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On the Optimal Use of Fictitious Time in Variation of Parameters Methods with Application to BG14

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SUMMARY

There are a number of variation of parameter methods found in the literature that use fictitious time, s, defined by the Sundmann transformation $r = dt/ds$ as independent variable. The most notable being KS, Kustaanheimo and Stiefel [1],[2], Burdet [3], BG14, Bond and Gottlieb [4], and BG14δ, Bond[12]. There are two problems that arise when fictitious time is used as independent variable. The first is that stopping on time becomes an iterative process, and the second, more serious, problem is that, without care, the methods mentioned above may cease to work altogether for certain classes of problems. In this paper a simple derivation of several variation of parameters methods using universal variables is given. A schematic interpretation of variation of parameters is given which points out the cause of the failure of previous methods. It is then shown how the independent variable (ficitious time) and other constants may be chosen so that stability and accuracy may be maintained even for those classes of problems that caused previous methods to fail. Augmenting the energy variable all the way to the Jacobi constant as was done in [4] is postponed until the end of the paper because it was felt that it would distract the reader from the issue of interest, namely, the optimal use of fictitious time. In addition, it is shown how to handle multiple vehicles and constant time steps. The use of constant time steps in conjunction with BG14 was found to be as accurate as fictitious time alone for near circular orbits. Non circular orbits were not tested.

INTRODUCTION

This work was prompted by the discovery of two classes of problems that caused the variation of parameters methods KS, and BG14 to fail, in the sense that the computer programs would either stop producing output, or would produce erroneous output. The two classes of problems were long term low thrust to escape, and reentry to the ground. It was found that restarting these problems at intermediate points caused them to proceed to a solution. This indicated that the difficulty in getting them to work without restarting was in some sense related to either the size of the increasing independent variable (secular terms), or the fixing of the origin, or both. Bond [12] suspected that the elemination of secular terms in the original formulation [4] would effect a cure and proposed BG14δ. Although it is stated in [12] that the removal of secular terms cures the instability problem, further experimentation has shown that, although they run a little further, the long term low thrust to escape, and reentry to the ground problems cause BG14δ to fail just like the others. This meant that restarting was the answer, but a total restart would mean a loss of accuracy. This is true because redundant variables are carried, and a total restart would mean that some of them would have to be ignored. Therefore, this paper examines how to restart in such a way as to lose no information. That is to say full use is made of the redundant variables, and none is recomputed. The way to do this is motivated by examining what is going on in the derivation of a variation of parameter method. This restart without loss of information is what is referred to as the optimal use of fictitious time. It turns out that this feature alone, without concern for secular terms, cures the numerical problems, mentioned above,

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that plagued previous implementations of BG14 and by inference KS.

DERIVATION OF EQUATIONS

The differential equation for perturbed two-body motion is

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = \vec{f} = -\frac{\partial V(\vec{r},t)}{\partial \vec{r}} + \vec{\phi} \tag{1}$$

where $\vec{r}$ is the position vector of one of the bodies with respect to the other, μ is the gravitational parameter, $\vec{f}$ is the vector of all perturbing accelerations, $V(\vec{r},t)$ is the perturbing potential, and $\vec{\phi}$ is the vector of all perturbing accelerations not derivable from a potential. Although eq(1) is easily written down, and is seen often, it is difficult to integrate accurately for long periods. The problem is the $\frac{\dot{r}}{r^3}$ term. This term causes eq(1) to be nonlinear, and to have a singularity at $r = 0$. These objections can be removed in two steps. The first step is to make a change in independent variable from time, t, to fictitious time, s, where s is defined by

$$\frac{dt}{ds} = t' = r \tag{2}$$

This is known as the Sundmann transformation, and was first used in a theoretical investigation of the three-body problem [5]. Using the chain rule for differentation, and denoting differentation of ( ) with respect to s by ( )′, results in

$$\vec{r}\,' = \dot{\vec{r}}\,t' = \dot{\vec{r}}\,r \tag{3}$$

$$r' = \dot{r}\,t' = \dot{r}r \tag{4}$$

also, since $r^2 = \vec{r}\cdot\vec{r}$

$$r' = r\dot{r} = \vec{r}\cdot\dot{\vec{r}} \tag{5}$$

From eq(3) and eq(4)

$$\vec{r}\,'' = r^2\ddot{\vec{r}} + r\dot{r}\,\dot{\vec{r}} = r^2\ddot{\vec{r}} + (\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}} \tag{6}$$

The second step is to replace the $(\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}$ term in eq(6). It was evidently Sperling [6], who first removed the $(\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}$ term by using the Laplace or perigee vector, $\vec{p}$, of two-body motion, where

$$\vec{p} = \frac{(\dot{\vec{r}}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\dot{\vec{r}})\dot{\vec{r}}}{\mu} - \frac{\vec{r}}{r} \tag{7}$$

Therefore, using eq(1) and eq(7) in eq(6) results in

$$\vec{r}\,'' = r^2\vec{f} - \mu\vec{p} + \left[\dot{\vec{r}}\cdot\dot{\vec{r}} - \frac{2\mu}{r}\right]\vec{r} \tag{8}$$

The coefficient of $\vec{r}$ is easily recognized as twice the energy integral of Keplerian motion, so, defining

$$\alpha_k \equiv \dot{\vec{r}}\cdot\dot{\vec{r}} - \frac{2\mu}{r} \tag{9}$$

eq(8) may be written

$$\vec{r}\,'' = \alpha_k\vec{r} + r^2\vec{f} - \mu\vec{p} \tag{10}$$

The introduction of the Laplace vector linearizes the $\vec{r}\,''$ equation when $\vec{f} = 0$. The real advantage of Sperling's insight will be apparent later.

Now, assuming λ and ∇ are arbitrary constants, add $\lambda\vec{r}$ and ∇ to eq(10), and subtract $\lambda\vec{r}$

and $\vec{\nabla}$ from eq(10). This will leave $\vec{r}''$ unchanged no matter what λ or $\vec{\nabla}$ are. These constants are introduced at this point in anticipation of being able to choose them such that the integrability of the variation of parameters equations (developed in the next section) is improved. Re-writing eq(10)

$$\vec{r}'' = \alpha_k \vec{r} + r^2 \vec{f} - \mu \vec{r} + \lambda \vec{r} - \lambda \vec{r} + \vec{\nabla} - \vec{\nabla} \tag{11}$$

Now, defining

$$\begin{aligned} \alpha &\equiv \alpha_k + \lambda \\ \vec{\varepsilon} &\equiv \mu \vec{r} + \vec{\nabla} \\ \vec{Q} &\equiv r^2 \vec{f} - \lambda \vec{r} \\ \vec{T} &\equiv \vec{Q} + \vec{\nabla} \end{aligned} \tag{12}$$

eq(11) becomes

$$\vec{r}'' = \alpha \vec{r} + \vec{T} - \vec{\varepsilon} \tag{13}$$

Using eq(5)

$$r'' = (\dot{\vec{r}} \cdot \dot{\vec{r}} + \vec{r} \cdot \ddot{\vec{r}})r \tag{14}$$

Now, using eq(1), eq(9), and the definitions of α and $\vec{Q}$ in eq(14) results in

$$r'' = \alpha r + \mu + \hat{r} \cdot \vec{Q} \tag{15}$$

where $\hat{r} \equiv \frac{\vec{r}}{r}$. At this point it is informative to examine the derivatives of α and $\vec{\varepsilon}$. Starting with α, and recalling that λ is a constant

$$\alpha' = r\dot{\alpha} = 2r\dot{\vec{r}} \cdot \ddot{\vec{r}} + 2\frac{\mu}{r}\dot{r} \tag{16}$$

Using eq(1) and eq(5), α' very easily reduces to

$$\alpha' = 2(\vec{r}' \cdot \vec{f}) \tag{17}$$

The derivative of $\vec{\varepsilon}$ is

$$\vec{\varepsilon}' = r\dot{\vec{\varepsilon}} = r\left[2(\dot{\vec{r}} \cdot \ddot{\vec{r}})\vec{r} + (\dot{\vec{r}} \cdot \dot{\vec{r}})\dot{\vec{r}} - (\dot{\vec{r}} \cdot \dot{\vec{r}})\dot{\vec{r}} - (\vec{r} \cdot \ddot{\vec{r}})\dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}})\ddot{\vec{r}} - \mu\frac{\dot{\vec{r}}}{r} + \mu\frac{\vec{r}}{r^2}\dot{r}\right] \tag{18}$$

Simplifying and using eq(1) and eq(5), $\vec{\varepsilon}'$ reduces to

$$\vec{\varepsilon}' = 2(\vec{r}' \cdot \vec{f})\vec{r} - (\vec{r} \cdot \vec{f})\vec{r}' - (\vec{r} \cdot \vec{r}')\vec{f} \tag{19}$$

Note that if $\vec{f} = \vec{0}$, both α and $\vec{\varepsilon}$ are constant, otherwise, they vary with s. Having come this far, it is possible to define a very good (but not yet elegant) method of solving the original set of differential equations. Using s as independent variable, and realizing that λ and $\vec{\nabla}$ can be arbitrarily chosen to be 0, integrate the set

$$\begin{aligned} t' &= r \\ \vec{r}'' &= \alpha \vec{r} + \vec{T} - \vec{\varepsilon} \\ r'' &= \alpha r + \mu + \hat{r} \cdot \vec{Q} \\ \alpha' &= 2(\vec{r}' \cdot \vec{f}) \\ \vec{\varepsilon}' &= 2(\vec{r}' \cdot \vec{f})\vec{r} - (\vec{r} \cdot \vec{f})\vec{r}' - (\vec{r} \cdot \vec{r}')\vec{f} \end{aligned} \tag{20}$$

The redundant equations are integrated because it has been shown that it is more accurate to integrate them than to calculate r, α and $\vec{\varepsilon}$ from the integrated values of $\vec{r}$ and $\vec{r}'$. In fact, as a general rule, which shall be applied again later, if at any level of knowledge (i.e.,

at any level of integration) integrated variables are combined in a single operation, it is more accurate to integrate a variable defined to be that combination. Alternatively, operations involving identities at a given level are probably equally accurate, except that an identity with the fewest number of integrated variables will most likely be the most accurate. The fundamental problem with integrating eqs(20) is that $\vec{r}$ is being integrated directly, and no use is being made of the fact that a conic solution could be used to remove the effect of the central force term. Ideally then, one would compute a conic solution, and then compute the variation in the constants of that solution caused by the perturbing acceleration, $\vec{f}$. This approach will yield greater accuracy than the direct approach, and is known as variation of parameters. Different variation of parameters methods vary different parameters, but they all begin with a solution to the unperturbed problem. A very elegant conic solution can be developed by expanding t and $\vec{r}$ in Taylor series about the initial point, and assuming that $\vec{f} = \vec{Q} = \vec{T} = \vec{0}$. The first few of the higher derivatives needed in the expansion for t are given below. Note that a pattern quickly develops, so that any of the derivatives could be written down by inspection

$$
\begin{aligned}
t_o' &= r_o \\
t_o'' &= r_o' \\
t_o''' &= r_o'' = \alpha r_o + \mu \\
t_o'''' &= \alpha r_o' \\
t_o''''' &= \alpha r_o'' = \alpha^2 r_o + \alpha\mu
\end{aligned}
\tag{21}
$$

etc.

The Taylor series expansion for t is, of course

$$t = t_o + t_o' s + t_o'' \frac{s^2}{2!} + t_o''' \frac{s^3}{3!} + \ldots \tag{22}$$

Using eqs(21) in eq(22) and collecting coefficients results in

$$
\begin{aligned}
t = t_o &+ r_o\left(s + \alpha\frac{s^3}{3!} + \alpha^2\frac{s^5}{5!} + \alpha^3\frac{s^7}{7!} + \ldots\right) \\
&+ r_o'\left(\frac{s^2}{2!} + \alpha\frac{s^4}{4!} + \alpha^2\frac{s^6}{6!} + \ldots\right) \\
&+ \mu\left(\frac{s^3}{3!} + \alpha\frac{s^5}{5!} + \alpha^2\frac{s^7}{7!} + \ldots\right)
\end{aligned}
\tag{23}
$$

Following the same approach,

$$
\begin{aligned}
\vec{r}_o'' &= \alpha\vec{r}_o - \vec{\varepsilon} \\
\vec{r}_o''' &= \alpha\vec{r}_o' \\
\vec{r}_o'''' &= \alpha\vec{r}_o'' = \alpha^2\vec{r}_o - \alpha\vec{\varepsilon} \\
\vec{r}_o''''' &= \alpha^2\vec{r}_o' \\
\vec{r}_o'''''' &= \alpha^2\vec{r}_o'' = \alpha^3\vec{r}_o - \alpha^2\vec{\varepsilon}
\end{aligned}
\tag{24}
$$

etc.

Note how the linearization resulting from the introduction of the Laplace vector makes the computation of these derivatives trivial. The Taylor series expansion for $\vec{r}$ is, of course

$$\vec{r} = \vec{r}_o + \vec{r}_o' s + \vec{r}_o'' \frac{s^2}{2!} + \vec{r}_o''' \frac{s^3}{3!} + \ldots \tag{25}$$

Using eqs(24) in eq(25) and collecting similar coefficients results in

$$\vec{r} = \vec{r}_o(1 + \alpha\frac{s^2}{2!} + \alpha^2\frac{s^4}{4!} + \alpha^3\frac{s^6}{6!} + \dots)$$
$$+ \vec{r}_o'(s + \alpha\frac{s^3}{3!} + \alpha^2\frac{s^5}{5!} + \alpha^3\frac{s^7}{7!} + \dots) \tag{26}$$
$$- \vec{\varepsilon}(\frac{s^2}{2!} + \alpha\frac{s^4}{4!} + \alpha^2\frac{s^6}{6!} + \dots)$$

The results obtained thus far can be simplified greatly by defining the following transcendental functions given by Goodyear [7] which are a form of Stumpff functions [8].

$$S_o \equiv 1 + \alpha\frac{s^2}{2!} + \alpha^2\frac{s^4}{4!} + \alpha^3\frac{s^6}{6!} + \dots$$
$$S_1 \equiv s + \alpha\frac{s^3}{3!} + \alpha^2\frac{s^5}{5!} + \alpha^3\frac{s^7}{7!} + \dots \tag{27}$$
$$S_2 \equiv \frac{s^2}{2!} + \alpha\frac{s^4}{4!} + \alpha^2\frac{s^6}{6!} + \dots$$
$$S_3 \equiv \frac{s^3}{3!} + \alpha\frac{s^5}{5!} + \alpha^2\frac{s^7}{7!} + \dots$$

A number of identities can be established by inspection of the S series. For example,

$$S_o = 1 + \alpha S_2\,, \quad S_1 = s + \alpha S_3\,, \quad etc. \tag{28}$$

Also, the derivatives with respect to s can be deduced by inspection (recall that when $\vec{f}$ is zero, α is constant)

$$S_o' = \alpha S_1$$
$$S_1' = S_o \tag{29}$$
$$S_2' = S_1$$
$$S_3' = S_2$$

A summary of the identities used in this paper are given in Appendix I.

Using the new transcendental functions, the previous results can be written

$$t = t_o + r_o S_1 + r_o' S_2 + \mu S_3$$
$$r = t' = r_o S_o + r_o' S_1 + \mu S_2$$
$$r' = (\mu + \alpha r_o) S_1 + r_o' S_o \tag{30}$$
$$\vec{r} = \vec{r}_o S_o + \vec{r}_o' S_1 - \vec{\varepsilon} S_2$$
$$\vec{r}\,' = (\alpha\vec{r}_o - \vec{\varepsilon}) S_1 + \vec{r}_o' S_o$$

This then is the solution to the unperturbed conic case. Eqs(30) may be used to determine t, r, r', $\vec{r}$, and $\vec{r}\,'$ as a function of s and the initial conditions, t_o, r_o, r_o', $\vec{r}_o$, $\vec{r}_o'$, α, and $\vec{\varepsilon}$. These equations are valid on any conic, (i.e., elliptic, parabolic, or hyperbolic), without exception and without singularity. Also, very efficient routines exist for computing the Stumpff functions for any value of s or α. A very elegant method of computing the series is outlined in [7].

The appearance of the terms $\mu + \alpha r_o$ and $\alpha\vec{r}_o - \vec{\varepsilon}$ in eqs(30) result from taking the derivative of S_o with respect to s. In the variation of parameters method that is to follow, all parameters that are constant now, for the unperturbed case, will be considered to be variables. This means that the sum and differences of integrated variables will be used to

compute $\vec{r}\,'$ and r'. This is not a particularly good thing to do for the following reason.

Each of the variables, $\vec{\varepsilon}$, α, and $\vec{r}_o$ will have some integration error associated with them. Forming the difference as indicated in eqs(30) could lead to a magnification of this error due to a loss of leading significant digits. This problem can be avoided by defining $\vec{\delta}_o \equiv \alpha\vec{r}_o - \vec{\varepsilon}$, and $\gamma_o \equiv \mu + \alpha r_o$. Then, with the use of eqs(28), eqs(30) can be re-written as follows

$$
\begin{aligned}
t &= t_o + r_o S_1 + r_o' S_2 + \mu S_3 \\
r = t' &= r_o S_o + r_o' S_1 + \mu S_2 \\
r' &= r_o' S_o + \gamma_o S_1 \\
\vec{r} &= \vec{r}_o + \vec{r}_o' S_1 + \vec{\delta}_o S_2 \\
\vec{r}\,' &= \vec{r}_o' S_o + \vec{\delta}_o S_1
\end{aligned}
\tag{30a}
$$

This form of eqs(30) should be as accurate as possible since each member depends on the fewest number of (what will be) integrated parameters.

Variation of Parameters

The essence of the variation of parameters method is to develop equations for the variation of the parameters used in the unperturbed solution. In the perturbed case, when $\vec{f}$, $\vec{Q}$, and $\vec{T}$ are not zero, eqs(30) or eqs(30a) can still be used to compute t, r, r', $\vec{r}$, and $\vec{r}\,'$, provided the constants or parameters of the motion, i.e., t_o, r_o, r_o', $\vec{r}_o$, $\vec{r}_o'$, $\vec{\delta}_o$, γ_o, α, and $\vec{\varepsilon}$ are considered to be functions of the independent variable, s. Because the parameters should vary so much more slowly than the state, a significant improvement in the accuracy of computing the state can be expected.

Differential equations for these parameters are developed below. It doesn't really matter whether eqs(30) or eqs(30a) are used in the development, the end results differ only by identities. Therefore, eqs(30) will be used now, and the computation of $\vec{\delta}$ and γ will be postponed until later.

Defining $\vec{\rho} \equiv \vec{r}_o$ and $\vec{\beta} \equiv \vec{r}_o'$, eqs(30) can be used to compute $\vec{r}$ and $\vec{r}\,'$ as

$$\vec{r} = \vec{\rho} S_o + \vec{\beta} S_1 - \vec{\varepsilon} S_2 \tag{31}$$

and

$$\vec{r}\,' = (\alpha\vec{\rho} - \vec{\varepsilon}) S_1 + \vec{\beta} S_o \tag{32}$$

provided the derivative of eq(31) with respect to s is the same as eq(32), and the derivative of eq(32) with respect to s is the same as eq(13) which is repeated below

$$\vec{r}\,'' = \alpha\vec{r} + \vec{T} - \vec{\varepsilon} \tag{13}$$

The imposition of these two criteria are sufficient to develop the variation of parameter equations for $\vec{\rho}$ and $\vec{\beta}$. Differentiating eq(31)

$$\vec{r}\,' = \vec{\rho}\,' S_o + \vec{\beta}\,' S_1 - \vec{\varepsilon}\,' S_2 + \vec{\rho} S_o' + \vec{\beta} S_1' - \vec{\varepsilon} S_2' \tag{33}$$

In this case, the computation of the S_i' variables must take into account the fact that α is no longer constant. Denoting $\frac{\partial(\,)}{\partial\alpha}$ by $(\,)^\alpha$ and recalling eqs(29)

$$
\begin{aligned}
S_o' &= \alpha S_1 + S_o^\alpha \alpha' \\
S_1' &= S_o + S_1^\alpha \alpha' \\
S_2' &= S_1 + S_2^\alpha \alpha' \\
S_3' &= S_2 + S_3^\alpha \alpha'
\end{aligned}
\tag{34}
$$

The evaluation of the partials with respect to α will be addressed later. For now, using eqs(34) in eq(33), and requiring that eq(32) still be valid, results in

$$\vec{0} = \vec{p}\,'S_o + \vec{\beta}\,'S_1 - \vec{\varepsilon}\,'S_2 + \left[\vec{p}S_o^{\alpha} + \vec{\beta}S_1^{\alpha} - \vec{\varepsilon}S_2^{\alpha}\right]\alpha' \tag{35}$$

Similarly, taking the derivative of eq(32) with respect to s, and requiring that eq(13)still be valid, and using eq(28), results in

$$\vec{T} = (\alpha\vec{p}\,' - \vec{\varepsilon}\,')S_1 + \vec{\beta}\,'S_o + \left[(\alpha\vec{p} - \vec{\varepsilon})S_1^{\alpha} + \vec{\beta}S_o^{\alpha} + \vec{p}S_1\right]\alpha' \tag{36}$$

Now, multiplying eq(35) by S_o, and eq(36) by S_1 and subtracting, results in

$$-S_1\vec{T} = (S_o^2 - \alpha S_1^2)\vec{p}\,' - (S_2S_o - S_1^2)\vec{\varepsilon}\,' + \tag{37}$$
$$\left[(S_oS_o^{\alpha} - \alpha S_1S_1^{\alpha} - S_1^2)\vec{p} + (S_oS_1^{\alpha} - S_1S_o^{\alpha})\vec{\beta} - (S_oS_2^{\alpha} - S_1S_1^{\alpha})\vec{\varepsilon}\right]\alpha'$$

Using the identity

$$S_o^2 - \alpha S_1^2 = 1 \tag{38}$$

which is proven in Goodyear [7] by comparing terms in the expansicn, and eq(28), it is easy to show that

$$S_1^2 - S_oS_2 = S_2 \tag{39}$$

Also, using the identities $2S_o^{\alpha} = sS_1$ and $2S_1^{\alpha} = sS_2 - S_3$, it is easy to show that $S_oS_o^{\alpha} - \alpha S_1S_1^{\alpha} = S_1^2/2$. Goodyear [7] proves the identities $S_1S_o^{\alpha} - S_oS_1^{\alpha} = \left[S_1S_2 + S_3\right]/2$ and $S_1S_1^{\alpha} - S_oS_2^{\alpha} = S_2^2/2$. Using these identities in eq(37) results in

$$\vec{p}\,' = -S_1\vec{T} - S_2\vec{\varepsilon}\,' + \left[S_1^2\vec{p} + (S_1S_2 + S_3)\vec{\beta} - S_2^2\vec{\varepsilon}\right]\frac{\alpha'}{2} \tag{40}$$

Now, multiplying eq(35) by αS_1, and eq(36) by S_o and subtracting, results in

$$S_o\vec{T} = (S_o^2 - \alpha S_1^2)\vec{\beta}\,' - (S_o - \alpha S_2)S_1\vec{\varepsilon}\,' + \tag{41}$$
$$\left[(S_1S_o + \alpha(S_oS_1^{\alpha} - S_o^{\alpha}S_1))\vec{p} + (S_o^{\alpha}S_o - \alpha S_1^{\alpha}S_1)\vec{\beta} - (S_1^{\alpha}S_o - \alpha S_2^{\alpha}S_1)\vec{\varepsilon}\right]\alpha'$$

Using the identities given previously, and the following two, also given by Goodyear [7], $2S_1^{\alpha} + S_oS_3 = S_1S_2$, and $2S_2^{\alpha} + S_1S_3 = S_2^2$, it is easy to show that

$$\vec{\beta}\,' = S_o\vec{T} + S_1\vec{\varepsilon}\,' - \left[(s + S_oS_1)\vec{p} + S_1^2\vec{\beta} - (S_1S_2 - S_3)\vec{\varepsilon}\right]\frac{\alpha'}{2} \tag{42}$$

Following a similar procedure, define $\rho \equiv r_o$ and $\beta \equiv r_o'$. Then r and r' from eq(30) can be written

$$r = \rho S_o + \beta S_1 + \mu S_2 \tag{43}$$

and

$$r' = (\mu + \alpha\rho)S_1 + \beta S_o \tag{44}$$

Taking a derivative of eq(43) with respect to s, and insisting that eq(44) be valid leads to

$$S_o\rho' + S_1\beta' + (\rho S_o^{\alpha} + \beta S_1^{\alpha} + \mu S_2^{\alpha})\alpha' = 0 \tag{45}$$

Now, taking a derivative of eq(44) with respect to s, and insisting that eq(15) and eq(43) be valid leads to

$$\alpha S_1\rho' + S_o\beta' + \left[(\mu + \alpha\rho)S_1^{\alpha} + \beta S_o^{\alpha} + S_1\rho\right]\alpha' = \hat{r}\cdot\vec{Q} \tag{46}$$

Now, multiply eq(45) by S_o and eq(46) by S_1 and subtract to get

$$(S_o^2 - \alpha S_1^2)\rho' +$$

$$\left[(S_o S_o^\alpha - S_1^2 - \alpha S_1 S_1^\alpha)\rho + (S_o S_1^\alpha - S_1 S_o^\alpha)\beta + (S_o S_2^\alpha - S_1 S_1^\alpha)\mu\right]\alpha' = -\hat{r} \cdot \vec{Q} S_1 \qquad (47)$$

Using the identities given earlier, eq(47) can be reduced to

$$\rho' = -\hat{r} \cdot \vec{Q} S_1 + \left[S_1^2 \rho + (S_1 S_2 + S_3)\beta + \mu S_2^2\right]\frac{\alpha'}{2} \qquad (48)$$

Now, multiply eq(45) by αS_1 and eq(46) by S_o and subtract to get

$$\beta'(S_0^2 - \alpha S_1^2) + \qquad (49)$$

$$\left[(S_1 S_o + \alpha(S_o S_1^\alpha - S_1 S_o^\alpha))\rho + (S_o S_o^\alpha - \alpha S_1 S_1^\alpha)\beta + (S_o S_1^\alpha - \alpha S_1 S_2^\alpha)\mu\right]\alpha' = \hat{r} \cdot \vec{Q} S_o$$

Using the identities given earlier, eq(49) can be reduced to

$$\beta' = \hat{r} \cdot \vec{Q} S_o - \left[(s + S_o S_1)\rho + S_1^2 \beta + (S_1 S_2 - S_3)\mu\right]\frac{\alpha'}{2} \qquad (50)$$

The last step is to develop a differential equation for t_o. Defining $\tau \equiv t_o$ and taking a derivative of t in eq(30), and still requiring $t' = r = \rho S_o + \beta S_1 + \mu S_2$ leads to

$$\tau' + S_1 \rho' + S_2 \beta' + (S_1^\alpha \rho + S_2^\alpha \beta + S_3^\alpha \mu)\alpha' = 0 \qquad (51)$$

Now, substituting eq(48) and eq(50) into eq(51) and simplifying leads to

$$\tau' = \hat{r} \cdot \vec{Q}\, S_2 - \left[(S_1 S_2 - S_3)\rho + S_2^2 \beta + (S_2 S_3 + s S_4 - 3S_5)\mu\right]\frac{\alpha'}{2} \qquad (52)$$

Finally then, instead of integrating the set, eq(20), the set

$$\begin{aligned}
\alpha' &= 2(\vec{r}\,' \cdot \vec{f}\,) \\
\vec{\varepsilon}\,' &= 2(\vec{r}\,' \cdot \vec{f}\,)\vec{r} - (\vec{r} \cdot \vec{f}\,)\vec{r}\,' - (\vec{r} \cdot \vec{r}\,')\vec{f} \\
\vec{\rho}\,' &= -S_1 \vec{T} - S_2 \vec{\varepsilon}\,' + \left[S_1^2 \vec{\rho} + (S_1 S_2 + S_3)\vec{\beta} - S_2^2 \vec{\varepsilon}\right]\frac{\alpha'}{2} \\
\vec{\beta}\,' &= S_o \vec{T} + S_1 \vec{\varepsilon}\,' - \left[(s + S_o S_1)\vec{\rho} + S_1^2 \vec{\beta} - (S_1 S_2 - S_3)\vec{\varepsilon}\right]\frac{\alpha'}{2} \qquad (53) \\
\rho' &= -\hat{r} \cdot \vec{Q} S_1 + \left[S_1^2 \rho + (S_1 S_2 + S_3)\beta + \mu S_2^2\right]\frac{\alpha'}{2} \\
\beta' &= \hat{r} \cdot \vec{Q} S_o - \left[(s + S_o S_1)\rho + S_1^2 \beta + (S_1 S_2 - S_3)\mu\right]\frac{\alpha'}{2} \\
\tau' &= \hat{r} \cdot \vec{Q}\, S_2 - \left[(S_1 S_2 - S_3)\rho + S_2^2 \beta + (S_2 S_3 + s S_4 - 3S_5)\mu\right]\frac{\alpha'}{2}
\end{aligned}$$

could be integrated, with, of course, eqs(30) used to compute $t, \vec{r}, \vec{r}\,', r, r'$.

As was mentioned earlier, eqs(30a) are probably a better set to use to compute $t, \vec{r}, \vec{r}\,', r, r'$, since combinations of integrated variables are not used together when multiplying a single Stumpff function. If this is done, $\vec{\delta}$ and γ must be integrated directly. Since there is no need to integrate both $\vec{\varepsilon}\,'$ and $\vec{\delta}\,'$, $\vec{\varepsilon}$ might as well be removed from eqs(53). This can be done easily by first examining only terms involving $\vec{\rho}$ and $\vec{\varepsilon}$ in $\vec{\rho}\,'$.

$$S_1^2 \vec{\rho} - S_2^2(\alpha\vec{\rho} - \vec{\delta}) = (S_1^2 - \alpha S_2^2)\vec{\rho} + S_2^2 \vec{\delta}$$

But, from Appendix I, $S_1^2 - \alpha S_2^2 = 2S_2$, hence $\vec{\rho}\,'$ becomes

$$\vec{\rho}\,' = -S_1 \vec{T} - S_2 \vec{\varepsilon}\,' + \left[2S_2 \vec{\rho} + (S_1 S_2 + S_3)\vec{\beta} + S_2^2 \vec{\delta}\right]\frac{\alpha'}{2}$$

Turning now to the coefficients of $\vec{\rho}$ and $\vec{\varepsilon}$ in $\vec{\beta}\,'$ leads to

$$(s + S_o S_1)\vec{p} - (S_1 S_2 - S_3)(\alpha\vec{p} - \vec{\delta}) = (s + S_o S_1 - \alpha S_1 S_2 + \alpha S_3)\vec{p} + (S_1 S_2 - S_3)\vec{\delta}$$

Using eqs(28) to replace s and S_o, leads easily to

$$\vec{\beta}\,' = S_o \vec{T} + S_1 \vec{\varepsilon}\,' - \left[2S_1\vec{p} + S_1^2\vec{\beta} + (S_1 S_2 - S_3)\vec{\delta}\right]\frac{\alpha'}{2}$$

Since $\vec{\delta} \equiv \alpha\vec{p} - \vec{\varepsilon}$, it's derivative is simply

$$\vec{\delta}\,' = \alpha\vec{p}\,' + \vec{p}\alpha' - \vec{\varepsilon}\,' = -\alpha S_1 \vec{T} - S_o \vec{\varepsilon}\,' + \left[2S_o\vec{p} + \alpha(S_1 S_2 + S_3)\vec{\beta} + \alpha S_2^2\vec{\delta}\right]\frac{\alpha'}{2}$$

Considerable numerical experience seems to indicate that, contrary to what one might expect, computing γ makes no appreciable contribution to accuracy, consequently it will not be considered further.

In summary then, in lieu of integrating eqs(53), one could integrate eqs(53a) below

$$\begin{aligned}
\alpha' &= 2(\vec{r}\,' \cdot \vec{f}\,) \\
\vec{p}\,' &= -S_1\vec{T} - S_2\vec{\varepsilon}\,' + \left[2S_2\vec{p} + (S_1 S_2 + S_3)\vec{\beta} + S_2^2\vec{\delta}\right]\frac{\alpha'}{2} \\
\vec{\delta}\,' &= \alpha\vec{p}\,' + \vec{p}\alpha' - \vec{\varepsilon}\,' = -\alpha S_1 \vec{T} - S_o \vec{\varepsilon}\,' + \left[2S_o\vec{p} + \alpha(S_1 S_2 + S_3)\vec{\beta} + \alpha S_2^2\vec{\delta}\right]\frac{\alpha'}{2} \\
\vec{\beta}\,' &= S_o \vec{T} + S_1 \vec{\varepsilon}\,' - \left[2S_1\vec{p} + S_1^2\vec{\beta} + (S_1 S_2 - S_3)\vec{\delta}\right]\frac{\alpha'}{2} \qquad (53a) \\
\rho' &= -\hat{r} \cdot \vec{Q} S_1 + \left[S_1^2\rho + (S_1 S_2 + S_3)\beta + \mu S_2^2\right]\frac{\alpha'}{2} \\
\beta' &= \hat{r} \cdot \vec{Q} S_o - \left[(s + S_o S_1)\rho + S_1^2\beta + (S_1 S_2 - S_3)\mu\right]\frac{\alpha'}{2} \\
\tau' &= \hat{r} \cdot \vec{Q}\, S_2 - \left[(S_1 S_2 - S_3)\rho + S_2^2\beta + (S_2 S_3 + sS_4 - 3S_5)\mu\right]\frac{\alpha'}{2}
\end{aligned}$$

The integration of eqs(53a) where $\vec{\varepsilon}\,'$ is computed but not integrated, allows the use of eq(30a) for computation of the state without forming $\vec{\delta}$ from other integrated terms. Also, it is probably equally accurate to integrate either form of $\vec{\delta}\,'$, since they differ only by identities at the same level of knowledge.

The Optimal Use of Fictitious Time

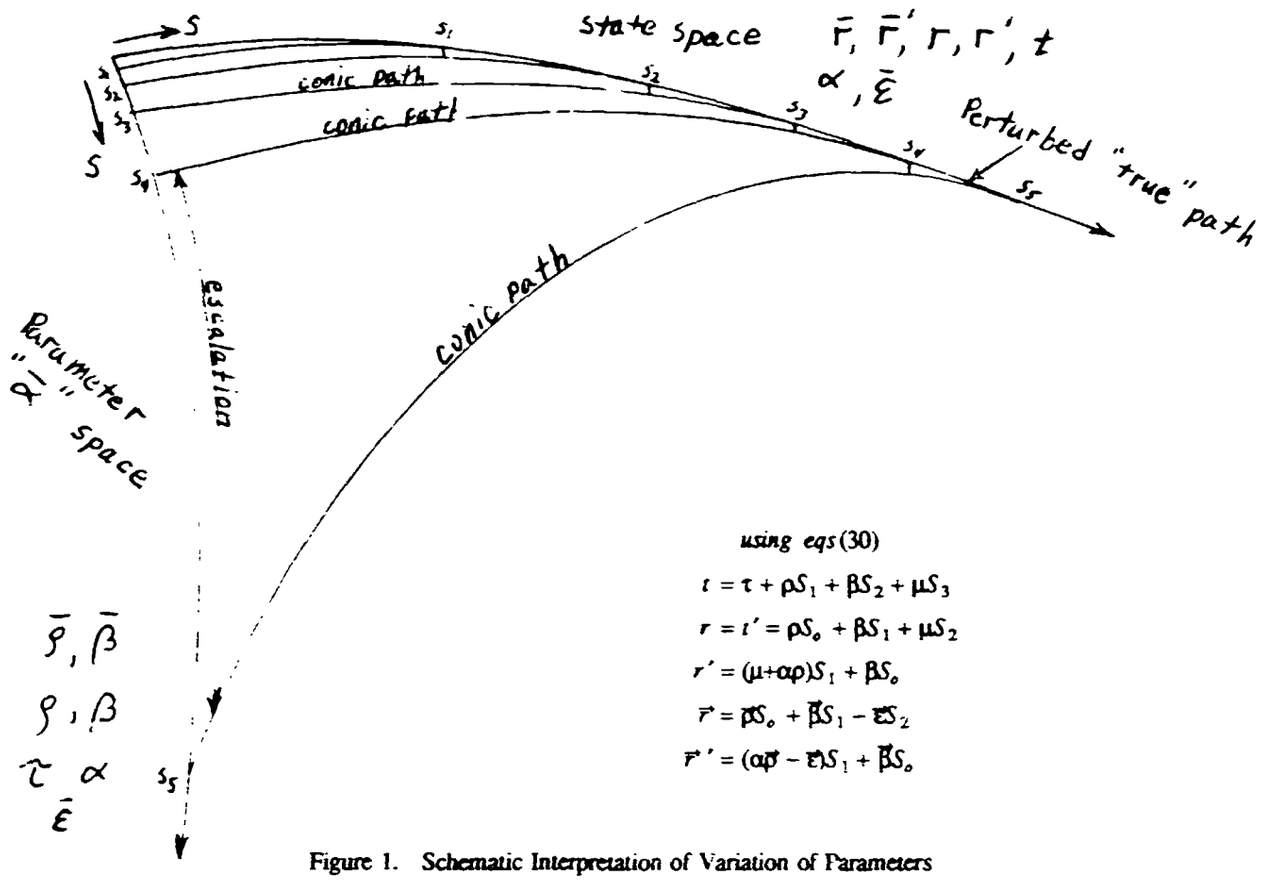
The interesting point to note in all this is that, whichever set of parameters is used, $\vec{r}$, $\vec{r}\,'$, r, , r' and t are all computed from the solution set $\vec{p}$, $\vec{\beta}$, ρ, β, α, and $\vec{\varepsilon}$ (*or* $\vec{\delta}$) and τ, using the current value of s.

Most people seeing these complex equations for the first time tend to be overwhelmed by the derivation and don't really see through all the algebra just what is happening physically. However, the equations developed above can be interpreted schematically so that an intuitive feel for what variation of parameters is up to can be obtained.

A real understanding of variation of parameters may be obtained by assuming that a "true", i.e. perturbed, trajectory path is generated by integrating eqs(20) forward from t_o. Note that this "true" path includes all perturbations and is a function of s. At every point along this "true" path, $\vec{r}$ and $\vec{r}\,'$ are known, and may be used to define a two-body conic, i.e. the osculating conic. Now, assume that at each point along the "true" path the osculating conic is used to coast *backwards* a distance -s, and that having arrived at -s, the osculating parameters, called "$\bar{\alpha}$", are re-computed such that now a forward coast of +s brings us back to the "true" $\vec{r}$ and $\vec{r}\,'$. One could imagine then the unfolding of "$\bar{\alpha}$" as s is increased from

zero, and that given s, one would know "$\bar{\alpha}$", and b could compute the "true" $\bar{r}$ and $\bar{r}'$. But this is exactly what eqs(30) or eqs(30a) imply. Th equations for α and $\bar{\varepsilon}$ are truly osculating, and they are the same on the "true" path and after coasting back -s. The variables $\bar{\rho}$, $\bar{\beta}$, ρ, and β may be thought of as functions of -s, since they are used to coast forward +s to define $\bar{r}$, $\bar{r}'$, r, and r' along the "true" path. It should be obvious that there are an infinite number of these that could be used depending on the choice of -s. The simplest choice is to choose s = 0, which implies

$$\bar{\rho} = \bar{r}_{true}, \quad \rho = r_{true}, \quad \bar{\beta} = \bar{r}_{true}', \quad \rho' = r_{true}'$$



using eqs (30)

$$t = \tau + \rho S_1 + \beta S_2 + \mu S_3$$
$$r = t' = \rho S_o + \beta S_1 + \mu S_2$$
$$r' = (\mu + \alpha\rho) S_1 + \beta S_o$$
$$\bar{r} = \bar{\rho} S_o + \bar{\beta} S_1 - \bar{\varepsilon} S_2$$
$$\bar{r}' = (\alpha\bar{\rho} - \bar{\varepsilon}) S_1 + \bar{\beta} S_o$$

Figure 1. Schematic Interpretation of Variation of Parameters

These ideas are depicted in Figure 1, in which the "true" trajectory unfolds to the right, while the variation of parameters integration occurs in the "$\bar{\alpha}$" space. The problem with keeping the initial point fixed can now be seen rather easily. If the perturbations are periodic, there will not be much gross motion in the "$\bar{\alpha}$" space since the basic orbit is not being changed excessively. However, when persistant perturbations occur, such as long term drag or low thrust, there will be continual motion in the "$\bar{\alpha}$" space. If the "true" path changes such that a wildly different conic is needed to coast to that point from the *initial-point*, rapid escalation in the "$\bar{\alpha}$" space is possible, and breakdown of the method can occur. In two specific cases, low thrust to escape and re-entry to the ground, breakdown of the original K-S, BG14, and BG14δ methods occurred. Now, consider what happens if the "$\bar{\alpha}$" space is not hooked to the initial point, but rather moves with the trajectory. Now the

only motion that occurs in "$\vec{\alpha}$" space occurs during a single step, and there is no accumulation of effect that must be coasted out with an accumulated independent variable. This is also shown in Figure 1. Consequently, by sliding the origin, the escalation can be reduced considerably, or eliminated altogether. Another advantage of not having s increase monotonically, is that previously, aside from the initial point, nothing could be done to improve the right hand side of the variational equations. This point has not been addressed yet, although it was alluded to earlier with the introduction of λ and ∇. The idea is this. If the variational equations change slowly then larger integration steps may be taken. By choosing λ and ∇ carefully, it is possible to drive the forcing terms on the right hand side of eqs(53) to zero, at least at t_o. However, if s is set to zero at the beginning of each integration step, λ and ∇ may be chosen again and again, so that the right hand side can constantly be set back to zero. Also, if s = 0, all the S_i are zero except $S_o = 1$. This means that τ, $\vec{p}$, $\vec{\beta}$, ρ, β can be reset to exactly t, $\vec{r}$, $\vec{r}\,'$, r, and r', with absolutely no error introduced.

Note that this is not the same as a re-initialization, since α and $\vec{\varepsilon}$ are not altered (other than changing λ and ∇). Re-initialization would result in a loss of accuracy which can be observed quite readily. In fact, the difference in the energy and radius as integrated, compared to those computed from $\vec{r}$ and $\vec{r}\,'$ is used as a measure of integration accuracy. Consequently, advancing the origin, i.e., setting s = 0, accomplishes several objectives. It makes the program faster. It removes sensitivity to a distant initial point. It removes the need to worry about secular terms, since the independent variable is always near zero. It allows the free parameters λ and ∇ to be chosen at each step in order to improve accuracy and increase step size. Since the independent variable, s, is going to be set to zero at the beginning of each step, it is interesting to look at the differential equations for the variation in the parameters, and see if a choice can be made for λ and ∇ that will make the right hand side zero. The hope is that this will cause the parameters to change more slowly, and consequently allow larger integration steps. Although only numerical experience will indicate what strategy, if any, is beneficial, what follows assumes that minimizing the right hand side of eqs(53) is the best strategy. Examining eqs(53) shows that β' will be zero if $\hat{r} \cdot \vec{Q}$ is zero. Choosing λ to cause this to happen leads to (recall that $\hat{r} = \vec{r}/r$)

$$\lambda = \vec{r} \cdot \vec{f} \tag{55}$$

Since α_k is continuous even if λ changes, from eqs(12), it follows that

$$\alpha_k = \alpha_{old} - \lambda_{old} = \alpha_{new} - \lambda_{new} \tag{56}$$

Consequently,

$$\alpha_{new} = \alpha_{old} + (\lambda_{new} - \lambda_{old}) \tag{57}$$

Again examining eqs(53) shows $\vec{\beta}\,'$ will be zero at s = 0 if $\vec{T}$ is zero. Consequently, choose

$$\nabla = \lambda_{new}\vec{r} - r^2\vec{f} \tag{58}$$

Also, since the perigee vector, $\vec{p}$, is continuous even if ∇ changes, from eqs(12), it follows that

$$\mu\vec{p} = \vec{\varepsilon}_{old} - \nabla_{old} = \vec{\varepsilon}_{new} - \nabla_{new} \tag{59}$$

Consequently,

$$\vec{\varepsilon}_{new} = \vec{\varepsilon}_{old} + (\nabla_{new} - \nabla_{old}) \tag{60}$$

If $\vec{\delta}$, and γ are used, and $\vec{p}$ and ρ are updated, it is worse, since $\vec{\delta}$ must be jumped to keep $\vec{\varepsilon}$ constant, and γ must be jumped to keep μ constant. If thereafter, λ is changed, they will both be jumped again because α will change. If further, ∇ is changed, $\vec{\delta}$ must be jumped an equivalent amount. This process can be thought of as jumping from surface to surface in multi-dimensional space in order to keep the parameters changing as slowly as possible.

Note that λ_{new} and ∇_{new} are held constant for the duration of an integration step, and are only changed at the beginning of the next integration step (when they become "old"). Note also that neither λ nor ∇ must be changed, it is just *possible* to change them if numerical experiments indicate that there is some benefit in doing so.

Relation to Bond and Gottlieb's Fourteen Element Method - BG14

The objective of this paper has been to present a simple derivation of a variation of parameters method, using universal variables and fictitious time, and to point out that the optimal use of fictitious time is to set it to zero at the start of each integration step. In other words, advance the origin. Advantages include faster running time and greater accuracy, secular terms are not important, and there is no build up of sensitivity to a distant starting point. Also, it possible to choose constants to drive the right hand side of the variational equations to zero at the beginning of each integration step.

The introduction of the ∇ associated with $\vec{\mathfrak{C}}$ is also new in this paper. The use of the Keplerian energy constant, α_k, in this paper was motivated more by simplicity of presentation than any desire to use the "best" energy constant. An observation of eqs(53) or eqs(53a) shows that α' is a major contributor to the right hand side of those equations. Consequently it seems reasonable to try to use for α a constant of the motion that, in fact, remains constant for the greatest possible range of perturbing accelerations. Total energy would be a better choice than Keplerian energy, since total energy is constant in the presence of a perturbing acceleration that is due to a time invarient potential. The Keplerian energy was used in [3], and the total energy was used in [9]. However, the Jacobian integral of [4] is best of all, since it remains constant in the presence of a perturbing acceleration resulting from a time varying potential. Therefore, in lieu of defining α_k as in eq(9), define the Jacobian integral

$$\alpha_j \equiv \dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{2\mu}{r} + 2(V(\vec{r},t) - \vec{\varpi} \cdot \vec{c}) \tag{61}$$

where $\vec{c} = \vec{r} \times \dot{\vec{r}}$ is the angular momentum, and $\vec{\varpi}$ is a constant vector which characterizes the time rate of change of the potential. Now, defining $\sigma \equiv \vec{\varpi} \cdot \vec{c}$, solve for the Keplerian energy as

$$\dot{\vec{r}} \cdot \dot{\vec{r}} - \frac{2\mu}{r} = \alpha_j - 2(V(\vec{r},t) - \sigma) \tag{62}$$

and substitute into eq(8) to get

$$\vec{r}\,'' = r^2\vec{f} - \mu\vec{r} + \left[\alpha_j - 2(V(\vec{r},t) - \sigma)\right]\vec{r} \tag{63}$$

as before, add and subtract $\lambda\vec{r}$ and ∇ to get

$$\vec{r}\,'' = \alpha_j\vec{r} + r^2\vec{f} - 2(V(\vec{r},t) - \sigma)\vec{r} - \mu\vec{r} + \lambda\vec{r} - \lambda\vec{r} + \nabla - \nabla \tag{64}$$

Now define

$$\begin{aligned} &\alpha \equiv \alpha_j + \lambda \\ &\vec{\mathfrak{C}} \equiv \mu\vec{r} + \nabla \quad (\textit{ same as before }) \\ &\vec{Q} \equiv r^2\vec{f} - \lambda\vec{r} - 2(V(\vec{r},t) - \sigma)\vec{r} \\ &\vec{T} \equiv \vec{Q} + \nabla \end{aligned} \tag{65}$$

With these definitions, eqs(53) and eqs(53a), given previously, are still valid with the exception that now, from eq(61), α' becomes

$$\alpha' = r\dot{\alpha}_j = r\left[2\dot{\vec{r}} \cdot \ddot{\vec{r}} + \frac{2\mu}{r^2}\dot{r} + 2V^{\vec{r}} \cdot \dot{\vec{r}} - 2\vec{\varpi} \cdot \dot{\vec{c}} + 2V^{t}\right] \tag{66}$$

where superscripts on V denote partial derivatives by $\vec{r}$ and time. Recall from eq(5) that $\dot{r} = \vec{r} \cdot \dot{\vec{r}}/r$. Also, [4] derives the fact that $V^t = -\vec{\omega} \cdot \vec{r} \times V^{\vec{r}}$. Using these, it is easy to show that eq(66) reduces to

$$\alpha' = 2(\vec{r}\,' - r\vec{\omega} \times \vec{r}) \cdot \vec{\phi} \tag{67}$$

Note that now α' depends only on $\vec{\phi}$ and not $\vec{f}$. Also, because σ was introduced into the problem, it is integrated from it's initial point rather than being calculated. This is done because it is more accurate to do so. The differential equation for σ is

$$\sigma' = r\dot{\sigma} = r\vec{\omega} \cdot \vec{r} \times \ddot{\vec{r}} = r\vec{\omega} \cdot \vec{r} \times \vec{f} \tag{68}$$

The only other difference is that now the optimal λ is

$$\lambda = \vec{r} \cdot \vec{f} + 2(\sigma - V(\vec{r},t)) \tag{69}$$

and ∇ is

$$\nabla = -\vec{Q} = (\lambda + 2(V - \sigma))\vec{r} - r^2\vec{f} \tag{70}$$

The major advantage in changing energy constants in the derivation is that greater accuracy can be attained by using an energy integral that is constant for a greater variety of perturbing accelerations. The cost for this is the introduction of the differential equation for σ. However, the advantage that accrues from the use of the Jacobian integral is more than worth the extra differential equation.

Numerical Results

The formulae developed above have been coded in the Ada Simulation Development System (ASDS) [13], and tested on a number of problems. One problem run was the 288 day J2 plus moon problem given on page 121 of [10]. This problem was run with and without ∇ and advancing the origin. The answer in [10] was obtained in both cases, when 50 steps per rev were used in conjunction with Runge_Kutta_Fehlberg_78 fixed step. When 30 steps per rev was used, advancing the origin and choosing ∇ resulted in a slightly smaller root mean square error. Although this problem contains a perturbing force outside the potential term, it is not a particularly severe test of the algorithm. Two severe test cases were run. The first was nominally a reentry to the ground. It was actually set up as a fixed final time problem, with the final time being chosen to bring the spacecraft essentially to the ground. Essentially is used because the final time was truncated to eight digits, 29.422577 days. The reentry problem: The reentry problem starts at $\vec{r}$ = (6677833.0 -62810.0 -27301.0)(m), $\dot{\vec{r}}$ = (79.0 6821.0 3627.0)(m/s). The ballistic number was 78.0(kg/m**2). The gravity model was GEM_10, 2 x 2. The atmosphere routine was Jacchia '70, with the density held constant below 90km. The results of this problem are given in Table I. All runs were made with Runge_Kutta_Fehlberg 7/8 fixed step, and λ and ∇ were held constant except in those runs where λ or v appear. The second severe test was a low thrust spiral from a circular to a hyperbolic orbit. The low thrust problem: Steve Sponaugle [11] first attempted to solve this problem using the K-S method [2]. He noted that the program could not start in a circular orbit and integrate all the way to hyperbolic. However, he also noted that if he reinitialized about two-thirds of the way out the program would integrate on through the parabolic region, albeit poorly. It was this one clue coupled with the desire to initialize properly, that led to the sliding origin concept. The low-thrust problem starts at $\vec{r}$ = (6783140.0, 0.0, 0.0)(m), $\dot{\vec{r}}$ = (0.0, 6736.774, 3657.770)(m/s). The thrust and flow rate are 122.36657(N) and 2.4955962e-03(kg/s) respectively. The initial mass is 375000.0(kg). Thrust is along the velocity vector. The gravity model was GEM-9. J2 only. Several runs of 240 days were made. The first two were made with the sliding origin method of this report using 80 steps/rev. Final $\vec{r}$ = (-634757780.495, 898972406.577, 20257150.2835)(m). This is shown in Table II. The other runs in the table compare the BG14δ and BG14ε sliding origin (Ω) methods to each other as well as to the BG14δ method in [12]. Note that

again the simple removal of secular terms failed to cure the instability. The addition of the sliding origin to the BG14δ of [12] results in a method whose characteristics are identical to those of the BG14δ of this report. Further, the effect of causing λ to change at each step is shown to have a dramatic effect in the low thrust problem, but very little for the reentry problem. Also, changing ∇ at each step as well as λ actually caused things to get slightly worse. Note that in every case the ε form is more accurate than the δ form, although not by much.

Conclusions

An examination of Tables I and II indicates that simply removing secular terms does not correct the instability in the original BG14. However, when the sliding origin was added to Bond's BG14δ method (BG14$\delta\Omega$ in the tables), that formulation and the BG14$\delta\Omega$ formulation herein give essentially identical results. The BG14$\varepsilon\Omega$ method gave essentially identical results for the low thrust problem, and, contrary to what was expected, gave slightly superior results for both the reentry problem and the low thrust problem. Also, choosing λ at each step improved the low thrust problem, but had little effect on the reentry problem. Previous studies had indicated that the $\delta\Omega$ form of BG14 was slightly superior to the $\varepsilon\Omega$ form. The results here do not show that. However, previous studies used 4th order Runge_Kutta rather than the more accurate 8th order used herein. The conclusion one should draw is that the methods are essentially the same and perhaps other criteria such as simplicity and speed should enter into the decision for a choice of formulation. Since the methods differ by only a few terms, one could code them both or arbitrarily choose one. It is suggested that studies be carried out with specific types of problems to see if λ or ∇ should be changed at each step, and whether the $\delta\Omega$ or the $\varepsilon\Omega$ form gives better results. It is obvious from [12] that either the change of variables or the removal of secular terms improves the original BG14 [4] if the origin remains fixed. However, since that method failed to solve either the low-thrust problem or re-entry to the ground, one must conclude that the real culprit is the fixed origin and not secular terms per se. When the sliding origin was added to Bond's $\vec{\delta}$ method, the instability vanished, and it solved the problem getting the same answer as obtained by the formulation represented by eqs(53a). The formulation represented by eqs(53a) is essentially the same as in[12], with the exception that γ does not appear on the right hand side of the differential equations and no particular effort was made to remove secular terms. Since eqs(53a) and the formulation in [12] (with sliding origin added) give essentially identical results, one must conclude that it is not the form of the right hand side of the differential equations that really matters, it is rather the choice of dependent variable that actually makes the difference. Finally then, the optimal use of fictitious time, i.e., moving the origin renders secular terms innocuous, since they can never grow, and also overcomes the sensitivity to the initial point that results from long term persistant forces. Any element method using fictitious time can be stabilized by setting fictitious time back to zero as indicated in this note. The addition of this concept to any of the Jacobian integral methods, i.e., BG14 in any of its forms ([4], [12], or as developed herein), makes them highly accurate, long term propagators.

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Appendix I

Useful identities, which are either given in [7] or which can be derived from them are

$$S_o^2 - \alpha S_1^2 = 1$$

$$S_o = 1 + \alpha S_2$$

$$S_1 = s + \alpha S_2$$

$$S_1^2 - S_o S_2 = S_2$$

$$S_1^2 = 2S_2 + \alpha S_2^2$$

$$2S_o^{\alpha} = sS_1$$

$$2S_1^\alpha = sS_2 - S_3 = S_1S_2 - S_oS_3$$
$$2S_2^\alpha = sS_3 - 2S_4 = S_2^2 - S_1S_3$$
$$2S_3^\alpha = sS_4 - 3S_5$$
$$2(S_1S_o^\alpha - S_oS_1^\alpha) = S_1S_2 + S_3$$
$$2(S_2S_1^\alpha - S_1S_2^\alpha) = S_2S_3$$
$$2(S_1S_1^\alpha - S_oS_2^\alpha) = S_2^2$$

Classical Stumpff Functions

If one defines the variable $z \equiv \alpha s^2$, and forms the following transcendental functions

$$c_o \equiv \frac{1}{0!} + \frac{z}{2!} + \frac{z^2}{4!} + \frac{z^3}{6!} + \ldots$$
$$c_1 \equiv \frac{1}{1!} + \frac{z}{3!} + \frac{z^2}{5!} + \frac{z^3}{7!} + \ldots$$
$$c_2 \equiv \frac{1}{2!} + \frac{z}{4!} + \frac{z^2}{6!} + \frac{z^3}{8!} + \ldots$$
$$c_3 \equiv \frac{1}{3!} + \frac{z}{5!} + \frac{z^2}{7!} + \frac{z^3}{9!} + \ldots$$

It is rather easy to show that the c's and S's are related by

$$S_o = c_o$$
$$S_1 = sc_1$$
$$S_2 = s^2c_2$$
$$S_3 = s^3c_3$$
$$S_4 = s^4c_4$$
$$S_5 = s^5c_5$$

The c's are known as Stumpff functions and are used in [4] except that α in [4] is the negative of α in this report. The S form of the Stumpff functions is used by Goodyear [7], and tend to make the equations seem less cluttered.

APPENDIX II Multiple Vehicles and Constant Time Steps

Since the independent variable is fictitious time, and since fictitious time is defined by $\frac{dt}{ds} = r$, it is obvious that since,in general, the radius of one vehicle will not be identical to the radius of another vehicle, time will unfold differently along the two trajectories. This is inconvenient when looking at relative motion between two vehicles, since almost all relevent measurements require time simultaneity. It is possible to gain time simultaneity the following way. First of all, one of the vehicles must be chosen to be the master vehicle. This vehicle will have fictitious time, s, truly independent. Assume this is vehicle one. We would like time on the second vehicle to unfold at the same rate it does on the first vehicle. That is to say, we would like

$$dt = r_1ds_1 = r_2ds_2 \qquad \text{(II-1)}$$

or

$$ds_2 = \frac{r_1}{r_2} ds_1 \tag{II-2}$$

This can also be written

$$\frac{ds_2}{ds_1} = \frac{r_1}{r_2} \tag{II-3}$$

Therefore, if eqs(53) for the second vehicle are multiplied by $\frac{r_1}{r_2}$, the independent variable is shifted to s_1. In addition however, whenever s_2 is needed in order to compute r, r', etc., it must be computed as the integral of eq(II-2), i.e., as

$$s_2 = \int \frac{r_1}{r_2} ds \tag{II-4}$$

Now that this is understood, it is easy to see that vehicle one is not even needed if a constant time step is desired. In this case *pretend* that vehicle one exists, and that this imaginary vehicle is in a perfectly circular orbit, and that it is being integrated with a step size, ds, that is chosen to be

$$ds = \frac{dt}{r_i} \tag{II-5}$$

where dt is the desired time step, and r_i is just a reasonable value for the radius of the imaginary orbit. Then, eqs(53) can be modified by multiplying by $\frac{r_i}{r_2}$, and s_1 is computed from

$$s_1 = \int \frac{r_i}{r_1} ds \tag{II-6}$$

In this way a single vehicle or many vehicles can be integrated at a constant time step size, with no iteration what-so-ever and with great accuracy. Note that the equation for τ is no longer needed since time is known absolutely. If τ is integrated time can be computed and used as a measure of integration accuracy.

A test case for a near circular orbit perturbed by a 4x4 gravity model was integrated for 30 days using both the standard BG14 and this constant time step approach. The step size was varied from 25 steps per rev to 8 steps per rev. In every case, the rss error away from a high precision run was slightly less with the constant time step method. An interpretation of this result is that the error made in integrating for s results in less error than that resulting from the computation of time using τ. The method above is convenient and seems to be quite accurate, at least for near circular orbits. However, since the computation of fictitious time involves the integral of 1/r, this approach may not be as accurate for elliptic orbits as the standard method that computes time analytically. A variable step size approach would be required as a minimum. In addition, in the absence of perturbations, an error would be introduced by the integration of 1/r, that would not be present with the standard method. A variation of this first approach would be to integrate τ, and solve for s from the time equation at every function call rather than integrate 1/r. This would add iterations for s, but not in terms of function calls. The advantage of this approach is that no error would be introduced by the integration of 1/r. Information from the next section can be used to speed convergence.

A SECOND APPROACH TO CONSTANT TIME STEP

A second approach to stopping on a specific time (with s as independent variable) can be developed by assuming that s is a function of t, and forming the Taylor series

$$s = s_o + \frac{ds}{dt}\Delta t + \frac{1}{2!}\frac{d^2s}{dt^2}\Delta t^2 + \frac{1}{3!}\frac{d^3s}{dt^3}\Delta t^3 + \ldots \tag{II-7}$$

The derivatives of s that are needed come from the fact that s is defind by

$$\frac{dt}{ds} = r \tag{II-8}$$

hence

$$\frac{ds}{dt} = \frac{1}{r} \tag{II-9}$$

and

$$\frac{d^2s}{dt^2} = -\frac{\dot{r}}{r^2} \tag{II-10}$$

where

$$\dot{r} = \frac{\vec{r}\cdot\dot{\vec{r}}}{r} \tag{II-11}$$

also

$$\frac{d^3s}{dt^3} = -\frac{\ddot{r}}{r^2} + \frac{2\dot{r}^2}{r^3} \tag{II-12}$$

Now,

$$\ddot{r} = \frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{r} + \frac{\vec{r}\cdot\ddot{\vec{r}}}{r} - \frac{\vec{r}\cdot\dot{\vec{r}}}{r^2}\dot{r} \tag{II-13}$$

Since the velocity vector, $\vec{v} = \dot{\vec{r}}$, and remembering the definition of $\dot{r}$, $\ddot{r}$ becomes

$$\ddot{r} = \frac{v^2}{r} + \frac{\vec{r}\cdot\ddot{\vec{r}}}{r} - \frac{\dot{r}^2}{r} \tag{II-14}$$

Now, from eq(1)

$$\vec{r}\cdot\ddot{\vec{r}} = \vec{r}\cdot\vec{f} - \frac{\mu}{r} \tag{II-15}$$

Consequently,

$$\frac{d^3s}{dt^3} = \frac{1}{r^3}(3\dot{r}^2 + \frac{\mu}{r} - v^2 - \vec{r}\cdot\vec{f}) \tag{II-16}$$

Defining $\delta \equiv \frac{\Delta t}{r}$, there results

$$\Delta s = \delta - \frac{1}{2}\dot{r}\delta^2 + \frac{1}{6}(3\dot{r}^2 + \frac{\mu}{r} - v^2 - \vec{r}\cdot\vec{f})\delta^3 \tag{II-17}$$

Whenever, δ is a reasonable number, eq(II-17) can be used to estimate Δs to land on a specific time. Since a third order Taylor series is being used, the estimate of Δs should be very accurate, and the time should be hit with very few iterations, probably only one.

TABLE I - Comparison of BG14εΩ, BG14δΩ and BG14δ Methods

Final Value of Position Vector - Reentry to the Ground (T = 29.422577Days)

Method	X (km)	Y (km)	Z (km)	RSS (km)
Gottlieb				
BG14εΩ - 100spr	2517.0799480	-5737.8247524	1188.3706594	------
BG14δΩ - 100spr	2517.0799488	-5737.8247526	1188.3706592	------
BG14εΩ - 10spr	2523.8693524	-5739.4783332	1186.2762008	7.29500
BG14δΩ - 10spr	2523.9860855	-5739.4725808	1186.2269801	7.41656
BG14δΩ - 8spr	2524.905855	-5739.6584337	1185.919466	8.40330
BG14εΩ - 8spr	2524.82146	-5739.649530	1185.949942	8.31388
BG14δΩ - 8spr λ	2524.90541	-5739.658044	1185.919490	8.40280
BG14εΩ - 8spr λ	2524.82098	-5739.64925	1185.950022	8.31333
BG14δΩ - 8spr λ $\bar{v}$	2524.90195	-5739.65710	1185.920422	8.39909
Bond [12]				
BG14δ - 100spr	Failed	Failed	Failed	N/A
BG14δΩ - 100spr	2517.0799489	-5737.8247527	1188.3706592	-------
BG14δΩ - 10spr	2523.9860855	-5739.4725807	1186.2269801	7.41656

TABLE II - Comparison of BG14εΩ, BG14δΩ and BG14δ Methods

Final Value of Position Vector - Low Thrust Spiral to Escape (T = 240.0Days)

Method	X (km)	Y (km)	Z (km)	RSS (km)
Gottlieb	-634757.780495	898972.406577	20257.150284	
BG14εΩ - 30spr	-634757.776479	898972.409549	20257.147576	0.00568
BG14δΩ - 30spr	-634757.776507	898972.409528	20257.147595	0.00564
BG14εΩ - 10spr	-634690.244994	899021.597745	20211.855408	95.0391
BG14δΩ - 10spr	-634690.245016	899021.597729	20211.855424	95.0391
BG14δΩ - 8spr	-634262.290049	899331.417762	19925.408943	696.026
BG14εΩ - 8spr	-634265.16938	899326 609579	19928.146539	690.195
BG14δΩ - 8spr λ	-634788.200341	898950.438100	20277.518474	42.695
BG14εΩ - 8spr λ	-634788.196759	898950.436222	20277.51741	42.693
BG14δΩ - 8spr λ $\bar{v}$	-634788.648135	898950.114106	20277.818275	43.324
Bond [12] BG14δ - 30spr	Failed	Failed	Failed	N/A
BG14δΩ - 30spr	-634757.776479	898972.409549	20257.147576	0.00568
BG14δΩ - 10spr	-634690.244994	899021.597745	20211.855409	95.0392

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16 Abstract

The optimal way to use fictitious time in variation of parameters methods is presented. Setting fictitious time to zero at the end of each step is shown to cure the instability associated with some types of problems. Only some parameters are reinitialized, thereby retaining redundant information.

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