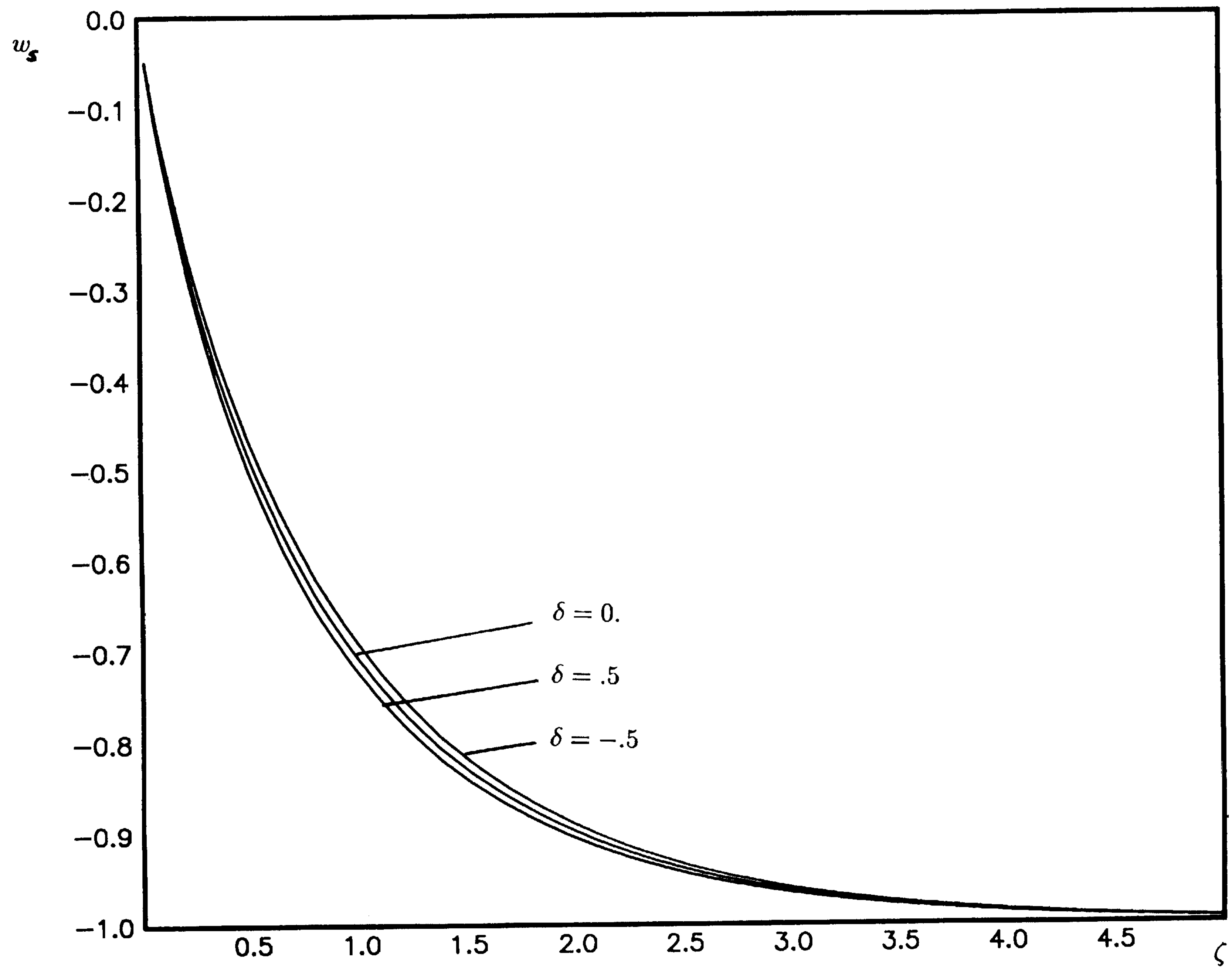


Figure (5.1) The function $w_s(\zeta)$ for $w_\infty = -1, \gamma = 0, .5, -.5.$

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Final Report

16. Abstract

A class of exact steady and unsteady solutions of the Navier Stokes equations in cylindrical polar coordinates is given. The flows correspond to the motion induced by an infinite disc rotating with constant angular velocity about the z-axis in a fluid occupying a semi-infinite region which, at large distances from the disc, has velocity field proportional to (x, -y,0) with respect to a Cartesian coordinate system. It is shown that when the rate of rotation is large Karman's exact solution for a disc rotating in an otherwise motionless fluid is recovered. In the limit of zero rotation rate a particular form of Howarth's exact solution for three-dimensional stagnation point flow is obtained. The unsteady form of the partial differential system describing this class of flow may be generalized to time-periodic equilibrium flows. In addition the unsteady equations are shown to describe a strongly nonlinear instability of Karman's rotating disc flow. It is shown that sufficiently large perturbations lead to a finite time breakdown of that flow whilst smaller disturbances decay to zero. If the stagnation point flow at infinity is sufficiently strong the steady basic states become linearly unstable. In fact there is then a continuous spectrum of unstable eigenvalues of the stability equations but, if the initial value problem is considered, it is found that, at large values of time, the continuous spectrum leads to a velocity field growing exponentially in time with an amplitude decaying in time.

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