A RECURSIVE APPROACH TO THE EQUATIONS OF MOTION FOR THE MANEUVERING AND CONTROL OF FLEXIBLE MULTI-BODY SYSTEMS

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Work supported in part by the AFOSR Research Grant F49620-89-C-0049DEF monitored by Dr. Spencer T. Wu; the support is greatly appreciated.
OVERVIEW

• The interest lies in a mathematical formulation capable of accommodating the problem of maneuvering a space structure consisting of a chain of articulated flexible substructures.

• Simultaneously, any perturbations from the "rigid-body" maneuvering and any elastic vibration must be suppressed.

• The equations of motion for flexible bodies undergoing rigid-body motions and elastic vibrations can be obtained conveniently by means of Lagrange's equations in terms of quasi-coordinates.

• The advantage of this approach is that it yields equations in terms of body axes, which are the same axes that are used to express the control forces and torques.
OVERVIEW (CONT'D)

- The equations of motion are nonlinear hybrid (ordinary and partial) differential equations.

- The partial differential equations can be discretized (in space) by means of the finite element method or the classical Rayleigh-Ritz method.

- The result is a set of nonlinear ordinary differential equations of high order.

- The nonlinearity can be traced to the rigid-body motions and the high order to the elastic vibration.

- Elastic motions tend to be small when compared with rigid-body motions.

- A perturbation approach permits breaking the problem into one for the rigid-body motions, which is nonlinear and of relatively low order, and for the elastic motions and the perturbations caused by them, which is linear and of relatively high order.
OVERVIEW (CONT'D)

• The rigid-body problem, which is associated with the maneuvering, is referred to as the zero-order (in a perturbation sense) problem and the control tends to be open loop.

• The perturbation suppression, which is associated with control, is referred to as the first-order problem and the control is closed-loop.

• The equations of motion are first derived for each individual substructure and then assembled into a single set for the fully interacting structure.

• The above is carried out by means of a kinematical synthesis eliminating the surplus coordinates.

• The kinematical synthesis, based on recursive relations, is carried out both for the zero-order and first-order problems.
HYBRID EQUATIONS FOR THE SUBSTRUCTURES

Figure 1 - The Articulated Chain of Substructures

\[ X_I Y_I Z_I = \text{inertial axes, } x_s y_s z_s = \text{body axes} \]
HYBRID EQUATIONS FOR THE SUBSTRUCTURES

Derivation of the equations of motion by the Lagrangian approach requires the Lagrangian, and hence the kinetic and potential energy.

Position Vector of Point $P_s$ in Substructure $s$:

$$W_s(t) = R_s(t) + r_s(P_s) + u_s(P_s, t), \quad s = 1, 2, ..., N$$

$R_s$ = radius vector from I to S; in terms of inertial coordinates

$r_s$ = radius vector from S to $P_s$; in terms of body coordinates

$u_s$ = elastic displacement vector of $P_s$; in terms of body coordinates

Velocity Vector of $P_s$:

$$\dot{W}_s(t) = V_s(t) + \ddot{\omega}_s(t)(r_s(P_s) + u_s(P_s, t)) + v_s(P_s, t)$$

$$= V_s(t) + (\ddot{r}_s(P_s) + \ddot{u}_s(P_s, t))^T \omega_s(t) + v_s(P_s, t), \quad s = 1, 2, ..., N$$

$V_s$ = velocity vector of S; in terms of body coordinates

$\omega_s$ = absolute angular velocity vector of $x_s y_s z_s$; in terms of body coordinates

$\ddot{\omega}_s$ = skew symmetric matrix formed from $\omega_s$

$v_s = \dot{u}_s$ = elastic velocity vector of $P_s$; in terms of body coordinates
HYBRID EQUATIONS FOR THE SUBSTRUCTURES (CONT'D)

Relation Between Inertial and Body-Axes Velocity Vectors:

\[ \mathbf{V}_s = C_s \dot{\mathbf{R}}_s, \quad \omega_s = D_s \dot{\theta}_s \]

\[ C_s = C_s(\theta_{s1}, \theta_{s2}, \theta_{s3}) = \text{matrix of direction cosines between } x_s y_s z_s \]
\[ \text{and } X_I Y_I Z_I \]

\[ D_s = D_s(\theta_{s1}, \theta_{s2}, \theta_{s3}) = \text{transformation matrix} \]

\[ \theta_{s1}, \theta_{s2}, \theta_{s3} = \text{angles defining the orientation of } x_s y_s z_s \text{ and referred to } X_I Y_I Z_I \]

\[ \mathbf{V}_s, \omega_s \text{ can be regarded as time derivatives of quasi-coordinates} \]
HYBRID EQUATIONS FOR THE SUBSTRUCTURES (CONT'D)

Kinetic Energy:

\[ T_s = \frac{1}{2} \int_{D_s} \rho_s \mathbf{\dot{W}}_s^T \mathbf{\dot{W}}_s dD_s \]
\[ = \frac{1}{2} m_s \mathbf{V}_s^T \mathbf{V}_s + \mathbf{V}_s^T \mathbf{\bar{S}}_s^T \mathbf{\bar{\omega}}_s + \mathbf{V}_s^T \int_{D_s} \rho_s \mathbf{v}_s dD_s \]
\[ + \frac{1}{2} \mathbf{\bar{\omega}}_s^T \mathbf{J}_s \mathbf{\bar{\omega}}_s + \mathbf{\bar{\omega}}_s^T \int_{D_s} \rho_s (\mathbf{\bar{r}}_s + \mathbf{\bar{\omega}}_s) \mathbf{v}_s dD_s + \frac{1}{2} \int_{D_s} \rho_s \mathbf{v}_s^T \mathbf{v}_s dD_s \]

\[ m_s = \int_{D_s} \rho_s dD_s, \quad \mathbf{\bar{S}}_s = \int_{D_s} \rho_s (\mathbf{\bar{r}}_s + \mathbf{\bar{\omega}}_s) dD_s, \quad \mathbf{J}_s = \int_{D_s} \rho_s (\mathbf{\bar{r}}_s + \mathbf{\bar{\omega}}_s)(\mathbf{\bar{r}}_s + \mathbf{\bar{\omega}}_s)^T dD_s \]

Potential Energy:

\[ V_s = \frac{1}{2} [\mathbf{u}_s, \mathbf{u}_s] \]

\( \rho_s = \) mass density; \( D_s = \) domain of substructure \( s \)
\([ \ , \ ] = \) energy inner product
HYBRID EQUATIONS FOR THE SUBSTRUCTURES (CONT'D)

General Hybrid Lagrange's Equations in Terms of Quasi-Coordinates:

\[
\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{V}_s} \right) + \tilde{\omega}_s \left( \frac{\partial L_s}{\partial \dot{V}_s} \right) - C_s \left( \frac{\partial L_s}{\partial \dot{R}_s} \right) = F_s
\]

\[
\frac{d}{dt} \left( \frac{\partial L_s}{\partial \dot{\omega}_s} \right) + \tilde{V}_s \left( \frac{\partial L_s}{\partial \dot{V}_s} \right) + \tilde{\omega}_s \left( \frac{\partial L_s}{\partial \dot{\omega}_s} \right) - \left( D_s^T \right)^{-1} \left( \frac{\partial L_s}{\partial \theta_s} \right) = M_s
\]

\[
\frac{\partial}{\partial t} \left( \frac{\partial \hat{L}_s}{\partial V_s} \right) - \left( \frac{\partial \hat{T}_s}{\partial u_s} \right) + \mathcal{L}_s u_s = \hat{U}_s
\]

\[ L_s = T_s - V_s = \text{Lagrangian}; \quad \hat{L}_s = \text{Lagrangian density} \]

\[ \hat{T}_s = \text{kinetic energy density}; \quad \mathcal{L} = \text{(stiffness) differential operator matrix} \]

\[ F_s, M_s = \text{resultant force and torque vectors} \]

\[ \hat{U}_s = \text{force density vector} \]
HYBRID EQUATIONS FOR THE SUBSTRUCTURES (CONT'D)

Explicit Hybrid Equations:

\[ m_s \ddot{V}_s + \ddot{S}_s^T \dot{\omega}_s + \int_{D_s} \rho_s \ddot{v}_s dD_s = (2\dot{S}_{us} + m_s \ddot{V}_s + \ddot{\omega}_s \ddot{S}_s) \omega_s + C_s \left( \frac{\partial L_s}{\partial \dot{R}_s} \right) + F_s \]

\[ \ddot{S}_s \dot{V}_s + J_s \dot{\omega}_s + \int_{D_s} \rho_s (\ddot{\bar{r}}_s + \ddot{\bar{u}}_s) \dot{v}_s dD_s = (\ddot{S}_s \dot{V}_s - \ddot{\omega}_s J_s - J_{us}) \omega_s - \ddot{\omega}_s \int_{D_s} \rho_s (\ddot{\bar{r}}_s + \ddot{\bar{u}}_s) v_s dD_s + (D_s^T)^{-1} \left( \frac{\partial L_s}{\partial \dot{\theta}_s} \right) + M_s \]

\[ \rho_s [\ddot{V}_s + (\ddot{\bar{r}}_s + \ddot{\bar{u}}_s)^T \ddot{\omega}_s + \ddot{v}_s] = \rho_s (\ddot{V}_s + 2\ddot{v}) \omega_s - \rho_s \ddot{\omega}_s^2 (r_s + u_s) - \mathcal{L}_s u_s + \ddot{U}_s \]

where

\[ S_{us} = \dot{S}_s = \int_{D_s} \rho_s \tilde{v}_s dD_s , \quad J_{us} = \dot{J}_s = \int_{D_s} \rho_s [\ddot{\bar{r}}_s (\ddot{\bar{r}}_s + \ddot{\bar{u}}_s)^T + (\ddot{\bar{r}}_s + \ddot{\bar{u}}_s) \ddot{v}_s^T] dD_s \]

Augmenting Equations:

\[ \dot{\dot{R}}_s = C_s^T V_s , \quad \dot{\theta}_s = D_s^{-1} \omega_s , \quad \dot{u}_s = v_s \]
ORDINARY DIFFERENTIAL EQUATIONS

Elastic Displacement Vector: \[ u_s(P_s, t) = \Phi_s(P_s)q_s(t), s = 1, 2, \ldots, N \]

\[ \Phi_s = \text{matrix of admissible functions (shape functions)} \]

\[ q_s = \text{vector of generalized displacements} \]

- Derive discretized \( T_s \) and \( V_s \)

Discretized State Equations:

\[
\begin{align*}
\dot{m}_s \dot{V}_s + \bar{S}_s \ddot{\omega}_s + \Phi_s \ddot{p}_s &= -m_s \ddot{\omega}_s V_s - \ddot{\omega}_s \bar{S}_s \omega_s - 2\ddot{\omega}_s \Phi_s p_s - (\dddot{\omega}_s + \dot{\omega}_s^2) \Phi_s q_s + F_s \\
\bar{S}_s \dot{V}_s + J_0 \dot{\omega}_s + \Phi_s \dot{p}_s &= -\bar{S}_s \ddot{\omega}_s V_s - \ddot{\omega}_s J_0 \omega_s - 2\ddot{\omega}_s \Phi_s p_s - ([\tilde{V}_s \omega_s] - \tilde{V}_s) \Phi_s + 2\dot{\Phi}_s + 2\ddot{\omega}_s \Phi_s - (\dddot{\omega}_s + \dot{\omega}_s^2) \Phi_s) q_s + M_s \\
\Phi_s^T \dot{V}_s + \bar{\Phi}_s \dot{\omega}_s + M_s \dot{p}_s &= -\Phi_s^T \ddot{\omega}_s V_s + \dot{\Phi}_s^T \omega_s - 2\bar{H}_s p_s - [K_s + \bar{H}_s (\omega_s) + \tilde{H}_s)] q_s + Q_s
\end{align*}
\]

Various terms involve integrals over \( D_s \)

Augmenting Equations: \[ \dot{R}_s = C_s^T V_s, \quad \dot{\theta}_s = D_s^{-1} \omega_s, \quad \dot{q}_s = p_s \]

PERTURBATION APPROACH

Perturbation Expansions:

\[ V_s = V_{s0} + V_{s1}, \quad \omega_s = \omega_{s0} + \omega_{s1}, \quad F_s = F_{s0} + F_{s1}, \quad M_s = M_{s0} + M_{s1} \]

- Subscript 0 denotes zero-order (in a perturbation sense) quantities
- Subscript 1 denotes first-order quantities
- First-order terms are one order of magnitude smaller than zero-order terms
- Elastic displacements and velocities are by definition of first order.
PERTURBATION APPROACH (CONT'D)

Introduce perturbation expansion into state equations and separate orders of magnitude.

Zero-Order State Equations:  \[ \mathcal{M}_{s0} \dot{x}_{s0} = \mathcal{C}_{s0} x_{s0} + B_{s0} f_{s0} + D_{s0} d_{s0} \]

Zero-Order State and Excitation Vectors:
\[ x_{s0}(t) = [R_{s0}^T(t) \theta_{s0}^T(t) V_{s0}^T(t) \omega_{s0}^T(t)]^T, \quad f_{s0}(t) = [F_{s0}^T(t) M_{s0}^T(t)]^T \]

Coefficient Matrices:

\[
\mathcal{M}_{s0} = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & m_s I & \tilde{S}_{s0}^T \\
0 & 0 & \tilde{S}_{s0} & J_{s0}
\end{bmatrix}, \quad \mathcal{C}_{s0} = \begin{bmatrix}
0 & 0 & C_{s0}^T & 0 \\
0 & 0 & 0 & D_{s0}^{-1}
\end{bmatrix}
\]

\[
B_{s0} = D_{s0} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
I & 0 \\
0 & I
\end{bmatrix}, \quad d_{s0} = \begin{bmatrix}
-m_s \omega_{s0} V_{s0} - \tilde{\omega}_{s0} \tilde{S}_{s0}^T \omega_{s0} \\
-\tilde{S}_{s0} \tilde{\omega}_{s0} V_{s0} - \tilde{\omega}_{s0} J_{s0} \omega_{s0}
\end{bmatrix}
\]
PERTURBATION APPROACH (CONT'D)

First-Order State Equations: \( \mathbf{M}_{s1} \dot{\mathbf{x}}_{s1} = \mathbf{C}_{s1} \mathbf{x}_{s1} + \mathbf{B}_{s1} \mathbf{f}_{s1} + \mathbf{D}_{s1} \mathbf{d}_{s1} \)

First-Order State and Excitation Vectors:
\[
\mathbf{x}_{s1}(t) = \begin{bmatrix} \mathbf{U}_{s1}^T(t) & \mathbf{\beta}_{s1}^T(t) & \mathbf{q}_{s}^T(t) & \mathbf{V}_{s1}^T(t) & \mathbf{\omega}_{s1}^T(t) & \mathbf{p}_{s}^T(t) \end{bmatrix}^T, \quad \mathbf{f}_{s1}(t) = \begin{bmatrix} \mathbf{F}_{s1}^T(t) & \mathbf{M}_{s1}^T(t) & \mathbf{Q}_{s}^T(t) \end{bmatrix}^T
\]

\( \mathbf{U}_{s1} = \) body-axes vector of perturbations in translational displacements

\( \mathbf{\beta}_{s1} = \) body-axes vectors of perturbations in rotational displacements

Coefficient Matrices:

\[
\mathbf{M}_{s1} = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & m_{s}I & \tilde{\mathbf{S}}_{s0}^T & \tilde{\Phi}_{s} \\
0 & 0 & 0 & \tilde{\mathbf{S}}_{s0} & J_{s0} & \tilde{\Phi}_{s} \\
0 & 0 & 0 & \tilde{\Phi}_{s}^T & \tilde{\Phi}_{s}^T & \mathbf{M}_{s} \\
\end{bmatrix}
\]
PERTURBATION APPROACH (CONT'D)

Coefficient Matrices:

\[
c_{s1} = \begin{bmatrix}
-\bar{\omega}_{s0} & -\bar{V}_{s0} & 0 & I & 0 & 0 \\
0 & -\bar{\omega}_{s0} & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & -(\bar{\omega}_{s0} + \bar{\omega}_{s0}^2)\bar{\Phi}_{s0} & -m_s\bar{\omega}_{s0} & -\Gamma_s & -2\bar{\omega}_{s0}\bar{\Phi}_{s0} \\
0 & 0 & -\bar{\Xi}_s & -\bar{S}_s\bar{\omega}_{s0} & -\Delta_s & -2\bar{\Phi}_s \\
0 & 0 & -\bar{K}_s & -\bar{\Phi}_s^T\bar{\omega}_{s0} & -\bar{\Upsilon}_s & -2\bar{H}_s
\end{bmatrix}
\]

\[
b_{s1} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}, \quad d_{s1} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
I
\end{bmatrix}
\]

in which

\[
\Gamma_s = \bar{S}_{s0}\bar{\omega}_{s0} - 2\bar{\omega}_{s0}\bar{S}_{s0} - m_s\bar{V}_{s0}, \quad \Delta_s = 2\bar{\omega}_{s0}J_{s0} + J_{s0}\bar{\omega}_{s0} - (trJ_{s0})\bar{\omega}_{s0} - \bar{S}_{s0}\bar{V}_{s0}
\]

\[
\bar{\Xi}_s = (\bar{V}_{s0}\bar{\omega}_{s0} - \bar{V}_{s0})\bar{\Phi}_s + 2\dot{\bar{\Phi}}_s + 2\bar{\omega}_s\dot{\bar{\Phi}}_s - (\bar{\omega}_{s0} + \bar{\omega}^2_{s0})\bar{\Phi}_s
\]

\[
\bar{\Upsilon}_s = \bar{\Phi}_s^T\bar{\omega}_{s0}^T - 2\bar{\Phi}_s^T + \bar{\Phi}_s^T\bar{V}_{s0}, \quad \bar{K}_s = K_s + \bar{H}_s + \dot{\bar{H}}_s
\]

\[
d_{s1} = -\bar{\Phi}_s^T(\dot{\bar{V}}_{s0} + \bar{\omega}_{s0}V_{s0}) - \bar{\Phi}_s\bar{\omega}_{s0} - \bar{\Phi}_s^T\bar{\omega}_{s0}
\]
Figure 2 - Two Adjacent Substructures in the Chain
Kinematical Constraints (linking the substructure together):
\[ \mathbf{R}_{q0} = \mathbf{R}_{p0} + C_{p0}^T \mathbf{r}_{pq} , \quad \theta_{q0} = \theta_{q0} \]
\[ \mathbf{V}_{q0} = C_{qp} \mathbf{V}_{p0} - C_{qp} \tilde{\mathbf{r}}_{pq} \omega_{p0} , \quad \omega_{q0} = \omega_{q0} \]

Recursive Relations:
\[ \mathbf{r}_{s0} = \mathbf{r}_{10} + \sum_{j=1}^{s} \sum_{i=1}^{j} C_{i,j-1} \mathbf{r}_{j-1,i} \]
\[ \mathbf{v}_{s0} = \prod_{i=s}^{2} C_{j,i-1} \mathbf{v}_{10} - \prod_{i=s}^{2} C_{j,i-1} \tilde{\mathbf{r}}_{12} \omega_{10} - \prod_{i=s}^{3} C_{j,i-1} \tilde{\mathbf{r}}_{23} \omega_{20} - \cdots - C_{s,s-1} \tilde{\mathbf{r}}_{s-1,s} \omega_{s0} \]

Relation Between the State of Substructure s and Part of Constrained State of Structure (Substructures 1 through s):
\[ \mathbf{x}^u_{s0} = T_{s0} \mathbf{x}^c_{s0} + \bar{\mathbf{C}}_s \mathbf{r}_{s0} , \quad s = 1, 2, \ldots, N \]
\[ \mathbf{x}^u_{s0} = \mathbf{x}_{s0} , \quad \mathbf{x}^c_{s0} = [\mathbf{R}_{10}^T \theta_{10}^T \theta_{20}^T \cdots \theta_{s0}^T \mathbf{V}_{10}^T \omega_{10}^T \omega_{20}^T \cdots \omega_{s0}^T]^T , \quad s = 1, 2, \ldots, N \]
\[ \mathbf{r}_{s0} = [\mathbf{r}_{12}^T \mathbf{r}_{23}^T \cdots \mathbf{r}_{s-1,s}^T]^T , \quad s = 1, 2, \ldots, N \]
KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS
(CONT'D)

\[ \mathcal{M}_0 \dot{x}_0^u = \mathcal{C}_0 x_0^u + \mathcal{B}_0 f_0 + \mathcal{D}_0 d_0 \]

Disjoint State Equations:

\[
x_0^u = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{N0} \end{bmatrix}, \quad f_0 = \begin{bmatrix} f_{10} \\ f_{20} \\ \vdots \\ f_{N0} \end{bmatrix}, \quad d_0 = \begin{bmatrix} d_{10} \\ d_{20} \\ \vdots \\ d_{N0} \end{bmatrix}
\]

Completion of the State Dimension:

\[
x_{10}^* = \begin{bmatrix} x_{10} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_{20}^* = \begin{bmatrix} 0 \\ x_{20} \\ \vdots \\ 0 \end{bmatrix}, \ldots, x_{N0}^* = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ x_{N0} \end{bmatrix}
\]

Unconstrained State:

\[ x_0^u = \sum_{s=1}^{N} x_{s0}^* \]

Full-Dimension Constraint Equation:

\[ x_0^u = \sum_{s=1}^{N} x_{s0}^* = \sum_{s=1}^{N} T_{s0} x_0^c + \bar{C} r_0 = T_0 x_0^c + \bar{C} r_0 \]

Constrained State:

\[ x_0^c = x_{N0}^c = [R_{10}^T \theta_{10}^T \theta_{20}^T \cdots \theta_{N0}^T V_{10}^T \omega_{10}^T \omega_{20}^T \cdots \omega_{N0}^T]^T, \quad r_0 = r_{N0} \]
KINEMATICAL SYNTHESIS FOR ZERO-ORDER EQUATIONS (CONT'D)

State Equations for Zero-Order Problems:

\[ \dot{x}_0 = A_0 x_0 + B_0^* f_0 + D_0^* d_0 + R_0^* r_0 \]

\[ A_0 = (T_0^T M_0 T_0)^{-1} T_0^T (C_0 T_0 - M_0 \dot{T}_0), \quad B_0^* = (T_0^T M_0 T_0)^{-1} T_0^T B_0 \]

\[ D_0^* = (T_0^T M_0 T_0)^{-1} T_0^T D_0, \quad R_0^* = (T_0^T M_0 T_0)^{-1} T_0^T (C_0 \ddot{C} - M_0 \ddot{\dot{C}}) \]

Note: Superscript c was dropped for simplicity
KINEMATICAL SYNTHESIS FOR FIRST-ORDER EQS.

Kinematical Constraints Yield Recursive Relations:

\[ U_{q1} = C_{qp}[U_{p1} - \tilde{r}_{pq}\beta_{p1} + \Phi_{pq}q_p], \quad \beta_{q1} = C_{qp}(\beta_{p1} + \Psi_{pq}q_p), \quad q_q = q_q \]

\[ V_{q1} = [\tilde{V}_{p0}C_{qp} + C_{qp}(\tilde{r}_{pq}\omega_{p0} - \tilde{V}_{p0})]\beta_{p1} + [\tilde{V}_{p0}C_{qp}\Psi_{pq} + C_{qp}\tilde{\omega}_{p0}\Phi_{pq}]q_p + C_{qp}V_{p1} - C_{qp}\tilde{r}_{pq}\omega_{p1} + C_{qp}\Phi_{pq}p_p \]

\[ \omega_{q1} = C_{qp}(\tilde{\omega}_{p0}\Psi_{pq}q_p + \omega_{p1} + \Psi_{pq}p_p), \quad p_q = p_q \]

where

\[ \Phi_{pq} = \Phi_p(r_{pq}), \quad \Psi_{pq} = \nabla \times \Phi_p(r_{pq}) \]

Matrix Form of Recursive Relations:

\[ x^u_{s1} = T_{s1}x^e_{s1}, \quad s = 2, 3, \ldots, N \]

Disjoint Perturbation State Equations:

\[ \mathcal{M}_1x^u_1 = C_1x^u_1 + B_1f_1 + D_1d_1 \]

\[ x^u_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{N1} \end{bmatrix}, \quad f_1 = \begin{bmatrix} f_{11} \\ f_{21} \\ \vdots \\ f_{N1} \end{bmatrix}, \quad d_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{N1} \end{bmatrix} \]
Completion of the Perturbation State Dimension:

\[
x_{11}^* = \begin{bmatrix}
  x_{11} \\
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}, \quad x_{21}^* = \begin{bmatrix}
  0 \\
  x_{21} \\
  0 \\
  \vdots \\
  0
\end{bmatrix}, \quad x_{31}^* = \begin{bmatrix}
  0 \\
  0 \\
  x_{31} \\
  \vdots \\
  0
\end{bmatrix}, \quad x_{N1}^* = \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  x_{N1}
\end{bmatrix}
\]

Constrained Perturbation State Vector:

\[x_i^c = [U_{11}^T \beta_{11}^T q_1^T q_2^T \cdots q_N^T V_{11}^T \omega_{11}^T p_1^T p_2^T \cdots p_N^T]^T\]

Full-Dimensional Constraint Equation:

\[x_i^u = \sum_{s=1}^{N} x_{s1}^* = \sum_{s=1}^{N} T_{s1}^* x_i^c = T_1 x_i^c\]

State Equations for First-Order Problem:

\[\dot{x}_1 = A_1 x_1 + B_1^* f_1 + D_1^* d_1\]

\[A_1 = (T_1^T M_1 T_1)^{-1} T_1^T (C_0 T_0 - M_0 \dot{T}_0), \quad B_1^* = (T_1^T M_1 T_1)^{-1} T_1^T B_1\]

\[D_1^* = (T_1^T M_1 T_1)^{-1} T_1^T D_1\]
SUMMARY AND CONCLUSIONS

- The equations of motion for a structure in the form of a collection of articulated flexible substructures can be derived conveniently by means of Lagrange's equations in terms of quasi-coordinates for flexible bodies.

- For practical reasons, the set of nonlinear hybrid (ordinary and partial) differential equations is transformed into a set of nonlinear ode's of high dimension.

- Due to the nature of the problem, a perturbation approach can be used to divide the equations into two sets containing terms differing in magnitude.

- The zero-order problem is nonlinear and of relatively low order. It is associated with the "rigid-body" maneuvering and the control is open loop.
SUMMARY AND CONCLUSIONS (CONT'D)

- The first-order problem is linear and of relatively high order. It is associated with the elastic vibration and the perturbation it causes in the rigid-body motions and the control is closed loop.

- The equations of motion are derived for each substructure separately.

- A given kinematical synthesis is used to link the various substructures together.

- The constraint equations lead to recursive relations that are used to eliminate the surplus coordinates.

- The procedure is used to derive state equations both for the zero-order and first-order problems.

- The formulation is particularly well suited for control design.