Optimal Chebyshev Polynomials on Ellipses in the Complex Plane

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Abstract

The design of iterative schemes for sparse matrix computations often leads to constrained polynomial approximation problems on sets in the complex plane. For the case of ellipses, we introduce a new class of complex polynomials which are in general very good approximations to the best polynomials and even optimal in most cases.

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§1. Introduction

We consider complex Chebyshev approximation problems of the type

\[ E_n(r, c) = \min_{p \in \Pi_n: p(c) = 1} \|p\|_{\mathcal{E}_r}, \quad \|p\|_{\mathcal{E}_r} := \max_{z \in \mathcal{E}_r} |p(z)|. \]  

(1)

Here \( \Pi_n \) denotes the space of all complex polynomials of degree at most \( n \), \( \mathcal{E}_r \) is any ellipse with foci \( \pm 1 \) and semi-axes \( (r \pm r^{-1})/2, \ r > 1, \) and \( c \in \mathbb{C} \setminus \mathcal{E}_r \). It will be convenient to express \( c \) as a point on the boundary \( \partial \mathcal{E}_R \) of the ellipse \( \mathcal{E}_R, \ R > r, \) i.e. \( c = c(R, \gamma) = ((R + R^{-1})\cos(\gamma) + i(R - R^{-1})\sin(\gamma))/2, \ \gamma \in [0, 2\pi) \). Since Haar's condition is satisfied, there always exists a unique optimal polynomial \( p_n(z; r, c) \) of (1).

Problems (1), in general with \( \mathcal{E} \subset \mathbb{C} \) any compact set instead of \( \mathcal{E}_r \), arise in numerical linear algebra. E.g. the design of iterative methods for the solution of large sparse non-Hermitian linear systems \( Ax = b \) with best possible convergence rates [2], the computation of optimal polynomial preconditioners for conjugate gradient type algorithms for \( Ax = b \) [7], or the acceleration of eigenvalue methods for \( A \) [6] all lead to problems of this type. However, for arbitrary sets \( \mathcal{E} \) the optimal polynomials are in general not known explicitly and therefore the methods are usually based on polynomials which are only asymptotically optimal. A popular choice for the set \( \mathcal{E} \) are ellipses, and then the scaled Chebyshev polynomials \( t_n(z; c) := T_n(z)/T_n(c) \) are used as approximations to the optimal polynomials of (1) [4, 6]. Clayton [1] showed that even \( t_n(z; c) \equiv p_n(z; r, c) \) if \( c \) is real, and in general \( t_n \) is nearly optimal for (1) as long as \( n \) is large. However, in some of the applications we mentioned, polynomials with small degree are used and typically the distance between \( c \) and \( \mathcal{E}_r \) is small. Depending on the position of \( c \) on \( \partial \mathcal{E}_R \), \( \|t_n(z; c)\|_{\mathcal{E}_r} > 1 \) can occur, and then \( t_n \) yields no useful approximation (cf. Example 1 given below).

In this note, we introduce a new class of asymptotically optimal polynomials \( q_n \) for Problem (1) which always satisfy \( \|q_n(z; c)\|_{\mathcal{E}_r} \leq \|t_n(z; c)\|_{\mathcal{E}_r} \) and \( \|q_n(z; c)\|_{\mathcal{E}_r} < 1 \). Moreover, they are even optimal in most cases.

§2. Results

The \( q_n \) are defined by

\[ q_n(z; c) = \frac{T_n(z) + \alpha_n}{T_n(c) + \alpha_n}, \quad \alpha_n = 2i\frac{\sin(n\gamma)}{(R^n - R^{-n})}. \]  

(2)

Here \( \alpha_n \) is the solution of the extremal problem

\[ M_n(r, c) = \min_{\alpha \in \mathbb{C}} \max_{z \in \mathcal{E}_r} \left| \frac{T_n(z) + \alpha}{T_n(c) + \alpha} \right|. \]  

(3)

We summarize the important properties of \( q_n(z; c) \) in the following
Theorem 1. [3]
(a) \( q_n(z; c) \) has precisely \( 2n \) extremal points \( z_j, j = 1, 2, \ldots, 2n \), on \( \partial \mathcal{E}_r \) with \( \| q_n(z; c) \|_{\mathcal{E}_r} = M_n(r, c) = (r^n + r^{-n})/(R^n + R^{-n}) \).
(b) There exists a number \( R_0(n, r) \) such that \( q_n(z; c) \equiv p_n(z; r, c) \) for all \( c \in \partial \mathcal{E}_R \) with \( R \geq R_0(n, r) \).
(c) Let \( c \in \partial \mathcal{E}_R \) be such that \( R > r(9r^4 - 1)/(r^4 - 1) \). Then, there exists an integer \( n_0(r, R) \) such that \( q_n(z; c) \equiv p_n(z; r, c) \) for all \( n \geq n_0(r, R) \).

Discussion: Supported by numerical tests (c.f. Example 2), we conjecture that (c) is true for arbitrary \( R > r > 1 \). \( \| q_n(z; c) \|_{\mathcal{E}_r} \) does not depend on the position of \( c \) on \( \partial \mathcal{E}_R \) and \( \| q_n(z; c) \|_{\mathcal{E}_r} \leq \| t_n(z; c) \|_{\mathcal{E}_r} \), where equality holds iff \( \sin(n\gamma) = 0 \), e.g. for \( c \in \mathbb{R} \) (cf. Example 1). The proof of Theorem 1 is based on the following characterization [5]: \( q_n(z; c) \equiv p_n(z; r, c) \) iff the linear system

\[
\sum_{j=1}^{2n} \sigma_j q_n(z_j; c)(z_j - c)p(z_j) = 0 \quad \text{for all } p \in \Pi_{n-1}
\]

has a nontrivial and nonnegative solution. See [3] for the explicit solution of (4).

Example 1. We compare \( \| q_n(z; c) \|_{\mathcal{E}_r} \) (continuous curve) and \( \| t_n(z; c) \|_{\mathcal{E}_r} \) (dashed curve) where \( r = 1.1, R = 1.2 \) for \( \gamma \in [0, \pi] \) and \( n = 3, 4 \) (cf. Figure 1).

![Figure 1. Maximum norm of q_n and t_n](image.png)

The nontrivial solutions of (4) always lead to a lower bound for the minimal deviation of problem (1), which is sharp in a certain sense:

Theorem 2. Let \( \sigma_j, j = 1, 2, \ldots, 2n \), be any nontrivial real solution of (4), normalized such that \( \sum_{j=1}^{2n} |\sigma_j| = 1 \), then

\[
L_n(r, c) = \frac{1}{M_n(r, c)} \left| \sum_{j=1}^{2n} \sigma_j q_n(z_j; c) \right| \leq E_n(r, c),
\]
where equality holds iff $q_n(z; c) \equiv p_n(z; r, c)$.

**Proof:** Let $p \in \Pi_{n-1}$. From (4) we obtain

$$\left| \sum_{j=1}^{2n} \sigma_j q_n(z_j; c) \right| = \left| \sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j; c)} \right| = \left| \sum_{j=1}^{2n} \sigma_j \overline{q_n(z_j; c)}(1 - (z_j - c)p(z_j)) \right|$$

$$\leq \left( M_n(r, c) \sum_{j=1}^{2n} |\sigma_j| \right) \left\| 1 - (z - c)p(z) \right\|_{\mathcal{C}},$$

and the result follows. □

We illustrate that $q_n$ is in general nearly optimal in the following

**Example 2.** In this example we compute the relative deviation $D_n(r, c) = (M_n(r, c) - L_n(r, c))/M_n(r, c)$ where $r = 2$ for $c \in [-2.1, 2.1] \times [-i2.1, i2.1]$ (here $D_n(r, c) := 0$ if $c \in \mathcal{E}_r$) and $n = 2, 3, 4, 5$. Note that $D_n(r, c) = 0$ if $q_n(r, c)$ is optimal (cf. Figure 2). We obtain $\max_{c \in \mathcal{C}} D_n(2, c) < 0.1024, 0.0498, 0.0336, 0.0210$ for $n = 2, 3, 4, 5$ resp.
Figure 2. Relative deviation of \( q_n \)

References