SYSTEM CHARACTERIZATION OF
POSITIVE REAL CONDITIONS

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SYSTEM CHARACTERIZATION OF POSITIVE REAL CONDITIONS
Preface

This research was conducted under the auspices of the Research Institute for Computing and Information Systems by Q. Wang, J.L. Speyer, and H. Weiss of the University of California, Los Angeles. Dr. A. Glen Houston, Director of RICIS, served as RICIS research representative.

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System Characterization of Positive Real Conditions

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Abstract

Necessary and sufficient conditions for positive realness in terms of state space matrices are presented under the assumption of complete controllability and complete observability of square systems with independent inputs. As an alternative to the positive real lemma and to the s-domain inequalities, these conditions provide a recursive algorithm for testing positive realness which result in a set of simple algebraic conditions. By relating the positive real property to the associated variational problem, the paper outlines a unified derivation of necessary and sufficient conditions for optimality of both singular and nonsingular problems.

1. Introduction

Positive real systems play a major role in control theory, especially in adaptive control, and in stability analysis. The impressive development of adaptive control and self-turning regulation over the last two decades [1,2] is hinged on satisfaction of some positive realness conditions. Alternatively, considerable initial knowledge about the controlled plant must be given. The prior knowledge is used to implement reference models, identifiers, or observer-based controllers of about the same order as the plant. Since the prior assumptions about the controlled plant may never be entirely satisfied, the stability properties of the related adaptive schemes are debatable. Therefore, a direct adaptive control procedure which does not use identifier or observer-based controllers in the feedback loop is preferred. The implementation of such an algorithm requires positive real controlled plants or alternatively, a synthesis of a positive real plant on the basis of the actual plant.

The existing tools for analysis and synthesis of positive real systems are based in the s-domain on complex variable inequalities which are inconvenient or in the state space requiring the positive real lemma equations. These tools are computationally complex and there is a need for an easily used complementary tool. In Sections 2 and 3, necessary and sufficient conditions for positive real systems with independent inputs are developed using optimal control theory for the associated partially singular problem. It is shown that in the totally singular case, these conditions are consistent with the generalized Legendre-Clebsch condition [3,4]. The new conditions are associated with the state space matrices of a minimal realization of a square system. The resulting test for positive realness reduces to recursively testing certain square matrices for positive definiteness and the solution to an algebraic Riccati equation. As an immediate result of the new necessary and sufficient conditions, we also show that the zeros of a positive real system lie in the closed left half complex plane. Some examples are given in Section 4 to illustrate the theory. Concluding marks are given in Section 5.

The derivation of the above results is related to dissipative systems. Basic definitions and physical characteristics are presented below.

1.1 Dissipative System

Consider the system input-output description: $H: U \rightarrow Y$ where $U = L_2^1(R_+)$ and $Y = L_2^2(R_+)$. The notation $L_2^1(R_+)$ is used to denote the space of square integrable functions $f: R_+ \rightarrow R^1$ where $R_+ = (0, \infty)$. The supply rate associated with this system is defined as a function $w: R^1 \times R^n \rightarrow R$ where

$$w(u,y) = y'Qy + 2y'Su + u'Ru$$

and $Q \in R^{n\times n}$, $S \in R^{n\times n}$, $R \in R^{d\times d}$ are constant matrices, with $Q$ and $R$ symmetric.

Definition 1.1 [5]: A dynamical system $H$ is dissipative with respect to the supply rate $w(u,y)$ if and only if

$$\int_{t_0}^{t_1} w[u(t), y(t)] dt \geq 0$$

for all $t_1 \geq t_0$ and all $u \in L_2^1$, whenever the initial state satisfies $x(t_0) = 0$. 

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1.2 Energy, Power and Information Relationships in
Dissipative Systems
The class of dissipative systems which has a finite
dimensional internal state is completely described in terms of energy
storage and power dissipation. Considering this class, the various
facets of the standard state space model can be associated with the
concepts of energy, power and information. 
Assume that the system under consideration is described by
a linear, time-invariant system
\[ \begin{align*}
    x &= Ax + Bu \\
    y &= Cx + Du
\end{align*} \tag{1.3} \tag{1.4}
\]
where \( x \in R^n \), \( u \in R^l \), \( y \in R^m \) and \( A, B, C \) and \( D \) are constant
matrices with appropriate dimension. Then, following [6], the
system matrices can be regarded as representing:
1. an energy-transformation and dissipation map, associated
   with the matrix \( A \).
2. a power injection map, associated with the matrices \( B \)
   and \( D \).
3. an information-extraction map, associated with the
   matrix \( C \).
Figure 1 describes the energy-power-information maps
associated with the system matrices.

![Energy-Power-Information maps associated with the System Matrices](image)

The matrix \( B \) represents the input coupling between the
information represented by the applied input signals and the power
available for injection into the system states. The matrix \( C \)
represents the output coupling between the energy in the system
states and the information in the available output signals. The matrix
\( D \) represents the output coupling between the information
represented by the applied input signals and the injected power into
the available output signals.

1.3 Review of the Positive Real Property
The positive real property is related directly to the transfer
function matrix description of the system. The positive real lemma,
presented in Section 2, connects the positive realizability to the
parameters of a system realization with complete controllability and
complete observability.
The Positive Real Property [7]: Let \( G(s) \) be an \( m \times m \)
matrix of functions of a complex variable \( s \), then \( G(s) \) is termed
positive real if the following conditions are satisfied:
(i) All the elements of \( G(s) \) are analytic in \( Re[s] > 0 \).
(ii) \( G(s) \) is real for real positive \( s \).
(iii) \( G^*(s) + G(s) \geq 0 \) for \( Re \{ s \} > 0 \),
where \( (\cdot)^* \) denotes complex conjugate transpose.
Remark 1.4: If \( G(s) \) is a real rational matrix of functions
of \( s \), then necessary and sufficient conditions for the positive real
property to hold are given by the following theorem.
Theorem 1.1 [7]: Let \( G(s) \) be a real rational matrix of functions
of \( s \). Then, \( G(s) \) is positive real if and only if:
(i) No element of \( G(s) \) has a pole in \( Re[s] > 0 \).
(ii) \( G^*(s) + G(s) \geq 0 \) for all real \( \omega \), with \( \omega \) not a pole of
any element of \( G(s) \).
(iii) If \( s = \text{finite pole} \) of \( G(s) \), it is at most a
simple pole, and the residue matrix,
\[
    K_0 = \lim_{s \to j\omega_0} (s - j\omega_0) G(s) \quad \text{if } j\omega_0 \text{ is finite,}
\]
\[
    K_\infty = \lim_{s \to j\omega_0} G(s)/s \quad \text{if } j\omega_0 \text{ is infinite,}
\]
is nonnegative definite Hermitian.
Following Definition 1.1, if the system is positive real, the
angle between the output vector \( y(t) \) and the input vector \( u(t) \) is
bounded below by -90 deg. and above by +90 deg.

2. Relations Between Optimal Control
and Positive Realness
2.1 The Related Variational Problem
Consider the cost functional
\[
    V[x, u(t)] = \int_{t_0}^{t_1} w(u(t), y(t)) \, dt \tag{2.1}
\]
where the supply rate
\[
    w(u, y) = y' u = u'D' u + x'C' u \tag{2.2}
\]
is associated with system (1.3) and (1.4), where the dimensions of
\( u \) and \( y \) are \( m \). The problem is to find necessary and sufficient
conditions for optimality of \( u^* (\cdot) \in U \) to minimize \( V[x_0, u(t)] \).
denoted $V^*[x_0,t_0]$, subject to the dynamic equation of (1.3) where $x(t_0) = x_0$ is prescribed.

Remark 2.1: Since only the symmetric part of $D$ contributes to $w(u,y)$, then

$$ w(u,y) = \frac{1}{2} ( uR' + 2x'C' ) 
(2.3) $$

where

$$ R = D + D' 
(2.4) $$

Remark 2.2: If $R \geq 0$, and rank $(R) = r < m$, there exists an orthogonal transformation $\Gamma = [ \Gamma_1, \Gamma_2 ]$ such that

$$ \begin{bmatrix} \Gamma_1' \\ \Gamma_2' \end{bmatrix} R [ \Gamma_1, \Gamma_2 ] = \begin{bmatrix} R, 0 \\ 0, 0 \end{bmatrix} 
(2.5) $$

where $R_1$ is positive. For instance, $\Gamma_1$ and $\Gamma_2$ may consist of normalized eigenvectors of $R$ associated with nonzero and zero eigenvalues, respectively [8]. There is a natural partitioning of the control vector associated with this transformation, a $r$-dimensional nonsingular control and an $(m-r)$-dimensional singular control.

2.2 Positive Real Lemma Equations

Necessary and sufficient condition for $V^*[x_0,t_0]$ to be bounded below over a finite time interval $[t_0, t_1]$ are presented in Theorem II.3.3 of [9]. The required positive real conditions are obtained via the extension of the optimality condition to the time-invariant, infinite-time case [10].

Under the complete controllability and complete observability assumption of system (1.3), necessary and sufficient conditions for the nonnegativity of $V[0, t_0, u'(\cdot)]$ are that there exist $\pi < 0$, $L$, and $W$ such that

$$ \begin{bmatrix} \pi A + A' \pi & \pi B + C' \\ B' \pi + C & R \end{bmatrix} = [ \begin{bmatrix} L' \\ W \end{bmatrix} ] [ \begin{bmatrix} L \\ W \end{bmatrix} ] \geq 0, 
(2.6) $$

where $W$ and $L$ are matrices with proper dimension.

By identifying $P = -\pi$, the positive real Lemma is stated.

The Positive Real Lemma [7]: Let $G(s)$ be an $m \times m$ matrix of real rational functions of a complex variable $s$, with $G(\infty) = \infty$. Let $\{ A, B, C, D \}$ be a minimal realization of $G(s)$. Then, $G(s)$ is positive real if and only if there exist real matrices $P$, $L$, and $W$ with $P$ positive definite and symmetric, such that:

$$ PA + A'P = -L'L
(2.7) $$

$$ BP = C - WL
(2.8) $$

$$ WW = D + D'
(2.9) $$

Remark 2.2: The generalized Legendre-Clebsch condition, which is a necessary condition for $V^*[x_0,t_0] > \cdot \cdot \cdot$ in the totally singular case, given in [3] for a linear time-invariant system can be written as

$$ \frac{\partial}{\partial u} ( H_u ) = CB - (CB)' = 0 
(2.10) $$

$$ \frac{\partial}{\partial u} ( H_u ) = CAB + (CAB)' \leq 0 
(2.11) $$

where $H$ is the variational Hamiltonian and $x \in R^n$ is the associated Lagrange multiplier

$$ H = uC + x'(Ax + Bu), \quad \lambda' = -H_x. $$

By letting $R = 0$, the necessary conditions (2.10) and (2.11) are also obtained from the positive real lemma.

3. Positive Real Conditions in Terms of State-Space Matrices

Necessary and sufficient conditions for the nonnegativity of $V[0, t_0, u'(\cdot)]$ are given by the existence of $\pi < 0$, $L$, and $W$ which satisfy (2.6). Let $G(s)$ be an $m \times m$ matrix of degree $n$. Consider a minimal realization $\{ A, B, C, D \}$ representing the finite-dimensional linear time-invariant dynamic equations given by (1.3) and (1.4). In terms of state space matrices $A$, $B$, $C$, and $D$, (2.6) gives necessary and sufficient conditions for a positive real system. In this section, new necessary and sufficient conditions are developed.

3.1 Standard Formulation of the Partially Singular Problem

Assume that $G(s)$ is a square matrix of proper rational function with independent columns. For any realization, the matrices $C$ and $B$ are full rank. Without loss of generality, we consider a minimal realization $\{ A, B, C, D \}$ of the form that $A$ is an $n \times n$ matrix which is partitioned as

$$ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} $$

where $A_{11}$ is a lock matrix, and $A_{22}$ is an $(n-k) \times (n-k)$ matrix, where $k$ is the dimension of the singular control.

$$ B = [ B_n, B_s ], \quad C = [ C_r, C_s ] $$

and

$$ D + D' = \begin{bmatrix} R, 0 \\ 0, 0 \end{bmatrix} $$

where $R_n$ is a $n \times r$ nonsingular matrix corresponding to the nonsingular control. $B_n$ is an $n \times r$ matrix, $B_s$ is an $n \times k$ matrix related to the singular control, $C_r$ is a $n \times n$ matrix, and $C_s$ is a $k \times n$ matrix, where $r = m - k$ is the dimension of the nonsingular control. If $n > k$, then $C_s$ has the following form

$$ C_s = [ C_{s1}, 0 ] $$

where $C_{s1}$ is a nonsingular matrix. Correspondingly, $B_s$ is written as $[ B_{s1} ]$. We define this as a standard realization.

Notice that the realization can be obtained by choosing suitable bases for the state space and the input/output space. For example, suppose $\{ A, B, C, G(\omega) \}$ is a minimal realization of $G(s)$. Let the column vectors of $\Gamma$, where $\Gamma$ is described in Remark 2.2, be a basis of the input/output space, then the following transformation $y = \Gamma_1 u = \Gamma v$ is defined. Furthermore, let $q_1, q_2, ..., q_{n-k}, q_{n-k+1}, ..., q_n$ be a basis of the state space, where $q_{n-k+1}, ..., q_n$ span the null space of $\Gamma_1 C_s$, and $q_1, q_2, ..., q_n$ are arbitrary vectors such that $Q = [ q_1, q_2, ..., q_{n-k}, q_{n-k+1}, ..., q_n ]$ is nonsingular. This defines a transformation $x = Q \xi$. The resulting dynamic equation can be written as

$$ \dot{\xi} = A \xi + B v 
(3.1) $$

$$ \eta = C \xi + D v, 
(3.2) $$

where $A = Q^{-1} \tilde{Q}, B = Q^{-1} \tilde{B} \Gamma, C = \Gamma \tilde{C} Q, \text{ and } D = \Gamma G(\omega) \Gamma$. The transfer function matrix of this system is $\Gamma G(s) \Gamma$, the positive
realness of \( G(s) \) is equivalent to the positive realness of \( \Gamma G(s) \Gamma \).

The application of (2.6) and development of the new necessary and sufficient conditions for the partially singular problem will be discussed under assumption of a standard realization as discussed.

3.2 Derivation of New Necessary and Sufficient Conditions

Necessary and sufficient condition for nonnegative of \( V(0,t_0,0,0) \) as given by condition (2.6) can be restated in the following equivalent forms: There exist a \( \pi < 0 \) and a matrix \( V \) such that

\[
\begin{bmatrix}
\pi A + A' \pi & \pi B + C' \\
B' \pi + C & R
\end{bmatrix} = VV',
\]

(3.3)

Furthermore, \( R \) being positive semi-definite is a necessary condition for satisfying (3.3). If \( R \succ 0 \), then (3.3) can be reduced to a condition based upon a Riccati equation. That is, there exists a negative definite solution \( \pi \) to the algebraic Riccati equation

\[
\pi (A - BR^{-1}C) + (A - BR^{-1}C)\pi = \pi BR^{-1}B' + CR^{-1}C = 0.
\]

(3.4)

If \( R \) is singular, (3.3) can be written as

\[
\begin{bmatrix}
\pi A + A' \pi & \pi B + C' \\
B' \pi + C & R_t
\end{bmatrix} = V V',
\]

(3.5)

or, equivalently, there exist a \( \pi B + C' = 0 \) and a matrix \( V_t \) such that

\[
\begin{bmatrix}
\pi A + A' \pi & \pi B + C' \\
B' \pi + C & R_t
\end{bmatrix} = V_t V_t.
\]

(3.6)

If the dimension of the state is less than or equal to the dimension of the singular control, i.e., \( n \leq k \), \( \pi \) can be determined from equation (3.5). If only if \( \pi < 0 \) is solvable from (3.5) and the same \( \pi \) satisfies (3.6), the system is positive real. If \( n > k \), the fact \( \pi < 0 \) and equation (3.5) imply that

\[
\pi B_t = (C_t B_t)' = -B_t \pi B_t > 0,
\]

(3.7)

Since \( C_t = [C_t, 0] \), and \( C_t \) is nonsingular. Equation (3.7) also implies that \( B_t \) is nonsingular. Furthermore, (3.5) provides a linear constraint on \( \pi \) which is discussed in Lemma 3.1 below.

Lemma 3.1: \( \pi < 0 \), \( \pi B_t + C_t' = 0 \) if and only if \( \pi B_t > 0 \) and

\[
\pi = \begin{bmatrix}
-\pi_{11}(\pi_{12})^{-1} & \pi_{12} \\
\pi_{12} & \pi_{11}
\end{bmatrix}
\]

(3.8)

for some \( \pi_{11} < 0 \).

Proof: Denote \( \pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{12} & \pi_{11} \end{bmatrix} \). To prove

sufficiency, we assume that \( \pi_{11} < 0 \), \( \pi B_t > 0 \), and

\[
\pi_{11} = -(\pi_{12})^{-1} C_t + (B_t) \pi B_t (B_t)^{-1}
\]

(3.9)

\[
\pi_{12} = (B_t) \pi B_t (B_t)^{-1}
\]

(3.10)

Define \( F = \begin{bmatrix} 1 & -\pi_{12}(\pi_{11})^{-1} \\ 0 & 1 \end{bmatrix} \), then \( F \) is nonsingular and

\[
F \pi F' = \begin{bmatrix}
1 & -\pi_{12}(\pi_{11})^{-1} \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\pi_{11} & \pi_{12} \\
\pi_{12} & \pi_{11}
\end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \pi_{11} \pi_{12}(\pi_{11})^{-1} \pi_{12}' I
\]

\[
= \pi_{12}(\pi_{11})^{-1} I
\]

\[
= -(B_t)\pi B_t (B_t)^{-1} C_t
\]

\[
= (B_t)\pi B_t (B_t)^{-1} C_t = 0.
\]

Since \( C_t B_t = C_t B_t > 0 \).

(3.11)

Therefore \( F \pi F' < 0 \), and it also implies that \( \pi < 0 \). Furthermore, by using \( \pi_{11} \) and \( \pi_{12} \) defined in (3.9) and (3.10), we get

\[
\pi_{11} B_t + \pi_{12} B_t = 0
\]

(3.12)

\[
\pi_{12} B_t + \pi_{11} B_t = 0
\]

(3.13)

By solving (3.12) and (3.13), the expressions of \( \pi_{11} \) and \( \pi_{12} \) are obtained which are the same as shown in (3.9) and (3.10).

Q.E.D.

Let the matrix shown in (3.6) be denoted as \( M(\pi, R_t) \)

\[
M(\pi, R_t) = \begin{bmatrix}
\pi A + A' \pi & \pi B_t + C_t' \\
B_t' \pi + C_t & R_t
\end{bmatrix}
\]

For any nonsingular matrix \( T \), (3.6) is equivalent to \( T M(\pi, R_t) T \)

\[
= V_t V_t,
\]

where \( V_t \) is a matrix with proper dimension. By defining

\[
T_t = \begin{bmatrix} 0 & B_t \\ B_t' & 0 \end{bmatrix}
\]

and using \( \pi \) defined in (3.8) as a function of \( \pi_{11} \), then \( T_t \) is nonsingular, and

\[
T_t M(\pi, R_t) T_t = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{12}' & M_{22} & M_{23} \\
M_{13}' & M_{23}' & M_{33}
\end{bmatrix}
\]

where

\[
M_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \pi_t (A_{12} - B_{12}(B_{11})^{-1} A_{11}) + (A_{12} - B_{12}(B_{11})^{-1} A_{11})' \pi_t
\]

\[
M_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \pi_t (A_{21} B_{11} + A_{22} B_{12}) - C_t A_{12}
\]

\[
M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = - (C_t B_t + B_t' A_t C_t)
\]

\[
M_{33} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = R_t
\]
By defining
\[ A_1 = A_{22} \cdot B_{22}(B_{11})^{-1} A_{12} \]  
(3.14)
\[ B_1 = \left[ A_{21} B_{11} + A_{22} B_{22} \cdot B_{21} \cdot A_{11} B_{11}^{-1} \right] 
B_{22}(B_{11})^{-1} A_{12} B_{21} \]  
(3.15)
\[ C_1 = \left[ -C_1 A_{12} \cdot 0 \right] \]  
(3.16)
\[ R_1 = \left[ \begin{array}{c}
(C_1 A_{12} + B_1 C_1') - C_1 B_1 + B_1 C_1' \\
A_1 B_1 B_1 + C_1' \\
B_1 + C_1 + R_1
\end{array} \right]. \]  
(3.17)
a condition which is equivalent to (3.6) can be stated as follows:
There exist a \( \pi_1 < 0 \) and a matrix \( V_1 \) such that
\[ \pi_1 A_1 + A_1' \pi_1 = V_1 V_1. \]  
(3.18)

According to the positive real lemma, Equation (3.18) implies that 
\[ \{ A_1, B_1, C_1, R_1 \} \] is positive real.

3.3 Necessary and Sufficient Conditions for Positive Realness

The results in Section 3.2 are summarized in the next theorem as an alternative necessary and sufficient condition for testing positive realness of a square system.

**Theorem 3.1:** The necessary and sufficient condition for 
\( \{ A, B, C, D \} \) to be positive real is that
(i) \( R \geq 0 \);
(ii) If \( R > 0 \), there exists a positive definite solution \( P \) to the following algebraic Riccati equation
\[ P (A - B R C) + (A - B R C)^2 P + P B R C B^t C R + C R C = 0; \]
(iii) If rank \( R = r < m \), and \( n \leq m-r \), there exists
\[ P = C B_r (B_r C_r)^{-1} > 0 \]
satisfying
\[ \begin{bmatrix} P & A_1' \circ P & -B_r P + C_r' \\
A_1 P + A_1' C_r & C_r & R_r \end{bmatrix} \geq 0; \]
v and \( \{ A_1, B_1, C_1, R_1 \} \) are defined in equations (3.14) to (3.17).

Condition (ii) is obtained by identifying \( P \) with \( -\pi \) in equation (3.4). Condition (iii) is the interpretation of (3.5) and (3.6) for the case \( n \leq m-r \). If \( P = \pi > 0 \) exists, then \( PB_r = C_r' \), \( PB_r \cdot C_r' = C_r B_r \), and \( P = C B_r (B_r C_r)^{-1} > 0 \) . Condition (iv) corresponds to the situations discussed through (3.7) to (3.13).

**Remark 3.1:** Alternative transformation approaches to the singular problem using the Kelley transformation for the linear quadratic problem are given in [9] for the matrix case. The approach here is different via the structure of \( \pi \) given by Lemma 3.1.

**Remark 3.2:** If \( \{ A, B, C, D \} \) is a minimal realization, then it is required for a positive real system that there exists a positive definite matrix \( P \) such that
\[ PA + AP \leq 0 \]
Therefore, it is required that \( \text{Re} \lambda_i[A] \leq 0 \) and the Jordan form of \( A \) has no blocks of size greater than 1x1 with pure imaginary diagonal elements.

**Remark 3.3:** If \( G(s) \) is strictly proper, the minimal realization is totally singular, then the characteristic polynomial of 
\( A_1 = A_{22} \cdot B_{22}(B_{11})^{-1} A_{12} \) is equal to the zero polynomial of the system up to a nonzero scalar factor.

Proof: Let \( \det G(s) = \det (C (sI - A) B) = \frac{\psi(s)}{\Delta(s)} \) where
\[ \Delta(s) = \det (s I - A) \]
and \( \psi(s) \) is the zero polynomial of the system. Since state feedbacks do not change the numerator of the transfer function matrix, for any matrix \( K \),
\[ \det (G_k(s)) = \det (C (sI - A - BK) B) = \frac{\psi(s)}{\Delta_k(s)} \]
where \( \Delta_k(s) = \det (s I - A - BK) \)
Let \( K = \{ 0, (B_{11})^{-1} A_{12} \} \), then
\[ A + BK = \left[ \begin{array}{c}
A_{11} \\
A_{21} A_{22} + B_{22}(B_{11})^{-1} A_{12} \\
A_{11} 0
\end{array} \right] \]
\[ \Delta_k(s) = \det (s I - A - BK) = \det \left[ \begin{array}{c}
(s I - A_{11}) \\
A_{21} 0
\end{array} \right] \]
\[ = \det (s I - A_{11}) \det (s I - A_1) \]
\[ \det (G_k(s)) = \det (C (sI - A - BK) B) = \det (C_{11} (sI - A_{11})^{-1} B_{22}) \]
\[ = -\det (C_{11}) \det (B_{11}) \]
\[ \det (s I - A_{11}) \]
Therefore,
\[ \psi(s) = \Delta_k(s) \det (G_k(s)) = \det (C_{11}) \det (B_{11}) \det (s I - A_1) \]
Q.E.D.

**Remark 3.4:** From (3.18) and Remark 3.3, we conclude that there are \( n-m \) finite zeros for a positive real system and all the zeros lie in the closed left half complex plane. In other words, the system is minimum phase.

4. Examples

**Theorem 3.1** introduces a recursive procedure for testing positive real systems, requests only for testing a series of matrices \( C_{11} B_{11} > 0 \), for \( i = 0, 1, 2, \ldots, l \), and the solution to an algebraic Riccati equation \( P_i > 0 \), where \( i \) is the index associated with the new system obtained from the \( i \)-th iteration, and \( i = 0 \) corresponds to \( B_{11} \), \( C_{11} \), and \( P \). The testing stops when \( R_i \) becomes nonsingular, or the dimension of the state is less or equal to the dimension of the singular control.

The following examples illustrate the application of **Theorem 3.1**.

**Example 4.1:** Given \( G(s) = \frac{(s + 2)^2}{s(s + 1)(s + 3)} \), an observable realization of \( G(s) \) is
\[ A = \begin{bmatrix}
-4 & 1 & 0 \\
-3 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \\
4 \\
4 \end{bmatrix}, \quad C = \begin{bmatrix} 1, 0, 0 \end{bmatrix}, \quad D = 0. \]
**First iteration:**
\( R = 0 \)
\( CB = (CB)' = 1 > 0 \)
A_1 = \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = [1, 0, 0], \quad R_1 = 0.

Second iteration:
R_1 = 0
C_1B_1 = (C_1B_1)^\dagger = -1 < 0
Therefore, the system is not positive real.

Example 4.2: Given \( G(s) = \frac{(s + 1)^2}{(s + 2)(s + 4)} \), an observable realization of \( G(s) \) is
\[
A = \begin{bmatrix} -6 & 1 \\ -8 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad C = [1, 0, 0], \quad D = 0.
\]

First iteration:
R = 0
CB = (CB)^\dagger = 1 > 0
\[
A_1 = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad C_1 = [-1, 0], \quad R_1 = 8.
\]

Second iteration:
R_1 = 8 > 0, the algebraic Riccati equation is
\[
P_1 \begin{bmatrix} -15 & 8 \\ -8 & 0 \end{bmatrix} + \begin{bmatrix} -15 & -4 \\ 0 & 8 \end{bmatrix}P_1 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0
\]
which has a positive definite solution
\[
P_1 = \begin{bmatrix} 0.0394 & -0.0225 \\ -0.225 & 0.1557 \end{bmatrix} > 0
\]
Therefore, the system is positive real.

Example 4.3: \( G(s) = \frac{s^2 + z^2}{s(s^2 + p^2)} \), a minimal realization of \( G(s) \) is
\[
A = \begin{bmatrix} 0 & 1 \\ -p^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ z^2 \end{bmatrix}, \quad C = [1, 0, 0], \quad D = 0.
\]

First iteration:
R = 0
CB = (CB)^\dagger = 1 > 0
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} z^2 \end{bmatrix}, \quad C_1 = [-1, 0],
\]
R_1 = 0.
Second iteration:
R_1 = 0
\[
C_1B_1 = (C_1B_1)^\dagger = p^2 - z^2 > 0 \quad \text{if and only if} \quad p^2 > z^2
\]
A_2 = 0, B_2 = z^2(z^2 - p^2), C_2 = 1, R_2 = 0.
Third iteration:
R_2 = 0.
\[
P_2 = C_2B_2(C_2B_2)^\dagger = \frac{1}{z^2(z^2 - p^2)} > 0 \quad \text{if} \quad p^2 > z^2
\]
- \( p_2A_2 \cdot A_2^TP_2 = 0 \).
Therefore, the system is positive real if and only if \( p^2 > z^2 \).

5. Summary and Conclusions

This paper reviews positive real system as a subclass of dissipative systems and states the positive real lemma equations. By using the variational problem associated with the partially singular problem, necessary and sufficient conditions for a system to be positive real are derived. These conditions are particularly transparent by using Lemma 3.1 which provides a uniquely structure for the matrix \( \xi \). These positive realness conditions are expressed in terms of the state space matrix inequalities and algebraic Riccati equations and do not deal with inequalities in the \( s \) domain or with solutions of the positive real lemma equations. These tests are direct, and a system either satisfies these conditions or not. There is no requirement to search over all matrices to determine if a condition can be satisfied as in the positive real lemma. Examples are given which demonstrate the power of this approach.

References


