Current Collection in an Anisotropic Plasma
Wei-wei Li
Center for Astrophysics and Space Sciences
University of California at San Diego, La Jolla, CA 92093

Abstract. A general method is given to derive the current-potential relations in anisotropic plasmas. Orbit limit current is assumed. The collector is a conductive sphere or an infinite cylinder. Any distribution which is an arbitrary function of the velocity vector can be considered as a superposition of many mono-energetic beams whose current-potential relations are known. The results for two typical pitch angle distributions are derived and discussed in detail. The general properties of the current potential relations are very similar to that of a Maxwellian plasma except for an effective temperature which varies with the angle between the magnetic field and the charging surface. The conclusions are meaningful to generalized geometries.

The Introduction

The current collection from incoming particle is a fundamental problem in charging theory. In this paper the current collection in anisotropic plasmas is studied. The analytic expressions for current potential relations which have been used so far are derived from Maxwellian distribution.

Here the basic current potential relation for Maxwellian plasma are repeated briefly. These well known results will be compared to that of anisotropic plasma frequently in rest part of this paper. The orbit limited current to a spherical conductor or cylindrical conductor for repelled particles is (Mott-Smith and Langmuir 1926; Prokopenko and Laframboise 1977,1980):

\[ I = I_0 e^{-\beta} \]  (1)

For attractive particle, the charging current is:

\[ I = I_0 (1 + \beta) \]  (2)

for a spherical conductor and the following:

\[ I = I_0 \left( \frac{2}{\sqrt{\pi}} \right) \sqrt{\beta} + \frac{1}{2} \sqrt{\pi} e^{\beta} \int_{\sqrt{\beta}}^{\infty} e^{-t^2} dt \]  (3)

for a cylinder.

Charging currents to satellites in anisotropic plasmas have not been discussed systematically. The real distributions of the plasmas at synchronous orbit may be very different from Maxwellian. For example, the pitch angle distributions in the earth magnetic field are very common. In this paper, the current collection in
anisotropic plasmas is studied to see how it would be deviate from that in isotropic plasma.

In order to discuss the variety of distributions, all other conditions are assumed as simple as possible:
1) The effect of space charge is neglected. Orbit limited current is assumed.
2) The distribution function can be an arbitrary function of the velocity vector but it does not depend on the spacial location. The currents from two distributions are derived in this paper:
   a.
   \[ f(v) = N \left( \frac{m}{2\pi k} \right)^{\frac{3}{2}} T_{\parallel} \frac{3}{2} \frac{1}{2} e^{-\frac{1}{2} m v^2_{\parallel}} \frac{1}{2} m v^2_{\perp} \]
   \( T_{\parallel} \) and \( T_{\perp} \) mean the directions related to the magnetic field.
   b. A source cone or a loss cone in a Maxwellian distribution function.
3) The probe is a conducting sphere or an infinite cylinder.
4) VXB force is neglected. The role played by the magnetic field is only to indicate the direction of the anisotropy.
5) Plasma is collisionless.
The above assumptions are reasonable for satellites at the synchronous orbit.

A distribution function of plasma and its current to a charged conductor can be separated and superposed. the plasma are separated into many monoenergetic beams whose charging currents are known. The currents to a sphere conductor \( I_p \) from such a beam are (Mott-Smith and Langmuir 1926):
   \[ I_p = I_0 (1 + e\phi/E) = \pi R^2 i (1 + e\phi/E) \]
   for attracted particle, the following:
   \[ I_p = I_0 (1 - e\phi/E) = \pi R^2 i (1 - e\phi/E) \]
   for repelled particle with \( E > e\phi \). When \( E < e\phi \), \( I_p = 0 \)
Here
\[ \beta = \frac{e\phi}{kT} \]
The "e" is the magnitude of a electron charge.
\( e\phi \) is the potential energy at the surface.
k is the Boltzmann’s constant.
T is the temperature of the plasma.
\( I_0 \) is the current to a uncharged probe. I and \( I_0 \) are define as total current for the sphere and refer to the current per unit length for the cylinder.
\[ I_0 = \text{area} \times \sqrt{\frac{kT}{2\pi m}} \]
m is the mass of a ion.
E is the kinetic energy of a particle at infinity.
The i is the current density of the beam.
R is the radius of the sphere.
The equation (4) and (5) simply comes from the energy conservation and the angular momentum conservation.

The basic assumption is that particles which carry maximum angular momentum and still reach the sphere are the grazing particle. This may not be true for attracted particles if the space charge effect is included (Laframboise 1965 Fig 4d). Therefore the condition for equation (4) and (5) is that current be orbit limited.

Similar relations hold for the cylinder:

\[ I = I_0 \sqrt{1 + e\phi/E} = 2R \ i \sqrt{1 + e\phi/E} \] (7)

for attracted particle and the following:

\[ I = I_0 \sqrt{1 - e\phi/E} = 2R \ i \sqrt{1 - e\phi/E} \] (8)

for repelled particles with \( E > e\phi \) when \( E < e\phi, \ I_p = 0 \)

The assumption and derivation of (7) and (8) is the same as equation (5) and (6).

Caution should be observed: If the beam is not perpendicular to the axis of the cylinder, the \( i \) in (7) and (8) is not the current density of the beam but is the component perpendicular to the axis at infinity. Also the \( E \) is not the total kinetic energy of the particle but the kinetic energy in the direction perpendicular to the axis. Integrating of these beam with weight of distribution function leads to the total charging current of incoming plasma. The integral is carried out in the plasma frame. The superposition method will be illustrated in more detail in next section.

The Result of the Current-potential Relations

It is more convenient to use dimensionless quantities:

- \( I_0 \): the current to an uncharged conductor
- \( I/I_0 \): Dimensionless charging current
- \( \beta \): Nondimensional potential \( \beta = e\phi/kT \) or \( e\phi/kT \)
- \( k \): the Boltzmann's constant.
- \( T \): the temperature of the plasma

The current to a attracting sphere from a two temperature plasma.

The "i" in equation (4) is the current density of the beam at infinity. It is equal to the density times the velocity. Now the density should be replaced by the number of particles in a infinitesimal volume in velocity space \( f(\vec{v})d^3(\vec{v}) \). \( f(\vec{v}) \) is the distribution function (4)

The current to the charged sphere is:

\[ I = \iiint f(\vec{v})V(1 + e\phi/E)d^3(\vec{v})S \] (9)
Here $S = \pi R^2$

$$d^3(\mathbf{v}) = V^2 \sin \theta \ d\theta \ dV$$

The angle $\theta$ and $\phi$ indicate the directions of particles at infinite. $\theta$ is the polar angle of the velocity. $\phi$ is the azimuthal angle of the velocity. All quantities are defined in the plasma frame. The distribution function $f(\mathbf{v})$, the velocity $V$, and the kinetic energy $E$ are the values at infinity (i.e., in the plasma frame). The integral is carried out in the plasma frame.

$$I = N e \pi R^2 \left( \frac{m}{2 \pi k} \right)^{\frac{3}{2}} \left[ T_{\parallel} \int_0^\infty dV \int_0^{2\pi} d\phi \int_0^\pi \sin \theta \ d\theta \right]$$

$$(1 + \frac{e\phi}{2mV^2})V^3 e^{-\frac{1}{2}mV^2(e^2/kT_{\parallel} + kT_{\perp})}$$

$$K(\theta) = \frac{\sin^2 \theta}{kT_{\parallel}} + \frac{\cos^2 \theta}{kT_{\perp}}$$

$$E = \frac{1}{2}mV^2$$

$$I = B \int_0^\infty dE \int_0^\pi \sin \theta d\theta \ E (e^{-E K(\theta)} + e\phi e^{-E K(\theta)})$$

Define $I_0$ as the current to an uncharged sphere.

$$I_0 = 2\pi R^2 N \sqrt{\frac{kT_{\perp}}{2\pi m}} \left( 1 + \frac{1}{2} \sqrt{\frac{CD}{1-C}} \right)$$

Here $C = \frac{T_{\perp}}{T_{\parallel}}$

$$D = 2 \ t g^{-1} \sqrt{C - 1} \ \text{if} \ \ T_{\perp} > T_{\parallel}$$

$$D = ln \ \frac{1 + \sqrt{1-C}}{1 - \sqrt{1-C}} \ \text{if} \ \ T_{\perp} > T_{\parallel}$$

Define the effective Temperature as:

$$T_f = T_{\perp} \left( \frac{\sqrt{|1-C|}}{CD} + 1 \right)$$

The current to a charged sphere becomes

$$I = I_0 (1 + e\phi/kT_f)$$

The dimensionless current are plotted in Fig. 1 as the function of the dimensionless potential for different values of $T_{\perp}/T_{\parallel}$ which is the indication of the
anisotropy. The lower curves correspond to lower $T_{\parallel}$. The curves are simply straight lines. The different slopes of the curves is calculated by equation (14). The form of current-potential relation is similar to the one of the Maxwellian plasma equation (2). The relation between the current and the potential is linear. From equation (5), the current potential relation is linear for a beam to a sphere. Therefore the superposition of the beams is also linear. This is clear according to equation (13). The change of the charging current is:

$$\Delta I = I - I_0 = e\phi \int f(V) \frac{V}{E} d^3\bar{v}$$

The current to a uncharged probe is:

$$I_0 = \int f(V) V d^3\bar{v}$$

$$\Delta I / I = e\phi \frac{\int \frac{V}{E} f(V) d^3\bar{v}}{\int f(V) V d^3\bar{v}}$$

If we write

$$\Delta I / I = \frac{e\phi}{T_f}$$

The change of the current is proportional to the potential. The quantity $T_f$ is independent of the potential and has the unit of energy. So we have the definition of $T_f$:

$$T_f = \frac{\int f(V) V d^3\bar{v}}{\int \frac{1}{m V^2} f(V) V d^3\bar{v}}$$

If the distribution function is Maxwellian, $T_f$ identifies with the temperature. In an anisotropic plasma, $T_f$ is related to the energy in the direction perpendicular to the surface. It is a combination of $T_\perp$ and $T_{\parallel}$. Its values lie between $T_\perp$ and $T_{\parallel}$ and is more close to $T_{\perp}$ since $T_{\perp}$ corresponds to two dimensions while $T_{\parallel}$ only corresponds to one dimension.

The current to a repelling sphere of two temperature plasma

Starting from equation (12). I changed the lower limit of the integral and the sign before $e\phi$

$$I = B \int_{e\phi}^{\infty} dE E (e^{-E K(\theta)} - e^{-E K(\theta)} \sin \theta) d\theta$$

Here B is defined by equation (10) $K(\theta)$ is defined by equation (11). The charging current is:

$$I = 2\pi R^2 \sqrt{\frac{k T_\perp}{2\pi m (T_{\parallel} / T_\perp)^{1/2}}} \int_{-1}^{1} dX \frac{e^{-e \phi G X^2}}{(1 - G k T_\perp X^2)^2} e^{-X^2}$$

(16)
Here \( G = \frac{1}{kT_\perp} - \frac{1}{kT_\parallel} \), Define \( \beta = e\phi / kT_\perp \)

\[
I/I_0 = e^{-\beta} \frac{C^3}{1 + \frac{1}{2} \sqrt{1-C^2}} \times \int_{-1}^{1} dX \frac{e^{-e\phi G X^2}}{(1 - G kT_\perp X^2)^2}
\]  

(17)

\( I_0 \) is the current to an uncharged sphere defined by equation (12)

In Fig. 2 the dimensionless current \( I/I_0 \) is plotted as the function of dimensionless potential \( e\phi / kT_\perp \). Each figure correspond to a different ratio of \( T_\perp/T_\parallel \). The shapes of the curves are very similar to the exponential form of Maxwellian. The logarithm scale is used for dimensionless current to show this similarity.

The current collected by a sphere from a plasma with a loss cone

For repelled particles,

\[
I = N e\pi R^2 \left( \frac{m}{2\pi kT} \right) \frac{3}{2} \int_{\text{app}}^{\infty} dV_r \times e^{-\frac{1}{2} \frac{m v^2}{kT}} \int_{0}^{2\pi} d\phi \int_{\theta_0}^{\pi-\theta_0} \sin \theta d\theta
\]

\( \theta_0 \) is the angle of loss cone

\[
I = I_{om} e^{-\frac{e\phi}{kT}} \cos \theta_0
\]

Here \( I_{om} \) is the current to a uncharged sphere in a Maxwellian plasma.

\[
I_{om} = I = 4\pi R^2 \sqrt{\frac{kT_\perp}{2\pi m}}
\]

The Current to an uncharged sphere \( I_0 \) is

\[
I_0 = I_{om} \cos \theta_0.
\]

\[
I/I_0 = e^{-\frac{e\phi}{kT}}
\]

(18)

For attracted Particles

\[
I = N e\pi R^2 \left( \frac{m}{2\pi kT} \right) \frac{3}{2} \int_{\text{app}}^{\infty} dV_r \times e^{-\frac{1}{2} \frac{m v^2}{kT}} \left( 1 + \frac{e\phi}{kT} \right) V_r^3
\]

\[
\int_{0}^{2\pi} d\phi \int_{\theta_0}^{\pi-\theta_0} \sin \theta d\theta
\]

\[
I = I_{om} (1 + \frac{e\phi}{kT}) \cos \theta_0
\]

\[
I_0 = I_{om} \cos \theta_0
\]

\[
I/I_0 = 1 + \frac{e\phi}{kT}
\]

(19)

The \( I - \phi \) relation is exactly the same as Maxwellian plasma. This conclusion can be generalized to any pitch angle distribution if the angle dependence is separated from energy dependence. i.e. \( f(\bar{v}) = f(\theta, \phi) \times f(E) \) This result
supports the condition that the angle preference of a distribution function does not change the \( I - \phi \) relation for a spherical conductor if the energy dependence of the distribution is the same as the Maxwellian distribution.

The current to a cylinder in a two temperature plasma.

Define the coordinates as shown in Fig. 3. The axis of the cylinder is defined as the Z direction. X axis is in the plane of B field and axis of the cylinder. The magnetic field is parallel to the plane with azimuthal zero. The polar angle of B field is \( \theta_0 \). The azimuthal angle of the velocity is \( \phi \). The polar angle of the velocity is \( \theta \). || \( \quad \parallel \quad \perp \) in The \( V_z \) is the velocity component in the direction of axis of the cylinder. \( V_r \) is the velocity perpendicular to the axis.

\[
I = 2NeR \left( \frac{m}{2\pi k} \right)^{\frac{3}{2}} / T_{\parallel} T_{\perp} \int_0^\infty dV_z dV_r \int_0^{2\pi} d\phi \int_0^{\infty} \sqrt{\frac{2\pi}{m}} V_r^2 \sqrt{1 + \frac{\epsilon \phi}{\frac{1}{2} m V_r^2}} V_r^2 e^{-\frac{\epsilon \phi}{m V_r^2}}
\]

in \( \int_0^{\infty} \) use \( \sqrt{\frac{2\pi}{m}} \) for repelled particles, use 0 for attracted particles.

\[
I = 2NeR \sqrt{\frac{2k}{m}} / T_{\parallel} T_{\perp} \int_0^{\infty} \sqrt{\frac{2\pi}{m}} \left( \frac{\cos^2 \theta_0}{kT_{\parallel}} + \frac{\sin^2 \theta_0}{kT_{\perp}} \right) \int_e^{\infty} \sqrt{\epsilon \phi \pm \frac{E_r e^{-\epsilon \phi}}{kT_{\perp}} I_0(b)} \epsilon d\epsilon
\]

Here \( I_0 \) is the zero order first kind of Bessel function \( E_r = \frac{1}{2} m V_r^2 \)

\[
b = E_r \sin^2 \theta_0 \left( \frac{1}{T_{\parallel}} - \frac{1}{T_{\perp}} \right) \times \left( 1 + \frac{\frac{1}{2} \cos^2 \theta_0 \left( \frac{1}{T_{\parallel}} - \frac{1}{T_{\perp}} \right)}{\frac{\cos^2 \theta_0}{kT_{\parallel}} + \frac{\sin^2 \theta_0}{kT_{\perp}}} \right)
\]

\[
I_0 = 2NeR \sqrt{\frac{2k}{m}} / kT_{\parallel} T_{\perp} \int_0^{\infty} \sqrt{\epsilon \phi \pm \frac{E_r e^{-\epsilon \phi}}{kT_{\perp}} I_0(HU)} \epsilon d\epsilon
\]

For \( H > 0 \) . Here, \( U = E_r / T_{\perp} E_r \) is the kinetic energy perpendicular to the surface of the cylinder.

\[
H = 1 + \frac{\sin^2 \theta_0 (1 - C)}{(C - 1) \cos^2 \theta_0 + \sin^2 \theta_0}
\]

\[
C = \frac{T_{\parallel}}{T_{\perp}}
\]

\( \theta_0 \) is the angle between the B field and the axis of the cylinder

\[
I / I_0 = \frac{\int_0^{\infty} \sqrt{\epsilon \phi \pm \beta} \epsilon e^{-\epsilon \phi} I_0(HU) \epsilon d\epsilon}{\int_0^{\infty} \sqrt{ue^{-u(1+H)} I_0(HU)} \epsilon d\epsilon}
\]

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Here

\[
\beta = \frac{e\phi}{kT_{\perp}} \\
C = \frac{T_{\perp}}{T_{\parallel}}
\]

in \( \sqrt{U \pm \beta} \) use + for attracted particles, use - for repelled particles.

See Fig.4 and Fig.5 In each figure the curves which change slower with potential correspond to higher \( T_{\parallel} \). The angle effect is indicated by the H factor in equation (22). When the B field is parallel to the the axis of the cylinder, \( H=0 \) and the \( I0(Hu) \) in equation (23) approaches 1. The \( I - \phi \) relation is reduced to the form of Maxwellian plasma. In this situation the velocity parallel to the axis will not be changed by electric field and does not contribute to the charging current at all. Therefore \( T_{\parallel} \) does not appear in equation (23). While \( \theta_o \) decreases from \( 90^\circ \) to 0, the motion parallel to B contributes less and less to the charging. \( T_{\parallel} \) becomes less important.

**Current in a Maxwellian plasma within a small source cone**

\[
\frac{I_{\text{attrac}}}{I_0} = U e^U K_1(U) 
\]
\[
\frac{I_{\text{repet}}}{I_0} = U e^{-U} K_1(U) 
\]

Here \( U = e \phi/2T \sin^2 \theta_o \) \( K_1 \) is the Third kind Bessel function of order one. If \( \theta_o \) is small current-potential relation behaves like a Maxwellian distribution with a lower temperature. If plasma come from a direction which is almost perpendicular to the surface, The curves behave like a Maxwellian plasma with a higher temperature.

See Fig.6 and Fig.7 \( I/I_0 \) is plotted as the function of \( e\phi/kT \). A curves of Maxwellian distribution is plotted in each figure to be compared with source cone. The U factor in equation (25) and (26) shows that \( 2T \sin^2 \theta_o \) is the effective temperature. \( T_{\text{effect}} \) is less than \( T \) when \( \theta_o < 45^\circ \). \( T_{\text{effect}} \) is greater than \( T \) when \( \theta_o > 45^\circ \). The temperature of a Maxwellian plasma can be understood as the average of these \( T_{\text{effect}} \) over all direction.

I only calculated the \( I - \phi \) relation for a flow from a small solid angle, but the behavior of a wider source cone can be estimated from the result of the small solid angle. If the maximum angle between the axis and a beam within the wide solid angle is \( 60^\circ \) and the minimum angle is \( 30^\circ \). The \( I - \phi \) curve of such a source cone should lie between the two curves corresponding to \( \theta_0 = 30^\circ \) and \( \theta_0 = 60^\circ \).

The Discussion
Arbitrary Distribution

There are other distributions such as a double Maxwellian and a monoenergetic beam plus a Maxwellian distribution function, for which charging currents are very easy to be obtained by using superposition. In general the method in this paper applies to any homogeneous distribution function. The only problem left is a mathematical one which refers to integrals of equation (5)-(8) in three dimensional space of canonical momenta. Sometime integrals have to be carried out numerically.

The similarity to the Maxwellian plasma

The general properties of the current-potential curves of pitch angle distributions are very similar to that of the Maxwellian plasma. The current-potential relations of attractive particle for a sphere is exactly linear as shown in Fig.1. For repelled particles the relation are almost exponential for both sphere and cylinder as shown in Fig.2,5 and 7.

The importance of the effective temperature

While the energy of Maxwellian plasma is indicated by the temperature, the energy of a pitch angle distribution in the charging problem refers to an effective energy in the direction perpendicular to the charged surface.

The value of the effective temperature

For a pitch angle distribution defined by equation (1), The value of effective temperature is an average of $T_{\perp}$ and $T_{\parallel}$. If the surface has no angle preference (a conducting sphere) $T_{\text{effect}}$ is defined by equation (9). $T_{\perp}$ contributes more to the average since $T_{\perp}$ indicates the kinetic energy in two dimensions while $T_{\parallel}$ is only related to the motion in one dimension. When the surface is parallel to the B field, $T_{\text{effect}}$ will deviates from the value of equation (9) and move closer to the $T_{\perp}$. When the surface is perpendicular to the B field, $T_{\text{effect}}$ will approach $T_{\parallel}$.

For the charging of a cylinder from a source cone of a Maxwellian plasma the $T_{\text{effect}}$ equals the temperature of the Maxwellian distribution times an angle factor. The factor is less than 1 when the source cone makes a small angle with the surface. When the source cone is perpendicular to the surface, the $T_{\text{effect}}$ will be greater than T. $T_{\text{effect}}$ can not exceed 2T.

Current to a uncharged surface element

The current to a uncharged surface has not been studied in section 2. All for-
mula and Figures are shown in terms of dimensionless currents $I/I_0$. There is not much physics involved in the calculation of $I_0$. It is just the random flux along the normal direction of the surface. Obviously the current varies with the angle between the surface and the preferred direction of the anisotropic distribution disregarding the geometry of the whole satellite. The change of the electron current and ion current due to $I_0$ will affect the equilibrium potential.

The equilibrium potential varies with the orientation of the surface.

The equilibrium potential varies with the angle between the magnetic field and the charging surface. The surface which is parallel to a source cone feel that the plasma has less energy; therefore, it would be charged to a lower equilibrium potential than the surface perpendicular to the source cone. The equilibrium potential depends on the effective temperature. The $T_f$ of a source cone has a upper limit which is about 2T.

The Conclusion

In case of (4) and source cone, the properties of current-potential relation of a conductor in a anisotropic plasma are qualitatively similar to that in Maxwellian since the energy distribution is similar. The difference caused by anisotropy is that a surface tangential to the preferring direction of anisotropy starts charging with a less charging current and reaches a lower equilibrium potential than a surface perpendicular to the preferred direction.

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References


Prokopenko, S.M.L., and J. G. Laframboise, High-voltage differential charge of geo-

\[
T_2/T_1 = 16, 8, 4, 2, 1, 1/2, 1/4, 1/8, 1/16 \text{ FROM TOP}
\]

Fig. 1 The current to a sphere from attracted particles of two temperature plasma.

\[
T_2/T_1 = 16, 1/8, 1/4, 1/2, 1, 2, 4, 8, 16 \text{ FROM TOP}
\]

\[
\ln(I/I_0)
\]

\[
\text{REPELLING POTENTIAL IN UNIT OF } kT_\perp^2
\]

Fig. 2 The current to a sphere from repelled particles of a two temperature plasma.
Fig. 3 A cylinder in a spherical coordinate

Fig. 4 The current to a cylinder from attracted particles of a two temperature plasma. $T_*/T_|| = 16, 8, 4, 2, 1$ from top

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Fig. 5 The current to a cylinder from repelled particle of a two temperature plasma $T_1/T_\parallel = 1, 2, 4, 8, 16$ from top

Fig. 6 Current to a cylinder from attracted particles of a Maxwellian plasma within a small source cone
Fig. 7 Current to a cylinder from repelled particles of a Maxwellian plasma within a small source cone