ANISOTROPIC ELLIPTIC OPTICAL FIBERS

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ANISOTROPIC ELLIPTIC OPTICAL FIBERS

BY

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THESIS

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SUMMARY

The instability of the fabrication process of optical fibers introduces both ellipticities and stress anisotropy. These perturbations are the causes for birefringence in single-mode optical fibers which have been researched extensively. In this research, the mode propagations in optical fibers with anisotropic elliptical core have been investigated.

The exact characteristic equation for anisotropic elliptical optical fibers can be obtained for odd and even hybrid modes in terms of infinite determinants utilizing Mathieu and modified Mathieu functions. The exact characteristic equation is applicable to elliptical fibers that have any ellipticity. A simplified characteristic equation can then be obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small. It has been shown that significant simplification can be achieved under this approximation.

The simplified characteristic equation is used to compute the propagation constants for the anisotropic elliptical fiber. The expression for the power carried by the fiber is also obtained and numerical results are presented. These results may be used to approximate a number of different shapes of fibers.
1. INTRODUCTION

The circular optical fiber is one of the most studied media for long distance communication. An optical fiber consists of a core of a dielectric material in which the refractive index is higher than the refractive index of the cladding. However, a cladded fiber is often times modeled as a dielectric rod when the cladding radius is large enough such that the guided mode fields will decay to insignificant values at the outer boundary of cladding. The theory of optical fibers of this type is well understood and has been described in detail in the previously published research[1,2].

In an effort to obtain a low-loss fiber, Monerie[3] carried out an experimental study of doubly clad fibers in which the refractive index of inner cladding was less than that of core and outer cladding. This study shows that the optimum doping levels in the core of doubly clad fibers are less than those required by dispersion-free singly clad fibers. This leads to a smaller propagation loss, since the scattering losses decreased with a decreasing dopant concentration in fibers.

It is interesting to note, however, that the instability in the fabrication process may introduce ellipticities in the optical fibers. This lack of circular symmetry is one of the causes for birefringence in single-mode optical fibers; such a birefringent fiber is also called a single-polarization single-mode fiber[4]. These birefringent fibers are important for systems utilizing such fibers as fiber optic sensors and for predicting the transmission bandwidth reduction caused by group-delay differences between orthogonally polarized modes.

The birefringence due to ellipticity has been studied experimentally[5,6] and the measured data have been compared with those
equation for an uniaxially anisotropic circular rod for hybrid modes of excitation. Analytic solutions for the circular fiber when both core and cladding consist of uniaxial material was presented by Tonning[32]. This study indicates that the cut-off frequency for the lowest-order mode is not affected by the cladding anisotropy.

When the circular cross-section of the fiber is deformed into a noncircularly symmetric profile, a single mode in a circular fiber may split into two modes with different polarizations and propagation velocities[33]. This has been experimentally verified by employing the near-field method[34] and the spectral polarization method[35]. Cozen and Dyott[36] obtained the cut-off frequency of the first higher order mode in an elliptical fiber from an approximate characteristic equation. However, the limitation of their results is described by Citerne[37] and Rengarajan[38]. The cut-off characteristic has also been obtained by solving the exact characteristic equation in terms of Mathieu functions and modified Mathieu functions[38,39] and by applying the mode-matching method[27] or the critical wavelength shift formual method[40]. However, there exists a disagreement in the interpretation of their results, especially in the region where the ratio between minor axis and major axis is small; Saad[41] presented possible reasons for these differences.

For most of the practical fibers used as optical communication lines, the simplification of the characteristic equation is possible by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small[26,33,42,43]. It is shown that the error introduced by this simplification is less than 10% even when the difference in the refractive indices is equal to two. The perturbation method
can also be applied to study the polarization effects in multi-modeled fibers when the fiber is weakly guiding and/or weakly anisotropic[44]. It is also possible for multi-mode fibers to simplify the characteristic equation by applying a perturbation method based on the far-from-cut-off approximation as shown by Paul and Shevgaonkar[45] in a study of circular fiber with uniaxial anisotropy. This approximation is useful for the multi-mode propagation in optical waveguides since the lower order modes that carry most of the power could be considered in the far-from-cut-off region.

For the more general case of biaxial anisotropic waveguide, an analytic solution of the field equations is not possible even for waveguides with simple geometries. However, the fields and propagation constants can be obtained by applying the numerical techniques discussed above. The propagation constants can also be computed by using a coupled mode theory. This coupled mode approach has been applied for the study of mode propagation in rectangular guides[46] and cylindrical fibers[47].

As it has been shown through the previous discussion, the instability of the fabrication process of optical fibers introduces both ellipticities and stress anisotropy. Also, the results obtained for an elliptical optical fiber may be used to approximate a number of different shapes of fibers. It can take the shape of a circular fiber or that of a flat tape fiber depending upon the eccentricity of the elliptical fiber. Hence, it is proposed to investigate the mode propagation in elliptical optical fibers containing uniaxial anisotropic media. In this study, the fiber will be modeled as a dielectric elliptical rod, since the departure of the cladding’s cross-section from circular form can be ignored in the case of large dimension of cladding radius. The exact characteristic equation for the anisotropic elliptical
fiber having any ellipticity will be obtained using the series of Mathieu and modified Mathieu functions. A simplified characteristic equation will then be obtained by applying the weakly guiding approximation and the computed results will be presented.
2. WAVE EQUATION IN ELLEIPTICAL COORDINATES

In solving Maxwell's equations, the wave equation in the waveguide can be expressed in the orthogonal curvilinear coordinates \((\varphi_1, \varphi_2, z)\) as

\[
\begin{align*}
(1/\ell_1^2) \left( \frac{\partial^2 E_z}{\partial \varphi_1^2} \right) + (1/\ell_2^2) \left( \frac{\partial^2 E_z}{\partial \varphi_2^2} \right) \\
+ \left( \frac{1}{\ell_1 \ell_2} \right) \left( \frac{\partial}{\partial \varphi_1} \right) \left( \frac{\partial E_z}{\partial \varphi_1} \right) + \left( \frac{\partial}{\partial \varphi_2} \right) \left( \frac{\partial E_z}{\partial \varphi_2} \right)
\end{align*}
\]

(2.1) 

\[+ k_1^2 E_z = 0 \]

where \(k_1\) is a constant and \(\ell_1\) and \(\ell_2\) are multiplying factors depending upon the particular coordinates [48]. \(\varphi\) is replaced by \(-\varphi\). An identical equation can be obtained for \(H_z\).

For the elliptical coordinates shown in Figure 1,

\[
\begin{align*}
\varphi_1 &= \varepsilon, \quad \varphi_2 = \gamma \\
\end{align*}
\]

(2.2) 

and

\[
\begin{align*}
\ell_1 = \ell_2 = q \left( \left( \cosh 2\varepsilon - \cos 2\gamma \right)/2 \right)^{1/2}.
\end{align*}
\]

(2.3) 

By substituting Eqs. (2.2) and (2.3) into Eq. (2.1), the following equation is obtained

\[
\begin{align*}
\frac{\partial^2 E_z}{\partial \varepsilon^2} + \frac{\partial^2 E_z}{\partial \gamma^2} + 2k^2 \left( \cosh 2\varepsilon - \cos 2\gamma \right) E_z = 0
\end{align*}
\]

(2.4) 

with \(2k = k_1 q\). Then Eq. (2.4) is the two-dimensional wave equation in elliptical coordinates.

If we let the solution of Eq. (2.4) be \(E_z(\varepsilon, \gamma) = \Psi(\varepsilon) \Phi(\gamma)\), Eq. (2.4) becomes

\[
\begin{align*}
\frac{\partial^2 \Psi}{\partial \varepsilon^2} + \frac{\partial^2 \Phi}{\partial \gamma^2} + 2k^2 \left( \cosh 2\varepsilon - \cos 2\gamma \right) \Psi \Phi = 0
\end{align*}
\]

(2.5) 

Dividing Eq. (2.5) by \(\Psi \Phi\) yields

\[
\begin{align*}
\left( \frac{1}{\Phi} \frac{\partial^2 \Psi}{\partial \varepsilon^2} + 2k^2 \cosh 2\varepsilon \right) + \left( \frac{1}{\Psi} \frac{\partial^2 \Phi}{\partial \gamma^2} - 2k^2 \cos 2\gamma \right) = 0.
\end{align*}
\]

(2.6) 

Since the equations in the parentheses are independent to each other, we obtain

\[
\begin{align*}
\frac{\partial^2 \Phi}{\partial \gamma^2} + \left( a - 2k^2 \cos 2\gamma \right) \Phi = 0
\end{align*}
\]

(2.7) 

and
Figure 1. Elliptical coordinate system
where $a$ is the separation constant. Then the solutions for Eq. (2.4) are

\begin{align}
(2.9) \quad E_z &= \begin{cases} 
C_m(\Xi, k^2) c_m(\gamma, k^2) & \text{(even)} \\
S_m(\Xi, k^2) s_m(\gamma, k^2) & \text{(odd)} 
\end{cases} \\
\text{for } k^2 > 0
\end{align}

\begin{align}
(2.10) \quad E_z &= \begin{cases} 
F_{ek}(\Xi, k^2) c_e(\gamma, k^2) & \text{(even)} \\
G_{ek}(\Xi, k^2) s_e(\gamma, k^2) & \text{(odd)} 
\end{cases} \\
\text{for } k^2 < 0.
\end{align}

Similarly, the solutions for $H_z$ can be obtained using the method discussed above.
3. CHARACTERISTIC EQUATION

The geometry shown in Figure 1 consists of an uniaxially anisotropic elliptical rod with a permittivity tensor

\[
\mathbf{\varepsilon}_1 = \begin{bmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_1 & 0 \\
0 & 0 & g\varepsilon_1
\end{bmatrix}
\]

which is embedded in a lossless dielectric medium of permittivity \( \varepsilon_0 \). The anisotropic parameter \( g \) indicates the effect of anisotropic dielectric. The anisotropic parameter is unity for isotropic case.

It has been shown that in order to satisfy the boundary conditions completely both longitudinal electric and magnetic fields must be present, thus only hybrid type modes exist in elliptical fibers[2]. Furthermore, due to the asymmetry of the elliptical cylinder, two types of modes exist and they are designated as an even type modes and an odd type modes.

3.1 EVEN MODES

Assuming the t-z dependence of \( e^{i(\omega t - Bz)} \) for all field components, where \( B \) is the propagation constant and \( \omega \) is the angular frequency, the axial components of the field for even modes are

\[
E_{z1} = \sum_{m=1}^{\infty} A_{1m} S_m(\xi, \gamma_1 e^2) S_m(\gamma, \gamma_1 e^2)
\]

\[
H_{z1} = \sum_{m=1}^{\infty} B_{1m} C_m(\xi, \gamma_1 h^2) C_m(\gamma, \gamma_1 h^2)
\]

for \( 0 \leq \xi \leq \xi_0 \)

\[
E_{z2} = \sum_{m=1}^{\infty} A_{2m} G_{m}(\xi, \gamma_2 e^2) S_m(\gamma, \gamma_2 e^2)
\]

\[
H_{z2} = \sum_{m=1}^{\infty} B_{2m} F_m(\xi, \gamma_2 h^2) C_m(\gamma, \gamma_2 h^2)
\]

for \( \xi_0 \leq \xi < \infty \)
where $A_{1m}$ and $B_{1m}$, $i = 1, 2$ are arbitrary constants, and

$$\gamma_{1e} = q^2/4(v^2\mu_0 - \beta^2)$$

(3.4) $$\gamma_{1h} = q^2/4(v^2\mu_0 - \beta^2)$$

The transverse field components are for $0 \leq \xi \leq \xi_0$

$$E_{11} = -1/(v^2\mu_0 - \beta^2)L$$

(3.5) $$E_{11} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\beta \frac{\partial}{\partial \xi_1} A_{1m} \sin'^2(\xi, \gamma_{1e}) \cos(\gamma, \gamma_{1e})$$

$$+ \gamma_{1h} \frac{\partial}{\partial \xi_1} B_{1m} \cos(\gamma, \gamma_{1h})]$$

(3.6) $$H_{11} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\beta \frac{\partial}{\partial \xi_1} A_{1m} \sin'^2(\xi, \gamma_{1e}) \cos(\gamma, \gamma_{1e})$$

$$- \gamma_{1h} \frac{\partial}{\partial \xi_1} B_{1m} \cos(\gamma, \gamma_{1h})]$$

(3.7) $$H_{11} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\gamma_{1h} \frac{\partial}{\partial \xi_1} A_{1m} \sin'^2(\xi, \gamma_{1e}) \cos(\gamma, \gamma_{1e})$$

$$+ \beta \frac{\partial}{\partial \xi_1} B_{1m} \cos(\gamma, \gamma_{1h})]$$

(3.8) $$H_{11} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\gamma_{1h} \frac{\partial}{\partial \xi_1} A_{1m} \sin'^2(\xi, \gamma_{1e}) \cos(\gamma, \gamma_{1e})$$

$$- \beta \frac{\partial}{\partial \xi_1} B_{1m} \cos(\gamma, \gamma_{1h})]$$

for $\xi_0 \leq \xi \leq \xi_0$

(3.9) $$E_{12} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\beta \frac{\partial}{\partial \xi_1} A_{2m} \sin'^2(\xi, \gamma_{2e}) \cos(\gamma, \gamma_{2e})$$

$$+ \gamma_{1h} \frac{\partial}{\partial \xi_1} B_{2m} \cos(\gamma, \gamma_{1h})]$$

(3.10) $$E_{12} = -1/(v^2\mu_0 - \beta^2)L$$

$$[\beta \frac{\partial}{\partial \xi_1} A_{2m} \sin'^2(\xi, \gamma_{2e}) \cos(\gamma, \gamma_{2e})$$

$$- \gamma_{1h} \frac{\partial}{\partial \xi_1} B_{2m} \cos(\gamma, \gamma_{1h})]$$

$q$ is the semifocal length of the ellipse and $\mu$ is the permeability.
where
\[ L = q \left( \frac{1}{\cos^2 \theta} - \frac{1}{\cos^2 \gamma} \right) / 2 \]

and the derivative with respect to \( \theta \) or \( \gamma \) is denoted by the prime.

The boundary conditions require that the tangential \( E \) and \( H \) fields be continuous at the dielectric discontinuities. Equating the tangential fields at the boundary surface, \( \xi = \xi_0 \), gives

\[
(3.13) \quad \sum_{m=1}^{\infty} A_m \text{Se}_m(\xi_0) \text{Se}_m(\gamma) = \sum_{m=1}^{\infty} A_m \text{Ge}_m(\xi_0) \text{Se}_m(\gamma)
\]

\[
(3.14) \quad \sum_{m=1}^{\infty} B_m \text{Ce}_m(\xi_0) \text{Ce}_m(\gamma) = \sum_{m=1}^{\infty} B_m \text{Fe}_m(\xi_0) \text{Ce}_m(\gamma)
\]

\[
(3.15) \quad \frac{1}{(\omega^2 \mu \xi_1 - \beta^2)} \left[ \beta \sum_{m=1}^{\infty} A_m \text{Se}_m(\xi_0) \text{Se}_m(\gamma)ight] - \sum_{m=1}^{\infty} B_m \text{Ce}_m(\xi_0) \text{Ce}_m(\gamma)
\]

\[
= \frac{1}{(\omega^2 \mu \xi_0 - \beta^2)} \left[ \beta \sum_{m=1}^{\infty} A_m \text{Ge}_m(\xi_0) \text{Se}_m(\gamma) - \sum_{m=1}^{\infty} B_m \text{Fe}_m(\xi_0) \text{Ce}_m(\gamma) \right]
\]

\[
(3.16) \quad \frac{1}{(\omega^2 \mu \xi_1 - \beta^2)} \left[ \omega \mu \sum_{m=1}^{\infty} A_m \text{Se}_m(\xi_0) \text{Se}_m(\gamma) \right] + \sum_{m=1}^{\infty} B_m \text{Ce}_m(\xi_0) \text{Ce}_m(\gamma)
\]

\[
= \frac{1}{(\omega^2 \mu \xi_0 - \beta^2)} \left[ \omega \mu \sum_{m=1}^{\infty} A_m \text{Ge}_m(\xi_0) \text{Se}_m(\gamma) + \sum_{m=1}^{\infty} B_m \text{Fe}_m(\xi_0) \text{Ce}_m(\gamma) \right]
\]

where the following abbreviations have been used,

\[
(3.17) \quad \text{Se}_m(\xi_0) = \text{Se}_m(\xi_0, \gamma_1 \xi^2)
\]

\[
(3.18) \quad \text{se}_m(\gamma) = \text{se}_m(\gamma, \gamma_1 \xi^2)
\]

\[
(3.19) \quad \text{Ce}_m(\xi_0) = \text{Ce}_m(\xi_0, \gamma_1 \xi^2)
\]

\[
(3.20) \quad \text{ce}_m(\gamma) = \text{ce}_m(\gamma, \gamma_1 \xi^2)
\]
Multiplying both sides of Eqs. (3.13) and (3.16) by $\text{se}_n(\gamma)$ and Eqs. (3.14) and (3.15) by $\text{ce}_n(\gamma)$, integrating with respect to $\gamma$ from 0 to $2\pi$, and applying the orthogonality relations of the angular Mathieu functions leads to

\begin{align}
\int_0^{2\pi} \text{ce}_m \text{ce}_n \, d\gamma &= 0 \quad \text{if } m \neq n
\end{align}

leads to

\begin{align}
A_{1m}\text{se}_n &= \sum_{m=0}^{\infty} \sum_{n=-m}^{m} A_{2m}\text{Gek}_m \beta_{m,n} \\
B_{1m}\text{ce}_n &= \sum_{m=0}^{\infty} \sum_{n=-m}^{m} B_{2m}\text{Fek}_m \alpha_{m,n}
\end{align}

The prime over the summation sign is used to indicate that either odd or even values of $m$ are used accordingly as to whether $n$ is odd or even.

\begin{align}
\alpha_{m,n}, \beta_{m,n}, \psi_{m,n} \text{ and } \nu_{m,n} \text{ are given by the following}
\end{align}

\begin{align}
\alpha_{m,n} &= \int_0^{2\pi} \text{se}_m(\gamma) \text{ce}_n(\gamma) \, d\gamma / \int_0^{2\pi} \text{ce}_n^2(\gamma) \, d\gamma \\
\beta_{m,n} &= \int_0^{2\pi} \text{ce}_n(\gamma) \text{se}_m(\gamma) \, d\gamma / \int_0^{2\pi} \text{se}_m^2(\gamma) \, d\gamma \\
\psi_{m,n} &= \int_0^{2\pi} \text{ce}_m(\gamma) \text{se}_n(\gamma) \, d\gamma / \int_0^{2\pi} \text{se}_n^2(\gamma) \, d\gamma \\
\nu_{m,n} &= \int_0^{2\pi} \text{se}_m(\gamma) \text{ce}_n(\gamma) \, d\gamma / \int_0^{2\pi} \text{ce}_n^2(\gamma) \, d\gamma.
\end{align}

Making use of Eqs. (3.26) and (3.27), Eqs. (3.28) and (3.29) yields two sets of infinite homogeneous equations

\begin{align}
\sum_{m=-\infty}^{\infty} A_{2m}\gamma_{m,n} + \sum_{m=-\infty}^{\infty} B_{2m}\eta_{m,n} &= 0 \\
\sum_{m=-\infty}^{\infty} A_{2m}\nu_{m,n} + \sum_{m=-\infty}^{\infty} B_{2m}\nu_{m,n} &= 0
\end{align}
where

\[ q_{m,n} = -(1 - \gamma_{1h^2/\gamma_2^2}) G_{mk}(\xi_0)^{m\nu} \theta_{m,\nu} \xi, n \]

\[ h_{m,n} = \nu \varphi_{m,n}/\beta \left[ \frac{G_{mk}(\xi_0)}{\varphi_{m,k}}(\xi_0)/\varphi_{m,\nu} - \frac{G_{mk}'(\xi_0)\gamma_{1h^2/\gamma_2^2}}{\varphi_{m,k}} \right] \]

\[ s_{m,n} = \nu \varphi_{m,n}/\beta \left[ \xi_1 G_{mk}(\xi_0)^{m\nu} \varphi_{m,k} - \frac{G_{mk}'(\xi_0)\gamma_{1h^2/\gamma_2^2}}{\varphi_{m,k}} \right] \]

\[ t_{m,n} = (1 - \gamma_{1h^2/\gamma_2^2}) G_{mk}(\xi_0)^{m\nu} \theta_{m,\nu} \xi, n \]

For a nontrivial solution, the infinite determinant of Eq. (3.34) must vanish. The propagation constant \( \beta \) can then be determined from the roots of this infinite determinant.

The infinite determinant for odd values of \( m \) and \( n \) is

\[
\begin{vmatrix}
s_{11} & t_{11} & s_{31} & t_{31} & - & - \\
g_{11} & h_{11} & g_{31} & h_{31} & - & - \\
s_{13} & t_{13} & s_{33} & t_{33} & - & - \\
g_{13} & h_{13} & g_{33} & h_{33} & - & - \\
: & : & : & : - & - & - \\
: & : & : & : - & - & - \\
\end{vmatrix} = 0
\]

and for even values of \( m \) and \( n \) is

\[
\begin{vmatrix}
h_{00} & g_{20} & h_{20} & g_{40} & - & - \\
t_{02} & s_{22} & t_{22} & s_{42} & - & - \\
h_{02} & g_{22} & h_{22} & g_{42} & - & - \\
t_{04} & s_{24} & t_{24} & s_{44} & - & - \\
: & : & : & : - & - & - \\
: & : & : & : - & - & - \\
\end{vmatrix} = 0
\]
3.2 **ODD MODES**

The axial components of the field for odd modes are

\[ E_z1 = \frac{\partial}{\partial \theta} A_{-m} \text{Ce}_1(\xi, \gamma_{1e}^2) \text{Ce}_1(\gamma, \gamma_{1e}^2) \]

\[ H_z1 = \frac{\partial}{\partial \theta} B_{-m} \text{Se}_1(\xi, \gamma_{1h}^2) \text{Se}_1(\gamma, \gamma_{1h}^2) \]  

(3.41)

for \( 0 \leq \xi \leq \xi_0 \)

\[ E_z2 = \frac{\partial}{\partial \theta} A_m \text{Fe}_m(\xi, \gamma_{2e}^2) \text{Ce}_1(\gamma, \gamma_{2e}^2) \]

\[ H_z2 = \frac{\partial}{\partial \theta} B_m \text{Ge}_m(\xi, \gamma_{2h}^2) \text{Se}_1(\gamma, \gamma_{2h}^2) \]

(3.42)

for \( \xi_0 \leq \xi < \infty \)

where \( A_{-m} \) and \( B_{-m} \), \( m = 1, 2 \) are arbitrary constants, and \( \gamma_{1e}^2, \gamma_{1h}^2, \) and \( \gamma_{2e}^2 \)

are given in Eq.(3.4).

The transverse field components are

for \( 0 \leq \xi \leq \xi_0 \)

\[ E_{\xi 1} = -1/(w^2 \mu \xi_1 - \beta^2) L \]

\[ \left[ \beta \frac{\partial}{\partial \phi} A_{-m} \text{Ce}_m'(\xi, \gamma_{1e}^2) \text{Ce}_1(\gamma, \gamma_{1e}^2) \right. \]

\[ + w\mu \frac{\partial}{\partial \phi} B_{-m} \text{Se}_m(\xi, \gamma_{1h}^2) \text{Se}_1(\gamma, \gamma_{1h}^2) \]

(3.43)

\[ E_{\gamma 1} = -1/(w^2 \mu \xi_1 - \beta^2) L \]

\[ \left[ \beta \frac{\partial}{\partial \phi} A_{-m} \text{Ce}_m'(\xi, \gamma_{1e}^2) \text{Ce}_1(\gamma, \gamma_{1e}^2) \right. \]

\[ - w\mu \frac{\partial}{\partial \phi} B_{-m} \text{Se}_m'(\xi, \gamma_{1h}^2) \text{Se}_1(\gamma, \gamma_{1h}^2) \]

(3.44)

\[ H_{\xi 1} = -1/(w^2 \mu \xi_1 - \beta^2) L \]

\[ \left[ -w\xi_1 \frac{\partial}{\partial \phi} A_{-m} \text{Ce}_m'(\xi, \gamma_{1e}^2) \text{Ce}_1(\gamma, \gamma_{1e}^2) \right. \]

\[ + \beta \frac{\partial}{\partial \phi} B_{-m} \text{Sn}_m'(\xi, \gamma_{1h}^2) \text{Sn}_1(\gamma, \gamma_{1h}^2) \]

(3.45)

\[ H_{\gamma 1} = -1/(w^2 \mu \xi_1 - \beta^2) L \]

\[ \left[ w\xi_1 \frac{\partial}{\partial \phi} A_{-m} \text{Ce}_m'(\xi, \gamma_{1e}^2) \text{Ce}_1(\gamma, \gamma_{1e}^2) \right. \]

\[ + \beta \frac{\partial}{\partial \phi} B_{-m} \text{Sn}_m'(\xi, \gamma_{1h}^2) \text{Sn}_1(\gamma, \gamma_{1h}^2) \]

(3.46)
for $\xi_0 \leq \xi < \epsilon$

(3.47) $E\xi_2 = -1/(w^2\mu\epsilon_0 - \beta^2)L$

\[
\begin{align*}
\beta \sum_{m=0}^{\infty} A_{2m} Fek^\prime (\xi, \gamma^2) \text{ce} \left( \gamma, \gamma^2 \right) \\
+ \mu \sum_{m=1}^{\infty} B_{2m} Gek^\prime (\xi, \gamma^2) \text{se} \left( \gamma, \gamma^2 \right) 
\end{align*}
\]

(3.48) $H\xi_2 = -1/(w^2\mu\epsilon_0 - \beta^2)L$

\[
\begin{align*}
\beta \sum_{m=0}^{\infty} A_{2m} Fek^\prime (\xi, \gamma^2) \text{ce} \left( \gamma, \gamma^2 \right) \\
- \mu \sum_{m=1}^{\infty} B_{2m} Gek^\prime (\xi, \gamma^2) \text{se} \left( \gamma, \gamma^2 \right) 
\end{align*}
\]

(3.49) $H\eta_2 = -1/(w^2\mu\epsilon_0 - \beta^2)L$

\[
\begin{align*}
\beta \sum_{m=0}^{\infty} A_{2m} Fek^\prime (\xi, \gamma^2) \text{ce} \left( \gamma, \gamma^2 \right) \\
+ \mu \sum_{m=1}^{\infty} B_{2m} Gek^\prime (\xi, \gamma^2) \text{se} \left( \gamma, \gamma^2 \right) 
\end{align*}
\]

(3.50) $H\eta_2 = -1/(w^2\mu\epsilon_0 - \beta^2)L$

\[
\begin{align*}
\beta \sum_{m=0}^{\infty} A_{2m} Fek^\prime (\xi, \gamma^2) \text{ce} \left( \gamma, \gamma^2 \right) \\
- \mu \sum_{m=1}^{\infty} B_{2m} Gek^\prime (\xi, \gamma^2) \text{se} \left( \gamma, \gamma^2 \right) 
\end{align*}
\]

The derivative with respect to $\xi$ or $\gamma$ is denoted by the prime.

Equating the tangential fields at the boundary surface, $\xi = \xi_0$, gives

(3.51) $E\xi_0 = \sum_{m=0}^{\infty} A_{1m} Cek (\xi_0) \text{ce} \left( \gamma, \gamma^2 \right)$

(3.52) $B_{1m} \text{se} \left( \xi_0 \right)$

(3.53) $1/(w^2\mu\epsilon_1 - \beta^2) \left[ \beta \sum_{m=0}^{\infty} A_{1m} Cek (\xi_0) \text{ce} \left( \gamma, \gamma^2 \right) \\
+ \mu \sum_{m=1}^{\infty} B_{1m} \text{se} \left( \xi_0 \right) \text{se} \left( \gamma, \gamma^2 \right) 
\right]

(3.54) $1/(w^2\mu\epsilon_1 - \beta^2) \left[ \beta \sum_{m=0}^{\infty} A_{1m} Cek (\xi_0) \text{ce} \left( \gamma, \gamma^2 \right) \\
+ \mu \sum_{m=1}^{\infty} B_{1m} \text{se} \left( \xi_0 \right) \text{se} \left( \gamma, \gamma^2 \right) 
\right]
The abbreviations

(3.55) \[ C_m(\xi_0) = C_m(\xi_0, \gamma_1e^2) \]
(3.56) \[ c_m(\gamma) = c_m(\gamma, \gamma_1e^2) \]
(3.57) \[ s_m(\xi_0) = s_m(\xi_0, \gamma_1e^2) \]
(3.58) \[ s_m(\gamma) = s_m(\gamma, \gamma_1e^2) \]
(3.59) \[ \text{Fek}_m(\xi_0) = \text{Fek}_m(\xi_0, \gamma_1e^2) \]
(3.60) \[ \text{Fek}_m(\gamma) = \text{Fek}_m(\gamma, \gamma_2e^2) \]
(3.61) \[ \text{Gek}_m(\xi_0) = \text{Gek}_m(\xi_0, \gamma_2e^2) \]
(3.62) \[ \text{Gek}_m(\gamma) = \text{Gek}_m(\gamma, \gamma_2e^2) \]

have been used.

Multiplying both sides of Eqs. (3.51) and (3.54) by \( c_m(\gamma) \) and Eqs. (3.52) and (3.53) by \( s_m(\gamma) \), integrating with respect to \( \gamma \) from 0 to \( 2\pi \), and applying the orthogonality relations of the angular Mathieu functions leads to

\[ A_{1m}C_m = \sum_{\xi_0} A_{2m}\text{Fek}_{m, n} \]
\[ B_{1m}S_m = \sum_{\xi_0} B_{2m}\text{Gek}_{m, n} \]
\[ \delta \sum_{\xi_0} A_{1m}C_m \nu_{n,m} = - \delta \sum_{\xi_0} B_{1m}S_m \nu_{n,m} = \beta Y_{1h}^2/\gamma_{12} \sum_{\xi_0} A_{2m}\text{Fek}_{m, n} \nu_{n,m} + \beta Y_{1h}^2/\gamma_{22} \sum_{\xi_0} B_{2m}\text{Gek}_{m, n} \nu_{n,m} \]

The prime over the summation sign is used to indicate that either odd or even values of \( m \) are used accordingly as to whether \( n \) is odd or even.

\( a_{m,n}, b_{m,n}, \psi_{m,n} \) and \( \lambda_{m,n} \) are given by the following

\[ a_{m,n} = \int_0^{\pi} \text{se}_{m}^{*}(\xi) \text{se}_{n}(\xi) \, d\xi / \int_0^{\pi} \text{se}_{n}^{2}(\xi) \, d\xi \]
\[ b_{m,n} = \int_0^{\pi} \text{ce}_{m}^{*}(\xi) \text{ce}_{n}(\xi) \, d\xi / \int_0^{\pi} \text{ce}_{n}^{2}(\xi) \, d\xi \]
\[ \psi_{m,n} = \int_0^{\pi} \text{se}_{m}^{*}(\xi) \text{ce}_{n}(\xi) \, d\xi / \int_0^{\pi} \text{ce}_{n}^{2}(\xi) \, d\xi \]
\[ \lambda_{m,n} = \int_0^{\pi} \text{ce}_{m}^{*}(\xi) \text{se}_{n}(\xi) \, d\xi / \int_0^{\pi} \text{se}_{n}^{2}(\xi) \, d\xi \]
Making use of Eqs. (3.63) and (3.64), Eqs. (3.65) and (3.66) yields two
sets of infinite homogeneous equations

\[ \begin{align*}
\frac{d}{d\omega} \lambda_{2m} \phi_{m,n} + \frac{d}{d\omega} \nu_{2m} \phi_{mn,n} &= 0 \\
\frac{d}{d\omega} \lambda_{2m} \phi_{m,n} + \nu_{2m} \phi_{mn,n} &= 0
\end{align*} \]

where

\[ \begin{align*}
g_{m,n} &= -(1 - \gamma_{1h}^2/\gamma_{2}^2) \frac{\Phi_{m,n}}{\tau_{m,n}} \delta_{m,n} \\
\tau_{m,n} &= \omega \mu_{m,n} \beta \left( \frac{\epsilon_{m}(\epsilon_{s})a_{m,n}}{\epsilon_{m}(\epsilon_{s})} - \frac{\epsilon_{m}(\epsilon_{s})b_{m,n}}{\epsilon_{m}(\epsilon_{s})} \right) \\
\tau_{m,n} &= (1 - \gamma_{1h}^2/\gamma_{2}^2) \frac{\Phi_{m,n}}{\tau_{m,n}} \delta_{m,n}
\end{align*} \]

For a nontrivial solution, the infinite determinant of Eq. (3.71) must
vanish. The propagation constant \( \beta \) can then be determined from the roots of
this infinite determinant.

The infinite determinant for odd values of \( m \) and \( n \) is

\[ \begin{vmatrix}
s_{11} & t_{11} & s_{31} & t_{31} & \cdots \\
g_{11} & h_{11} & g_{31} & h_{31} & \cdots \\
s_{13} & t_{13} & s_{33} & t_{33} & \cdots \\
g_{13} & h_{13} & g_{33} & h_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix} = 0 \]
and for even values of \( m \) and \( n \) is

\[
\begin{bmatrix}
    s_{00} & s_{20} & t_{20} & s_{40} & - & - \\
    s_{02} & s_{22} & t_{22} & s_{42} & - & - \\
    q_{02} & q_{22} & h_{22} & q_{42} & - & - \\
    s_{04} & s_{24} & t_{24} & s_{44} & - & - \\
\end{bmatrix}
\]

(3.77) $= 0$

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 & - & - \\
    1 & 1 & 1 & 1 & - & - \\
    1 & 1 & 1 & 1 & - & - \\
\end{bmatrix}
\]
4. WEAKLY GUIDING APPROXIMATION

The exact characteristic equations obtained in Chapter 3 are valid for an anisotropic elliptical fiber with any eccentricities. These equations are also applicable to the fiber with any refractive index differences between the core and cladding material. However, for most of practical fibers, the difference in the refractive indices of the core and the cladding is typically very small. The simplified characteristic equations can be obtained under this condition which is known as the weakly guiding approximation.

Applying the weakly guiding approximation results in the following

\(\gamma_2^2 = \gamma_1 h^2 + \kappa v^2 \mu e_1 (1 - \epsilon_0/\epsilon_1) = \gamma_1 h^2\) \hspace{1cm} (4.1)
\[1 - \frac{\gamma_1 h^2}{\gamma_2^2} = 0.\] \hspace{1cm} (4.2)

4.1 EVEN MODES

Applying Eqs. (4.1) and (4.2) into Eqs. (3.20) and (3.30) yields the equations

\(\text{ce}_n(\gamma, \gamma_2^2) = \text{ce}_n(\gamma, \gamma_1 h^2)\) \hspace{1cm} (4.3)
\(a_{m,n} = \int_0^{2\pi} \text{ce}_m(\gamma, \gamma_1 h^2) \text{ce}_n(\gamma, \gamma_1 h^2) d\gamma / \int_0^{2\pi} \text{ce}_m^2(\gamma, \gamma_1 h^2) d\gamma\) \hspace{1cm} (4.4)

where \(\Delta_{m,n}\) is the Kronecker delta which is zero when \(m \neq n\) and is unity when \(m = n\).

Substituting Equations (4.1) - (4.4) into Equations (3.35) - (3.38), the infinite determinants for even modes become

\(\prod \left( g_{m,a} t_{m,a} - h_{m,a} s_{m,a} \right) = 0\) \hspace{1cm} (4.5)

or

\(g_{m,a} t_{m,a} - h_{m,a} s_{m,a} = 0\) \hspace{1cm} (4.6)

for \(m = 0, 1, 2, \ldots\).
By substituting Equations (3.35) - (3.38) into Eq.(4.6), the following equation is obtained

\[-(1 - \gamma_{1h}^2/\gamma_{2}^2)(\varepsilon_{1}/\varepsilon_{0} - \gamma_{1h}^2/\gamma_{2}^2)(\varepsilon'_{1}/\varepsilon_{0})_{m,n} = \psi_{n,m} (\varepsilon'_{m}/\varepsilon_{0})_{m,n} \]

(4.7)

\[= \left[ \frac{\varepsilon_{1}/\varepsilon_{0}}{\varepsilon_{0}/\varepsilon_{m}} \right] \frac{Gek_{m}'(\varepsilon_{0})/Gek_{m}(\varepsilon_{0})}{(\gamma_{1h}^2/\gamma_{2}^2)Gek_{m}'(\varepsilon_{0})/Gek_{m}(\varepsilon_{0})} \]

This is the simplified characteristic equation for even modes compared to the infinite determinants as given in Eqs.(3.39) and (3.40). When the elliptical rod degenerates to a circular rod, the simplified characteristic equation becomes that of the anisotropic circular fiber.

4.2 ODD MODES

Applying Eqs.(4.1) and (4.2) in Eqs.(3.58) and (3.67) yields the equations

\[s_{m,n}(\gamma, \gamma_{2}^2) = s_{m,n}(\gamma, \gamma_{1h}^2) \]

(4.8)

\[a_{m,n} = \int_{0}^{\pi} s_{m,n}(\gamma, \gamma_{1h}^2) s_{m,n}(\gamma, \gamma_{1h}^2) d\gamma/\int_{0}^{\pi} s_{m,n}^2(\gamma, \gamma_{1h}^2) d\gamma \]

(4.9)

where \(\Delta_{m,n}\) is the Kronecker delta which is zero when \(m \neq n\) and is unity when \(m = n\).

Substituting Equations (4.1) - (4.2) and (4.8) - (4.9) into Equations (3.72) - (3.75), the infinite determinants for odd modes become

\[-(g_{m,n} t_{m,n} - h_{m,n} s_{m,n}) = 0 \]

(4.10)

or

\[g_{m,n} t_{m,n} - h_{m,n} s_{m,n} = 0 \]

(4.11)

for \(m = 0, 1, 2, - - - \).

By substituting Equations (3.72) - (3.75) into Eq.(4.11), the following equation is obtained
\[-(1 - \frac{Y_1^2}{Y_2^2})(\varepsilon_1/\varepsilon_0 - \frac{Y_1^2}{Y_2^2})(\varepsilon_n, m, n \nu, m, n \psi, m, n \phi, m, n \phi, m, n)\]

\[(4.12) = \left(\frac{8n'(\varepsilon_0)/8n(\varepsilon_0) - (\frac{Y_1^2}{Y_2^2})Ge_n'(\varepsilon_0)/Ge_n(\varepsilon_0)}{(\varepsilon_1/\varepsilon_0)Ge_n'(\varepsilon_0)/Ge_n(\varepsilon_0) - (\frac{Y_1^2}{Y_2^2})Fe_n'(\varepsilon_0)/Fe_n(\varepsilon_0)}\right).\]

This is the simplified characteristic equation for odd modes compared to the infinite determinants given in Eqs. (3.76) and (3.77). When the elliptical rod degenerates to a circular rod, the simplified characteristic equation becomes that of the anisotropic circular fiber.
5. NUMERICAL RESULTS FOR PROPAGATION CONSTANTS

5.1 ISOTROPIC ELLIPTICAL FIBERS

When the anisotropic parameter \( g \) in Eq. (3.1) is equal to unity, the simplified characteristic equations in Eq. (4.7) and Eq. (4.12) become that of an isotropic elliptic guide. In Figures 2 through 5, the normalized guide wavelength \( \lambda / \lambda_0 \) for the isotropic elliptical fibers is plotted as a function of the normalized cross-section area and normalized major axis for \( \varepsilon_1 / \varepsilon_0 = 2.5 \) and for the various values of \( \varepsilon_0 \). These results are compared with those given by Yeh[2] which are indicated by symbols; the results are in close agreement.

For the \( \text{eHE}_{11} \) mode, it can be seen in Figure 2 that the normalized guide wavelength is almost equal to unity for the very small value of the cross-section area. This indicates that the geometry of the waveguide has no effect on the normalized guide wavelength when the wavelength is much larger than the physical dimension of the core of fibers. For a fixed value of cross-section area, the normalized guide wavelength is smaller for larger the value of \( \varepsilon_0 \). This indicates that more energy is carried inside of the circular core than the elliptical core. As the normalized cross-section area becomes larger, the difference in the normalized guide wavelengths for varying \( \varepsilon_0 \) becomes small again. This is since most of the energy is carried inside of the core and the geometry of waveguide has no effect on the normalized guide wavelength.

However, as observed in Figure 3, the \( \text{oHE}_{11} \) mode is different from the \( \text{eHE}_{11} \) mode in that the difference in the normalized guide wavelengths for varying \( \varepsilon_0 \) is smaller than that of \( \text{eHE}_{11} \) when the value of normalized cross-section area is fixed. This small difference is due to the fact that the electric lines are being compressed such that the field density is more
Figure 2. Normalized wavelength for isotropic fiber as a function of normalized cross-section area for even modes. Symbols are from Yeh[2].
Figure 3. Normalized wavelength for isotropic fiber as a function of normalized cross-section area for odd modes. Symbols are from Yeh[2].
Figure 4. Normalized wavelength for isotropic fiber as a function of normalized major axis for even modes. Symbols are from Yeh(2).
Figure 5. Normalized wavelength for isotropic fiber as a function of normalized major axis for odd modes. Symbols are from Yeh[2].
concentrated inside the waveguide. For a fixed value of cross-section area, more energy is carried inside of the elliptical core than the circular core since the normalized guide wavelength is smaller for smaller the value of $\varepsilon_0$.

In Figures 4 and 5, the normalized guide wavelength is plotted against the normalized major axis for various values of $\varepsilon_0$ and for $\varepsilon_1/\varepsilon_0 = 2.5$. In these figures, the difference in the normalized guide wavelengths for varying $\varepsilon_0$ is larger than those in Figures 2 and 3 since there is more binding dielectric material in a circular core than in a flatter rod (i.e. smaller $\varepsilon_0$) when the value of normalized major axis is fixed.

5.2 ANISOTROPIC ELLIPTICAL FIBERS

In Figures 6 and 7, the normalized guide wavelength $\lambda/\lambda_0$ for an anisotropic elliptical fiber is plotted as a function of the normalized cross-section area for various values of anisotropy and for $\varepsilon_1/\varepsilon_0 = 2.5$ and $\varepsilon_0 = 0.5$. These figures indicate that the geometry of the waveguide and anisotropy of the core have no effect when the wavelength is much larger than the physical dimension of the core of fibers which indicate that most of the energy is carried outside of the core. For a fixed value of cross-section area, the normalized guide wavelength is smaller for larger the value of anisotropy. This condition indicates that the field intensity is more concentrated in the core, thus indicating that more energy is carried inside of the core. As the normalized cross-section area becomes larger, the difference in the normalized guide wavelengths for the varying anisotropy becomes smaller again. This indicates that the geometry and anisotropy of waveguide have a smaller effect on the normalized guide wavelength.
Figure 6. Normalized wavelength for anisotropic fiber as a function of normalized cross-section area for even modes.
Figure 7. Normalized wavelength for anisotropic fiber as a function of normalized cross-section area for odd modes.
The normalized guide wavelength in Figures 8 and 9 is plotted against the normalized major axis for the various values of anisotropy and for $\varepsilon_1/\varepsilon_0 = 2.5$ and $\varepsilon_0 = 0.5$. The effect of anisotropy on the normalized guide wavelength is similar to those in Figures 6 and 7.
Figure 8. Normalized wavelength for anisotropic fiber as a function of normalized major axis for even modes.
Figure 9. Normalized wavelength for anisotropic fiber as a function of normalized major axis for odd modes.
6. POWER CONSIDERATIONS

The power along the z axis in the medium i of the fiber may be obtained by integrating the poynting vector over the surface area,

\[ P_i = \frac{1}{2} \int_0^L \left( \vec{E}_i \times \vec{H}_i^* \right) \cdot \hat{z} \, ds \]

\[ = \frac{1}{2} \int_0^L \int_0^{2\pi} \left( E_{i1} H_{i1}^* - E_{i1} H_{i1}^* \right) L^2 \, d\theta \, d\phi \]

where

\[ L = q(\cosh 2\xi - \cos 2\xi)^{1/2} \]

and \( \xi_0 = 0 \) and \( \xi_2 = -1 \).

6.1 EVEN MODES

Substituting Equations (3.5) through (3.8) into Eq. (6.1) and integrating over the core area yields

\[ P_{\text{core}} = \left( \frac{1}{2} \gamma_1^2 \right) \int_{\xi_0}^{\xi_2} B\psi_{1n} \left( \psi_{1n}^* \right)^2 \left( xS_{m} \right)^2 + S_{m}^* S_{m}^* \right) \]

\[ + \left( \frac{1}{2} \gamma_1^2 \psi_n \right) \int_{\xi_0}^{\xi_2} B_{1n} \psi_{1n}^* \left( xS_{m} \right)^2 + S_{m}^* S_{m}^* \right) \psi_n d\xi \]

\[ + \left( \frac{1}{2} \gamma_2^2 \psi_n \right) \int_{\xi_0}^{\xi_2} B_{2n} \psi_{2n}^* \left( xS_{m} \right)^2 + S_{m}^* S_{m}^* \right) \psi_n d\xi \]

Similarly, the power carried in the cladding is

\[ P_{\text{clad}} = \left( \frac{1}{2} \gamma_2^2 \right) \int_{\xi_0}^{\xi_2} B\psi_{0n} \left( \psi_{0n}^* \right)^2 \left( xG_{m} \right)^2 + G_{m}^* G_{m}^* \right) \]

\[ + \left( \frac{1}{2} \gamma_2^2 \psi_n \right) \int_{\xi_0}^{\xi_2} B_{2n} \psi_{2n}^* \left( xG_{m} \right)^2 + G_{m}^* G_{m}^* \right) \psi_n d\xi \]

\[ + \left( \frac{1}{2} \gamma_3 \psi_n \right) \int_{\xi_0}^{\xi_2} B_{3n} \psi_{3n}^* \left( xG_{m} \right)^2 + G_{m}^* G_{m}^* \right) \psi_n d\xi \]
In Eqs. (6.2) and (6.3), $a_n$ and $b_n$ are the characteristic values of the even and odd Mathieu functions of order $n$, respectively. The prime over the summation sign is used to indicate that $n = n$ is excluded. Also, $\gamma_1^2 = 4\gamma_1^2/q^2$ and $\gamma_2^2 = 4\gamma_2^2/q^2$ are used.

The following abbreviations have been used,

\begin{align*}
(6.4) & \quad C_{an} = \int_0^{2\pi} c_{en}'(\tau) c_{en}(\tau) \, d\tau \\
(6.5) & \quad S_{an} = \int_0^{2\pi} s_{en}'(\tau) s_{en}(\tau) \, d\tau \\
(6.6) & \quad T_{an} = \int_0^{2\pi} c_{en}'(\tau) s_{en}(\tau) \, d\tau \\
(6.7) & \quad C_{an}^* = \int_0^{2\pi} c_{en}''(\tau) c_{en}''(\tau) \, d\tau \\
(6.8) & \quad S_{an}^* = \int_0^{2\pi} s_{en}''(\tau) s_{en}''(\tau) \, d\tau \\
(6.9) & \quad T_{an}^* = \int_0^{2\pi} c_{en}''(\tau) s_{en}''(\tau) \, d\tau.
\end{align*}

The power distribution characteristics for the eHE mode is given in Figure 10. The fractional power carried by the core and cladding is plotted against the normalized major axis for the various values of anisotropy and for $\varepsilon = 2.5$ and $\varepsilon = 1.0$. Most of the power is carried in the cladding near the cut-off and in the core far from the cut-off. For a fixed value of the normalized major axis, the more energy is concentrated inside of the core for larger the value of anisotropy and far from the cut-off.

6.2 ODD MODES

Substituting Equations (3.43) through (3.46) into Eq. (6.1) and integrating over the core area yields

\begin{align*}
(6.10) \quad P_{core} &= (1/2\gamma_1^4) \int_0^{2\pi} \beta \psi_1 \epsilon_1 A_{lm}^2 \left[ x c_{en}^2 + C_{an} c_{en}^2 \right] \\
& \quad + \beta \psi_1 \epsilon_1 B_{lm}^2 \left[ x s_{en}^2 + S_{an} s_{en}^2 \right] \, d\epsilon \\
& \quad + \left( \beta^2 + \psi_1^2 \mu \right) / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} B_{lm} A_{en} T_{en} \left[ S_{en} c_{en} \right] \, d\epsilon \\
& \quad + \beta \psi_1 / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} B_{lm} A_{en} T_{en} \left[ S_{en} s_{en} - S_{en} s_{en} \right] \, d\epsilon \\
& \quad + \beta \psi_1 / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} \left[ A_{en} c_{en} + C_{en} c_{en} \right] \, d\epsilon \\
& \quad + \beta \psi_1 / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} \left[ A_{en} c_{en} + C_{en} c_{en} \right] \, d\epsilon \\
& \quad + \beta \psi_1 / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} \left[ A_{en} c_{en} + C_{en} c_{en} \right] \, d\epsilon \\
& \quad + \beta \psi_1 / 2\gamma_1^4 \int_0^{2\pi} \sum_{en} \left[ A_{en} c_{en} + C_{en} c_{en} \right] \, d\epsilon.
\end{align*}
Figure 10. Power distribution characteristics for elliptical fiber as a function of normalized major axis for even modes.
Similarly, the power carried in the cladding is

\[
P_{\text{clad}} = (1/2Y_2^4) \int_{\xi_0}^{\xi_0} \left( \beta \omega_0 E_{2m} A_{2m}^2 (nPek_m^2 + a_{mn}^2 Pek_m^2) \ight) \text{d}\xi
\]

\[+ \beta \mu \sum B_{2m}^2 \left( mGek_m^2 + a_{mn}^2 Gek_m^2 \right) \text{d}\xi
\]

\[+ \left( \beta^2 + \omega^2 \mu \xi_0 / 2Y_2^4 \right) \sum \frac{\xi_0}{B_{2m} B_{2n} S_{2m} S_{2n}} \left( Gek_m^2 Gek_n^2 \right)
\]

\[= Gek_m Gek_n \left( \frac{\xi_0}{b_m - b_n} \right)^{-1}
\]

\[+ \beta \omega_0 / 2Y_2^4 \sum \frac{\xi_0}{B_{2m} B_{2n} S_{2m} S_{2n}} \left( Pek_m^2 Pek_n^2 \right)
\]

\[= Pek_m Pek_n \left( \frac{\xi_0}{a_m - a_n} \right)^{-1}
\]

In Eqs. (6.10) and (6.11), \( a_m \) and \( b_m \) are the characteristic values of the even and odd Mathieu functions of order \( m \), respectively. The prime over the summation sign is used to indicate that \( m = n \) is excluded. Also, \( Y_1^2 = 4Y_{1}^2/q^2 \) and \( Y_2^2 = 4Y_{2}^2/q^2 \) are used.

The following abbreviations have been used,

\[
(6.12) \quad C_{mn} = \int_{\xi_0}^{\xi_0} c_{m} \left( \xi \right) c_{n} \left( \xi \right) \text{d}\xi
\]

\[
(6.13) \quad S_{mn} = \int_{\xi_0}^{\xi_0} s_{m} \left( \xi \right) s_{n} \left( \xi \right) \text{d}\xi
\]

\[
(6.14) \quad T_{mn} = \int_{\xi_0}^{\xi_0} t_{m} \left( \xi \right) t_{n} \left( \xi \right) \text{d}\xi
\]

\[
(6.15) \quad C_{mn}^* = \int_{\xi_0}^{\xi_0} c_{m} \left( \xi \right) c_{n}^* \left( \xi \right) \text{d}\xi
\]

\[
(6.16) \quad S_{mn}^* = \int_{\xi_0}^{\xi_0} s_{m} \left( \xi \right) s_{n}^* \left( \xi \right) \text{d}\xi
\]

\[
(6.17) \quad T_{mn}^* = \int_{\xi_0}^{\xi_0} t_{m} \left( \xi \right) t_{n}^* \left( \xi \right) \text{d}\xi
\]

The power distribution characteristics for the \( \text{oHE}_{11} \) mode is given in Figure 11. The fractional power carried by the core and cladding is plotted against the normalized major axis for the various values of anisotropy and for \( E_1/E_0 = 2.5 \) and \( E_0 = 1.0 \). Most of the power is carried in the cladding near the cut-off and in the core far from cut-off. However, the difference in the power distribution for \( \text{oHE}_{11} \) for the varying anisotropy is smaller than that of \( \text{eHE}_{11} \) for a fixed value of normalized major axis.
Figure 11. Power distribution characteristics for elliptical fiber as a function of normalized major axis for odd modes.
7. DESCRIPTION OF COMPUTER PROGRAMS

In this chapter, the computer programs for the anisotropic elliptical fiber will be considered. These programs are written in the language of FORTRAN IV and in order to present the complete computer programs the subroutines to calculate the Mathieu and modified Mathieu functions developed by Rengarajan and Lewis[49] will be included. The theory and notations used in the computer programs are the same as those employed by McLachlan[48].

The normalized propagation constant and power distribution characteristics as a function of the normalized cross section area or major axis for the given value of $\delta$ and anisotropy have been determined by utilizing these programs. These computer programs consist of a main program and user called subroutines: CHAVAL and POWER. These subroutines CHAVAL and POWER call nine subroutines in order to compute the Mathieu and modified Mathieu functions.

In subroutine CHAVAL, an initial guess for the given mode is chosen and used to evaluate either Eq.(4.7) or Eq.(4.12). Next, Muller's method is used iteratively to determine the normalized wavelength that will minimize the function; an error criterion has been used to terminate the iteration. In subroutine POWER, the power distribution characteristics are calculated using the normalized wavelength obtained in subroutine CHAVAL. The algorithm was run on an CYBER 990 using only a single processor.

The sequence of the called subroutines is illustrated in the following flow chart.
Modified Mathieu function

The first time computation

Yes

STORE

No

CERAD  SERAD  FERAD  GERAD
8. CONCLUSION

The exact characteristic equation for anisotropic elliptical optical fibers is obtained for the odd and even hybrid modes in terms of infinite determinants employing Mathieu and modified Mathieu functions. The exact characteristic equation is applicable to elliptical fibers with any ellipticity. A simplified characteristic equation can then be obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small. Under this approximation, it can be shown that significant simplification can be achieved.

The simplified characteristic equation is used to compute the normalized wavelength for an anisotropic elliptical fiber. When the anisotropy parameter is equal to unity, the characteristic equation becomes that of isotropic fiber. The results are compared to the previous research and they are in close agreement. For a fixed value of the normalized cross-section area or major axis, the normalized wavelength $\lambda/\lambda_0$ is small for larger the value of anisotropy. This condition indicates that more energy is carried inside of the fiber. However, the geometry and anisotropy of fiber have a smaller effect when the normalized cross-section area or major axis is very small or very large.

An exact solution for the wave equation can not be determined when the thermoelastic stress causes a transverse anisotropy over the core of fibers. One possibility is that the propagation characteristics in the biaxial anisotropic fibers could be obtained by applying the numerical or approximation techniques given in Chapter 1 and this could be a subject for further study.
REFERENCES


APPENDIX

The following is a listing of the computer programs, MAIN, CHAVAL, POWER, CHVAL2, EXPAND, ANGMFC, FACTOR, STORE, CERAD, SERAD, FERAD AND GERAD written in FORTRAN IV language.
THIS PROGRAM CALCULATES THE NORMALIZED WAVELENGTH AND POWER DISTRIBUTION FOR ELLIPTICAL FIBER AS FUNCTION OF NORMALIZED CROSS-SECTION AREA OR MAJOR AXIS.

PROGRAM MAIN
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION RES(56), X(56)
OPEN (6,FILE="OUTPUT")

C C C C C C C C C C C
C C C C C C C C C C C
C C C C C C C C C C C
C NEVOD = 1 FOR ODD MODE,
C = 2 FOR EVEN MODE.
C KASE = 1 FOR NORMALIZED CROSS-SECTIONAL AREA,
C = 2 FOR NORMALIZED MAJOR AXIS.
C ETA = INDEPENDENT VARIABLE IN MATHIEU FUNCTION
C PHI = INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION
C MODE = WAVE MODE NUMBER
C P = EP1/EP0
C G = ANISOTROPY EPZ/EPIX
C NEVOD=1
KASE=2
MODE=1
P=2.500
G=1.500
PHI=1.000
GO TO (11,12), NEVOD
11 WRITE(6,111) MODE
GO TO 13
12 WRITE(6,125) MODE
13 WRITE(6,132) P
WRITE(6,134) PHI
C CALCULATE THE NORMALIZED WAVELENGTH
C CALL CHARVAL (NEVOD,P,G,MODE,PHI,KASE,RES)
C DO 14 I=1,56
14 X(I)=RES(I)
C CALCULATE POWER DISTRIBUTION FOR THE GIVEN MODE
C CALL POWER (NEVOD,P,G,MODE,PHI,KASE,X)

111 FORMAT(1X, 'THIS IS RESULT FOR ODD MODE, M =', I2)
122 FORMAT(1X, 'RATIO OF COFÉ AND CLADDING PERMITTIVITY =',
* D12.5)
123 FORMAT(1X, 'ANISOTROPY =', D12.5)
124 FORMAT(1X, 'VARIABLE IN MATHIEU FUNCTION =', D12.5)
SUBROUTINE CHARVAL (NEVOD,P,G,MODE,PHI,KASE,RES)

PURPOSE : CALCULATE THE NORMALIZED WAVELENGTH FOR ELLIPTICAL GUIDE.
INPUT : NEVOD -(INTEGER) SPECIFIES = 1 FOR ODD MODE
                = 2 FOR EVEN MODE
          P -(DOUBLE PRECISION) IS THE RATIO BETWEEN CORE AND CLADDING PERMEABILITY.
          G -(DOUBLE PRECISION) IS THE ANISOTROPY, EZ/EX.
          MODE -(INTEGER) IS THE MODE OF CHARACTERISTIC EQUATION.
          PHI -(DOUBLE PRECISION) IS INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION.
          KASE -(INTEGER) = 1 FOR NORMALIZED CROSS-SECTION AREA,
                           = 2 FOR NORMALIZED MAJOR AXIS.
OUTPUT : RES -(DOUBLE PRECISION) CONTAINS THE NORMALIZED WAVELENGTH.

IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION CHV1(23), CHV2(23), A8(25), QV(4), ETA(21),
          SE1(21),SE1D(21),SE0(21),CE1(21),CE1D(21),
          CEC(21),CEC1(21),SEC(21),CED(21),SED(21),
          SE1S(21),CE1S(21),CEGMZ(21),SECM(21),CEIM(21),
          SEIM(21),RES(56)
DATA PI/3.141592653589793DC/
ETA : INDEPENDENT VARIABLE IN MATHIEU FUNCTION
XI : INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION
ORDER : WAVE MODE NUMBER
P : EPS/EPX
G : ANISOTROPY EPZ/EPX

M=20
LAST=56
IF(KASE.EQ.2) LAST=56
XI=PHI
DT=DTANH(XI)
DCOS2=DCOSH(XI)*DCOSH(XI)
X2=PI*DT*DCOS2
WRITE(6,210) X2
FORMAT(1X, D12.5)

SUBDIVIDE ETA FOR INTEGRATION, ALFA, BETA, GAMMA AND ANU.

H=2.000*PI/20.0DC
ETA(1)=0.0D0
DO 11 I=2,21
```
      ETA(I)=ETA(I)+(I-1)*H
      CSA - NORMALIZED CROSS SECTION AREA OR MAJOR AXIS
      CSA=0.000
      XINC=2.5D-2
      VARX=0.999999999D
      DO 70 K=1,LAST
      CSA=CSA+XINC
      GO TO (12,13), KASE
      WRITE(6,201) CSA
      FORMAT(1X,*NORMALIZED CROSS SECTION AREA = *,D12.5)
      X1=(PI*PI*CSA)/((.000000000D)*DCOS2)
      GO TO 14
      WRITE(6,202) CSA
      FORMAT(1X,*NORMALIZED MAJOR AXIS = *,D12.5)
      X1=(PI*PI*CSA**2)/((.000000000D)*DCOS2)
      FIRST GUESS OF VAR = LANDA/LANDAO
      IF(K.EQ.1) VAR=VARX
      NC=1
      NS=0
      VAR2=1.0D0/(VAR*VAR)
      EVALUATE Q : INDEPENDANT VARIABLE
      GAMMAE = QV(1), GAMMAH = QV(2), GAMMA2 = QV(3)
      QV(1)=X1*(P*G-G*VAR2)
      QV(2)=X1*(P-VAR2)
      QV(3)=X1*(1.0D0-VAR2)
      QV(4)=X1*(1.0D0-VAR2)
      CALCULATE MATHIEU AND MODIFIED MATHIEU FUNCTIONS
      DO 50 KQ=1,4
      IEVOD=1
      IF(MOD(KQ,2).EQ.1) IEVOD=2
      ORDER=MODE
      IF(MOD(KQ,2).EQ.1) ORDER=MODE
      Q=QV(KQ)
      CALL CHVAL2(M,Q,CHV1,CHV2,J)
      CV=CHV1(ORDER)
      IF(MOD(KQ,2).EQ.1) CV=CHV2(ORDER+1)
      GO TO 17
      IEVOD=1
      IF(MOD(KQ,2).EQ.0) IEVOD=2
      ORDER=MODE
```

IF(MOD(KQ, 2) .EQ. 0) IORDER = MODE
Q = QV(KQ)
CALL CHVAL2(M, Q, CHV1, CHV2, J)
CV = CHV1(IORDER)
IF(MOD(KQ, 2) .EQ. 0) CV = CHV2(IORDER + 1)

OBVIOUS EXPANDING COEFFICIENT, ABXX

CALL EXPAND(Q, IEVOD, IORDER, CV, 3, AB, N)

CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES,
ORDER = MODE

KQEO = KQ
IF(MOD(IEVOD, 2) .EQ. 1) KQEO = KQ + 4
GO TO (21, 22, 23, 24, 22, 21, 24, 23), KQEO
DO 41 I = 1, 21
SEI(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), C, AB, N)
SEI(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), 1, AB, N)
GO TO 45
DO 42 I = 1, 21
SEQ(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), 0, AB, N)
GO TO 45
DO 43 I = 1, 21
CE1(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), 0, AB, N)
CE1(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), 1, AB, N)
GO TO 45
DO 44 I = 1, 21
CEO(I) = ANGMFC(Q, IEVOD, IORDER, ETA(I), 0, AB, N)

NORMALIZATION FACTOR FOR MODIFIED MATHIEU FUNCTION

CALL FACTOR(IEVOD, IORDER, Q, AB, N, PS)

COMPUTE AND STORE THE VALUES OF BESSEL FUNCTIONS

CALL STORE(Q, X, N)

CALCULATE MODIFIED MATHIEU FUNCTIONS

GO TO (51, 52, 53, 54, 52, 51, 54, 53), KQEO
SE = SERAD(Q, IORDER, 0, PS, AB, N)
SED = SERAD(Q, IORDER, 1, PS, AB, N)
GO TO 50
GE = GERAD(Q, IORDER, 0, PS, AB, N)
GED = GERAD(Q, IORDER, 1, PS, AB, N)
GO TO 50
CE = GERAD(Q, IORDER, 0, PS, AB, N)
CED = GERAD(Q, IORDER, 1, PS, AB, N)
GO TO 50
FE = FERAD(Q, IORDER, 0, PS, AB, N)
```plaintext
FED=FERAD(Q, IORDER, P, AB, N)
CONTINUE

CALCULATE MTH TERM( = MODE) OF
ALFA, BETA, GAMMA AND NU.

WRITE(6,107) SE, SED, GE, GED, CE, CED, FE, FED
DO 56 I=1,21
CE0M(I)=CE0(I)
SE0M(I)=SE0(I)
CE1M(I)=CE1(I)
SE1M(I)=SE1(I)
CE01(I)=CE0(I)*CE1(I)
SE01(I)=SE0(I)*SE1(I)
CE1SQ(I)=CE1(I)*CE1(I)
CEDSE(I)=CE1D(I)*SE1(I)

56 SEDCE(I)=SE1D(I)*CE1(I)
S1=SIMPSN(CEO1, 20, H)
S2=SIMPSN(SEO1, 20, H)
S3=SIMPSN(CEDSE, 2C, H)
S4=SIMPSN(SEDCE, 20, H)
S5=SIMPSN(SE1SQ, 20, H)
S6=SIMPSN(CE1SQ, 2C, H)
S5M=S5
S6M=S6
GO TO (57, 58), NEVOD

57 ALFAM=S2/S5
BETAM=S1/S6
GAMMAM=S4/S6
ANUM=S3/S5
GO TO 59

58 ALFAM=S1/S6
BETAM=S2/S5
GAMMAM=S3/S5
ANUM=S4/S6

59 XMSQD=ALFAM*BETAM
RHC=QV(12)/QV(13)
WRITE(6,1C3) ALFAM, BETAM, GAMMA, ANU, RHC
FORMAT(1X, 5D12.5)

CALCULATE MATHIEU FUNCTION INTEGRALS
ORDER = N, N+2 - -

ALFA=0.0D0
BETA=0.0D0
GAMMA=0.0D0
ANU=0.0D0
XMSQN1=0.0D0
XMSQN2=0.0D0
DO 90 IM=1,4
```

IORDER=2*IM-1  
IFI(MODE,2).EQ.,C) IORDER=2*IM-2  
IFI(IORDER.NE.MODE) GO TO 61  
ALFA=ALFAM  
BETA=BETAM  
GAMMA=GAMMAM  
ANUm=ANUM  
XMSQN1=XMSQN1+BETA*ANUM  
XMSQN2=XMSQN2+ALFA*GAMMA  
XMSQ=(XMSQN1*XMSQN2)/XMSQD  
GO TO 92  
61  
DO 80 KQ=1,4  
GO TO (62,63), NEVOD  
62  
IEVOD=1  
IFI(MOD(KQ,2).EQ.1) IEVOD=2  
Q=QV(KQ)  
CALL CHVAL2(M,Q,CHV1,CHV2,J)  
CV=CHV1(IORDER)  
IFI(MOD(KQ,2).EQ.1) CV=CHV2(IORDER+1)  
GO TO 64  
63  
IEVOD=1  
IFI(MOD(KQ,2).EQ.0) IEVOD=2  
Q=QV(KQ)  
CALL CHVAL2(M,Q,CHV1,CHV2,J)  
CV=CHV1(IORDER)  
IFI(MOD(KQ,2).EQ.0) CV=CHV2(IORDER+1)  
C  
OBTAIN EXPANDING COEFFICIENT, ABXX  
C  
64  
CALL EXPAND(Q,IEVOD,IORDER,CV,3,ABN)  
C  
CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES  
C  
KQEO=KQ  
IFI(MOD(NEVOD,2).EQ.1) KQEO=KQ+4  
GO TO (71,72,73,74,72,71,74,73,71), KQEO  
71  
DO 81 I=1,21  
SE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),C,AB,N)  
81  
SE10(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)  
GO TO 8C  
73  
DO 82 I=1,21  
SE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),C,AB,N)  
GO TO 8C  
72  
DO 83 I=1,21  
CE1(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),C,AB,N)  
83  
CE10(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)  
GO TO 8C  
74  
DO 84 I=1,21  
CE0(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),C,AB,N)  
85  
CONTINUE  
C
CALCULATE SUM (BETA * NU) AND SUM (ALFA * GAMMA)

DO 86 I=1,21
CE01(I)=CEOM(I)*CE1(I)
SE01(I)=SEOM(I)*SE1(I)
SE1SQ(I)=SE1(I)*SE1(I)
CE1SQ(I)=CE1(I)*CE1(I)
CEDSE(I)=CE1D(I)*CE1M(I)
SEDE(I)=SE1D(I)*CE1M(I)
S1=SIMPSN(CEO1,20,H)
S2=SIMPSN(SE01,20,H)
S3=SIMPSN(CEDSE,20,H)
S4=SIMPSN(SEDE,20,H)
S5=SIMPSN(SE1SQ,20,H)
S6=SIMPSN(CE1SQ,20,H)

GO TO (87,88), NEVOD

87 ALFA=S2/S5
BETA=S1/S6
GAMMA=S4/S6M
ANU=S3/S5M
GO TO 89

88 ALFA=S1/S6
BETA=S2/S5
GAMMA=S3/S5M
ANU=S4/S6M

89 XMSQN1=XMSQN1+BETA*ANU
XMSQN2=XMSQN2+ALFA*GAMMA
XMSQ=(XMSQN1*XMSQN2)/XMSQD

92 WRITE(6,103) ALFA,BETA,GAMMA,ANU,XMSQ
96 CONTINUE

EVALUATE CHARACTERISTIC EQUATION

Y1=-(XMSQ*(1.00D0-RHC)**2)
Y2=VAR*VAR
GO TO (93,94), NEVOD

93 Y3=(CSED/SE)-(RHC*GED/GE)
Y4=(CSED/CE)-(RHC*FED/FE)
WRITE(6,101) Y1,Y2,Y3,Y4
GO TO 95

94 Y3=(CSED/CE)-(RHC*FED/FE)
Y4=(PSED/SE)-(RHC*GED/GE)
WRITE(6,101) Y1,Y2,Y3,Y4

95 YX=Y2*Y3*Y4
YZ=YX/Y1
Y=YZ-1.00D0
WRITE(6,203) YX, Y1, YZ, Y, VAR

203 FORMAT(1X,5D12.5)

DESIDE ON TOLERANCES
IF(NC.*NE.*1.*AND.*DABS(Y).*LE.*2.*0D-3) GO TO 39
IF(NC.*EQ.*1) GO TO 32
IF(NC.*EQ.*2) GO TO 34
IF(NS.*EQ.*0) GO TO 34
IF(Y.*YS1) 36,36,31

YS1=Y
VARS1=VAR
VAR=(VARS1+VARS2)/2.0DC
NS=NS+1
IF(NS.*LE.*20) GO TO 10
GO TO 47

1 ST CALCULATION OF Y, DECREMENT VAR BY 0.01

YS1=Y
VARS1=VAR
32
VAR=VAR-1.0D-2
NC=NC+1
YMIN=Y
VARMIN=VAR
IF(NC.*LE.*2) GO TO 10
GO TO 48

34
IF(Y.*YS1) 36,36,35
35
IF(DABS(Y)-DABS(YS1)) 32,33,33
36
YS2=Y
VARS2=VAR
VAR=(VARS1+VARS2)/2.0DC
NS=NS+1
YMIN=Y
VARMIN=VAR
IF(NS.*LE.*2) GO TO 10
WRITE(6,106)
WRITE(6,108) YMIN, VARMIN
GO TO 39

WRITE(6,102) VAR=X
WRITE(6,109) YMIN, VARMIN
39
RES(K) VAR
70 CONTINUE
RETURN
101 FORMAT(1X,4D12.5)
102 FORMAT(1X,*ERROR: RESULT HAS SAME SIGN FOR 10 TRIES*)
106 FORMAT(1X,*ERROR: 10 TRY FAILED TO OBTAIN RESOLUTION*)
107 FORMAT(1X,8D12.5)
108 FORMAT(1X,*OBTAINED RESOLUTION =", D12.5,
C MIN CALCULATED NORMALIZED WAVELENGTH =", D12.5)

END
SUBROUTINE POWER (NEVOD,P,G,MODE,BOUND,KASE,A)

PURPOSE: CALCULATE POWER DISTRIBUTION ON ELLIPTICAL
INPUT : NEVOD -(INTEGER) SPECIFIES = 1 FOR ODD MODE 
          = 2 FOR EVEN MODE
P -(DOUBLE PRECISION) IS THE RATIO BETWEEN CORE 
   AND CLADDING PERMEABILITY.
G -(DOUBLE PRECISION) IS THE ANISOTROPY, EZ/EX.
MODE -(INTEGER) IS THE MODE OF CHARACTERISTIC 
   EQUATION.
PHI -(DOUBLE PRECISION) IS INDEPENDENT VARIABLE 
   IN MODIFIED MATHIEU FUNCTION.
A -(DOUBLE PRECISION) IS THE NORMALIZED 
   WAVELENGTH.
KASE -(INTEGER) = 1 FOR NORMALIZED CROSS-SECTION 
   AREA, 
          = 2 FOR NORMALIZED MAJOR AXIS.
OUTPUT : RES -(DOUBLE PRECISION) CONTAINS THE RATIO OF 
          POWER DISTRIBUTION.

IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION CHV1(23), CHV2(23), AB(25), QV(4), A(56),
    ETA(21), PHI(41), SE1(21), SEID(21), SEO(21),
    CSE0D(21), CE1(21), CEID(21), CEO(21), CEOD(21),
    CSE(21), SEO(21), CE(21), CED(21),
    CFE(41), FED(41), GE(41), GED(41),
    CS1(21), S2(21), S3(21), S4(21), S5(21), S6(21),
    CS7(21), S8(21), S9(21), S10(21), S11(21),
    CS12(21), S13(21), S14(21),
    CSS21(21), S22(21), S23(21)
DATA PI/3.14159265358979300/

ETA : INDEPENDENT VARIABLE IN MATHIEU FUNCTION 
XI : INDEPENDENT VARIABLE IN MODIFIED MATHIEU FUNCTION 
ORDER : WAVE MODE NUMBER 
P : EP1/EP0 
G : ANISOTROPY EPZ/EPX 

M=20
EP=(1.0D-9)/(36.0D0*PI)
XNU=4.0D0*PI*1.0D-7
CONS=DSQRT(EP/XNU)
LAST=40
IF(KASE.EQ.2) LAST=56
PHI0=0.0D0
PHI1=BOUND
PHI2=5.0D0*BOUND
DT=DTANH(PHI1)
DCOS2=DCOSH(PHI1)*DCOSH(PHI1)
X2=PI*DT*DCOS2
WRITE(6,210) X2

SUBDIVIDE ETA AND PHI FOR INTEGRATION.
C
H1=2.000/PI/20.000
ETA(I)=0.000
DO 11 I=2,21
11 ETA(I)=ETA(I-1)+H1
H2=BOUND/20.000
PHI(I)=0.000
DO 12 I=2,21
12 PHI(I)=PHI(I-1)+H2
H3=(PHI2-PHI1)/20.000
DO 13 I=22,41
13 PHI(I)=PHI(I-1)+H3

C
CSA = NORMALIZED CROSS SECTION AREA OR MAJOR AXIS
C
CSA=0.000
XINC=2.500
DO 70 K=1,LAST
CSA=CSA+XINC
IF(A(K)*EQ.1.000) GO TO 81.
GO TO (14,15), KASE
14 WRITE(6,201) CSA
201 FORMAT(1X,"NORMALIZED CROSS SECTION AREA = ",F12.5)
C
X1=(PI*PI*CSA)/(4.000*DCOS2)
GO TO 16
15 WRITE(6,202) CSA
202 FORMAT(1X,"NORMALIZED MAJOR AXIS = ",F12.5)
C
X1=(PI*PI*CSA/4.000)*DCOS2
C
CALCULATE CONSTANTS
C
VAR2=1.500/A(K)*A(K))
C
EVALUATE Q : INDEPENDANT VARIABLE
C
GAMMAE = QV1, GAMMAH = QV2, GAMMA2 = QV(3)
C
QV(1)=X1*(P*G-G*VAR2)
QV(2)=X1*(P-VAR2)
QV(3)=X1*(1.000-VAR2)
QV(4)=X1*(1.000-VAR2)
C
C4=CONS/A(K)
C1=P*C4
C2=1.000/A(K)*CONS
C3=P*VAR2
C5=1.000+VAR2
C6=(QV(2)*QV(2))/(QV(3)*QV(3))
C
CALCULATE MATHIEU AND MODIFIED MATHIEU FUNCTIONS
DO 50 KQ=1,4
GO TO (17,18), NEVOD

17 IEVOD=1
IF(MOD(KQ,2).EQ.0) IEVOD=2
IORDER=MODE
IF(MOD(KQ,2).EQ.1) IORDER=MODE
Q=QM(KQ)
WRITE(6,301) X1,J
CALL CHVAL2(M,Q,CHV1,CHV2,J)
CV=CHV1(IORDER)
IF(MOD(KQ,2).EQ.1) CV=CHV2(IORDER+1)
GO TO 19

18 IEVOD=1
IF(MOD(KQ,2).EQ.0) IEVOD=2
IORDER=MODE
IF(MOD(KQ,2).EQ.1) IORDER=MODE
Q=QM(KQ)
WRITE(6,301) X1,J
CALL CHVAL2(M,Q,CHV1,CHV2,J)
CV=CHV1(IORDER)
IF(MOD(KQ,2).EQ.0) CV=CHV2(IORDER+1)

C
C OBTAIN EXPANDING COEFFICIENT, ABXX
C
19 CALL EXPAND(Q,IEVOD,IORDER,CV,3,AB,N)
C
C CALCULATE MATHIEU FUNCTIONS AND DERIVATIVES,
C IORDER = MODE
C
KQEO=KQ
IF(MOD(NEVOD,2).EQ.1) KQEO=KQ+4
GO TO (21,22,23,24,22,21,24,23), KQEO

21 DO 41 I=1,21
SEL(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
41 DO 45 I=1,21
SEI(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
GO TO 45

22 DO 43 I=1,21
CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
43 DO 45 I=1,21
CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),0,AB,N)
GO TO 45

24 DO 44 I=1,21
CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
44 DO 45 I=1,21
CEO(I)=ANGMFC(Q,IEVOD,IORDER,ETA(I),1,AB,N)
GO TO 45

C
C NORMALIZATION FACTOR FOR MODIFIED MATHIEU FUNCTION
CALL FACTOR(IEVOD, IORDER, Q, AB, N, PS)
CALL CALCULATE MODIFIED MATHIEU FUNCTIONS
GO TO (31,32,33,34,32,31,34,33,135,10)
DO 51 I=1,21
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
DO 51 I=1,21
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
GO TO 50
DO 51 I=1,21
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
GO TO 50
DO 52 I=21,41
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
GO TO 50
DO 53 I=1,21
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
GO TO 50
DO 54 I=21,41
CALL STORE(Q, PHI(I), N)
SE(I)=SERAD(Q, IORDER, 0, PS, AB, N)
GO TO 50
CONTINUE
GO TO 50
CALL CALCULATE INTEGRAND
DO 56 I=1,21
S21(I)=SEO(I)*SE1(I)
S22(I)=SE1(I)*SE1(I)
S23(I)=CE10(I)*SE1(I)
S1(I)=SE10(I)*SE10(I)
S2(I)=CE10(I)*CE10(I)
S3(I)=CE10(1)*SE1(I)
S4(I)=SEO(I)*SEO(I)
S5(I)=CEO(I)*CEO(I)
S6(I)=CEO(I)*SEO(I)
DO 56 I=1,21
S7(I)=SEO(I)*CEO(I)
S8(I)=SE1(I)*SE1(I)
S9(I)=CEO(I)*CEO(I)
S10(I)=CE10(I)*CE10(I)
DO 57 I=21,41
II=I-10
S11(I)=GED(I)*GED(I)
S12(I)=GE(I)*GE(I)
59

PERFORM THE INTEGRATION

ST1=SIMPSN(S21, 20, H1)
ST2=SIMPSN(S22, 20, H1)
ST3=SIMPSN(S23, 20, H1)
T11=PI*SIMPSN(S7, 10, H2)
T12=SIMPSN(S1, 20, H1)*SIMPSN(S8, 20, H2)
T21=PI*SIMPSN(S9, 20, H2)
T22=SIMPSN(S2, 20, H1)*SIMPSN(S10, 20, H2)
T31=SIMPSN(S3, 20, H1)
T32=(CE(21)*SE(21)-CE(1)*SE(1))
T41=PI*SIMPSN(S11, 20, H3)
T42=SIMPSN(S4, 20, H1)*SIMPSN(S12, 20, H3)
T51=PI*SIMPSN(S13, 20, H3)
T52=SIMPSN(S5, 20, H1)*SIMPSN(S14, 20, H3)
T61=SIMPSN(S6, 20, H1)
T62=(FE(41)*GE(41)-FE(21)*GE(21))

CALCULATE THE ARBITRARY CONSTANTS.

A11=(1.0D0-OV(2)/OV(3))*FE(21)*(ST3/ST2)
A12=CONS*AI(K1ST1/ST2)
A13=((-FE(21)*CE(21))/CE(21)-OV(2)*FE(21)/OV(3))
A21=A11/A12*A13
A1=A21*GE(21)/SE(21)
A1=A1*A1
B1=FE(21)*FE(21)
BA1=A21*FE(21)*GE(21)/SE(21)
A2=A21*A21
BA2=A21
T1=C1*A1*(T11+T12)
T2=C2*B1*(T21+T22)
T3=C3*BA1*T31*T32
WRITE(6, 302) T1, T2, T3
PCOR=T1+T2-T3
T4=C4*BA2*(T41+T42)
T5=C2*1.0D0*(T51+T52)
T6=C5*BA2*T61*T62
WRITE(6, 302) T4, T5, T6
PCLAD=C6*(T4+T5-T6)
RCOR=PCOR/(PCOR+PCLAD)
RCLAD=1.0D0-RCOR
WRITE(6, 211) RCOR, RCLAD, PCOR, PCLAD
GO TO 70

81 WRITE(6, 212)
7C CONTINUE
21C FORMAT(1X, 012.5)
211 FORMAT(11X, 0412.5)
DOUBLE PRECISION FUNCTION SIMPSN(Q,N,H)
DOUBLE PRECISION Q(21)

C C INTEGRATION BY SIMPSON'S RULE
C C
C SIMPSN=Q(1)+4*Q(2)+Q(N+1)
DO 1 I=4,N,2
1 SIMPSN=SIMPSN+2*Q(I-1)+Q(I)
SIMPSN=SIMPSN*H/3*C00
RETURN
END

SUBROUTINE CHVAL2(N,QQ,CHV1,CHV2,J)
PURPOSE: TO COMPUTE THE CHARACTERISTIC VALUES OF ODD
AND EVEN MATHIEU FUNCTIONS OF POSITIVE OR
NEGATIVE 'Q'
INPUT: N (INTEGER) SPECIFIES THAT CH. VALUES BE
OBTAINED FOR ORDERS 0 THRU N-1 FOR EVEN
FUNCTIONS AND FOR ORDERS 1 THRU N-1 FOR
ODD FUNCTIONS
QQ (DOUBLE PRECISION) THE PARAMETER 'Q' IN
MATHIEU'S DIFFERENTIAL EQUATION
OUTPUT: CHV1 (DOUBLE PRECISION) AN ARRAY OF LENGTH N
CONTAINING CH. VALUES OF ODD MATHIEU FUNCTIONS
OF ORDERS 1 THRU N-1.
CHV1(N) IS A DUMMY VARIABLE.
CHV2 (DOUBLE PRECISION) AN ARRAY OF LENGTH N
CONTAINING CH. VALUES OF EVEN MATHIEU FUNCTIONS
OF ORDERS 0 THRU N-1.
J (INTEGER) MAXIMUM ORDER UPTO WHICH CH. VALUES
HAVE BEEN SUCCESSFULLY COMPUTED
DOUBLE PRECISION CV1(6,25),CV2(6,25),CHV1(N),CHV2(N),QQ
DOUBLE PRECISION OABS,DABS

101 FORMAT('C',5X,'NOT ALL CH. VALUES AVAILABLE------WARNING')
IF(QQ.LT.0.DO) GO TO 20
IF(N.GT.1) CALL MFCVAL(N-1,N-1,QQ,CHV1,J1)
CALL MFCVAL(N,N-1,QQ,CHV2,J2)
IF(J1.LT.(N-1).OR.J2.LT.N) WRITE(6,1C1)
J=MIN(J1,J2-1)
DO 10 I=1,J
10 CONTINUE
CHV2(J+1)=CV2(1,J+1)
RETURN
20 OABS=DABS(QQ)
CALL MFCVAL(N,N-1,OABS,CHV1,CHV2,J2)
IF(N .NE. 1) GO TO 25

CHV2(1) = CV2(1,1)
RETURN

25  CALL MFCVAL(N-1, N-1, QABS, CV1, J1)
IF(J1 .NE. (N-1). OR. J2 .NE. N) WRITE(6,101)
J = MIN0(J1, J2-1)
DO 30 I = 1, J, 2
CHV2(I) = CV2(1, I)
CHV1(I) = CV2(I, I+1)
CHV2(I+1) = CV1(I, I)
IF((I+1) .LE. J) CHV1(I+1) = CV1(I, I+1)

30  CONTINUE
IF(MOD(N-1, 2). EQ. 0) CHV2(N) = CV2(1, N)
RETURN
END

SUBROUTINE MFCVAL(N, R, QQ, CV, J)

************
INTEGER J, K, KK, L, M, N, R, TYPE
DOUBLE PRECISION A, CV, DL, DR, DTM, Q, QQ, T, TM, TOL, TOLA
DOUBLE PRECISION FILL(3)
DIMENSION CV(6, N)
EQUIVALENCE (DL, DR, T)
COMMON/MF1/Q, TOL, TYPE, DUMMY(4)
COMMON/MF2/FILL
TOL = 1.0D-13
IF(N-R) 10, 10, 20

10  L = 1
GO TO 30

20  L = 2

30  Q = QQ
DO 50 C K = 1, N

J = K
IF(Q) 960, 490, 30

40  KK = MIN0(K, 4)
TYPE = 2 * MOD(K, 2) * MOD(K-L+1, 2)
C FIRST APPROXIMATION
GO TO(100, 200, 300, 400), KK

100 IF(Q-1.0DC) 110, 140, 140
110  GO TO(120, 130), L
120  A = 1.0DC - .125DO = QQ
GO TO 420

130  A = QQ
A = A*(-.5DC+.546875DC*A)
GO TO 420

140 IF(Q-2.0DO) 150, 180, 180
150  GO TO(160, 170), L
160  A = 1.033DO-1.07460C*Q-.569DC*QQ
GO TO 420

170  A = .23DO-.495DC*Q-.191DO*QQ
GO TO 420

130  A = -.25DO+2.0DO*QQ+2.0DO*C5QRT(Q)
GO TO 420

DL=L
IF(Q*DL-6.000) 210,350,350
GO TO 120,230,L
A=4.01521D0-Q*(1.046D0+.667857D0)*Q
GO TO 420
A=1.000+1.950C7D0*Q-.186143D0*Q
GO TO 420
IF(Q-8.000) 310,350,350
GO TO 130,230,330,L
A=8.93867D0+.178156D0*Q-.0252132D0*Q
GO TO 420
A=3.70017D0+.953485D0*Q-.0475065D0*Q
GO TO 420
DR=K-1
A=CV(1,K-1)-DR+4.0D0*DSQRT(Q)
GO TO 420
A=CV(1,K-1)-CV(1,K-2)
A=3.0D0*A+CV(1,K-3)
GO TO 420
IF(Q.GE.1.0D0) GO TO 440
IF(K.NE.1) GO TO 430
TOLA=DMAX1(DMIN1(TOLA,DABS(A)),1.0D-14)
GO TO 450
TOLA=TOL*DABS(A)
GO TO 450
TOLA=TOL*DMAX1(Q,DABS(A))
TOLA=DMAX1(DMIN1(TOLA,DABS(A)),.4D0*DSQRT(Q)),1.0D-14)
C CRUDE UPPER AND LOWER BOUNDS
CALL BOUNDS(KpAtTOLAoCVINtM)
IF(M,NE,0) IF(M-1) 470,910,900
C ITERATE
CALL MFITRB(TOLA,CV(1,K),CV(2,K),M)
IF(M.GT.0) GO TO 920
C FINAL BOUNDS AND FUNCTIONS, D
T=CV(1,K)-TOLA
CALL TMODA(T,T,M,DTH,M)
IF(M.GT.0) GO TO 940
T=CV(1,K)-TM/DTM
485 T=CV(1,K)-TOLA
CALL TMODF(T,T,M,DTH,M)
IF(M.GT.0) GO TO 950
C Q EQUALS ZERO
CV(1,K)=(K-L+1)*Q
CV(2,K)=0.0D0
CV(3,K)=CV(1,K)
CV(4,K)=0.0D0
CV(5,K)=CV(1,K)
CV(6,K)=0.000
500 CONTINUE
550 RETURN
C PRINT ERROR MESSAGES
900 WRITE(6,901) K
901 FORMAT(*0*,*CRUDE BOUNDS CANNOT BE LOCATED, NO OUTPUT*,
C * FOR K=9,12)
C GO TO 930
910 WRITE(6,911) K
911 FORMAT(*3*,*ERROR IN SUBPROGRAM TMOFAP VIA SUBPROGRAM
C BOUNDS, NO OUTPUT, FOR K=9,12)
C GO TO 930
920 WRITE(6,921) K
921 FORMAT(*0*,*ERROR IN SUBPROGRAM, TMOFAP VIA SUBPROGRAM,
C MFITR8, NO OUTPUT, FOR K=9,12)
930 J=J-1
GO TO 550
940 WRITE(6,941) K
941 FORMAT(*0*,*ERROR IN SUBPROGRAM, TMOFAP, NO LOWER BOUND,
C FOR K=9,12)
CV(3,K)=0.000
CV(4,K)=0.000
GO TO 450
950 WRITE(6,951) K
951 FORMAT(*0*,*ERROR IN SUBPROGRAM, TMOFAP, NO UPPER BOUND,
C FOR K=9,12)
CV(5,K)=0.000
CV(6,K)=0.000
GO TO 500
960 WRITE(6,961)
961 FORMAT(2H0Q GIVEN NEGATIVELY, 2OH USED ABSOLUTE VALUE)
Q=-Q
GO TO 40
END
SUBROUTINE BOUNDS(K,APPROX,TOLA,CV,N,MM)
INTEGER K,KA,M,MM,N
DOUBLE PRECISION A,APPROX,AC,A1,CV,DTM,DO,D1,O,TM,TOLA
DIMENSION CV(6,N)
COMMON/HF1/O,DFM47)
COMMON/MF2/AC,A,A1
KA=0
IF(K.EQ.1) GO TO 20
IF(APPROX-CV(1,K-1)) 10,10,20
10 AO=CV(1,K-1)+1.000
GO TO 30
20 AO=APPROX
30 CALL TMOFAP(AO,TO,TM,DTM,M)
IF(M.GT.0) GO TO 250
DO=TM/TDM
IF(DD) 100,300,50
C AO IS LOWER BOUND,
SEARCH FOR UPPER BOUND
A1 = A0 + D0 * 100
CALL THDF(A1, TM, DTM, M)
IF(M .GT. 0) GO TO 250
D1 = -TM/DTM
IF(D1) 200, 350, 60
A0 = A1
DO = D1
KA = KA + 1
IF(KA = 4) 50, 400, 400

A1 IS UPPER BOUND, SEARCH FOR LOWER BOUND
A1 = A0
D1 = D0
A0 = DMAX1(A1 + D1 - 100, -200, A0)
IF(K .EQ. 1) GO TO 110
IF(A0 = CV(1, K - 1)) 150, 150, 110
CALL THDF(A0, TM, DTM, M)
IF(D0) 250, 300, 200
KA = KA + 1
IF(KA = 4) 100, 400, 400
KA = KA + 1
IF(KA = 4) 100, 400, 400
AC = DMAX1(TOLA + DABS(D1))
GO TO 30
A = .50 * A0 + D0 + A1
IF(A .LE. A0 OR A .GE. A1) A = .50 * (A0 + A1)
GO TO 250
M = 1
GO TO 250
CV(1, K) = A0
CV(2, K) = 0.00
M = 2
GO TO 250
END

SUBROUTINE MFITR8(TOLA, CV, DCV, MM)
INTEGER M, MM, N
DOUBLE PRECISION A, A0, A1, A2, CV, D, DCV, DTM, TM, TOLA
LOGICAL LAST
COMMON/MF2/A0, A1
N = 0
LAST = .FALSE.
N = N + 1
CALL THDF(A, TM, DTM, M)
IF(M .GT. 0) GO TO 400
D = -TM/DTM
IS TOLERANCE MET
IF(N.EQ.40.OR.A-AC.LE.TOLA.OR.A1-A.LE.TOLA.OR.
C DABS(D)+L.T.TOLA)LAST=.TRUE.
IF(D) 110,100,120

100
CV=A
DCV=0.00
GO TO 320

110
A1=A
GO TO 200
C REPLACE UPPER BOUND BY A

120
A0=A
GO TO 300
C REPLACE LOWER BOUND BY A

200
A2=A+D
IF(LAST) GO TO 300
IF(A2.GT.A0.AND.A2.LT.A1) GO TO 250
A=.500*(A0+A1)
GO TO 50

250
A=A2
GO TO 50
300
IF(A2.LE.A0.OR.A2.GE.A1) GO TO 350
CALL TMODA(A2,MT,DTM,M)
IF(M.GT.0) GO TO 400
D=-TM/DTM
CV=A2

310
DCV=0
320
MM=M
RETURN

350
CV=A
GO TO 310
400
CV=0.00
DCV=0.00
GO TO 320
END

SUBROUTINE TMODA(ALFA,MT,DTM,ND)
INTEGER K,KK,KT,L,MF,MG,M1,M2S,ND,TYPE
DOUBLE PRECISION ALFA,B,DTM,STYPE
C STATEMENT FUNCTION
V(K)=(AA-DBLE(FLOAT(K))==2)/Q
ND=0
KT=0
AA=ALFA
STYPE=TYPE
QINV=1.0DC/Q
DO 10 L=1,2
   DO 5 K=1,2CC
EVALUATION OF THE TAIL OF A CONTINUED FRACTION

CONTINUE

IF(MOD(TYPE, 2)) 20, 30, 20

GO TO 40

M0 = TYPE + 2

K = 500 + DSQRT(DMAX1(3, CD) + QA, C) * D0)

M2S = MIN2(2*K + M0 + 4, 398 + MOD(M0, 2))

CONTINUE

A(1) = 1, 0, D0

A(2) = V(M2S + 2)

B(1) = V(M2S)

B(2) = A(2) * Q(1) - 1, 0, D0

Q1 = A(2) / B(2)

DO 50 K = 1, 200

MF = M2S + 2 + 2*K

T = V(MF)

A(3) = T*A(2) - A(1)

B(3) = T*B(2) - B(1)

Q2 = A(3) / B(3)

IF(DABS(Q1 - Q2) < LT, TOL) GO TO 70

Q1 = Q2

A(1) = A(2)

A(2) = A(3)

B(1) = B(2)

B(2) = B(3)

CONTINUE

K = 1

T = 1, 0, D0 / T

TT = -T*T*Q1**

L = MF - M2S

DO 80 K = 2, L + 2

T = 1, 0, D0 / (V(MF - K) - T)

TT = T*T*(TT - Q1**)

CONTINUE

KK = M2S / 2 + 1

IF(KT = EQ. 1) Q2 = T

G(KK, 2) = .500*Q2*T

DG(KK, 2) = TT

STAGE 1

G(2, 1) = 1, 0, D0

DO 140 K = M0, M2S + 2

KK = K / 2 + 1

IF(KL < 5) IF(K = 3) 100, 110, 120

G(KK, 1) = V(KK - 2) - 1*CD0 / G(KK - 1, 1)

DG(KK, 1) = Q1** + DG(KK - 1, 1) / G(KK - 1, 1)**2

GO TO 130

G(2, 1) = V(0)

DG(2, 1) = Q1**
GO TO 130
G(2,1)=V(1)+DTYPE/DG
DG(2,1)=QINV
GO TO 130
G(3,1)=V(2)+(DTYPE/DG)/G(2,1)
DG(3,1)=QINV+(2,DTYPE=DTYPE)/G(2,1)/G(2,1)*2
IF(TYPE.EQ.2) G(2,1)=0.00
IF(DABS(G(KK,1)),LT,1.00) GO TO 200
CONTINUE
C BACKTRACK
TM=G(KK,2)-G(KK,1)
DTM=DG(KK,2)-DG(KK,1)
M1=M2S
KT=M2S-M0
DO 180 L=2,KT,2
K=M2S-L
KK=K/2+1
G(KK,2)=1.00/(V(K)-G(KK+1,2))
DG(KK,2)=G(KK,2)+2*(QINV-DG(KK+1,2))
IF(K-2) 150,150,160
G(2,2)=2.00*D2G(2,2)
DG(2,2)=2.00*D2G(2,2)
T=G(KK,2)-G(KK,1)
IF(DABS(T)-DABS(TM)) 170,180,180
TM=T
DTM=DG(KK,2)-DG(KK,1)
M1=K
CONTINUE
GO TO 320
C STAGE 2
200 M1=K
K=M2S
KK=K/2+1
IF(K.EQ.M1) IF(K-2) 300,300,310
K=K-2
KK=KK-1
T=V(K)-G(KK+1,2)
IF(DABS(T)-1.00) 250,220,220
G(KK,2)=1.00/T
DG(KK,2)=(DG(KK+1,2)-QINV)/T**2
GO TO 210
C STAGE 3
250 IF(K.EQ.M1) IF(T) 220,290,220
HP=DG(KK+1,2)-QINV
260 G(KK,2)=FL
H(KK)=T
K=K-2
KK=KK-1
F=V(K)*T-1.00
IF(K.EQ.M1) IF(F) 280,290,280
IF(DABS(F)-DABS(T)) 270,280,280
SUBROUTINE EXPAND(Q0,FNC,R,CV,NORM,CD,N)
PURPOSE: TO GET EXPANDING COEFFICIENTS FROM ROUTINE
COEF. TERMINATE THE TERMS FOR REQD. ACCUTACY
AND DO THE NORMALIZATION
DIMENSION CD(25)
DOUBLE PRECISION A,CV,QD,Q0,TOL,T,AB,ERR,DABS,
CD,SUM,ti,DSQRT,SM1
INTEGER R,FNC,TYPE,CASE,NORM
COMMON DUM1(1600),A,T,DUM2(6),AB(200)
COMMON /MF1/Q,TOL,TYPE,M1,MD,M25,MF
101 FORMAT('0e0',THE # OF EXPANDING COEFFICIENT REQD. IS MORE
C THAN 25',5X,'WARNING')
102 FORMAT('0',ERROR IN SUBPROGRAM', 'TMOFA VIA COEF. VERIFY
C ARGUMENTS NO OUTPUT')
TOL=1.0D-13
C TO TEST THE LAST VALUE OF ARRAY CD ERR IS USED
ERR=1.0D-20
Q=QD
TYPE=2*MOD(FNC,2)+MOD(R,2)
C FOR NEGATIVE Q AND ODD ORDERS, EXP. COEFFS. FOR EVEN AND
C ODD FUNCTIONS ARE INTERCHANGED.
IF(Q,LT.0.0D0,AND.MOD(R,2).EQ.1)TYPE=2*MOD((FNC-1),2)+MOD(R,2)
M=O
A=CV
Q=DABS(Q)
CALL COEF(M)
IF(M,EQ.0) GO TO 5
WRITE(6,102)
RETURN
5 TYPE=2*MOD(FNC,2)MOD(R,2)
CASE=TYPE+1
C THE COEFFICIENTS PASSED THRU COMMON ARRAY AB IN DOUBLE
C PRECISION IS GIVEN TO AN ARRAY CD OF LENGTH 25 FOR
C FURTHER PROCESION
DO 10 I=1,25
CD(I)=AB(I)
10 CONTINUE
IF(DABS(CO(I))<LT.1.D-30) CD(I)=0.D0
CONTINUE
IF(CD(25).GT.ERR) WRITE(6,101)
N1=R/Z+1
DO 20 I=N1,25
IF(CD(I).EQ.0.D0) GO TO 25
N=I
20 CONTINUE
C NORMALISING THE CODES. PRESENTLY IN NEUTRAL NORM
SUM=0.D0
IF(NORM.EQ.1) GO TO 14C
C GETTING STRATTON NORMALISATION FACTOR
IF(QD.LT.0.D0) GO TO 91
GO TO (40,40,60,80),CASE
40 DO 50 J=1,N
SUM=SUM+CD(J)
50 CONTINUE
GO TO 100
60 DO 70 J=1,N
SUM=SUM+CD(J)*DBLE(FLOAT(2^*(J-1)))
70 CONTINUE
GO TO 100
80 DO 90 J=1,N
SUM=SUM+CD(J)*DBLE(FLOAT(2*J-1))
90 CONTINUE
GO TO 100
C GOT NEGATIVE Q STRATTON NORMALISATION FACTOR IS DIFFERENT
T1=-1.0D0
IF(MOD(R/Z,2).EQ.1)T1=-T1
GO TO (92,92,94,96),CASE
92 DO 93 J=1,N
T1=-T1
SUM=SUM+T1*CD(J)
93 CONTINUE
GO TO 100
94 DO 95 J=1,N
T1=-T1
SUM=SUM+CD(J)*T1*DBLE(FLOAT(2^*(J-1)))
95 CONTINUE
GO TO 100
96 DO 97 J=1,N
T1=-T1
SUM=SUM+CD(J)*T1*DBLE(FLOAT(2*J-1))
97 CONTINUE
100 IF(NORM.EQ.2) GO TO 12C
C GETTING INCE'S NORMALISATION FACTOR
SUM1=0.D0
DO 10 J=1,N
SUM1=SUM1+CD(J)*CD(J)
10 CONTINUE
IF(FNC.EQ.2.AND.MOD(R+2).EQ.C) SUM1=SUM1+CD(1)*CD(1)
SUM1=DSQRT(SUM1)
IF(NORM.EQ.3) SUM=DSIGN(SUM1,SUM)
DIVIDE ALL COEFS. BY NORMALISING FACTOR FOR 2 & 3 ONLY

DO 130 I=1,N
CD(I)=CD(I)/SUM
130 CONTINUE

FOR MATHIEU FUNCTIONS OF SE2N+2 TYPE(CASE=3) COEFS. SHOULD
BE 2, 8, ETC. BUT THE ROUTINE COEF RETURNS A BO=O ALSO.

THIS IS TO BE DROPPED.

DO 140 I=2,N
CD(I-1)=CD(I)
140 CONTINUE
CD(N)=0.DO
RETURN

SUBROUTINE COEF(M)
INTEGER K,KA,KB,KK,M,MF,ML,MM,M0,M1,M2S,TYPE
DOUBLE PRECISION A,AB,FL,G,H(200),Q,T,TOL,V,V2
COMMON GIZOC,2ItDUM,IBOO,AT,TT,KT,KAtKE,KK,HH,ML,AB(200)
COMMON /MFI/QpTOL,TYPE,M1,M2,M2S,MF
EQUIVALENCE IH(I)pGI,(I))
DATA FL,V2/1.O_30tl.D'15/
STATEMENT FUNCTION
V(K)=(A-DBLE(FLOAT(K))**2)/Q
CALL TMOPA(A,T,T,M)
IF(M.EQ.0) GO TO 300
DO 60 K=1,200
  AB(K)=0.DO
60 CONTINUE
KA=M1-MO+2
DO 90 K=2,KA,2
  KK=(M1-K)/2+1
  IF(K-2).GT.70,70,90
  AB(KK)=1.DO
  GO TO 90
70 AB(KK)=AB(KK+1)/G(KK+1,1)
90 CONTINUE
KA=0
DO 130 K=M1,M2S,2
  KK=K/2+1
  ML=K
  IF(G(KK,2),EQ,FL) GO TO 1CC
  AB(KK)=AB(KK-1)*G(KK,2)
  GO TO 110
1CC T=AB(KK-2)
  IF(K,EQ.4,AND,M1.EQ.2) T=T+T
  AB(KK)=T/(V(K-2)*H(KK-1,1,DC)
110 IF(IDABS(AB(KK)),GE.1,D-17) KA=CA
IF(KA.EQ.3) GO TO 260
KA=KA+1
CONTINUE
T=DBLQ(DBS(AH(KK))/VZ1/DBLQ1.DO/DBS(G(KK,2)))
KA=2*IDINT(T)
ML=KA+2+M2S
IF(ML.GT.399) GO TO 400
KB=KA+2+MF
T=1.DO/V(K8)
KK=MF-M2S
DO 150 K=2,KK,2
T=1.DO/(V(KB-K)-T)
150 CONTINUE
KK=ML/2+1
G(KK,2)=T
DO 200 K=2,KK,2
KK=(ML-K)/2+1
G(KK,2)=1.DO/(V(ML-K)-G(KK+1,2))
200 CONTINUE
KA=M2S+2
DO 250 K=KA,ML,2
KK=K/2+1
AB(KK)=AB(KK-1)*G(KK,2)
250 CONTINUE
C NEUTRAL NORMALIZATION
260 T=AB(1)
MM=MOD(TYPE,2)
KA=MM+2
DO 280 K=KA,ML,2
KK=K/2+1
IF(DBS(T)-DBS(AB(KK))) 270,280,280
270 T=AB(KK)
MM=K
280 CONTINUE
DO 290 K=1,KK
AB(K)=AB(K)/T
290 CONTINUE
300 RETURN
400 M=-1
GO TO 300
END
DOUBLE PRECISION FUNCTION ANGMFC(QD,FNC,R,XD,DERIV,AR,N)
C PURPOSE: TO COMPUTE A PERIODIC MATHIEU FUNCTION,
C ODD OR EVEN TYPE OR ITS DERIVATIVE
DOUBLE PRECISION PC,PS,DPC,DPS
EXTERNAL PC,PS,DPC,DPS
DIMENSION AR(25),AB(25)
INTEGER FNC,R,DERIV,TYPE,CASE,P
DOUBLE PRECISION AR,AB,KD,X,T1,SUM,DCOS,DSIN,QD
COMMON/NTERM/NLIMI
COMMON/ANG/AB,X,P
NLIMIT=N
Q=QD
X=10
TYPE=2*MOD(FNC,2)+MOD(R,2)
CASE=TYPE+1
DO 1 I=1,N
AB(I)=AR(S)
CONTINUE
C FOR NEGATIVE Q IN ALL SUMMATIONS ALTERNATE TERMS HAVE
C A MINUS SIGN
IF(QS)=90,35
20 T1=-1.000
IF(CASE.EQ.3)T1=1.000
IF(MOD(R/2,2).EQ.1)T1=-T1
DO 30 I=1,N
T1=-T1
AB(I)=T1*AB(I)
30 CONTINUE
35 P=-1
IF(CASE.EQ.1)P=-2
IF(CASE.EQ.3)P=0
IF(DERIV.EQ.1)GO TO 60
GO TO(40,4C,50,5C),CASE
40 CALL SIGMA(PC,SUM)
ANGMFC=SUM
RETURN
50 CALL SIGMA(PS,SUM)
ANGMFC=SUM
RETURN
60 GO TO(70,70,8C,8C),CASE
70 CALL SIGMA(DPC,SUM)
ANGMFC=SUM
RETURN
80 CALL SIGMA(DPS,SUM)
ANGMFC=SUM
RETURN
90 IF(DERIV.EQ.1)GO TO 120
GO TO(120,100,110,11C),CASE
100 ANGMFC=DCOS(DBLE(FLOAT(R)))*X
RETURN
110 ANGMFC=DSIN(DBLE(FLOAT(R)))*X
RETURN
120 GO TO(130,130,140,140),CASE
130 ANGMFC=-R*DSIN(DBLE(FLOAT(R)))*X
RETURN
140 ANGMFC=R*DCOS(DBLE(FLOAT(R)))*X
RETURN
END
DOUBLE PRECISION FUNCTION PC(K)
INTEGER P,K
DOUBLE PRECISION AB(25),X,DCOS
COMMON/ANG/AB,X,P
C EVALUATES ONE TERM OF THE EVEN PERIODIC SOLUTION
PC = AB(K) * DCOS(DBLE(FLOAT(2*K+P))*X)
RETURN
END

DOUBLE PRECISION FUNCTION PS(K)
INTEGER P, K
DOUBLE PRECISION AB(25), X, DSIN
COMMON/ANG/AB*X,P
C EVALUATES ONE TERM OF THE ODD PERIODIC SOLUTION
PS = AB(K) * DSIN(DBLE(FLOAT(2*K+P))*X)
RETURN
END

DOUBLE PRECISION FUNCTION DPC(K)
INTEGER P, K
DOUBLE PRECISION AB(25), X, T, DSIN
COMMON/ANG/AB*X,P
C EVALUATES ONE TERM OF THE DERIVATIVE OF THE EVEN PERIODIC
MATHIEU FUNCTION.
T = 2*K+P
DPC = -AB(K) * T * DSIN(T*X)
RETURN
END

DOUBLE PRECISION FUNCTION DPS(K)
INTEGER P, K
DOUBLE PRECISION AB(25), X, T, DCOS
COMMON/ANG/AB*X,P
C EVALUATES ONE TERM OF THE DERIVATIVE OF THE ODD PERIODIC
MATHIEU FUNCTION.
T = 2*K+P
DPS = AB(K) * T * DCOS(T*X)
RETURN
END

SUBROUTINE SIGMA(DUM, SUM)
PURPOSE: TO SUM N TERMS (SPECIFIED BY THE COMMON
COMMON BLOCK N TERM) OF A FUNCTION.
DOUBLE PRECISION DUM, SUM, ERR, T1, TERM, DABS
COMMON/NTERM/NLIMIT

1C1 FORMAT(*C*, *CONVERGENCE NOT TO SATISFACTION*, 10X, *WARNING*)
ERR = 1.0D-13
T1 = DUM(1)
SUM = T1
N = NLIMIT
IF(NLIMIT.GE.22) N = 22
IMIN = 5
DO 10 I = 2, N
TERM = DUM(I)
SUM = SUM + TERM
TERM = DABS(TERM)
IF(I.GT.IMIN) GO TO 1C
IF(DABS(SUM).GE.ERR*TERM) RETURN
CONTINUE
1C CONTINUE
12 IF(I.EQ.22) WRITE(6,101)
SUBROUTINE FACTOR(FNC,R,QO,AB,N,PS)
PURPOSE: TO COMPUTE THE NORMALIZATION FACTOR, P2N, P2N+1, S2N+1, OR S2N+2 FOR MODIFIED MATHIEU FUNCTIONS OF POSITIVE 'Q' AND P2N, P2N+1, S2N+1, OR S2N+2 FOR THOSE OF NEGATIVE 'Q'.

DIMENSION A9(25)
INTEGER FNC,R CASE
DOUBLE PRECISION AB,PS,PSP,ABS,DSQRT,RQ,ODEV,
SUM1,SUM2,T1,T2,QO
RQ=DSQRT(DABS(QO))
CASE=2*MOD(FNC,2)+MOD(R,2)+1
IF(QD.LT.0.0D0)CASE=CASE+1.
ODEV=1.0D0
IF(MOD(R/2,2).EQ.1)ODEV=-1.0D0
SUM1=0.0D0
SUM2=0.0D0
T1=1.0D0
GO TO (10,30,50,70,10,70,50,30),CASE
FOR ALL Q AND EVEN ORDER ---- P2N AND P2N+1 ---- FNC=2

10 DO 20 I=1,N
SUM1=SUM1+AB(I)
T1=-T1
SUM2=SUM2+T1*AB(I)
20 CONTINUE
PS=SUM1/SUM2/AB(1)
PSP=PS*ODEV
IF(QD.LT.0.0D0)PS=PSP
RETURN
FOR POSITIVE Q AND ODD ORDERS P2N+1, P2N+1 IF FNC=2
NEG Q AND ODD ORDER IF FNC=1

30 DO 40 I=1,N
SUM1=SUM1+AB(I)
SUM2=SUM2+T1*AB(I)*DBLE(FLOAT(2*I-1))
40 CONTINUE
PS=SUM1/SUM2/(RQ*AB(1))
PSP=PS*ODEV
IF(QD.LT.0.0D0)PS=PSP
RETURN
FOR ALL Q AND EVEN ORDER IF FNC=1 ---- S2N+2, S2N+2

50 T1=1.0D0
DO 60 I=1,N
T2=AB(I)*DBLE(FLOAT(2*I))
SUM1=SUM1+T2
T1=-T1
SUM2=SUM2+T1*T2
60 CONTINUE
PS=SUM1/SUM2/(RQ*RQ*AB(1))
PSP=PS*ODEV
SUBROUTINE STDREIQ(D, X, NMAX)

PURPOSE: TO COMPUTE AND STORE THE VALUES OF BESSEL FUNCTIONS AND DERIVATIVES REQUIRED IN MATHIEU FUNCTION CALCULATION.

DOUBLE PRECISION D, Q, ABS, R, A, B, SQRT, EXP, BSIV1, BSIV2, BSZZV2, BSIVZ, BSIV2, BSZV2, X, V, VB
COMMON/LOCAL/DUMMY(1), V, VB, DUMMY2(50)
COMMON/RADIAL/BSIV1(25), BSIV2(25), BSZV2(25), BSIV1(25),
* BSIV2(25), BSZV(25)

N = NMAX + 3
IF(N .GE. 25) N = 25
N1 = N - 1
ABS = ABS(D)
R = SQRT(ABS)
V1 = R * EXP(-X)
V2 = ABS / V1
IF(D .LT. 0.0D0) GO TO 20
CALL BESSEL(1, V1, BSIV1, N)
CALL BESSEL(1, V2, BSIV2, N)
CALL BESSEL(2, V2, BSIV2, N)
BSIV1 = BSIV1
BSIV2 = BSIV2
DO 10 I = 2, N1
BSIV1(I) = BSIV1(I - 1) - BSIV1(I + 1) * 0.5D0
BSIV2(I) = BSIV2(I - 1) - BSIV2(I + 1) * 0.5D0
BSZV2(I) = BSZV2(I - 1) - BSZV2(I + 1) * 0.5D0
10 CONTINUE
RETURN

20 CALL BSL2(1, V1, BSIV1, N)
CALL BSL2(1, V2, BSIV2, N)
CALL BSL2(2, V2, BSIV2, N)
BSIV1 = BSIV1
BSIV2 = BSIV2
DO 30 I = 2, N1
BSIV1(I) = BSIV1(I - 1) + BSIV1(I + 1) * 0.5D0
30 CONTINUE
RETURN
SUBROUTINE BESSEL(SOL,U,BSJY,N)
INTEGER N,N,N,SOL
DOUBLE PRECISION BSJY(N),U
N=N-1
IF(U.EQ.0.DO.AND.SOL.EQ.2) GO TO 80
IF(U.GE.8.DO) GO TO 30
GO TO(10,20),SOL
CALL JOJ1(U,BSJY)
GO TO 40
CALL YOYII(U,BSJY)
GO TO 40
CALL LUKE(U,SOL,BSJY)
GO TO 100
IF(N.LT.2) GO TO 100
GO TO(50,60),SOL
CALL JNS(BSJY,U,N)
GO TO 100
CONTINUE FORMULAR
DO 70 K=2,NN
BSJY(K+1)=2.DO*DBL(FLOAT(K-1))*BSJY(K)/U-BSJY(K-1)
70 CONTINUE
GO TO 100
NN=NN+1
DO 90 K=1,NN
BSJY(K)=-1.DO*37
90 CONTINUE
RETURN
END
SUBROUTINE JOJ1(X,BJ)
DOUBLE PRECISION BJ(2),T(5),X
T(1)=X/2.DO
BJ(1)=1.DO
BJ(2)=T(1)
T(2)=-T(1)**2
T(3)=1.DO
T(4)=1.DO
T(5)=T(4)**2+T(2)**2
BJ(1)=BJ(1)+T(4)
T(5)=T(4)**2+T(1)**2
BJ(2)=BJ(2)+T(5)
IF(DMAX1(ABS(T(4)),ABS(T(5)))).LT.1.DO-15) RETURN
T(3)=T(3)+1.DO
GO TO 10
END
SUBROUTINE YCY1(X,BY)
DOUBLE PRECISION T(10),X,BY(2)
T(1)=X/2.DO
CONTINUE
RETURN
END
SUBROUTINE LUKE(U,KIND,BSJY)
INTEGER KIND
DOUBLE PRECISION A(19),B(19),CS,C(19),D(19),G(3),BSJY(2),
C     WARNING - THE FOLLOWING DATA STATEMENTS ARE NOT IN ASA
C     STANDARD FORTRAN
DATA A/-.99959506476967287416D0,
    -.53807956139606913D-3,
    -.13179677123361576D-3,
    .151422497486444D-5,
    .15846861792063D-6,
    -856269559346D-8,
    -29572343355D-9,
    .6573556254D-10,
    -223749703D-11,
    -4482114CD-12,
    .6954827D-13,
    -151340D-14,
    -924220D-15,
    .15558D-15,
    -476D-17,
    -274D-17,
    .61D-18,
    -4D-19,
    -1D-19/;
DATA B/-.7769355569420532136D-2,
* -7748032309654476730-2,
* 25365411654307960-4,
* 394273598399711D-5,
* 10723498299129D-6,
* 721389799328D-8,
* 73764602893D-9,
* 150687811D-11,
* 574589537D-11,
* 45996574D-12,
* 2270323D-13,
* 88789D-14,
* 74497D-15,
* 5847D-16,
* 241D-16,
* 265D-17,
* 13D-18,
* 10D-18,
* 2D-19/
DATA C/1.00067753586591346234D0,
* 90100725195908183D-3,
* 22172634918599454D-3,
* 196575946319104D-5,
* 2088953114327D-6,
* 10281443508940D-7,
* 37597C54789D-9,
* 7638891358D-10,
* 238734670D-11,
* 51825489D-12,
* 7693969D-13,
* 114008D-14,
* 103294D-14,
* 168210D-15,
* 459D-17,
* 302D-17,
* 65D-18,
* 4D-19,
* 1D-19/
DATA D/233768299862858032D-1,
* 233468012235455533D-1,
* 3576010596901382D-4,
* 560863149492627D-5,
* 1327389408434D0-6,
* 916975845066D-8,
* 86838880371D-9,
* 378073035D-11,
* 663145586D-11,
* 50584390D-12,
* 2720782D-13,
* 985381D-14,
* 79398D-15,
* 6757D-16,
SUBROUTINE BSL2(SOL,U,BSIK,N)
  PURPOSE: TO COMPUTE MODIFIED BESSEL FUNCTIONS, "I" OR "K" TYPE FOR ORDERS 0 THROUGH N-1 IN DOUBLE PRECISION.

INTEGER N,SOL
DOUBLE PRECISION BSIK(N),U
IF(SOL.EQ.2)GO TO 30
IF(U.GE.8.*D0) GO TO 10
CALL IO11(U,BSIK)
GO TO 20
CALL LUKE2(U,SOL,BSIK)
IF(N.LT.2)RETURN
RETURN
IF(U.EQ.0.*D0) GO TO 70
IF(U.GE.5.*D0) GO TO 40
CALL KOK1(U,BSIK)
GO TO 50
CALL LUKE2(U,SOL,BSIK)
IF(N.LT.2)RETURN
RETURN
RECURRANCE FORMULA
NN=N-1
DO 60 K=2,NN
BSIK(K+1)=2.*D0*DOUBLE(FLOAT(K-1))*BSIK(K)/U+BSIK(K-1)
60 CONTINUE
RETURN
DO 80 K=1,N
BSIK(K)=1.*CD+75
80 CONTINUE
RETURN
END

SUBROUTINE IO11(X,BI)
  PURPOSE: TO EVALUATE "IC" AND "I1" BESSEL FUNCTIONS
  BY SUMMING THE SERIES.

DOUBLE PRECISION BI(2),T(5),X,DMAX1,DA3S
SUBROUTINE KO1(X, BK)

PURPOSE: TO EVALUATE *K0* AND *K1* BESSEL FUNCTIONS

DOUBLE PRECISION T(10), X, BK(2), DMAX1, DABS, DLOG

T(1) = X/2.000
T(2) = T(1)**2
T(3) = 1.000
T(4) = 1.000
T(4) = T(4)*T(2)/T(3)**2
B1(1) = B1(1) + T(4)
T(5) = T(4)*T(1)/(T(3) + 1.000)
B1(2) = B1(2) + T(5)
IF(DMAX1(DABS(T(4)), DABS(T(5))).LT.1.0D-15) RETURN
T(3) = T(3) + 1.000
GO TO 10

END

SUBROUTINE KOK1(X, BK)

PURPOSE: TO EVALUATE *KO* AND *K1* BESSEL FUNCTIONS

DOUBLE PRECISION T(10), X, BK(2), DMAX1, DABS, DLOG

T(1) = X/2.000
T(2) = T(1)**2
BK(1) = 1.000
BK(2) = T(1)
T(7) = 0.000
T(10) = -T(1)
T(3) = 0.000
T(4) = 0.000
T(5) = 1.000
T(3) = T(3) + 1.000
T(4) = T(4) + 1.000/T(3)
T(5) = T(5) - T(2)/T(3)**2
BK(1) = BK(1) + T(5)
T(6) = T(5)*T(4)
T(7) = T(7) + T(6)
T(8) = T(5)*T(10)/(T(3) + 1.000)
BK(2) = BK(2) + T(8)
T(9) = T(8) - 2.000*T(4) + 1.000/(T(3) + 1.000)
T(10) = T(10) + T(9)
IF(DMAX1(DABS(T(6)), DABS(T(9))).LT.1.0D-15) GO TO 10
T(2) = 577215664901532900 + DLOG(T(11))
BK(1) = BK(1) + T(2) + T(7)
BK(2) = BK(2)*T(2) + 1.000/X + T(11)/2.000
RETURN

END

SUBROUTINE LUKE2(U, KIND, SIK)

PURPOSE: TO EVALUATE MODIFIED BESSEL FUNCTIONS, J0 AND J1

DOUBLE PRECISION A(34), B(21), C(34), D(21), U, SIK(2), X, G(34),
R(2), S(2), DEXP, DSQRT

DATA A/1.0082792054587400, 8.4451226249209430-2,
3.1727006307775650-3, 7.247591099958960-5, 5.1358772687820-6,
5.6816965809120-7, 9.513091222850-8, 1.2384253640-8,
\[ 10 \]
\[ 15 \]
\[ Z_\]
DO 25 K=3,N
G(K)=((4.0D0*K-2.0D0)*G(K-1)-G(K-2))
25 CONTINUE
S(1)=0.0D0
S(2)=0.0D0
DO 30 K=1,N
I=N+1-K
S(1)=S(1)+B(I)*G(I)
S(2)=S(2)+D(I)*G(I)
30 CONTINUE
BSIK(1)=1.253314137315500*S(1)*DEXP(-U)/DSQRT(U)
BSIK(2)=1.253314137315500*S(2)*DEXP(-U)/DSQRT(U)
RETURN
END

SUBROUTINE INS(I,U,M)

PURPOSE= TO EVALUATE \*I* BESSEL FUNCTIONS OF HIGHER ORDERS BY A CONTINUED FRACTION EXPANSION

INTEGER K,KA,KK,M
DOUBLE PRECISION A,G,DZ,DG25,II(M),P(3),Q(3),U

IF(U.EQ.0.0D0) GO TO 50
DM=2*M
P(1)=2.0D0
Q(1)=1.0D0
P(2)=1.0D0
Q(2)=DM/U
D(1)=P(2)/Q(2)
A=2.0D0

10 B=(DM+A)/U
P(3)=3*P(2)+P(1)
Q(3)=8*Q(2)+Q(1)
D(2)=P(3)/Q(3)
IF(DABS(D(1)-D(2)).LT.1.0D-15) GO TO 20
P(1)=P(2)
P(2)=P(3)
Q(1)=Q(2)
Q(2)=Q(3)
D(1)=D(2)
A=A*2.0D0
GO TO 10

20 G(M)=D(2)
KA=M-2
DO 30 K=1,KA
KK=M-K
A=2*KK
G(KK)=U/(A+U*G(KK+1))
IF(DABS(G(KK)).LE.1.0D-35) G(KK)=1.0D-35
30 CONTINUE
DO 40 K=2,M
IF(DABS(II(K)).LT.1.0D-35) GO TO 35
II(K+1)=G(K)*II(K)
40 CONTINUE
GO TO 40
35
II(K+1)=0.000
40
CONTINUE
RETURN
50
DO 60 I=3,M
60
II(I)=0.000
RETURN
END

DOUBLE PRECISION FUNCTION CERAD(QD,R,DERIV,PS,AR,N)

PURPOSE: TO COMPUTE A MODIFIED MATHIEU FUNCTION

EXTERNAL C2NP,C2NP,C2NP,C2NP,C2NP,C2NP,C2NP,C2NP,C2NP

DOUBLE PRECISION AB(25),QD,PS,PSP,OUTPUT,AR(25)

INTEGER R,CASE,DERIV,FNC

COMMON/NTERM/N1

COMMON/LOCAL/DUMMY(8),AB

N1=N

PSP=PS

DO 5 I=1,N

AB(I)=AR(I)

5 CONTINUE

CASE=MOD(R,2)+1

IF(QD.LT.0.0D0) CASE=CASE+2

IF(DERIV.EQ.1) GO TO 50

GO TO(10,20,30,40),CASE

THE VALUE OF CE2N(Z,Q)

10 CALL SIGMA(C2NP,OUTPUT)

CERAD=PS*OUTPUT/AB(1)

RETURN

THE VALUE OF CE2N+1(Z,Q)

20 CALL SIGMA(C2NP,OUTPUT)

CERAD=PS*OUTPUT/AB(1)

RETURN

THE VALUE OF CE2N(Z,-Q)

30 CALL SIGMA(C2NP,OUTPUT)

CERAD=PSP*OUTPUT/AB(1)

RETURN

THE VALUE OF CE2N+1(Z,-Q)

40 CALL SIGMA(C2NP,OUTPUT)

CERAD=PSP*OUTPUT/AB(1)

RETURN

FOLLOWING ARE DERIVATIVES OF FUNCTIONS

50 GO TO(60,70,80,90),CASE

THE VALUE OF CE2N'(Z,Q)

60 CALL SIGMA(DC2NP,OUTPUT)

CERAD=PS*OUTPUT/AB(1)

RETURN

THE VALUE OF CE2N+1'(Z,Q)

70 CALL SIGMA(DC2NP,OUTPUT)

CERAD=PS*OUTPUT/AB(1)
RETURN
THE VALUE OF CE2N*(ZtQ)
CALL SIGMA(DC2NN,OUTPUT)
CERAD=PSP*OUTPUT/AB(1)
RETURN
THE VALUE OF CE2N+1*(ZtQ)
CALL SIGMA(DC2NN,OUTPUT)
CERAD=PSP*OUTPUT/AB(1)
RETURN
END
DOUBLE PRECISION FUNCTION C2NP(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CE2N(ZtQ).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,BSJV1,BSJV2, DBSYV2
COMMON/LOCAL/DUMMY(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),BSJV1(25), DBSJV2(25),DBSYV2(25)
C2NP=AB(K)*BSJV1(K)*BSJV2(K)
IF(MOD(K,2).EQ.0)C2NP=-C2NP
RETURN
END
DOUBLE PRECISION FUNCTION C2NIP(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CE2N+1(ZtQ).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,BSJV1,BSJV2, DBSYV2
COMMON/LOCAL/DUMMY(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),BSJV1(25), DBSJV2(25),DBSYV2(25)
C2NIP=AB(K)*BSJV1(K)*BSJV2(K+1)+BSJV1(K+1)*BSJV2(K)
IF(MOD(K,2).EQ.0)C2NIP=-C2NIP
RETURN
END
DOUBLE PRECISION FUNCTION C2NN(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CE2N(ZtQ).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSIV2,BSIV1,BSIV2, DBSKV2
COMMON/LOCAL/DUMMY(8),AB
C2NN=AB(K)*BSIV1(K)*BSIV2(K)
IF(MOD(K,2).EQ.0)C2NN=-C2NN
RETURN
END
DOUBLE PRECISION FUNCTION C2NIN(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES FOR CE2N+1(ZtQ)
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSIV2,BSIV1,BSIV2, DBSKV2
COMMON/LOCAL/DUMMY(8),AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),3SKV2(25),BSIV1(25), DBSIV2(25),DBSKV2(25),
DOUBLE PRECISION FUNCTION DCZNP(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CEZ\(k(Z,\Omega)\).

DOUBLE PRECISION AB(25), BSV1, BSV2, BSJV1, BSJV2, DSV1, DSV2,
DBSVV, V1, V2
COMMON/LOCAL/DUMMY(4), V1, V2, AB
COMMON/RADIAL/BSVV(25), BSJV(25), DSVV(25), DSV(25)
* DCZNP=AB(K)*(-DBSVV(K)*BSJVV(K)*V1+BSJVV(K)*DSVVV(K)*V2)
IF(MOD(K, 2).EQ.0)DCZNP=-DCZNP
RETURN
END

DOUBLE PRECISION FUNCTION DCZNP(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CEZ\(k+1(Z,\Omega)\).

DOUBLE PRECISION AB(25), BSV1, BSV2, BSJV1, BSJV2, DSV1, DSV2,
DBSVV, V1, V2
COMMON/LOCAL/DUMMY(4), V1, V2, AB
COMMON/RADIAL/BSVV(25), BSJV(25), DSVV(25), DSV(25)
* DCZNP=AB(K)*(-DBSVV(K)*BSJVV(K+1)*V1+BSJVV(K)*DSVVV(K+1)*V2)
RETURN
END

DOUBLE PRECISION FUNCTION DCZNN(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR CEZ\(k(Z,\Omega)\).

DOUBLE PRECISION AB(25), BSIV1, BSIV2, BSKV2, DBSV1, DBSV2,
BSIVV, V1, V2
COMMON/LOCAL/DUMMY(4), V1, V2, AB
COMMON/RADIAL/BSIV1(25), BSKV2(25), BSIV2(25), DBSV1(25),
DBSV2(25), DSKV(25)
* DCZNN=AB(K)*(-DBSV1(K)*BSIVV(K)*V1+BSIVV(K)*DBSVV(K)*V2)
IF(MOD(K, 2).EQ.0)DCZNN=-DCZNN
RETURN
END
DOUBLE PRECISION FUNCTION SERAD(QD, R, DERIV, PS, AR, N)

PURPOSE: TO COMPUTE A MODIFIED MATTHIEU FUNCTION (OR DERIVATIVE) OF FIRST KIND CORRESPONDING TO ODD MATTHIEU FUNCTION (SE FUNCTIONS)

EXTERNAL SZN2P, SZN1P, SZN2N, SZN1N, DSZN2P, DSZN1P, DSZN2N, DSZN1N

DOUBLE PRECISION AB(ZSNI, QD, AR, OUTPUT)
INTEGER R, CASE, DERIV

COMMON/NTERM/NI
COMMON/LOCAL/DUMMY(8), AB

N1=N

DO 5 I=1, N
AB(I)=AR(I)

5 CONTINUE
CASE=MOD(R, 2)+1
IF(QD.LT.0.000) CASE=CASE+2
IF(DERIV.EQ.1) GO TO 50
GO TO(10, 20, 30, 40), CASE
10 CALL SIGMA(SZN2P, OUTPUT)
SERAO=-PS*OUTPUT/AB(I)
RETURN

20 CALL SIGMA(SZN1P, OUTPUT)
SERAD=PS*OUTPUT/AB(I)
RETURN

30 CALL SIGMA(SZN2N, OUTPUT)
SERAD=PS*OUTPUT/AB(I)
RETURN

40 CALL SIGMA(SZN1N, OUTPUT)
SERAD=PS*OUTPUT/AB(I)
RETURN

FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50 GO TO(60, 70, 80, 90), CASE
60 CALL SIGMA(DSZN2P, OUTPUT)
SERAO=-PS*OUTPUT/AB(I)
RETURN

70 CALL SIGMA(DSZN1P, OUTPUT)
SERAD=PS*OUTPUT/AB(I)
RETURN
THE VALUE OF SE2N+2(Z,-Q)
CALL SIGMA(DS2N2N,OUTPUT)
SERAD=PSP*OUTPUT/AB(1)
RETURN
THE VALUE OF SE2N+1(Z,-Q)
CALL SIGMA(DS2N1N,OUTPUT)
SERAD=PSP*OUTPUT/AB(1)
RETURN
END
DOUBLE PRECISION FUNCTION S2N2P(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+2(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DSJ1V1,DSJ2V2,
* COMMON/LOCAL/DUMMY1(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DSJ1V1(25),
* DSJ2V2(25),DSYV2(25)
S2N2P=AB(K)*BSJV1(K)*BSJV2(K+2)-BSJV1(K+2)*BSJV2(K)
IF(MOD(K,2).EQ.0)S2N2P=-S2N2P
RETURN
END
DOUBLE PRECISION FUNCTION S2N1P(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+1(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DSJ1V1,DSJ2V2,
DSYV2
* COMMON/LOCAL/DUMMY1(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DSJ1V1(25),
DSJ2V2(25),DSYV2(25)
S2N1P=AB(K)*BSJV1(K)*BSJV2(K+1)-BSJV1(K+1)*BSJV2(K)
IF(MOD(K,2).EQ.0)S2N1P=-S2N1P
RETURN
END
DOUBLE PRECISION FUNCTION S2N2N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+2(Z,-Q).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DSIV1,DSIV2,
* DBSKV2
COMMON/LOCAL/DUMMY1(8),AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DSIV1(25),
DSIV2(25),DBSKV2(25)
S2N2N=AB(K)*BSIV1(K)*BSIV2(K+2)-BSIV1(K+2)*BSIV2(K)
IF(MOD(K,2).EQ.0)S2N2N=-S2N2N
RETURN
END
DOUBLE PRECISION FUNCTION S2N1N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+1(Z,-Q).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DSIV1,DSIV2,
* DBSKV2
COMMON/LOCAL/DUMMY1(8),AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKYV2(25),DBSIV1(25),

DBSIV2(25),DBSKYV2(25)
S2N1N=AB(K)*BSIV1(K)*BSIV2(K+1)-BSIV1(K+1)*BSIV2(K)
IF(MOD(K,2).EQ.0)S2N1N=-S2N1N
RETURN
END
DOUBLE PRECISION FUNCTION DS2N2P(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+2'(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DBSJVI,DSBJV2,

* DBSYV2,V1,V2
COMMON/LOCAL/DUMMY1(4),V1,V2,AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJVI(25),

* DBSJV2(25),DBSYV2(25)
DS2N2P=AB(K)*(-DBSJVI(K)*BSJV2(K+2)*V1+

* BSJV1(K)*DBSJV2(K+2)*V2+
* DBSJV1(K+2)*BSJV2(K)*V1-BSJV1(K+2)*DBSJV2(K)*V2)
IF(MOD(K,2).EQ.0)DS2N2P=-DS2N2P
RETURN
END
DOUBLE PRECISION FUNCTION DS2N1P(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+1'(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DSBJV1,DSBJV2,

* DBSYV2,V1,V2
COMMON/LOCAL/DUMMY1(4),V1,V2,AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DBSJVI(25),

* DBSJV2(25),DBSYV2(25)
DS2N1P=AB(K)*(-DBSJVI(K)*BSJV2(K+1)*V1+

* BSJV1(K)*DBSJV2(K+1)*V2+
* DBSJV1(K+1)*BSJV2(K)*V1-BSJV1(K+1)*DBSJV2(K)*V2)
IF(MOD(K,2).EQ.0)DS2N1P=-DS2N1P
RETURN
END
DOUBLE PRECISION FUNCTION DS2N2N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+(Z,Q).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKYV2,DSBSIV1,DSBSIV2,

* DBSKYV2,V1,V2
COMMON/LOCAL/DUMMY1(4),V1,V2,AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKYV2(25),DBSIV1(25),

* DBSIV2(25),DBSKYV2(25)
DS2N2N=AB(K)*(-DBSIV1(K)*BSIV2(K+2)*V1+

* BSIV1(K)*DBSIV2(K+2)*V2+
* DBSIV1(K+2)*BSIV2(K)*V1-BSIV1(K+2)*DBSIV2(K)*V2)
IF(MOD(K,2).EQ.0)DS2N2N=-DS2N2N
RETURN
END
DOUBLE PRECISION FUNCTION DS2N1N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR SE2N+1'(Z,Q).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DS2N1N,AB(25),BSIV1,BSIV2,
* DBSKV2,BSIV1,BSIV2
COMMON/LOCAL/DUMMY1(4),V1,V2,AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
* DBSIV2(25),DBSKV2(25)
DS2N1N=AB(K)*(-BSIV1(K)*BSIV2(K+1)*V1+
* BSIV1(K)*BSIV2(K+1)*V2+
* DBSIV1(K+1)*BSIV2(K)*V1-BSIV1(K+1)*DBSIV2(K)*V2)
IF(MOD(K,2).EQ.O)DS2N1N=-DS2N1N
RETURN
END

DOUBLE PRECISION FUNCTION FERAD(QD,R,DERIV,PS,AR,N)
PURPOSE: TO COMPUTE A MODIFIED MATHIEU FUNCTION
(OR DERIVATIVE) OF SECOND KIND CORRESPONDING TO
EVEN MATHIEU FUNCTION (CE FUNCTIONS)
EXTERNAL FY2N,FK2N,FY2N1,FK2N1,DFY2N,DFK2N,DFY2N1,DFK2N1
DOUBLE PRECISION AB(25),QD,PS,PS,PSP,OUTPUT,AR(25),PI
INTEGER R,CASE,DERIV
COMMON/NTERM/NI
COMMON/LOCAL/DUMMY(8),AB
DATA PI/3.141592653589793D0/
NI=N
PSP=PS
DO 1 I=1,N
AB(I)=AR(I)
CONTINUE
CASE=MOD(R+2.1+1)
IF(QD.LT.0.000) CASE=CASE+2
IF(DERIV.EQ.1) GO TO 50
GO TO(10,20,30,40),CASE
10 CALL SIGMA(FY2N,OUTPUT)
FERAD=PS*OUTPUT/AB(1)
RETURN
C THE VALUE OF FEY2N(Z,Q)
20 CALL SIGMA(FY2N1,OUTPUT)
FERAD=PS*OUTPUT/AB(1)
RETURN
C THE VALUE OF FEY2N+1(Z,Q)
30 CALL SIGMA(FK2N,OUTPUT)
FERAD=PS*OUTPUT/(AB(1)+PI)
RETURN
C THE VALUE OF FEK2N(Z,-Q)
40 CALL SIGMA(FK2N1,OUTPUT)
FERAD=PS*OUTPUT/(AB(1)+PI)
RETURN
C FOLLOWING ARE DERIVATIVES OF FUNCTIONS
50 GO TO(60,70,80,90),CASE
C THE VALUE OF FEY2N'(Z,Q)
60 CALL SIGMA(DFY2N,OUTPUT)
FERAD=PS*OUTPUT/AB(1)
RETURN
THE VALUE OF FEY2N+1(Z,Q)
CALL SIGMA(DFY2N1,OUTPUT)
FERAD=PS*OUTPUT/AB(1)
RETURN
THE VALUE OF FEK2N*(Z,-Q)
CALL SIGMA(DFK2N,OUTPUT)
FERAD=PS*OUTPUT/(AB(1)*PI)
RETURN
THE VALUE OF FEK2N+1*(Z,-Q)
CALL SIGMA(DFK2N,OUTPUT)
FERAD=PS*OUTPUT/(AB(1)*PI)
RETURN
END
DOUBLE PRECISION FUNCTION FYZN(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FEY2N(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DSBJV1,DSBJV2,
* DBSYV2
COMMON/LOCAL/DUMMYY1(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DSBJV1(25),
* DBSJ V2(25),DBSYV2(25)
FYZN=AB(K)*BSJV1(K)*BSYV2(K)
IF(MOD(K,2).EQ.0) FYZN=-FYZN
RETURN
END
DOUBLE PRECISION FUNCTION FYZN1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FEY2N+1(Z,Q).
DOUBLE PRECISION AB(25),BSJV1,BSJV2,BSYV2,DSBJV1,DSBJV2,
* DBSYV2
COMMON/LOCAL/DUMMYY1(8),AB
COMMON/RADIAL/BSJV1(25),BSJV2(25),BSYV2(25),DSBJV1(25),
* DBS JV2(25),DBSYV2(25)
FYZN1=AB(K)*BSJV1(K)*BSYV2(K+1)*BSJV1(K+1)*BSYV2(K)
IF(MOD(K,2).EQ.0) FYZN1=-FYZN1
RETURN
END
DOUBLE PRECISION FUNCTION FKZN(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FEK2N(Z,Q).
DOUBLE PRECISION AB(25),BSIV1,BSIV2,BSKV2,DSIV1,DSIV2,
* DBSVK2
COMMON/LOCAL/DUMMYY1(8),AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DSIV1(25),
* DBSIV2(25),DBSKV2(25)
FKZN=AB(K)*BSIV1(K)*BSKV2(K)
RETURN
END
DOUBLE PRECISION FUNCTION FKZNI(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FEK2N+1(Z,Q).
DO DOUBLE PRECISION AB(25), BSV1, BSV2, BSKV2, DBSIV1, DBSIV2,
* DBSKV2
COMMON/LOCAL/DUMMY1(8), AB
COMMON/RADIAL/BSV1(25), BSV2(25), DBSIV1(25),
* DBSIV2(25), DSKV2(25)
FK2N1=AB(K) *(BSV1(K) * BSV2(K+1)-BSV1(K+1) * BSKV2(K))
RETURN
END
DOUBLE PRECISION FUNCTION DFY2N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FEY2N*(Z,Q).
DOUBLE PRECISION AB(25), BSJV1, BSJV2, BSJV2, DBSJV1, DBSJV2,
DBSV2, V1,*
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSJV1(25), BSJV2(25), DBSJV1(25),
DBSV2(25), DBSV2(25)
DFY2N=AB(K) *(DBSV1(K) * BSJV2(K) * V1+BSJV1(K) * DSVV2(K) * V2)
RETURN
END
DOUBLE PRECISION FUNCTION DFK2N(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FSK2N*(Z,Q).
DOUBLE PRECISION AB(25), BSV1, BSV2, BSKV2, DBSIV1, DBSIV2,
DBSKV2, V1,*
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSV1(25), BSV2(25), DBSIV1(25),
DBSKV2(25), DBSKV2(25)
DFK2N=AB(K) *(DBSV1(K) * BSV2(K) * V1+BSV1(K) * DSVK2(K) * V2)
RETURN
END
DOUBLE PRECISION FUNCTION DFK2N1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FSK2N+1*(Z,Q).
DOUBLE PRECISION AB(25), BSV1, BSV2, BSKV2, DBSIV1, DBSIV2,
DBSKV2, V1,*
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSV1(25), BSV2(25), DBSIV1(25),
DBSKV2(25), DBSKV2(25)
DFK2N1=AB(K) *(DBSV1(K) * BSV2(K) * V1+BSV1(K) * DSVK2(K) * V2)
RETURN
END
DOUBLE PRECISION FUNCTION DFK2N1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR FSK2N+1*(Z,Q).
DOUBLE PRECISION AB(25), BSV1, BSV2, BSKV2, DBSIV1, DBSIV2,
DBSKV2, V1,*
DOUBLE PRECISION FUNCTION GERAD(QD, R, DERIV, PS, AR, N)

PURPOSE: TO COMPUTE A MODIFIED MATHIEU FUNCTION (OR DERIVATIVE) OF SECOND KIND CORRESPONDING TO ODD MATHIEU FUNCTION (CE FUNCTIONS)

EXTERNAL GY2N2, GY2N1, G2N2, G2N1, DG2N2, DG2N1, DGK2N1

DOUBLE PRECISION AB(25), QD, PS, PSP, OUTPUT, AR(25), PI

INTEGER R, CASE, DERIV

COMMON/LOCAL/DUMMY(8), AB

DATA PI/3.141592653589793DC/

N1=N

PSP=PS

DO 5 I=1,N

5 CONTINUE

CASE=MOD(R,2)+1

IF(QD.LT.0.0001) CASE=CASE+2

IF(DERIV.EQ.1) GO TO 50

GO TO(10,20,30,40), CASE

10 THE VALUE OF GY2N+2(Z,Q)

CALL SIGMA(GY2N2, OUTPUT)

GERAD=-PS*OUTPUT/AB(I)

RETURN

20 THE VALUE OF GY2N+1(Z,Q)

CALL SIGMA(GY2N1, OUTPUT)

GERAD=PS*OUTPUT/AB(I)

RETURN

30 THE VALUE OF GEK2N+2(Z,-Q)

CALL SIGMA(GK2N2, OUTPUT)

GERAD=PSP*OUTPUT/(AB(I)^PI)

RETURN

40 THE VALUE OF GEK2N+1(Z,-Q)

CALL SIGMA(GK2N1, OUTPUT)

GERAD=PSP*OUTPUT/(AB(I)^PI)

RETURN

50 FOLLOWING ARE DERIVATIVES OF FUNCTIONS

GO TO(60,70,80,90), CASE

60 THE VALUE OF GY2N+2*(Z,Q)

CALL SIGMA(DG2N2, OUTPUT)

GERAD=-PS*OUTPUT/AB(I)

RETURN

70 THE VALUE OF GY2N+1*(Z,Q)

CALL SIGMA(DG2N1, OUTPUT)
DOUBLE PRECISION FUNCTION GYZN2(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEYZN=E(Z+Q).

DOUBLE PRECISION AB(25), BSJV1, BSJV2, BSYV2, DBSVJ1, DBSVJ2,
DBSYV2
* COMMON/LOCAL/DUMMY1(8), AB
* COMMON/RADIAL/BSJV1(25), BSJV2(25), BSYV2(25), DBSVJ1(25),
DBSYV2(25)
* GYZN2=AB(K)*BSJV1(K)*BSYV2(K+2)-BSJV1(K+2)*BSYV2(K)
IF(MOD(K,2).EQ.0)GYNZ2=-GYNZ2
RETURN
END

DOUBLE PRECISION FUNCTION GYZN1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEYZN+1(Z+Q).

DOUBLE PRECISION AB(25), BSJV1, BSJV2, BSYV2, DBSVJ1, DBSVJ2,
DBSYV2
* COMMON/LOCAL/DUMMY1(8), AB
* COMMON/RADIAL/BSJV1(25), BSJV2(25), BSYV2(25), DBSVJ1(25),
DBSYV2(25)
* GYZN1=AB(K)*BSJV1(K+1)*BSYV2(K+1)-BSJV1(K+1)*BSYV2(K)
IF(MOD(K,2).EQ.0)GYNZ1=-GYNZ1
RETURN
END

DOUBLE PRECISION FUNCTION GKZN2(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEKZN+2(Z-Q).

DOUBLE PRECISION AB(25), BSIV1, BSIV2, BSKV2, DBSIV1, DBSIV2,
DBSKV2
* COMMON/LOCAL/DUMMY1(8), AB
* COMMON/RADIAL/BSIV1(25), BSIV2(25), BSKV2(25), DBSIV1(25),
DBSKV2(25)
* GKZN2=AB(K)*BSIV1(K)*BSKV2(K+2)-BSIV1(K+2)*BSKV2(K)
RETURN
END

DOUBLE PRECISION FUNCTION GKZN1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEKZN+1(Z-Q).

DOUBLE PRECISION AB(25), BSIV1, BSIV2, BSKV2, DBSIV1, DBSIV2,
* DBSKV2
COMMON/LOCAL/DUMMY1(8), AB
COMMON/RADIAL/BSIV1(25), BSIV2(25), BSKV2(25), DBSIV1(25), DBSIV2(25),
* DBSKV2(25)
GK2N1=AB(K)*(BSIV1(K)*BSKV2(K+1)*BSIV1(K+1)*BSKV2(K))
RETURN
END

DOUBLE PRECISION FUNCTION DGY2N2(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEY2N+Z*(Z, Q).
DOUBLE PRECISION AB(25), BSJV1, BSJV2, BSJV1, DBSJV1, DBSJV2,
* DBSJV2, V1, V2
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSJV1(25), BSJV2(25), BSJV2(25), DBSJV1(25),
* DBSJV2(25), DBSJV2(25)
DGY2N2=AB(K)*(BSJV1(K)*BSJV2(K+2)*V1+*BSJV1(K)*DBSJV2(K+2)*V2+
* BSJV1(K)*BSJV2(K+2)*V1-BSJV1(K+2)*DBSJV2(K)*V2)
IF(MOD(K+2).EQ.0)DGY2N2=-DGY2N2
RETURN
END

DOUBLE PRECISION FUNCTION DGY2N1(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEY2N+Z*(Z, Q).
DOUBLE PRECISION AB(25), BSJV1, BSJV2, BSJV1, DBSJV1, DBSJV2,
* DBSJV2, V1, V2
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSJV1(25), BSJV2(25), BSJV2(25), DBSJV1(25),
* DBSJV2(25), DBSJV2(25)
DGY2N1=AB(K)*(BSJV1(K)*BSJV2(K+1)*V1+*BSJV1(K)*DBSJV2(K+1)*V2+
* BSJV1(K+1)*BSJV2(K)*V1-BSJV1(K+1)*DBSJV2(K)*V2)
IF(MOD(K+2).EQ.0)DGY2N1=-DGY2N1
RETURN
END

DOUBLE PRECISION FUNCTION DGKZN2(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEKZN+2*(Z, Q).
DOUBLE PRECISION AB(25), BSIV1, BSIV2, BSKV2, DBSIV1, DBSIV2,
* DBSKV2, V1, V2
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSIV1(25), BSIV2(25), BSKV2(25), DBSIV1(25),
* DBSIV2(25), DBSKV2(25)
DGKZN2=AB(K)*(DBSIV1(K)*BSKV2(K+2)*V1+*BSIV1(K)*DBSKV2(K+2)*V2+
* BSIV1(K+2)*BSKV2(K)*V1-BSIV1(K+2)*DBSKV2(K)*V2)
RETURN
END

DOUBLE PRECISION FUNCTION DGKZNI(K)
PURPOSE: TO CALCULATE K TH TERM IN SUM OF SERIES
FOR GEKZN+I*(Z, Q).
DOUBLE PRECISION AB(25), BSIV1, BSIV2, BSKV2, DBSIV1, DBSIV2,
* DBSKV2, V1, V2
COMMON/LOCAL/DUMMY1(4), V1, V2, AB
COMMON/RADIAL/BSIV1(25), BSIV2(25), BSKV2(25), DBSIV1(25),
* DBSIV2(25), DBSKV2(25)
DGKZNI=AB(K)*(DBSIV1(K)*BSKV2(K+1)*V1+*BSIV1(K)*DBSKV2(K+1)*V2+
* BSIV1(K+1)*BSKV2(K)*V1-BSIV1(K+1)*DBSKV2(K)*V2)
RETURN
END
* DBSKV2,V1,V2
COMMON/LOCAL/DUMMY1(4),V1,V2,AB
COMMON/RADIAL/BSIV1(25),BSIV2(25),BSKV2(25),DBSIV1(25),
   DBSIV2(25),DBSKV2(25)
* DGK2N1=AB(K)*(-BSIV1(K)*BSKV2(K+1)*V1+
   BSIV1(K)*DBSKV2(K+1)*V2-
* DBSIV1(K+1)*BSKV2(K)*V1+BSIV1(K+1)*DBSKV2(K)*V2
RETURN
END
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The exact characteristic equation for an anisotropic elliptic optical fiber is obtained for odd and even hybrid modes in terms of infinite determinants utilizing Mathieu and modified Mathieu functions. A simplified characteristic equation is obtained by applying the weakly guiding approximation such that the difference in the refractive indices of the core and the cladding is small.

The simplified characteristic equation is used to compute the normalized guide wavelength for an elliptical fiber. When the anisotropic parameter is equal to unity, the results are compared with the previous research and they are in close agreement.

For a fixed value of normalized cross-section area or major axis, the normalized guide wavelength $\lambda/\lambda_0$ for an anisotropic elliptic fiber is small for larger the value of anisotropy. This condition indicates that more energy is carried inside of the fiber. However, the geometry and anisotropy of the fiber have a smaller effect when the normalized cross-section area is very small or very large.