Turbulent Fluid Motion II —
Scalars, Vectors, and Tensors

Robert G. Deissler
Lewis Research Center
Cleveland, Ohio

July 1991
Turbulent Fluid Motion II -

Scalars, Vectors, and Tensors

Robert G. Deissler
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

Summary

Some of the mathematical apparatus used for the representation and study of turbulence is developed.

2.1 Introduction

In the study of turbulence one encounters physical quantities such as densities, temperatures, forces, velocities, velocity products, and stresses. We notice that these quantities are not all of the same type. For instance a density requires only one number to represent it, but a velocity requires three numbers, or it has three components. A quantity which is the product of two velocities, where each component of one velocity is multiplied in turn by each component of the other, has nine components.

A density is an example of a scalar, or of a tensor of order zero. It can be represented by a symbol with no subscripts, say $\rho$. A velocity is an example of a vector and is called a tensor of order one. It can be represented by a symbol with one subscript, say $u_i$, where $i = 1, 2, 3$. The three values of $i$ correspond to directions in space parallel to the directions of the three perpendicular coordinate axes $x_i$ ($i = 1, 2, 3$). The $x_i$ form a right-handed coordinate system, and are often written as $(x, y, z)$. A quantity which is the product of two velocities is an example of a second-order tensor. It can be represented by a symbol with two subscripts, say $T_{ij} = u_i u_j$, where $i = 1, 2, 3; j = 1, 2, 3$. Similarly, products of more than two velocities are tensors of higher order. Thus $u_i u_j u_k$ ($i, j, k = 1, 2, 3$) is a third order tensor, etc. Averaged values of velocity products, written as $\bar{u}_i u_j, \bar{u}_i u_j u_k$ etc. and called velocity correlations (the overbars indicate averaged values), are important quantities in the theory of turbulence.

2.2 Rotation of Coordinate Systems

It should not be assumed that all quantities represented by symbols with a given number of subscripts (0, 1, 2, . . . subscripts) are tensors as in the examples of the last paragraph (see e.g., section 2.8). To be called a tensor, it is necessary that a quantity obey a certain transformation law when referred to a rotated coordinate system. In considering the transformation laws of tensors, we first note that a rectangular coordinate system transforms under a rotation according to the law

---

\[ T_{ij}^{'} = \sum_{k=1}^{3} \frac{\partial x_j}{\partial x_k} \frac{\partial x_i}{\partial x_k} T_{ij} \]

We shall consider here only rectangular Cartesian coordinate systems. Tensors defined in terms of the transformation laws of such coordinate systems are called Cartesian tensors. When we use the term "tensor" in this series "Cartesian tensor" will be understood.
\[ x^*_i = a_{ij} x^*_j, \quad (2-1) \]

where \( x^*_i \) is a coordinate of a point in the rotated coordinate system \( x^*_i \), and \( x^*_i \) is a coordinate of the same point in the unrotated system \( x_i \). (Note that \( x^*_i \) or \( x^*_i \) can designate either a coordinate system or a coordinate, since a coordinate system is given by its coordinates.) The \( a_{ij} \) form a set of nine constants. The conditions under which equation (2-1) makes sense as a transformation law will be considered later in this section.

Equation (2-1) uses the Einstein summation convention, where a repeated subscript in a term designates a sum of terms, with the subscript successively taking on the values 1, 2, and 3. This convention will be used throughout the text, except where otherwise indicated. Note that the symbol used for a repeated subscript is immaterial, so that such a subscript is often called a dummy subscript. The expression for \( x^*_i \), written out, is

\[ x^*_i = a_{1i} x_1 + a_{2i} x_2 + a_{3i} x_3. \]

The symbols used for the subscripts which are unrepeated in a term of an equation (see, e.g., equation (2-1)) are also immaterial, so long as the same symbols are used in all terms, and so long as they differ from those used for other subscripts in the equation. Thus in substituting one equation into another, the symbols used for some of the subscripts must frequently be changed in order to avoid confusion. If a subscript occurs more than twice in a term the equation is generally ambiguous. Also, the same unrepeated subscripts must occur in all terms of an equation. These points, while possibly obvious once they have been mentioned, are important for carrying out meaningful tensor manipulations.

The square of the distance \( ds \) between two neighboring points is given in the \( x_i \)-coordinate system by

\[ ds^2 = dx_i dx_i = \delta_{ij} dx_i dx_j \quad (2-2) \]

where \( \delta_{ij} \) is called the Kronecker delta, defined by

\[
\begin{align*}
\delta_{ij} &= 1 \text{ for } i = j, \\
\delta_{ij} &= 0 \text{ for } i \neq j.
\end{align*}
\]

The truth of equation (2-2) can be seen by writing out the three terms of \( dx_i dx_j \) (summation on \( i \)) and the nine terms of \( \delta_{ij} dx_i dx_j \) (summation on \( i \) and \( j \)) and using (2-3). Note that \( \delta_{ij} = \delta_{ji} \). In order that the transformation (2-1) make physical or geometrical sense, it is necessary that the distance \( ds \) be the same in the \( x^*_i \)- and the \( x_i \)-coordinate systems. Thus

\[ ds^2 = dx_i^* dx_i^* = a_{ij} a_{lk} dx_j dx_k \quad (2-4) \]

where equation (2-1) in differential form \( (dx_i^* = a_{ij} dx_j) \) is applied twice, and equation (2-2) is used. Equation (2-4) gives
(a_{ij}a_{ik} - \delta_{jk})dx_j dx_k = 0

for all values of the $dx_1$, or

\[ a_{ij}a_{ik} = \delta_{jk}. \]  

(2-5)

Equation (2-5) gives the nine relations (six of which are different) which must be satisfied by the $a_{ij}$ if equation (2-1) is to be a sensible transformation law (lengths remain invariant under the transformation). Multiplication of equation (2-1) by $a_{ik}$ and use of equation (2-5) and (2-3) give

\[ a_{ik}x^*_i = a_{ik}a_{ij}x^*_j = \delta_{jk}x^*_j = x^*_k, \]

(2-6)

where the last step can be verified by writing out the terms and using equation (2-3). In general, multiplication of a quantity containing say a subscript $j$ or a subscript $k$ (or both) by $\delta_{jk}$ changes the subscript $j$ to $k$, or $k$ to $j$, in that quantity.

Considering only the first and last terms of equation (2-6), and changing the subscript $k$ to $j$ on both sides of the equation, gives

\[ x^*_j = a_{ij}x^*_i. \]

(2-7)

If we differentiate equation (2-1) with respect to $x_j$ and equation (2-7) with respect to $x^*_i$, we get

\[ \frac{\partial x^*_i}{\partial x_j} = \frac{\partial x^*_j}{\partial x^*_i} = a_{ij}. \]

(2-8)

A set of equations equivalent to (2-5), but of slightly different form, can be obtained as follows:

\[ ds^2 = dx_1 dx_i = a_{ij}dx^*_j a_{k1} dx^*_k = dx^*_1 dx^*_i = \delta_{jk} dx^*_j dx^*_k \]

where equation (2-7) in differential form is used. Therefore

\[ a_{ji}a_{k1} = \delta_{jk}. \]

(2-5a)

### 2.3 VECTORS (FIRST-ORDER TENSORS)

A quantity $u_i$ is said to be a vector, or a first-order tensor, if it obeys the transformation law

\[ u^*_i = a_{ij}u^*_j, \]

(2-9)

where $u^*_i$ is a component of a vector in the rotated coordinate system $x^*_i$ and $u^*_i$ is a component of the same vector in the unrotated system $x_i$. (Note that $u^*_i$ (or $u_i^*$) can designate either a vector component or the vector itself, since a vector can be specified by specifying its components.)

We note by comparing equations (2-1) and (2-9) that a vector transforms according to the same law as the coordinates of a point. Thus the coordinates of a point form a vector. They define a directed-line segment drawn from the origin to the
point $x_i$. That vector is usually called a position or displacement vector. Similarly any vector defined by equation (2-9) can be interpreted as a directed-line segment. So the definition given by equation (2-9) agrees with the perhaps more familiar definition of a vector as a directed-line segment, or as a quantity with both magnitude and direction (displacement, velocity, area, force, etc.).

It is easy to show that the sum or difference of two vectors, say $u_i$ and $v_i$, is a vector. For, from equation (2-9),

$$u_i + v_i = a_{ij}u_j + a_{ij}v_j = a_{ij}(u_j + v_j) \tag{2-10}$$

which shows that $u_i + v_i$ obeys the transformation law for a vector. Equation (2-10) is another way of stating the familiar addition law for vectors represented by directed-line segments; instead of adding directed-line segments geometrically, we add corresponding components in either the $x_i$ or $x^*$ coordinate system.

### 2.4 SECOND-ORDER TENSORS

#### 2.4.1 Definition and Simple Examples

A second-order tensor $u_{ij}$ is defined by generalization of equation (2-9) as a quantity that obeys the transformation law

$$u_{ij}^* = a_{ik}a_{lj}u_{kl}^* \tag{2-11}$$

where, as usual, repeated subscripts (in this case, $k$ and $l$) indicate summations. Thus equation (2-11) represents nine equations, each with nine terms on the right side. Tensor notation affords, among other things, considerable economy in writing.

From the definition given by equation (2-11) it follows that the product of two vectors $u_i$ and $v_j$ is a second-order tensor. For, from equation (2-9),

$$u_i^*v_j^* = a_{ik}a_{lj}u_{kl}v_{kl}^* \tag{2-12}$$

which shows that $u_i^*v_j^*$ obeys the transformation law for a second-order tensor (equation (2-11)). The product $u_i^*v_j^*$, where the subscript on one vector is not repeated on the other one, is called an outer product.

Another example of a second-order tensor is the spatial derivative, or gradient, of a vector $\partial u_j/\partial x_j$. We can show that $\partial u_i/\partial x_j^*$ is a second-order tensor as follows: Following the rules of partial differentiation, we obtain

$$\frac{\partial u_i^*}{\partial x_j^*} = \frac{\partial u_i}{\partial x_j^*} \frac{\partial x_k}{\partial x_j^*} = \frac{\partial x_k}{\partial x_j^*} \frac{\partial x_k}{\partial x_j^*} (a_{ij}u_{ij}) \tag{2-13}$$

where $u_{ij}^*$ is assumed to be a function of $x_k^*$, and equation (2-9) is used. Using equation (2-8) for $\partial x_k/\partial x_j^*$ we get

$$\frac{\partial u_i^*}{\partial x_j^*} = a_{ik}a_{jl} \frac{\partial x_k}{\partial x_j^*} \tag{2-14}$$
which shows that $\partial u_1/\partial x_j$ obeys the transformation law (2-11) and is thus a second-order tensor.

2.4.2 Stress and the Quotient Law

Still another second-order tensor is the stress $\sigma_{ij}$ defined by

$$\sigma_{ij} = \frac{\Delta A_{i}^{*} \Delta F_{j}}{\Delta A_{1}}$$

where $\Delta F_{j}$ is the force component in the $x_j$-direction acting on the small area element $\Delta A_1$, whose normal is in the $x_1$-direction.

To show that $\sigma_{ij}$ is a tensor, first write a sum of forces of the type $\Delta F_{j}$:

$$\sigma_{ij} \Delta A_{1} = \Delta F_{j}$$

A product of two tensors such as that in equation (2-16), where a subscript is repeated, is called an inner product. But the area $\Delta A_{1}$ (where we designate the area by its components) is a vector, since it can be represented by a directed-line segment normal to the plane of the area (it has magnitude and direction). Similarly the force $\Delta F_{j}$ has magnitude and direction and is thus a vector. The quantity $\Delta F_{j}$ is also a vector, since it is a sum of vectors of the type $\Delta F_{j}$ (equation (2-10)).

Next write equation (2-16) in the transformed coordinate system $x'$:

$$\sigma_{ij}^{*} \Delta A_{i}^{*} = \Delta F_{j}^{*}$$

Since $\Delta A_{1}$ and $\Delta F_{j}$ are vectors, we have, according to equation (2-9),

$$\sigma_{ij}^{*} a_{ik} \Delta A_{k} = \sigma_{jk}^{*} \Delta F_{k}$$

where equation (2-16) is used in the next to last term, and the dummy subscripts $i$ and $k$ are interchanged in the last term. Then, from the first and last terms,

$$\Delta A_{k} (\sigma_{ij}^{*} a_{ik} - \sigma_{ij}^{*} a_{ik}) = 0.$$  

Since equation (2-18) holds for all values of $\Delta A_{k}$,

$$a_{ik} \sigma_{ij}^{*} = a_{ji} \sigma_{k1}^{*}.$$  

To get this equation into the form of equation (2-11), multiply it by $a_{\ell k}$, or

$$a_{ik} a_{\ell k} \sigma_{ij}^{*} = a_{ji} a_{\ell k} \sigma_{k1}^{*}.$$  

Finally, using equation (2-5a),

$$\delta_{ij} \sigma_{ij}^{*} = a_{ji} a_{\ell k} \sigma_{k1}$$
or
Comparing equation (2-20) with (2-11) shows that $\sigma_{ij}$ is a second-order tensor.

We have shown, starting from equation (2-16), that if inner multiplication of a quantity $\sigma_{ij}$ by a vector with arbitrary components gives another vector, then $\sigma_{ij}$ is a second-order tensor. This is one form of the quotient law. Once it has been established, as has been done here, it provides in some cases a simple test for determining whether a quantity is a tensor. Further discussion of the quotient law will be given in section 2.5.2.

2.4.3 The Kronecker Delta, a Tensor

In the foregoing paragraphs we showed that a product of two velocities (or vectors), a gradient of a velocity, and a stress, while representing different physical entities, are all alike in that they are second-order tensors. Next we ask whether the Kronecker delta $\delta_{ij}$ is a second-order tensor. According to the definition in equation (2-3) the components of $\delta_{ij}$ do not depend on the orientation of coordinate axes. Thus

$$\delta_{ij} = \delta_{ij}.$$

Using equations (2-5a) and (2-3),

$$\delta_{ij} = \delta_{ij} = \delta_{i}^{i} \delta_{j}^{j} = \delta_{i j}.$$

Comparing the first and last members of equations (2-21) with (2-11) shows that $\delta_{ij}$ is a second-order tensor.

The Kronecker delta $\delta_{ij}$ is an example of an isotropic tensor. That is, its components remain invariant with rotation of coordinate axes. An isotropic tensor is sometimes called a numerical tensor, since its components have the same numerical values for all rotations of the coordinate axes.

We now show that the most general second-order isotropic tensor is $I\delta_{ij}$, or that any second-order isotropic (numerical) tensor can be written as $I\delta_{ij}$, where $I$ is a scalar. The transformation law for a second-order tensor is given by equation (2-11). Let $I^{*}_{ij}$ be any (the most general) second-order isotropic tensor, so that $I^{*}_{ij} = I_{ij}$. Then equation (2-11) becomes

$$I^{*}_{ij} = I_{ij} = \delta_{ik}a_{jk}I_{kl}.$$

Multiplying the last two members of this equation by $a_{jm}$ and using equation (2-5) give

$$a_{jm}I^{*}_{ij} = a_{jk}a_{jm}I_{kl} = a_{ik}\delta_{ml}I_{kl} = a_{ik}I_{km}.$$

The first and last members of this equation give
\[ \delta_{km} a_{jk} I_{ij} = \delta_{ij} a_{jk} I_{km}, \]

or

\[ a_{jk} (\delta_{km} I_{ij} - \delta_{ij} I_{km}) = 0. \]

Since the relation for \( I_{ij} \) cannot depend on the \( a_{kj} \) (on the orientation of the coordinate axes), the quantity in parentheses is zero, and we get, after contracting the indices \( k \) and \( m \) (setting \( m = k \)),

\[ I_{ij} = (I_{kk}/3) \delta_{ij}, \]

where \( I_{kk} \) is a scalar (see section 2.6). Any value of \( I_{kk} \) satisfies this equation, as can be seen by contracting the indices \( i \) and \( j \), so that

\[ I_{ij} = I \delta_{ij}. \]

That is, any (the most general) second-order isotropic tensor can be written as \( I \delta_{ij} \), where \( I \) is an arbitrary scalar.

### 2.5 THIRD- and HIGHER-ORDER TENSORS

The generalization of equation (2-11) to tensors of higher order is obvious. For instance a third-order tensor \( u_{ijk} \) is defined by

\[ u_{ijk} = a_{ij} a_{jm} a_{kn} u_{mn}, \tag{2-22} \]

which represents 27 equations, each with 27 terms on the right side. An example of a third-order tensor is the product of three velocities \( u_i u_j u_k \). The product of four velocities forms a fourth-order tensor \( u_i u_j u_k u_l \), etc.

#### 2.5.1 Vorticity and the Alternating Tensor

An important third-order tensor is the alternating tensor \( \epsilon_{ijk} \), where

\[ \epsilon_{ijk} = \delta_{1i} \delta_{2j} \delta_{3k} + \delta_{12} \delta_{j3} \delta_{k1} + \delta_{13} \delta_{j1} \delta_{k2} - \delta_{i1} \delta_{2j} \delta_{3k} - \delta_{i2} \delta_{j1} \delta_{3k} - \delta_{i3} \delta_{j2} \delta_{3k}. \tag{2-23} \]

(We call \( \epsilon_{ijk} \) a tensor in anticipation of showing that it is such later in this section.) Evaluation of equation (2-23) shows that

\[
\begin{align*}
\epsilon_{ijk} &= 0 \text{ if two subscripts are equal,} \\
\epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = 1, \\
\text{and} \quad \epsilon_{132} &= \epsilon_{321} = \epsilon_{213} = -1. 
\end{align*}
\tag{2-23a}
\]

\(^2\)If, however, \( I_{ij} \) were not an isotropic tensor, then \( a_{jk} (\delta_{km} I_{ij} - \delta_{ij} I_{km}) = 0 \), and we could not set the quantity in parentheses equal to zero; the relation between \( I_{ij} \) and \( I_{km} \) would not be independent of the \( a_{kj} \).
The alternating tensor \( \varepsilon_{ijk} \) is usually defined by equation (2-23a), but equation (2-23) is more convenient for our purposes (and may generally be preferable).

We define a quantity \( \omega_i \) as follows:

\[
\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j},
\]

where \( u_k \) is a vector and \( x_j \) is a coordinate (also a vector). Equation (2-24) becomes, on using (2-23),

\[
\omega_i = (\delta_{i1}\delta_{j2}\delta_{k3} - \delta_{i1}\delta_{j3}\delta_{k2} + \delta_{i2}\delta_{j3}\delta_{k1} - \delta_{i2}\delta_{j1}\delta_{k3}) \frac{\partial u_k}{\partial x_j}
\]

\[
+ \delta_{i3}\delta_{j1}\delta_{k2} - \delta_{i3}\delta_{j2}\delta_{k1})(\frac{\partial u_k}{\partial x_j})
\]

\[
= \delta_{i1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \delta_{i2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \delta_{i3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).
\]

Therefore, the three components of the quantity \( \omega_i \) are

\[
\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \quad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \quad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.
\]

If \( u_i \) is the velocity at a point in a fluid, \( \omega_i \) is called the vorticity and is a measure of the local swirl or rotation. Equations (2-26) show that each component \( \omega_i \) has the magnitude of the rotation of the fluid in a \( x_j - x_k \) plane \((j, k \neq i)\) and is perpendicular to that plane. Thus the quantity \( \omega_i \) has both magnitude and direction and so is a vector. The vector \( \omega_i \) (or \( \omega \)) is also called the curl of \( u_i \) and, in vector notation, is written as \( \nabla \times u \). The quantity \( \frac{\partial u_k}{\partial x_j} \) in equation (2-24) is the gradient of a vector and, as shown in section 2.4.1, is a second-order tensor. Since \( \omega_i \) is a vector, \( \varepsilon_{ijk} \) is a third-order tensor by a slightly more general form of the quotient law than that in section 2.4.2. (See the next section.) By equation (2-23) it is isotropic (or numerical), since its components remain invariant with rotation of coordinate axes.

2.5.2 A More General Quotient Law

In order to prove the quotient law used in the last section, we note that equation (2-24) is of the form
\[ v_i = v_{ij} v_{kj} \]  \hspace{1cm} (2-27)

where \( v_i \) is a vector and \( v_{kj} \) is an arbitrary second-order tensor (its components can have arbitrary values). We want to prove that \( v_{ij} \) is a third-order tensor. In a rotated coordinate system equation (2-27) becomes

\[ v^* = v^*_{ij} v^*_{kj} \]  \hspace{1cm} (2-28)

or, since \( v_i \) and \( v_{kj} \) are tensors, we can write, using equations (2-9) and (2-11),

\[ a_{ij} v^*_l = v^*_{ijk} a_{km} v^*_m. \]

Substituting for \( v^* \) from equation (2-27),

\[ v^*_{ijk} a_{km} v^*_m = a_{ij} v^*_{ijk} v^*_{kj} = 0. \]  \hspace{1cm} (2-29)

We factor out \( v^* \) after making a change of dummy subscripts in the second term of equation (2-29). This gives

\[ (a_{km} v^*_{ijk} - a_{ln} v^*_{nm}) v^*_m = 0. \]

Since \( v^*_m \) is arbitrary, the quantity in parentheses is zero, or

\[ a_{km} v^*_{ijk} = a_{ln} v^*_{nm}. \]

To get this equation into the form of equation (2-22) we multiply it by \( a_{rl} a_{qm} \) or

\[ a_{km} a_{rl} a_{qm} v^*_{ijk} = a_{ln} a_{qm} a_{rl} v^*_{nm}. \]  \hspace{1cm} (2-30)

Finally, using equation (2-5a) on the left side of (2-30) gives

\[ \delta_{kr} \delta_{jq} v^*_{ijk} = a_{ln} a_{qm} a_{rl} v^*_{nm}. \]

or

\[ v^*_{1qr} = a_{ln} a_{qm} a_{rl} v^*_{nm}. \]  \hspace{1cm} (2-31)

Comparison of equation (2-31) with (2-22) shows that \( v^*_{ijk} \) is a third-order tensor. Thus if, in equation (2-27), \( v_i \) is a vector (first-order tensor) and \( v_{kj} \) is an arbitrary tensor, then \( v^*_{ijk} \) is a tensor. This proves a rather general form of the quotient law. As mentioned earlier, a product such as that in equation (2-27), in which a subscript or subscripts in one factor is repeated in the other one, is called an inner product. In general the quotient law states that a quantity is a tensor if an inner multiplication of that quantity with an arbitrary tensor (its components can have arbitrary values) is itself a tensor.

### 2.6 ZERO-ORDER TENSORS AND CONTRACTION

We notice in the definitions of first-, second-, and third-order tensors (equations (2-9), (2-11), and (2-22)), that the number of \( a_{ij} \)'s in the transformation law equals the order of the tensor. Also, of course, the number of subscripts...
on a tensor equals its order. Thus for a tensor of order zero or a scalar, say \( u \), we should have in place of equation (2-9) or (2-11),

\[
\mathbf{u}^* = \mathbf{u}.
\]  

(2-32)

So a zero-order tensor, or a scalar, has the same value in the coordinate system \( x_i^* \) as in \( x_i \). For that reason it is often called an invariant. Since \( u \) in equation (2-32) can be any unsubscripted quantity, we can say that any unsubscripted quantity is a tensor of order zero.

Multiplication of a tensor by a scalar gives a tensor of the same order. For instance, multiplication of equation (2-11) (for a second-order tensor) by a scalar \( u \) gives

\[
\mathbf{u}^* \mathbf{u}_{ij} = a_{ik} a_{jl} u_{k'l'}.
\]  

(2-33)

where equation (2-32) was used. Thus \( uu_{k'l'} \) transforms as a second-order tensor.

In the second-order tensor \( u_{ij} \), we can set \( j = i \). That process is called contraction. Then, according to equation (2-11), we have

\[
u^*_{ii} = a_{ik} a_{il} u_{k'l'} = \delta_{kk} u_{kk} = u_{ii}.
\]  

(2-34)

where equation (2-5) is used. Comparison of equation (2-34) with (2-32) shows that \( u_{ii}^* \) is a zero-order tensor, or a scalar. Thus contraction of the subscripts \( i \) and \( j \) lowered the order of the tensor by two. In general the process of contraction lowers the order of a tensor by two. As another example we contract the second-order tensor \( \partial u_i / \partial x_i \) to form the scalar \( \partial u / \partial x_i \), which is called the divergence of \( u_i \), and which, according to equations (2-14) and (2-5) is a scalar. In vector notation the divergence of \( u_i \) is written as \( \nabla \cdot u \). As a final example of contraction, contract the second-order tensor \( u_{ij} v_j \) to form \( u_{j} v_{j} \). The quantity \( u_{j} v_{j} \) is a scalar, since

\[
u^*_{j} v_{j} = a_{ik} a_{jl} u_{k'l'} = \delta_{kk} u_{kk} = u_{j} v_{j}.
\]  

(2-35)

It is called the dot or inner product of the vectors \( u \) and \( v \) and is often written as \( u \cdot v \).

We can show that the gradient of a scalar, say \( u \), is a vector. For, proceeding as in obtaining equations (2-13) and (2-14),

\[
\frac{\partial u^*}{\partial x_j^*} = \frac{\partial u}{\partial x_k} a_{jk} \frac{\partial x_k}{\partial x_j}^* = a_{jk} \frac{\partial u}{\partial x_k}.
\]  

(2-36)

which is the transformation law for a vector. In vector notation the gradient of \( u \) is written as \( \nabla u \).

### 2.7 OUTER AND INNER PRODUCTS OF TENSORS OF HIGHER ORDER

It is shown in sections 2.4.1 and 2.6 respectively that outer and inner products of vectors are tensors of some order. It is straightforward to show that outer and inner products of tensors of any order are tensors. For example the outer product \( u_{i} u_{jk} \) is a third order tensor, since
Also, the inner product \( u^* u_{jk} \) is a first-order tensor (vector), since, using equation (2-5),

\[
    u^* u_{jk} = a^* a_{jk} a_{mn} u^*_m u_{mn} = \delta_{mn} a_{kn} u^* u_{mn} = a_{kn} u^*_m u_{mn}.
\]

2.8 SUBSCRIPTED QUANTITIES THAT ARE NOT TENSORS

In order to give a better understanding of what tensors are, we give here some examples of quantities which, although subscripted, are not tensors. First consider the quantity \( a_{ij} \). Recall that the \( a_{ij} \) form a set of nine constants defined by equation (2-1). Assume first that \( a_{ij} \) is a second-order tensor. Then

\[
    a^*_{ij} = a_{ik} a_{jk} = a_{ik} \delta_{jk} = a_{ij}
\]

where equations (2-12), (2-1a), and (2-3) were used. Thus if \( a_{ij} \) is a tensor, it must be isotropic. But we showed in section 2.4.3 that the most general second-order isotropic tensor is \( I \delta_{ij} \), where \( I \) is a scalar. So \( a_{ij} \) is a tensor only if it is equal to \( I \delta_{ij} \). If \( a_{ij} = I \delta_{ij} \), equation (2-1) becomes

\[
    x^* = I \delta_{ij} x^* = I x^*.
\]

But in contrast to the statement of this equation the \( x^* \) are not proportional to the \( x^* \) for arbitrary rotations, so that \( a_{ij} \neq I \delta_{ij} \), and according to our argument \( a_{ij} \) is not a tensor.

We showed in the last paragraph that \( a_{ij} \) is not a tensor, since \( a_{ij} \neq I \delta_{ij} \). More generally we can say that any quantity \( h^*_{ij} \) whose components have the same numerical values in all rectangular coordinate systems is a nontensor if \( h^*_{ij} \neq I \delta_{ij} \), since \( I \delta_{ij} \) is the most general isotropic (numerical) tensor (section 2.4.3).

As another example consider a quantity \( w_i = u^* v_i \) (no sum on \( i \)), where \( u_i \) and \( v_i \) are both vectors. One might imagine that \( w^* = u^{(1)} v_i \) is also a vector (first order tensor), since it has one assignable subscript. But \( u^* = a_{ij} u_j \) and \( v^* = a_{ik} v_k \), so that

\[
    w^* = u^{(1)} v_i = a^{(1)} a_{ik} u_j v_k.
\]

If \( w_i = u^{(1)} v_i \) were a first-order tensor it would transform as \( w^* = a_{ij} w^* = a_{ij} u^* v^* \), which is considerably different from equation (2-39). Since both of these expressions cannot be true, \( w_i = u^{(1)} v_i \) is not a tensor. Similarly quantities such as \( w^*_{ij} = u^{(1)} u^*_{ij} \) and \( w^*_{ijk} = u^{(1)} u^*_{ijk} \) are not tensors. Note that all of these quantities are products which are neither inner nor outer.
2.9 CLOSING REMARKS

It has been shown (eq. (2-10)) that the sum or difference of two vectors is a vector. Similarly the sum of any two tensors of the same order is a tensor of that order. No meaning is attached to the sum of tensors of different orders, say \( u_1 + u_{ij} \); that is not a tensor.

In general, an equation containing tensors has meaning only if all the terms in the equation are tensors of the same order, and if the same unrepeated subscripts appear in all the terms. These facts will be used in obtaining appropriate equations for fluid turbulence.

This explanation of Cartesian tensors should contain what is needed for our purposes. It is hoped that it has been reasonably clear. Other treatments of tensors are given, for instance, in books by Jeffreys, Spain, Arfken, Lass, Langlois, and Goodbody (refs. 1 to 6).

With the foregoing background the derivation of appropriate continuum equations for turbulence should be straightforward. Before deriving them, however, a justification for calling the fluid a continuum for the study of turbulence will be given.

REFERENCES


### Abstract

Some of the mathematical apparatus used for the representation and study of turbulence is developed.