MODELING THE PRESSURE-DILATATION CORRELATION

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ABSTRACT

It is generally accepted that the pressure-dilatation, which is an additional compressibility term in the turbulence transport equations, may be important for high-speed flows. Recent direct simulations of homogeneous shear turbulence have given concrete evidence that the pressure-dilatation is important insofar that it contributes to the reduced growth of turbulent kinetic energy due to compressibility effects. The present work addresses the problem of modeling the pressure-dilatation. We first isolate a component of the pressure-dilatation which exhibits temporal oscillations and, using direct numerical simulations of homogeneous shear turbulence and isotropic turbulence, show that it has a negligible contribution to the evolution of turbulent kinetic energy. Then, an analysis for the case of homogeneous turbulence is performed to obtain a model for the non-oscillatory pressure-dilatation. This model algebraically relates the pressure-dilatation to quantities traditionally obtained in incompressible turbulence closures. The model is validated by direct comparison with the pressure-dilatation data obtained from the simulations.

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1 Introduction

The pressure-dilatation appears as an explicit compressibility term in the equations for mean temperature and turbulent kinetic energy. It has been generally recognized that this term may be large in high-speed flows and therefore requires consideration in compressible turbulence closures. In Sarkar, Erlebacher, Hussaini and Kreiss\(^1\), it was found that the pressure-dilatation was smaller than the compressible dissipation in direct simulations of isotropic compressible turbulence which suggested that the pressure-dilatation can be absorbed into the model for the compressible dissipation derived therein. However, subsequent direct simulations of homogeneous shear flow by Blaisdell et al.\(^2\) and Sarkar, Erlebacher and Hussaini\(^3\) showed that the pressure-dilatation is comparable to the compressible dissipation and contributes to the reduced growth of turbulent kinetic energy induced by compressibility. This has motivated a revisit to the issue of modeling the pressure-dilatation in the present paper. Recently, Taulbee and VanOsdol\(^4\) and Zeman\(^5\) have also considered this problem. Taulbee and VanOsdol related the sum of the pressure-dilatation and compressible dissipation to a model involving the density variance and the divergence of the mean velocity. A separate modeled transport equation was proposed for the density variance. Zeman\(^5\) equated the pressure-dilatation to the time rate of change of pressure-variance and then, extending the work by Sarkar et al.\(^3\) on equilibration of the compressible pressure variance on the fast acoustic time scale, proposed a transport equation for the pressure variance. The objective of the present work is to deduce a model for the pressure-dilatation which is algebraically related to quantities obtained by incompressible turbulence closures, in contrast to the previous models, which require new transport equations.

We consider the behavior of the pressure-dilatation in homogeneous flows. Direct simulation results are presented and compared for two flows - homogeneous shear turbulence and decaying isotropic turbulence. The DNS results show that the contribution of the pressure-dilatation to the turbulent kinetic energy evolution is more important in the case of shear turbulence than in decaying isotropic turbulence. We also find the somewhat surprising result that the major contributor to the pressure-dilatation is not the compressible pressure,
but is the incompressible pressure associated with the solenoidal velocity. The governing equations are analyzed for the case of homogeneous flow to obtain a formal expression for the pressure-dilatation from which a model is deduced using scaling arguments.

2 DNS results

A spectral collocation method along with a third-order accurate Runge-Kutta time advancement was used to solve the compressible Navier-Stokes equations. Details of the calculations are available in Sarkar, Erlebacher and Hussaini\textsuperscript{3} for homogeneous shear flow, and in Erlebacher, Hussaini, Kreiss and Sarkar\textsuperscript{6} for decaying isotropic turbulence. A uniform 96\textsuperscript{3} grid was used for discretizing the flow domain. Table 1 shows some of the parameters of the three shear flow cases (S1,S2 and S3), and three decaying isotropic cases (D1,D2, and D3). In Tables 1 and 2, $S$ and $\nu$ denote the shear rate and kinematic viscosity used in the (nondimensional) compressible Navier-Stokes equations, while $R_{\lambda,0}$ and $M_{t,0}$ denote the initial values of the Taylor microscale Reynolds number and turbulent Mach number respectively. Note that $Re_{\lambda} = q \lambda / \nu$ where $q = \sqrt{u_i' u_i'}$ and $\lambda = q / \sqrt{\omega_i' \omega_i'}$, while $M_t = q / \bar{c}$ where $\bar{c} = \sqrt{\gamma R T}$ is the mean speed of sound. In order to minimize the introduction of compressible effects due to initial conditions, all the cases start with incompressible data, that is, the velocity field is divergence-free ($d' = \nabla \cdot u' = 0$), the pressure field satisfies the usual Poisson equation for incompressible flows, and density fluctuations $\rho' = 0$. The initial temperature is calculated from the equation of state using the known pressure and density fields.

The homogeneous flows considered here have temporally evolving turbulence statistics. Fig. 1 shows the evolution of the pressure-dilatation $p'd'$ as a function of normalized time $St$ for the two homogeneous shear cases S1 and S2. Two trends in the behavior of $p'd'$ are evident from Figs. 1a-b; first, the pressure-dilatation develops pronounced oscillations in time and second, it is more negative than positive. These trends are common to all the simulations of homogeneous shear flow that we have performed. Such oscillations in statistical quantities are neither expected nor encountered in incompressible flows and complicate issues in compressible turbulence modeling. By numerical experiments, it was found that
the nominal time period of the oscillations decreased approximately linearly with the speed of sound. This suggested that one could isolate the oscillatory part of $\bar{p}'d'$ by decomposing the fluctuating pressure $p'$ into the sum of an incompressible part $p''$ and a compressible part $pC'$. The component $p''$ is associated with the incompressible velocity field $u^I$ which is divergence-free ($\nabla \cdot u^I = 0$) and satisfies the usual Poisson equation

$$\nabla^2 p'' = -2\bar{\rho}\bar{u}_{m,n}u''_{n,m} - \bar{\rho}u''_{m,n}u'_{n,m}$$

and the remainder $p' - p''$ is the compressible pressure $pC'$. Since

$$p' = p'' + pC'$$

we have

$$\bar{p}'d' = \bar{p}''d' + \bar{p}C'd'$$

Fig. 2a shows the evolution of $\bar{p}C'd'$ and $\bar{p}''d'$ for case S1. The oscillations are substantial only for $\bar{p}C'd'$, and furthermore, the peaks and valleys in the evolution of $\bar{p}C'd'$ in Fig. 2a seem to be much more symmetric around the origin than those in $\bar{p}'d'$ in Fig. 1a.

One of the important effects of $\bar{p}'d'$ is its influence on the budget of the turbulent kinetic energy $K$. The equation for $K$ in homogeneous turbulence is

$$\frac{\partial}{\partial t}(\bar{p}'d')(K) = \bar{P} - \bar{\rho}\varepsilon_s - \bar{\rho}\varepsilon_c + \bar{p}'d'$$

where $\bar{P} = -\bar{\omega}_{i,j}u''_{i,j}''$ is the production, $\varepsilon_s$ the solenoidal dissipation, $\varepsilon_c$ the compressible dissipation, and $\bar{p}'d'$ the pressure-dilatation. The overbar denotes a conventional Reynolds average, while the overtilde denotes a Favre average. A single superscript $'$ represents fluctuations with respect to the Reynolds average, while a double superscript $''$ signifies fluctuations with respect to the Favre average. In order to gauge the relative importance of the two components $\bar{p}C'd'$ and $\bar{p}''d'$ of the pressure-dilatation in the evolution of the turbulent kinetic energy, we plot the integrals $\int \bar{p}'d' dt$ and $\int \bar{p}C'd' dt$ in Fig. 2b. Fig. 2b shows that the integrated contribution of $\bar{p}'d'$ is about an order of magnitude larger than that of $\bar{p}C'd'$. Thus, even though the extrema of $\bar{p}C'd'$ are larger than $\bar{p}'d'$, the time-integrated contribution of
these oscillations in $\overline{pc'd'}$ is negligible compared to the contribution of $\overline{pl'd'}$. Fig. 3 and Fig. 4, which contrast the behavior of $\overline{pl'd'}$ and $\overline{pc'd'}$ for two other homogeneous shear flow cases S2 and S3, confirm the aforementioned trends observed in case S1. (See Table 1 for the initial conditions of cases S2 and S3). It seems that for homogeneous shear flow, $\overline{pl'd'}$ dominates $\overline{pc'd'}$ in the kinetic energy equation. It is interesting that even though both the dilatation and the compressible pressure are associated with the same hyperbolic system of equations governing the compressible mode, the correlation of the dilatation with the compressible pressure plays a smaller overall role than the correlation of the dilatation with the incompressible pressure.

Figs. 5 and 6 show DNS results on the pressure-dilatation from the decaying isotropic turbulence cases D1 and D2 (see Table 2 for initial conditions). Both $\overline{pl'd'}$ and $\overline{pc'd'}$ in Fig. 5a and 6a show sharp transients initially whereby the pressure field is appropriately redistributed into incompressible and compressible parts. The integrated $\overline{pl'd'}$ is much larger than the integrated $\overline{pc'd'}$ in Figs. 5b and 6b just as in the homogeneous shear flow case. In Figs. 5a and 6a, the term $\overline{pc'd'}$ becomes approximately zero after a short initial transient. This is in agreement with the exact solution derived in Sarkar et al.\(^1\) for the linearized equations for the compressible mode which predicted that $\overline{pc'd'} \to 0$ after a transient on the acoustic time scale. The DNS reveal an important difference in the behavior of $\overline{pl'd'}$ in isotropic turbulence with respect to homogeneous shear. The term $\overline{pl'd'}$ is predominantly positive in the case of decaying isotropic turbulence, in contrast to its predominantly negative behavior in homogeneous shear turbulence. The different signs of the pressure-dilatation have been explained by a theoretical consideration of the equations of the pressure variance and density variance by Sarkar, Erlebacher and Hussaini\(^3\).

To summarize, the DNS results show that the pressure-dilatation is predominantly negative and has pronounced oscillations in homogeneous shear flow, while it is predominantly positive in decaying isotropic turbulence. We find that the oscillations in $\overline{pl'd'}$ are confined to the component $\overline{pc'd'}$ of the pressure-dilatation associated with the compressible pressure,
and furthermore that, because of self-cancellation, the contribution of this oscillatory component to the development of the kinetic energy is negligible. Therefore, it seems that only the component $p'^{I'}_{d'}$ of the pressure-dilatation requires modeling, which we proceed to do in the next section.

3 Modeling the pressure-dilatation

Consider the Poisson equation Eq. (1) for the incompressible pressure. As is often done for pressure-strain modeling (see Lumley⁷, Reynolds⁸) for incompressible flows, the incompressible pressure can be split into a rapid part $p'^{R'}$ which reacts instantaneously to a change in the mean velocity gradient and a slow part $p'^{S'}$. Thus

$$p'^{I'} = p'^{R'} + p'^{S'}$$

(3)

where

$$\nabla^2 p'^{R'} = -2\bar{\rho} u'^I_{m,n} u'^I_{n,m}$$

(4)

and

$$\nabla^2 p'^{S'} = -\bar{\rho} u'^I_{m,n} u'^I_{n,m}$$

(5)

Eq. (4) can be exactly solved by Fourier transforms for homogeneous turbulence to give

$$p'^{R'} = 2i\bar{\rho} \frac{k_m}{k^2} \hat{u}^I_{m,n} \hat{u}^I_n$$

(6)

Note that $\hat{\phi}$ denotes the Fourier transform of $\phi$. From Eq. (6) it follows that

$$\overline{p'^{R'} d'} = 2\bar{\rho} \hat{u}^I_{m,n} \int \frac{k_m k_j}{k^2} E^{IC}_{n j} dk$$

(7)

Here $E^{IC}_{n j}$ denotes the spectrum of the mixed Reynolds stress tensor $u'^I_{n} u'^C_{j}$. Eq. (7) is an exact expression for the rapid pressure-dilatation $\overline{p'^{R'} d'}$ and, though cumbersome, can be used for obtaining a simple model. By inspection of Eq. (7), it is clear that because of the dependence on the local energy spectrum tensors, a transport equation is required for a general representation of $\overline{p'^{R'} d'}$. However, we will attempt to obtain an algebraic rather
than a differential model for compressibility terms. This is possible if there are equilibrium scalings in the flow, and the ensuing model will be useful if the additional compressibility correlations do not dominate the incompressible terms in a given transport equation.

Define the tensor $A_{mn}$ by

$$A_{mn} = \int \frac{k_m k_i}{k^2} E_{nj}^{IC} dk$$

which enables the following compact representation of Eq. (7) of the rapid pressure-dilatation

$$\overline{p^{R'd'}} = 2\overline{\rho \bar{u}^{I}_{m,n} A_{mn}}$$

The simplest dimensionally consistent form for $A_{mn}$ which has the correct dependence on $u'_i$ and $u'^C_i$ is

$$A_{mn} = \alpha_2 \overline{u'^C_m u'_n}$$

where $\alpha_2$ is a dimensionless parameter which in general is a function of the actual shape of the energy spectrum tensors for the incompressible and compressible velocity. We have found through analysis and DNS that for homogeneous shear turbulence

$$u^C = O(M_t) u'$$

where $u^C$ and $u'$ respectively denote the $L_2$ norms of the compressible and incompressible velocity fields. In Sarkar et al., we had shown that the dilatational velocity has a fast time scale which is $O(M_t)$ times the solenoidal velocity's time scale, and therefore the correlation between $u'_m$ and $u'^C_n$ should be prorated by a factor of $M_t$. Using Eq. (11) and prorating the mixed correlation, we obtain from Eq. (10),

$$A_{mn} = \alpha_2 M_t^2 \overline{u'_m u'_n}$$

Substituting Eq. (12) into Eq. (9) gives

$$\overline{p^{R'd'}} = -\alpha_2 M_t^2 P$$

where $P$ is the production $-\overline{u'_m u'_n u'_n}$. Eq. (13) may be used as a model for the rapid-pressure dilatation when using a conventional Reynolds-averaged system of equations. We
prefer using a combination of Favre and Reynolds averages (eg. see Sarkar and Balakrishnan\(^9\)) for compressible flow calculations, and for such a system of equations we propose using Eq. (13) with the production being \(\mathcal{P} = -\bar{u}_{m,n} u_n^\prime u_n^\prime\). Since \(\rho^\prime/\bar{\rho} = O(M_t^2)\), changing from the Reynolds-averaged definition to the Favre-averaged definition of the production \(\mathcal{P}\) introduces terms of higher order in \(M_t\) which may be neglected.

Let us now consider the remaining part of the pressure-dilatation, the slow pressure-dilatation \(\overline{p_{d}^s}\). After using Fourier transforms to solve Eq. (5) for the slow pressure, we obtain the following expression for the slow pressure-dilatation,

\[
\overline{p_{d}^s} = \overline{\rho} \int \frac{k_m k_i k_j}{k^2} (i\hat{u}_i \hat{u}_j \hat{u}_m^C - i\hat{u}_i \hat{u}_j \hat{u}_m^C) \, dk
\]  

\[(14)\]

Here \(\hat{\phi}^*\) denotes the complex conjugate of the Fourier transform \(\hat{\phi}\). An order of magnitude analysis of the r.h.s. of Eq. (14) gives

\[
\overline{p_{d}^s} = \overline{\rho} O\left(\frac{u^C}{u^I}\right) O\left(\frac{u^3}{l}\right) M_t
\]  

\[(15)\]

In Eq. (15), \(l\) denotes the integral length scale of the turbulence, and the last factor \(M_t\) arises from the disparity between the scales of the incompressible and compressible fields. Using Eq. (11) for the scaling of \(u^C\), and noting that \(\epsilon_s = O(u^3/l)\), we obtain the model

\[
\overline{p_{d}^s} = \alpha_3 \overline{\rho} \epsilon_s M_t^2
\]  

\[(16)\]

Combining Eqs.(13) and (16), we have the following model for the incompressible pressure-dilatation

\[
\overline{p_{d}^d} = -\alpha_2 \overline{\rho} \mathcal{P} M_t^2 + \alpha_3 \overline{\rho} \epsilon_s M_t^2
\]  

\[(17)\]

We will now use the DNS data to verify the functional dependence stated in Eq. (17) and also to calibrate the model coefficients \(\alpha_2\) and \(\alpha_3\).

Because the production \(\mathcal{P} = 0\) in decaying isotropic turbulence, the variation of the incompressible pressure-dilatation with \(\epsilon_s\) can be verified using DNS of isotropic turbulence. The ratio \(\overline{p_{d}^d}/(\overline{\rho} \epsilon_s M_t^2)\) is shown as a function of non-dimensional time in Fig. 7a. This ratio reaches an equilibrium value by a time of 0.25, substantiating the validity of the second
Based on the DNS value of the equilibrium ratio, the model coefficient $a_3$ in Eq. (17) is taken to be 0.2. The remaining part of the model for the pressure-dilatation is calibrated against simulations of homogeneous shear flow. Fig. 7b shows that, in accord with our model, the rapid part of the pressure-dilatation scales as $\bar{\rho}PM_t^2$. The ratio $(\bar{P}d' - 0.2\bar{\epsilon}_s M_t^2)/(\bar{P}PM_t^2)$ reaches an approximate equilibrium value of $-0.4$, suggesting that the model coefficient $a_2 = 0.4$.

Finally, after using the arguments of section 2 to neglect $\bar{p}d''$ relative to $\bar{P}d'$, the model for the pressure-dilatation becomes

$$\bar{p}'d' = -a_2\bar{p}PM_t^2 + a_3\bar{\rho}\bar{\epsilon}_s M_t^2$$

(18)

where $P = -\bar{u}_i\bar{u}_j\bar{u}_i\bar{u}_j$ is the production of kinetic energy, $M_t = \sqrt{2K/\gamma R\bar{T}}$ the turbulent Mach number, $\epsilon_s$ the solenoidal dissipation, and the model coefficients are $a_2 = 0.4$, $a_3 = 0.2$.

We are now in the process of applying the pressure-dilatation model to the compressible shear layer and the flat plate boundary layer.

4 Conclusions

We have obtained a model for the pressure-dilatation after applying scaling arguments applied to a formal solution for homogeneous turbulence. This model has been validated by direct comparison with DNS results for pressure-dilatation in homogeneous shear flow and isotropic turbulence. Eq. (18) is a reasonable approximation for inhomogeneous flows without walls. However, future refinements may be necessary to capture different physical processes of importance in other flows. For example, a process of importance in shock-turbulence interaction and engine flows is a compressive mean velocity field. In such flows, the pressure-dilatation model Eq. (18) will have a contribution which is linear in the mean compression. Inspection of the dilatation equation for homogeneous compression, obtained by taking the divergence of the momentum equation, shows that there should be such a term in the pressure-dilatation. Of course, even though the form of the dependence on mean compression is already present in our model, in order to get the coefficient of the dependence...
right, it may be necessary to add a term like $\alpha_4 \tilde{\rho} K \tilde{u}_{m,m} M_i^2$ to the model, with $\alpha_4$ perhaps being calibrated with respect to DNS of homogeneous compression. In wall-bounded flows, the pressure-dilatation is probably smaller than the estimate given by Eq. (18) because, first, as shown by Kim and Lee\textsuperscript{10} the rms rapid pressure (except very near the wall) is a smaller fraction of $\tilde{\rho} q^2$ in wall bounded flows compared to homogeneous shear flow, and second, the velocity component normal to the wall is preferentially damped.

The model presented here for the pressure-dilatation is currently being applied in the calculation of inhomogeneous flows. As pointed out in the previous paragraph, some extensions may be necessary in order to incorporate different physical processes in other flows. We plan to further develop the pressure-dilatation model for more general flows.
Table 1: Parameters for the DNS cases of homogeneous shear flow

<table>
<thead>
<tr>
<th>Case</th>
<th>$S$</th>
<th>$\nu$</th>
<th>$M_{t,0}$</th>
<th>$R_{\lambda,0}$</th>
<th>$\rho'$</th>
<th>$d'$</th>
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<tr>
<td>S1</td>
<td>10</td>
<td>1/150</td>
<td>0.3</td>
<td>24</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S2</td>
<td>15</td>
<td>1/150</td>
<td>0.3</td>
<td>24</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>S3</td>
<td>15</td>
<td>1/125</td>
<td>0.4</td>
<td>20</td>
<td>0</td>
<td>0</td>
</tr>
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Table 2: Parameters for the DNS cases of decaying isotropic turbulence

<table>
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<tr>
<th>Case</th>
<th>$\nu$</th>
<th>$M_{t,0}$</th>
<th>$R_{\lambda,0}$</th>
<th>$\rho'$</th>
<th>$d'$</th>
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<tr>
<td>D1</td>
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<td>0.4</td>
<td>31</td>
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<tr>
<td>D2</td>
<td>1/175</td>
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<tr>
<td>D3</td>
<td>1/175</td>
<td>0.6</td>
<td>27</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
References


4D. Taulbee and J. VanOsdol, AIAA Paper 91-0524.


Fig. 1a. Time evolution of pressure–dilatation for Case S1.

Fig. 1b. Time evolution of pressure–dilatation for Case S2.
Fig. 2a. Comparison of $\overline{p'\delta'}$ and $\overline{\delta'\delta'}$ for Case S1.

Fig. 2b. Comparison of the time integrals of $\overline{p'\delta'}$ and $\overline{\delta'\delta'}$ for Case S1.
Fig. 3a. Comparison of $\overline{p'd'}$ and $\overline{p^2d'}$ for Case S2.

Fig. 3b. Comparison of the time integrals of $\overline{p'd'}$ and $\overline{p^2d'}$ for Case S2.
Fig. 4a. Comparison of $\overline{p'd'}$ and $\overline{p'^c}$ for Case S3.

Fig. 4b. Comparison of the time integrals of $\overline{p'd'}$ and $\overline{p'^c}$ for Case S3.
Fig. 5a. Comparison of $p T d'$ and $p T d'$ for Case D1.

Fig. 5b. Comparison of the time integrals of $p T d'$ and $p T d'$ for Case D1.
Fig. 6a. Comparison of $\overline{p' d'}$ and $\overline{p' d''}$ for Case D2.

Fig. 6b. Comparison of the time integrals of $\overline{p' d'}$ and $\overline{p' d''}$ for Case D2.
Fig. 7a. Incompressible pressure–dilatation in decaying isotropic turbulence.

Fig. 7b. Incompressible pressure–dilatation in homogeneous shear turbulence.
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It is generally accepted that the pressure-dilatation, which is an additional compressibility term in the turbulence transport equations, may be important for high-speed flows. Recent direct simulations of homogeneous shear turbulence have given concrete evidence that the pressure-dilatation is important insofar that it contributes to the reduced growth of turbulent kinetic energy due to compressibility effects. The present work addresses the problem of modeling the pressure-dilatation. We first isolate a component of the pressure-dilatation which exhibits temporal oscillations and, using direct numerical simulations of homogeneous shear turbulence and isotropic turbulence, show that it has a negligible contribution to the evolution of turbulent kinetic energy. Then, an analysis for the case of homogeneous turbulence is performed to obtain a model for the non-oscillatory pressure-dilatation. This model algebraically relates the pressure-dilatation to quantities traditionally obtained in incompressible turbulence closures. The model is validated by direct comparison with the pressure-dilatation data obtained from the simulations.