Electromagnetic Finite Elements
Based on a Four-Potential
Variational Principle

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ELECTROMAGNETIC FINITE ELEMENTS BASED ON
A FOUR-POTENTIAL VARIATIONAL PRINCIPLE

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SUMMARY

We derive electromagnetic finite elements based on a variational principle that uses the electromagnetic four-potential as primary variable. This choice is used to construct elements suitable for downstream coupling with mechanical and thermal finite elements for the analysis of electromagnetic/mechanical systems that involve superconductors. The key advantages of the four-potential are: the number of degrees of freedom per node remain modest as the problem dimensionality increases, jump discontinuities on interfaces are naturally accommodated, and static as well as dynamics are included without any a priori approximations. The new elements are tested on an axisymmetric problem under steady-state forcing conditions. The results are in excellent agreement with analytical solutions.
1. INTRODUCTION

The present work is part of a research program for the numerical simulation of electromagnetic/mechanical systems that involve superconductors. The simulation involves the interaction of the following four components:

1. **Mechanical Fields**: displacements, stresses, strains and mechanical forces.
2. **Thermal Fields**: temperature and heat fluxes.
3. **Electromagnetic (EM) Fields**: electric and magnetic field strengths and fluxes, currents and charges.
4. **Coupling Fields**: the fundamental coupling effect is the constitutive behavior of the materials involved. Particularly important are the metallurgical phase change phenomena triggered by thermal, mechanical and EM fields.

1.1 Finite Element Treatment

The first three fields (mechanical, thermal and electromagnetic) are treated by the finite element method. This treatment produces the spatial discretization of the continuum into mechanical, thermal and electromagnetic meshes of finite number of degrees of freedom. The finite element discretization may be developed in two ways:

1. **Simultaneous Treatment**. The whole problem is treated as an indivisible whole. The three meshes noted above become tightly coupled, with common nodes and elements.

2. **Staged Treatment**. The mechanical, thermal and electromagnetic components of the problem are treated separately. Finite element meshes for these components may be developed separately. Coupling effects are viewed as information that has to be transferred between these three meshes.

The present research follows the staged treatment. More specifically, *we develop finite element models for the fields in isolation, and then treat coupling effects as interaction forces between these models*. This “divide and conquer” strategy is ingrained in the partitioned treatment of coupled problems [4,16], which offers significant advantages in terms of computational efficiency and software modularity. Another advantage relates to the way research into complex problems can be made more productive. It centers on the observation that some aspects of the problem are either better understood or less physically relevant than others. These aspects may be then temporarily left alone while efforts are concentrated on the less developed and/or more physically important aspects. The staged treatment is better suited to this approach.

1.2 Mechanical Elements

Mechanical elements for this research have been derived using general variational principles that decouple the element boundary from the interior thus providing efficient ways to work
out coupling with non-mechanical fields. The point of departure was previous research into the free-formulation variational principles reported in Ref. [5]. A more general formulation for the mechanical elements, which includes the assumed natural strain formulation, was established and reported in Refs. [5,6,14,15]. New representations of thermal fields have not been addressed as standard formulations are considered adequate for the coupled-field phases of this research.

2. ELECTROMAGNETIC ELEMENTS

The development of electromagnetic (EM) finite elements has not received to date the same degree of attention given to mechanical and thermal elements. Part of the reason is the widespread use of analytical and semianalytical methods in electrical engineering. These methods have been highly refined for specialized but important problems such as circuits and waveguides. Thus the advantages of finite elements in terms of generality have not been enough to counterweight established techniques. Much of the EM finite element work to date has been done in England and is well described in the surveys by Davies [1] and Trowbridge [21]. The general impression conveyed by these surveys is one of an unsettled subject, reminiscent of the early period (1960-1970) of finite elements in structural mechanics. A great number of formulations that combine flux, intensity, and scalar potentials are described with the recommended choice varying according to the application, medium involved (polarizable, dielectric, semiconductors, etc.) number of space dimensions, time-dependent characteristics (static, quasi-static, harmonic or transient) as well as other factors of lesser importance. The possibility of a general variational formulation has not apparently been recognized.

In the present work, the derivation of electromagnetic (EM) elements is based on a variational formulation that uses the four-potential as primary variable. The electric field is represented by a scalar potential and the magnetic field by a vector potential. The formulation of these variational principle proceeds along lines previously developed for the acoustic fluid problem [7,8].

The main advantages of using potentials as primary variables as opposed to the more conventional EM finite elements based on intensity and/or flux fields are, in order of importance:

1. Interface discontinuities are automatically taken care of without any special intervention.
2. No approximations are invoked a priori since the general Maxwell equations are used.
3. The number of degrees of freedom per finite element node is kept modest as the problem dimensionality increases.
4. Coupling with the mechanical and thermal fields, which involves derived fields, can be naturally evaluated at the Gauss points at which derivatives of the potentials are evaluated.

Following a recapitulation of the basic field equations, the variational principle is stated.
The discretization of these principle into finite element equations produces semidiscrete dynamical equations, which are specialized to the axisymmetric case. These equations are validated in a simulation of a cylindrical conductor wire.

3. ELECTROMAGNETIC FIELD EQUATIONS

3.1 The Maxwell Equations

The original Maxwell equations (1873) involve four spatial fields: \( B, D, E \) and \( H \). Vectors \( E \) and \( H \) represents the electric and magnetic field strengths (also called intensities), respectively, whereas \( D \) and \( B \) represent the electric and magnetic flux densities, respectively. All of these are three-vector quantities, that is, vector fields in three-dimensional space \( (x_1 \equiv x, x_2 \equiv y, x_3 \equiv z) \):

\[
E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}, \quad E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad H = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}.
\]

Other quantities are the electric current 3-vector \( j \) and the electric charge density \( \rho \) (a scalar). Units for these and other quantities of interest in this work are summarized in Tables 1-2.

With this notation, and using superposed dots to denote differentiation with respect to time \( t \), we can state Maxwell equations as*

\[
\begin{align*}
\dot{B} + \nabla \times E &= 0, \\
\nabla \times H - \dot{D} &= j, \\
\nabla \cdot D &= \rho, \\
\nabla \cdot B &= 0.
\end{align*}
\]

The first and second equation are also known as Faraday's and Ampère-Maxwell laws, respectively.

The system (2) supplies a total of eight partial differential equations, which as stated are independent of the properties of the underlying medium.

3.2 Constitutive Equations

The field intensities \( E \) and \( H \) and the corresponding flux densities \( D \) and \( B \) are not independent but are connected by the electromagnetic constitutive equations. For an electromagnetically isotropic, non-polarized material the equations are

\[
B = \mu H, \quad D = \varepsilon E
\]

* Some authors, for example Eyges [2], include \( 4\pi \) factors and the speed of light \( c \) in the Maxwell equations. Other textbooks, e.g. [19,20], follow Heaviside's advice in using technical units that eliminate such confusing factors.
Table 1  Electric and Magnetic Quantities

<table>
<thead>
<tr>
<th>Quantities</th>
<th>Symbol</th>
<th>MKS-Weber Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric charge density</td>
<td>( \rho )</td>
<td>coulomb/m²</td>
</tr>
<tr>
<td>Electric field intensity</td>
<td>( \mathbf{E} )</td>
<td>newton/coulomb</td>
</tr>
<tr>
<td>Electric flux density</td>
<td>( \mathbf{D} )</td>
<td>coulomb/m²</td>
</tr>
<tr>
<td>Electric resistance</td>
<td>( R )</td>
<td>ohm</td>
</tr>
<tr>
<td>Electric conductivity</td>
<td>( q )</td>
<td>mho</td>
</tr>
<tr>
<td>Displacement current density</td>
<td>( \mathbf{J} )</td>
<td>coulomb/(sec.m²)</td>
</tr>
<tr>
<td>Susceptibility*</td>
<td>( \varepsilon )</td>
<td>coulomb/(joule.m)</td>
</tr>
<tr>
<td>Current</td>
<td>( J )</td>
<td>coulomb/sec</td>
</tr>
<tr>
<td>Magnetic field intensity</td>
<td>( \mathbf{H} )</td>
<td>newton/weber or amperes/m</td>
</tr>
<tr>
<td>Magnetic flux density</td>
<td>( \mathbf{B} )</td>
<td>weber/m²</td>
</tr>
<tr>
<td>Magnetic permeability†</td>
<td>( \mu )</td>
<td>weber/(joule.m) or henry/m</td>
</tr>
</tbody>
</table>

* Also called capacitivity and permittivity
† Also called inductivity

Table 2  Equivalences Between Various MKS-Giorgi Units

| 1 newton | 1 kg.m/sec² |
| 1 joule  | 1 newton.m  |
| 1 watt   | 1 joule/sec |
| 1 coulomb| 1 ampere.sec|
| 1 ohm    | 1 volt/ampere|
| 1 farad  | 1 coulomb/volt|
| 1 henry  | 1 (volt.sec)/ampere|
| 1 weber  | 1 volt.sec  |
| 1 mho    | 1 ohm⁻¹     |

where \( \mu \) and \( \varepsilon \) are the permeability and susceptibility, respectively, of the material†. These coefficients are functions of position but (for static or harmonic fields) do not depend on time. In the general case of a non-isotropic material both \( \mu \) and \( \varepsilon \) become tensors. Even in isotropic media \( \mu \) in general is a complicated function of \( \mathbf{H} \); in ferromagnetic materials it depends on the previous history (hysteresis effect).

In free space \( \mu = \mu_0 \) and \( \varepsilon = \varepsilon_0 \), which are connected by

\[
\varepsilon_0^2 = \frac{1}{\mu_0 \varepsilon_0}
\]

where \( c_0 \) is the speed of light in a free vacuum. In MKS-A units, \( c_0 = 3.10^9 \) m/sec and

\[
\mu_0 = 4\pi \times 10^{-7} \text{ henry/m}, \quad \varepsilon_0 = \mu_0^{-1} c_0^{-2} = (36\pi)^{-1} \times 10^{-11} \text{ sec}^2/(\text{henry.m})
\]

† Other names are often used, see Table 1.
The condition $\mu = \mu_0$ holds well for most practical purposes in such media as air and copper; in fact $\mu_{\text{air}} = 1.0000004\mu_0$ and $\mu_{\text{copper}} = .99999\mu_0$.

The electrical field strength $E$ is further related to the current density $j$ by Ohm's law:

$$j = gE$$

where $g$ is the conductivity of the material. Again for an non-isotropic material $g$ is generally a tensor which may also contain real and imaginary components; in which case the above relation becomes the generalized Ohm's law. For good conductors $g >> \epsilon$; for bad conductors $g << \epsilon$. In free space, $g = 0$.

3.3 Maxwell Equations in Terms of $E$ and $B$

To pass to the four-potential considered in Section 4 it is convenient to express Maxwell's equations in terms of the electrical field strength $E$ and the magnetic flux $B$. In fact this is the pair most frequently used in electromagnetic work that involve arbitrary media. On eliminating $D$ and $H$ through the constitutive equations (3) we obtain

$$\dot{B} + \nabla \times E = 0, \quad \nabla \times B - \mu \epsilon \dot{E} = \mu j, \quad \nabla \cdot E = \rho / \epsilon, \quad \nabla \cdot B = 0.$$  \hspace{1cm} (7)

The second equation assumes that $\epsilon$ is independent of time; otherwise $\dot{E} = \epsilon dE/dt$ should be replaced by $d(\epsilon E)/dt$. In charge-free vacuum the equations reduce to

$$\dot{B} + \nabla \times E = 0, \quad \nabla \times B - \frac{1}{c^2} \dot{E} = 0, \quad \nabla \cdot E = 0, \quad \nabla \cdot B = 0.$$  \hspace{1cm} (8)

3.4 The Electromagnetic Potentials

The electric scalar potential $\Phi$ and the magnetic vector potential $A$ are introduced by the definitions

$$E = -\nabla \Phi - \dot{A}, \quad B = \nabla \times A.$$  \hspace{1cm} (9)

This definition satisfies the two homogeneous Maxwell equations in (7). The definition of $A$ leaves its divergence $\nabla \cdot A$ arbitrary. We shall use the Lorentz gauge [13]

$$\nabla \cdot A + \mu \epsilon \dot{\Phi} = 0.$$  \hspace{1cm} (10)

With this choice the two non-homogeneous Maxwell equations in terms of $\Phi$ and $A$ separate into the wave equations

$$\nabla^2 \Phi - \mu \epsilon \ddot{\Phi} = -\rho / \epsilon, \quad \nabla^2 A - \mu \epsilon \ddot{A} = -\mu j.$$  \hspace{1cm} (11)
4. THE ELECTROMAGNETIC FOUR-POTENTIAL

Maxwell's equations can be presented in a compact manner* in the four-dimensional spacetime defined by the coordinates

\[ x_1 \equiv x, \quad x_2 \equiv y, \quad x_3 \equiv z, \quad x_4 = \imath ct \quad (12) \]

where \( x_1, x_2, x_3 \) are spatial Cartesian coordinates, \( \imath^2 = -1 \) is the imaginary unit, and \( c = 1/\sqrt{\mu \varepsilon} \) is the speed of EM waves in the medium under consideration. In the sequel Roman subscripts will consistently go from 1 to 4 and the summation convention over repeated indices will be used unless otherwise stated.

4.1 The Field Strength Tensor

The unification can be expressed most conveniently in terms of the field-strength tensor \( \mathbf{F} \), which is a four-dimensional antisymmetric tensor constructed from the components of \( \mathbf{E} \) and \( \mathbf{B} \) as follows:

\[ \mathbf{F} = \begin{pmatrix} 0 & F_{12} & F_{13} & F_{14} \\ -F_{12} & 0 & F_{23} & F_{24} \\ -F_{13} & -F_{23} & 0 & F_{34} \\ -F_{14} & -F_{24} & -F_{34} & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \quad (13) \]

Here \( \beta \) is an adjustment factor to be determined later. Similarly, introduce the four-current vector \( \mathbf{J} \) as

\[ \mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \end{pmatrix} \quad (14) \]

Then, for arbitrary \( \beta \), the non-homogeneous Maxwell equations, namely \( \nabla \times \mathbf{B} - \mu \epsilon \dot{\mathbf{E}} = \mathbf{\mu j} \) and \( \nabla \cdot \mathbf{E} = \rho/\epsilon \), may be presented in the compact "continuity" form†

\[ \frac{\partial F_{ik}}{\partial x_k} = J_i \quad (15) \]

The other two Maxwell equations, \( \nabla \cdot \mathbf{B} = 0 \) and \( \nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0 \), can be presented as

\[ \frac{\partial F_{ik}}{\partial x_m} + \frac{\partial F_{mi}}{\partial x_k} + \frac{\partial F_{km}}{\partial x_i} = 0, \quad (16) \]

where the index triplet \((i, j, k)\) takes on the values \((1,2,3), (4,2,3), (4,3,1)\) and \((4,1,2)\).

* A form compatible with special relativity.
† The covariant form of these two equations.
4.2 The Four-Potential

The EM "four-potential" $\phi$ is a four-vector whose components are constructed with the electric and magnetic potential components of $A$ and $\Phi$:

$$\phi = \beta \begin{cases} 
\phi_1 \\
\phi_2 \\
\phi_3 \\
\phi_4
\end{cases} \overset{\text{def}}{=} \begin{cases} 
 cA_1 \\
 cA_2 \\
 cA_3 \\
i\Phi
\end{cases}.$$  

(17)

It may then be verified that $F$ can be expressed as the four-curl of $\phi$, that is

$$F_{ik} = \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k},$$  

(18)

or in more detail and using commas to abbreviate partial derivatives:

$$F = \begin{pmatrix} 
0 & \phi_{2,1} - \phi_{1,2} & \phi_{3,1} - \phi_{1,3} & \phi_{4,1} - \phi_{1,4} \\
\phi_{1,2} - \phi_{2,1} & 0 & \phi_{3,2} - \phi_{2,3} & \phi_{4,2} - \phi_{2,4} \\
\phi_{1,3} - \phi_{3,1} & \phi_{2,3} - \phi_{3,2} & 0 & \phi_{4,3} - \phi_{3,4} \\
\phi_{1,4} - \phi_{4,1} & \phi_{2,4} - \phi_{4,2} & \phi_{3,4} - \phi_{4,3} & 0
\end{pmatrix}.$$  

(19)

4.3 The Lagrangian

With these definitions, the basic Lagrangian of electromagnetism can be stated as:

$$L = \frac{1}{4} F_{ik} F_{ik} - J_i \phi_i = \frac{1}{4} \beta^2 \left( \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k} \right)^2 - J_i \phi_i$$  

(20)

$$= \frac{1}{2} \beta^2 (c^2 B^2 - E^2) - \frac{\beta^2}{\epsilon} (j_1 A_1 + j_2 A_2 + j_3 A_3 - \rho \Phi),$$

in which

$$B^2 = B^T B = B_1^2 + B_2^2 + B_3^2, \quad E^2 = E^T E = E_1^2 + E_2^2 + E_3^2.$$  

(21)

Comparing the first term with the magnetic and electric energy densities [2,19,20]

$$u_m = \frac{1}{2} B^T H = \frac{1}{2\mu} B^2, \quad u_e = \frac{1}{2} D^T E = \frac{1}{2\epsilon} E^2,$$  

(22)

we must have $\beta^2 c^2 = \beta^2/\mu\epsilon = 1/\mu$, from which

$$\beta = \sqrt{\epsilon}.$$  

(23)

† Lanczos [12] presents this Lagrangian for free space, but the expression (24) for an arbitrary material was found in none of the textbooks on electromagnetism listed in the References.
Consequently the required Lagrangian is

\[ L = \frac{1}{2\mu} B^2 - \frac{1}{2} \epsilon E^2 - (j_1 A_1 + j_2 A_2 + j_3 A_3 - \rho \Phi). \tag{24} \]

The associated variational form is

\[ R = \int_{t_0}^{t_1} \int_V L \, dV \, dt \tag{25} \]

where \( V \) is the integration volume considered in the analysis. In theory \( V \) extends over the whole space, but in the numerical simulation the integration is truncated at a distant boundary or special devices are used to treat the decay behavior at infinity.

4.4 The Four-Field Equations

On setting the variation of the functional (24) to zero we recover the field equations (15–16). Taking the divergence of both sides of (15) and observing that \( F \) is an antisymmetric tensor so that its divergence vanishes we get

\[ \frac{\partial J_i}{\partial x_i} = c\mu (\nabla \cdot j + \rho) = 0, \tag{26} \]

The vanishing term in parenthesis is the equation of continuity, which expresses the law of conservation of charge. The Lorentz gauge condition (10) may be stated as \( \nabla \cdot \phi = 0 \). Finally, the potential wave equations (11) may be expressed in compact form as

\[ \Box \phi_i = -J_i \tag{27} \]

where \( \Box \) denotes the "four-wave-operator", also called the D’Alembertian:

\[ \Box \overset{\text{def}}{=} \frac{\partial^2}{\partial x_k \partial x_k} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{c^2 \partial t^2}. \tag{28} \]

Hence each component of the four-potential \( \phi \) satisfies an inhomogeneous wave equation. In free space, \( J_i = 0 \) and each component satisfies the homogeneous wave equation.
5. THE AXISYMMETRIC TEST EXAMPLE

The simplest example for testing the finite element formulation based on the four-potential variational principle is provided by the axisymmetric magnetic field generated by a uniform, steady current flowing through a straight, infinitely long conducting wire of circular cross section. In the present Section we derive expressions for the magnetostatic fields outside and within the conductor. These analytical solutions will be later compared with the finite element numerical solutions.

5.1 The Free-Space Magnetic Field

To take advantage of the axisymmetric geometry we choose a cylindrical coordinate system with the wire centerline as the longitudinal z-axis. The vector components in the cylindrical coordinate directions \( r, \theta \) and \( z \) are denoted by

\[
A_1, B_1, E_1 \quad \text{in the } r \text{ direction}
\]

\[
A_2, B_2, E_2 \quad \text{in the } \theta \text{ direction}
\]

\[
A_3, B_3, E_3 \quad \text{in the } z \text{ direction}
\]

The electromagnetic fields will then vary in the radial direction (\( r \)) but not in the angular (\( \theta \)) and axial (\( z \)) directions. Similarly, the current density that flows in the wire has only one nonzero component acting in the positive or negative \( z \) direction; conventionally we select the positive direction.

In Cartesian coordinates the radial component of the electrostatic potential in free space can be calculated from the expression (see, e.g., [2,10,18,19,20])

\[
A_z = A_3 = \frac{\mu_0}{4\pi} \int_V \frac{j_3 \, dV}{|r|},
\]  

(29)

where \(|r|\) is the distance between the elemental charge \( j_3 \, dV \) and the point in space at which we wish to find the field potential. The integral extends over the volume containing charges. This expression serves equally well in cylindrical coordinates. In fact, the transformation of \( z \) components will be one to one if the center of the systems coincide.

As noted above the only non-vanishing component of the current vector is \( j_3 \, dS \) where \( dS \) is the elemental cross sectional area of the conductor and \( j_3 \) is the current density in the \( z \) direction. If \( dl \) represents the differential length of the wire, then \( \int_S j_3 \, dS = \int_S j_3 \, dS \, dl = I \, dl = I \, dz \) and \(|r| = \sqrt{r^2 + z^2} \). Substitution into Eq. (29) yields

\[
A_3(r) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{r^2 + z^2}}.
\]

(30)

This integral diverges, but this difficulty can be overcome by taking the wire to have a finite length \( 2L \) symmetric with respect to the field point, that is large with respect to its diameter. Integrating between \(-L\) and \(+L\) we get

\[
A_3(r) = \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz}{\sqrt{r^2 + z^2}} = \frac{\mu_0 I}{4\pi} \ln \left( z + \sqrt{r^2 + z^2} \right) \bigg|_{-L}^{+L}.
\]

(31)
Expanding this equation in powers of \( r/L \) and retaining only first-order terms gives

\[
A_3 = -\left( \frac{\mu_0 I}{2\pi} \right) \ln r + C. \tag{32}
\]

where \( C \) is an arbitrary constant. For subsequent developments it is convenient to select \( C = (\mu_0 I/2\pi) \ln R_T \), where \( R_T \) is the “truncation radius” of the finite element mesh in the radial direction. Then

\[
A_3 = -\left( \frac{\mu_0 I}{2\pi} \right) \ln \left( \frac{r}{R_T} \right). \tag{33}
\]

With this normalization \( A_3 = 0 \) at \( r = R_T \). Taking the curl of \( A \) gives the \( B \) field in cylindrical coordinates:

\[
B = \nabla \times A = \begin{cases} B_1 \\ B_2 \\ B_3 \end{cases} = \begin{cases} B_r \\ B_\theta \\ B_z \end{cases} = \begin{cases} \frac{1}{r} \frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z} \\ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r} \\ \frac{1}{r} \frac{\partial (r A_2)}{\partial r} - \frac{1}{r} \frac{\partial A_1}{\partial \theta} \end{cases} = \begin{cases} 0 \\ -\frac{\partial A_3}{\partial r} \\ 0 \end{cases}. \tag{34}
\]

It is seen that the only non-vanishing component of the magnetic flux density is

\[
B_\theta = B_2 = \mu_0 H_2 = -\frac{\partial A_3}{\partial r} = \frac{\mu_0 I}{2\pi r}. \tag{35}
\]

This expression is called the law of Biot-Savart in the EM literature.

5.2 Magnetic Field Within the Conductor

Again restricting our consideration to the static case, we have from Maxwell’s equations in their integral flux form

\[
\oint_C H \cdot ds = \oint_C \mu^{-1} B \cdot ds = \int_S j \cdot dS, \tag{36}
\]

where \( C \) is a contour around the field point traversed counterclockwise with an oriented differential arclength \( ds \) and \( dS \) is the oriented surface element inside the contour. The term for the electric field disappears in this analysis because \( \dot{E} = 0 \). From before we know that the right hand side of Eq. (35) is equal to the normal component of the current that flows through the cross sectional area evaluated by the integral. In the free space case, this is the total current that flows through the conductor. But in the conductor the amount of current is a function of the distance \( r \) from the center. Again using \( I \) to represent the total current carried by the conductor, and \( R \) the radius of the conductor, and assuming an uniform current density \( j_3 = I/(\pi R^2) \), the right hand side of (35) become

\[
\int_S j \cdot dS = \int_S j_3 dS = \frac{I}{\pi R^2} \int_S dS = I \frac{r}{R^2}. \tag{37}
\]
Evaluating the left hand side of the integral and solving for $B_2$ gives:

$$2\pi r \mu^{-1} B_2 = I \frac{r^2}{R^2}, \quad B_2 = \frac{\mu I r}{2\pi R^2}. \tag{38}$$

Comparing with (34) we see that if $\mu = \mu_0$ then $B_2$ is continuous at the wire surface $r = R$ and has the value $\mu_0 I/(2\pi R)$. But if $\mu \neq \mu_0$ there is a jump $(\mu - \mu_0) I/(2\pi R)$ in $B_2$.

The magnetic potential $A_3$ within the conductor is easily computed by integrating $-B_2$ with respect to $r$:

$$A_3 = -\frac{\mu I r^2}{4\pi R^2} + C. \tag{39}$$

The value of $C$ is determined by matching (33) at $r = R$, since the potential must be continuous. The result can be written

$$A_3 = \frac{I}{2\pi} \left[ \frac{1}{2} \mu \left( 1 - \frac{r^2}{R^2} \right) - \mu_0 \ln \left( \frac{R}{R_T} \right) \right]. \tag{40}$$

The preceding expressions (33)-(40) for $A_3$ could also be derived in a somewhat more direct fashion by integrating the ordinary differential equation $\nabla^2 A_3 = r^{-1} (\partial(\partial A_3/\partial r) \partial r) = \mu j_3$ to which the second of (11) reduces.

6. FINITE ELEMENT DISCRETIZATION

6.1 The Lagrangian in Cylindrical Coordinates

To construct finite element approximations we need to express the Lagrangian (24)

$$L = \frac{1}{2\mu} B^2 - \frac{1}{2} \epsilon E^2 - (j^T A - \rho \Phi), \tag{41}$$

in terms of the potentials written in cylindrical coordinates. For $B^2$ we can use the expression of the curl (33)

$$B^2 = \left( \frac{1}{r} \frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial z} \right)^2 + \left( \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial (r A_2)}{\partial r} - \frac{1}{r} \frac{\partial A_1}{\partial \theta} \right)^2. \tag{42}$$

For $E^2$ we need the cylindrical-coordinate gradient formulas

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_r \\ E_\theta \\ E_z \end{bmatrix} = \begin{bmatrix} E_r \\ E_\theta \\ E_z \end{bmatrix} = -\begin{bmatrix} \frac{\partial \Phi}{\partial r} + \dot{A}_1 \\ \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \dot{A}_2 \\ \frac{\partial \Phi}{\partial z} + \dot{A}_3 \end{bmatrix}, \tag{43}$$

so that

$$E^2 = E^T E = \left( \frac{\partial \Phi}{\partial r} + \frac{\partial A_1}{\partial t} \right)^2 + \left( \frac{1}{r} \frac{\partial \Phi}{\partial \theta} + \frac{\partial A_2}{\partial t} \right)^2 + \left( \frac{\partial \Phi}{\partial z} + \frac{\partial A_3}{\partial t} \right)^2. \tag{44}$$
In the axisymmetric case, $A_1 = A_2 = 0$; furthermore $A_z = A_3$ is only a function of the radial distance from the wire. Therefore $\partial A_3/\partial \theta = \partial A_3/\partial z = 0$. From symmetry considerations we also know that the electric field cannot vary in the $\theta$ and $z$ directions, which gives $\partial \Phi/\partial z = \partial \Phi/\partial \theta = 0$. Finally, the only nonvanishing current density component is $j_3$. Consequently the Lagrangian (41) simplifies to

$$L = \frac{1}{2\mu} \left( \frac{\partial A_3}{\partial r} \right)^2 - \frac{1}{2\varepsilon} \left[ \left( \frac{\partial \Phi}{\partial r} \right)^2 + \left( \frac{\partial A_3}{\partial t} \right)^2 \right] - \left( j_3 A_3 - \rho \Phi \right).$$

(45)

6.2 Constructing EM Finite Elements

To deal with this particular axisymmetric problem a two-node "line" finite element extending in the radial $r$ direction is sufficient. In the following we deal with an individual element identified by superscript $e$. The two element end nodes are denoted by $i$ and $j$. The electric potential $\Phi$ and the magnetic potential $A_3 \equiv A_z$ are interpolated over each element as

$$\phi^e = N_\phi^e \Phi^e, \quad A_3^e = N_A^e A_3^e,$$

(46)

Here row vectors $N_\phi^e$ and $N_A^e$ contain the finite element shape functions for $\Phi^e$ and $A_3^e$, respectively, which are only functions of the radial coordinate $r$:

$$N_\phi^e = \begin{pmatrix} N_{\phi i}(r) \\ N_{\phi j}(r) \end{pmatrix}, \quad N_A^e = \begin{pmatrix} N_{A i}(r) \\ N_{A j}(r) \end{pmatrix},$$

(47)

and column vectors $\Phi^e$ and $A_3^e$ contain the nodal values of $\Phi$ and $A_3$, respectively, which are only functions of time $t$:

$$\Phi^e = \begin{bmatrix} \Phi_i(t) \\ \Phi_j(t) \end{bmatrix}, \quad A_3^e = \begin{bmatrix} A_{3i}(t) \\ A_{3j}(t) \end{bmatrix}.$$  

(48)

Substitution of these finite element assumptions into the Lagrangian (45) and then into Eq. (25) yields the variational integral as sum of element contributions $R = \sum_e R^e$, where

$$R^e = \int_{t_0}^{t_1} \int_{V^e} \frac{1}{2\mu} \left( \frac{\partial N_A^e A_3^e}{\partial r} \right)^2 - \frac{1}{2\varepsilon} \left[ \left( \frac{\partial N_\phi^e \Phi^e}{\partial r} \right)^2 + \left( \frac{\partial N_A^e A_3^e}{\partial t} \right)^2 \right]$$

$$- \left( j_3 N_A^e A_3^e - \rho N_\phi^e \Phi^e \right) \, dV^e \, dt.$$  

(49)

where $V^e$ denotes the volume of the element. Taking the variation with respect to the element node values gives

$$\delta R^e = \int_{t_0}^{t_1} \int_{V^e} (\delta A_3^e)^T \left[ \frac{1}{\mu} \left( \frac{\partial N_A^e}{\partial r} \right)^T \frac{\partial N_A^e}{\partial r} A_3^e + \epsilon (N_A^e)^T N_A^e \dot{A}_3^e - j_3 (N_A^e)^T \right]$$

$$+ \int_{t_0}^{t_1} \int_{V^e} (\delta \Phi^e)^T \left[ -\epsilon (\frac{\partial N_\phi^e}{\partial r})^T \frac{\partial N_\phi^e}{\partial r} \Phi^e + \rho (N_\phi^e)^T \right] \, dV^e \, dt.$$  

(50)
On applying fixed-end initial conditions at $t = t_0$ and $t = t_1$ and the lemma of the calculus of variations, we proceed to equate each of the expressions in brackets to zero. From the first bracket we obtain for each element the following second-order dynamic equations for the magnetic potential at the nodes, which are purposely written in a notation resembling the mass-stiffness-force equations of mechanics:

$$M_A^e \ddot{A}_3^e + K_A^e A_3^e = f_A^e,$$

where

$$M_A^e = \int_{V_0} \epsilon (N_A^e)^T N_A^e \, dv, \quad K_A^e = \int_{V_0} \frac{1}{\mu} \left( \frac{\partial N_A^e}{\partial r} \right)^T \frac{\partial N_A^e}{\partial r} \, dv,$$  \hspace{1cm} (52)

$$f_A^e = \int_{V_0} j_3 (N_A^e)^T \, dv.$$  \hspace{1cm} (53)

From the second bracket we obtain for the electric potential a simpler relation which does not involve time derivatives, i.e., is static in nature:

$$K_\Phi^e \Phi^e = f_\Phi^e,$$  \hspace{1cm} (54)

where

$$K_\Phi^e = \int_{V_0} \epsilon \left( \frac{\partial N_\Phi^e}{\partial r} \right)^T \frac{\partial N_\Phi^e}{\partial r} \, dv, \quad f_\Phi^e = \int_{V_0} \rho (N_\Phi^e)^T \, dv.$$  \hspace{1cm} (55)

Assembling these equations in the usual way we obtain the semidiscrete master finite element equations:

$$M_A \ddot{A}_3 + K_A A_3 = f_A,$$

$$K_\Phi \Phi = f_\Phi.$$  \hspace{1cm} (56)

6.9 The Static Case

In time-independent problems, the term $\ddot{A}_3$ disappears from (56) and the master finite element equations of electromagnetostatics become

$$K_A A_3 = f_A,$$

$$K_\Phi \Phi = f_\Phi.$$  \hspace{1cm} (57)

If the current density and charge distributions are known a priori then these two equations may be solved separately. If only the charge distribution $\rho$ is known then the second equation should be solved first to obtain the electric field $E$ as gradient of the computed electric potential $\Phi$; then the current density $j$ can be obtained from Ohm's law (6) and used to computed the force vector $f_A$ of the first equation, which is then solved for the magnetic potential. Conversely, if only the current density distribution is known a priori the preceding steps are reversed.

For the present test problem the current distribution is assumed to be known, and we shall be content with solving the first equation for the magnetic flux.
6.4 An Alternative Semidiscretization

If upon setting the brackets of the variation (50) to zero we multiply them through by $\mu$ and $1/\epsilon$, respectively, the expressions for the mass, stiffness and force matrices become

$$
M^e_A = \int_V \frac{1}{c^2} (N_A^e)^T N_A^e dV^e, \quad K^e_A = \int_V \left( \frac{\partial N_A^e}{\partial r} \right)^T \frac{\partial N_A^e}{\partial r} dV^e, \quad f_A^e = \int_V \mu j \omega N_A^e^T dV^e, \\
K^e = \int_V \left( \frac{\partial N_A^e}{\partial r} \right)^T \frac{\partial N_A^e}{\partial r} dV, \quad f^e = \int_V \frac{1}{\epsilon} \rho (N_\phi^e)^T dV.
$$

(58)

The matrices $M$ and $K$ above are quite similar to the capacitance and reactance matrices, respectively, obtained in the potential analysis of acoustic fluids [7,8]. Another attractive feature of (58) is that $K_A = K_\phi$ if the shape functions of both potentials coalesce, as is natural to assume. These advantages are, however, more than counterbalanced by the fact that "jump forces" contributions to $f_A$ and $f_\phi$ arise on material interfaces where $\mu$ and $\epsilon$ change abruptly, and the proper handling of such forces substantially complicates the programming logic. Note that this issue does not arise in the treatment of homogeneous acoustic fluids.

6.5 Applying Boundary Conditions

The finite element mesh is necessarily terminated at a finite size, which for the test problem is defined as the truncation radius $R_T$ alluded to in Section 5.1. In static calculations the material outside the FE mesh may be viewed as having zero permeability $\mu$, or, equivalently, infinite stiffness or zero potential. It follows that the potential value at the node located on the truncation radius may be prescribed to be zero. This is the only essential boundary condition necessary for this particular problem.

7. NUMERICAL VALIDATION

7.1 Finite Element Model

The test problem consists of a wire conductor of radius $R$ transporting a unit current density. For this problem the finite element mesh is completely defined if we specify the radial node coordinates $r_i^e = r_n$ and $r_j^e = r_{n+1}$ for each element $e$. If the mesh contains $N_{ec}$ elements inside the conductor, those elements are numbered $e = 1, 2, \ldots N_{ec}$ and nodes $n = 1, 2, \ldots N_{ec} + 1$ starting from the conductor center outwards. The first node ($n = 1$) is at the conductor center $r = 0$ and node $n = N_{ec} + 1$ is placed at the conductor boundary $r = R$. The mesh is then continued with $N_{ef}$ elements into free space, with a double node at the conductor boundary. The last node is placed at $r = R_T$ at which point the free space mesh is truncated; usually $R_T = 4R$ to $5R$. Although the mesh appears to be one-dimensional, a typical element actually forms a "tube" of longitudinal axis $z$, internal radius $r_i^e$ and external radius $r_j^e$, extending a unit distance along $z$. 

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Figure 1. Magnetic potential $A_3$ vs. distance from center $r$, $\mu_{\text{wire}} = 10.0$: finite element values (triangles) and analytical values (squares).

Figure 2. Magnetic potential $A_3$ vs. distance from center $r$, $\mu_{\text{wire}} = 1.0$: finite element values (triangles) and analytical values (squares).
Figure 3. Magnetic flux density $B_2$ vs. distance from center $r$, $\mu_{wire} = 10.0$: finite element values (triangles) and analytical values (squares). Values shown on the interface $r = 1$ with dark symbols have been extrapolated from element center values to display the jump more accurately; this extrapolation scheme has not been used elsewhere.

Figure 4. Magnetic flux density $B_2$ vs. distance from center $r$, $\mu_{wire} = 1.0$: finite element values (triangles) and analytical values (squares).
For the present study the magnetic potential was linearly interpolated in $r$, using the linear shape functions

$$N_A^e = \left( \frac{1}{2}(1 - \xi), \frac{1}{2}(1 + \xi) \right), \quad (59)$$

where $\xi$ is the dimensionless isoparametric coordinate that varies from $-1$ at node $i$ to $+1$ at node $j$. This interpolation provides for $C^0$ continuity of the potential inside the conductor and in free space.

For the calculation of the element stiffnesses and force vectors, it was assumed that the permeability $\mu$ and the current density $j_3$ were uniform over the element. Then analytical integration over the element geometry gives

$$K_A^e = \frac{\mu r_m}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad f_A^e = j_3 \ell \left\{ \frac{1}{6}(2r_i^e + r_j^e) \right\}, \quad (60)$$

where $r_m = \frac{1}{2}(r_i^e + r_j^e)$ is the mean radius and $\ell = r_j^e - r_i^e$ the radial length. For the test problem, $\mu$ is constant inside the conductor whereas outside it $\mu = \mu_0$ was assumed to be unity. The longitudinal current density is $j_3 = I/(\pi R^2)$ inside the conductor whereas outside it $j_3$ vanishes.

The master stiffness matrix and force vector were assembled following standard finite element techniques. The only essential boundary condition was the setting of the nodal potential on the truncation boundary to zero, as explained in Section 6.5. The modified master equations were processed by a conventional symmetric skyline solver, which provided the value of the magnetic potential at the mesh nodes. The magnetic flux density $B_2$, which is constant over each element, was recovered in element by element fashion through the simple finite difference scheme

$$B_2^e = -\frac{\partial A_3}{\partial r} \approx \frac{A_3^{e_i} - A_3^{e_j}}{\ell} \quad (61)$$

This value is assigned to the center of element $e$.

### 7.2 Numerical Results

The numerical results shown in Figures 1 through 6 pertain to a unit radius conductor ($R = 1$), with the external (free space) mesh truncated at $R_T = 5$. The element radial lengths $r_j^e - r_i^e$ were kept constant and equal to 0.25, which corresponds to 4 internal and 16 external elements.

The computed values of the potential $A_3$ are compared with the analytical solution given by Eqs. (33) and (40). As can be seen the agreement is excellent. The comparison between computed and analytical values of the magnetic flux density $B_2$ shows excellent agreement except for the last element near the wire center, at which point the difference scheme (61) loses accuracy. The permeability of free space is conventionally selected to be unity. Figures 1, 3, and 5 illustrate the case where the wire permeability $\mu_{wire}$ is set to 10.0, whereas Figures 2, 4, and 6 are for the case in which $\mu_{wire}$ is 1.0, that is, same as in free space. (The value of the susceptibility $\epsilon$ does not appear in these magnetostatic
Figure 5. Restriction of Figure 3 to $r > R = 1$, $\mu_{\text{wire}} = 10.0$, showing free space magnetic flux density in more detail.

Figure 6. Restriction of Figure 4 to $r > R = 1$, $\mu_{\text{wire}} = 1.0$, showing free space magnetic flux density in more detail.
computations.) Figures 1 and 2 show computed and analytical magnetic potentials. The slope discontinuity at $r = 1$ in Figure 1 is a consequence of the change in permeability $\mu$ from the wire material to free space. Figures 3 and 4 show the computed and analytical magnetic flux densities. As discussed in Section 5.2, the jump at $r = 1$ in Figure 3 is due to the change in permeability $\mu$ from the material to free space. Figures 5 and 6 show the computed and analytical magnetic flux densities in free space with more detail. Note that Figures 5 and 6 for $r > 1$ are identical; this is the expected result because, as shown in Section 5.1, the free-space magnetic flux field depends only upon the current enclosed by a surface integral around the wire and not on the details of the interior field distribution.

In summary, the finite element model performed very accurately in the test problem and converged, as expected, to the analytical solution as the size of the elements decreased.

8. CONCLUSIONS

The results obtained in the one-dimensional steady-state case are encouraging, and appear to be extensible to two- and three-dimensional problems without major difficulties. The electric field remains effectively decoupled from the magnetic field except through Ohm's law. Care must be taken, however, in modeling the forcing function terms so as to avoid the appearance of discontinuity-induced forces at physical interfaces.

The next step in achieving the goal of a finite element model for a superconductor field is to study the time-dependent case, starting with harmonic currents and proceeding eventually to general transients. The code for this is currently written, but a suitable analytical solution for comparison with computed responses is still being developed.

If encouraging results are obtained in the dynamic case, thermocoupling effects will be added to the code. References [3,17,22] discuss several different approaches applicable to various contexts (e.g. eddy currents) and these will have to be investigated for suitability for capturing the couplings effects that are relevant to the superconducting problem.

After modeling the coupling effects, the next step will be to model the superconducting fields. The feasibility of using the current model for superconductor applications is great, as the current density of a superconductor can be approximated by the standard current density multiplied by a constant squared. This constant is called the London penetration depth. Other analytical models that possess similar characteristics have been developed and are presented in Ref. [11].

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APPENDIX: COMPUTER PROGRAM

This Appendix lists the computer program used to test the new electromagnetic elements on the axisymmetric test example. Sections of the program that pertain to the in-core skyline solver SKYFAC/SKY SOL and the command language reader TinyClip are not listed here. Their source code is presented in the following publications:


```
C=DECK AAWIRE
C=BLOCK FORTRAN

program WIRE

integer MUMEL, MUNNP, MDOF
parameter (MUMEL=100, MUNNP=MUMEL+1)
parameter (MDOF=MUNNP)
integer numel, numnp, ndof

character CCLVAL
character status*60
integer nodelm(2, MUMEL), bctag(MUNNP)
double precision kmuelm(MUMEL), kepselem(MUMEL)
integer dlp(0:MDOF)
double precision a(MUNNP), b(MUMEL)
double precision r(MUNNP), t(MUNNP), fbc(MUNNP)
double precision sm(MDOF+3)
double precision aex(MUNNP), bex(MUMEL), fex(MUNNP)
double precision vi(MUNNP), v2(MUMEL)
double precision kmu, keps, wrad, trad, inten
integer nelwir, nelex

1000 continue

call MATERIAL (kmu, keps)
call PRINTMAT (kmu, keps)
call DIMENSIONS (wrad, trad)
call PRINTDIM (wrad, trad)
call CURRENT (inten)
call PRINTCUR (inten)

1500 continue

call SUBDIVIDE (nelwir, nelex, numel, numnp, ndof)
call PRINSSUB (nelwir, nelex, numel, numnp, ndof)
call GENWLEMS
```
C 4000 continue
C
C call CLREAD . (' New FE subdivision (y/n)? ', ', ', )
if (CCLVAL(1) .eq. 'Y') go to 1500
C call CLREAD . (' New problem data (y/n)? ', ', ', )
if (CCLVAL(1) .eq. 'Y') go to 1000
stop
der C
C=END FORTRAN
C=DECL ASSEMFOR
C=PREP Purpose Assemble force vector
C=BLOCK FORTRAN

subroutine ASSEMFOR
$ (numel, nodelm, kmuelm, kepselm, inten,$ nummp, r, wrad, trad, ndof, bctag, f, fbc, status)
integer numel, nodelm(2,*), nummp
integer ndof, bctag(ndof)
integer eldof(2)
double precision r(*), kmuelm(*), kepselm(*)
double precision inten, wrad, trad
double precision f(*), fbc(*)
character(*) status
double precision re(2), fe(2), mu
integer i, j, n, ne

C
C $ status = 
C do 1500 j = 1, ndof
C

1500 continue

C

do 3000 ne = 1,numel
   do 2200 i = 1,2
      n = nodelm(i,ne)
      re(i) = r(n)
      edof(i) = n
   continue
   mu = kmuelm(ne)
   call FORCE (ne, re, inten, wrad, fe, status)
   if (status .ne. ' ') then
      call ERROR ('ASSEMFOR', status)
   end if
   do 2500 i = 1,2
      j = edof(i)
      f(j) = f(j) + fe(i)
   2500 continue
C

3000 continue

do 4000 j = 1,ndof
   fbc(j) = f(j)
   if (bctag(j) .ne. 0) fbc(j) = 0.0
   4000 continue

return
end
C=END FORTRAN
C=DECK ASSEMSTF
C= PURPOSE Assemble master stiffness matrix
C=BLOCK FORTRAN

subroutine ASSEMSTF
$ (numel, nodelm, kmuelm, kpselm, $ numnp, r, ndof, bctag, sm, dlp, status)
character*(*) status
integer numel, nodelm(2,numel), numnp
integer ndof, bctag(ndof), dlp(0:ndof)
double precision kmuelm(numel), kpselm(numel)
double precision r(ndof), sm(*)
double precision re(2), sme(2,2)
integer edof(2)
integer i, j, k, ii, jj, n, ne
C

status = ' ' 
C

call FORMDLP (numel, nodelm, ndof, bctag, dlp)
do 2500 i = 1,abs(dlp(ndof))
   sm(i) = 0.0
  2500 continue
C

do 4000 ne = 1,numel
   do 2200 i = 1,2
      n = nodelm(i,ne)
```fortran
C

! read current intensity
!

subroutine CURRENT (inten)
    double precision DCLVAL
    double precision inten
    call CLREAD ('Enter current intensity: ', inten)
    return
end
!

! read problem dimensions (wire and truncation radius)
!

subroutine DIMENSIONS (wrad, trad)
    double precision DCLVAL
    double precision wrad, trad
    call CLREAD ('Enter wire radius, trunc radius: ', wrad, trad)
    return
end
!

! fatal error termination subroutine
!

subroutine ERROR (name, message)
    character(*) name, message
```

2200

```fortran
C

! call STIFF (ne, re, knuelm(ne), sme, status)
! if (status ..ne. ' ') then
!     call ERROR ('ASSEMBLY', status)
! end if
!

C

do 3600 i = 1,2
    ii = eldof(i)
    do 3600 j = 1,2
        jj = eldof(j)
        if (jj .le. ii) then
            k = abs(dlp(ii)) - ii + jj
            sm(k) = sm(k) + sme(i,j)
        end if
    3600 continue
3600 continue
!

C

4000 continue
!

C

return
end
```

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integer i, l

C

l = len(message)
do 1200 i = len(message), i-1
   if (message(i:i) .ne. ' ') go to 1300
   l = i
1200 continue
1300 continue
   print *, ' '
   print *, '*** Fatal error condition detected ***'
   print *, message(1:l)
   print *, 'Error detected by ', name
   stop '*** Error stop ***'
end

C=END FORTRAN
C=DECK FORCE
C= PURPOSE Compute node forces for axisymm EM element due to j
C=BLOCK FORTRAN
subroutine FORCE

$ (ne, re, inten, wrad, fe, status)
   integer ne
   double precision re(2), inten, wrad, fe(2)
   character*(*) status
   double precision ri, rj, rm, fn

   status = ' '  
   ri = re(1)  
   rj = re(2)  
   if (rj .le. ri) then
      write (status, '(A,I5) ') $ 'FORCE: Negative or zero length, element',ne
      return
   end if
   rm = 0.5*(ri+rj)
   if (rm .lt. wrad) then
      fn = (inten/(3.14159*wrad**2))*(rj-ri)
      fe(1) = fn*(ri+ri+rj)/6.
      fe(2) = fn*(ri+rj+rj)/6.
   else
      fe(1) = 0.0
      fe(2) = 0.0
   end if
   return
end

C=END FORTRAN
C=DECK FORMDLP
C= PURPOSE Form diagonal location pointer (DLP) array
C=BLOCK FORTRAN
subroutine FORMDLP

$ (numel, ndelm, ndof, bctag, dlp)
   integer numel, ndelm(2,numel), ndof  
   integer bctag(ndof), dlp(0:*)
   integer i, j, k, n, ne, aldof(2), nsky
C
  do 1200  i = 0, ndof
        dlp(i) = 0
 1200  continue
  do 2000  ns = 1, nsxel
        do 1600  i = 1, 2
                 n = nodelm(i,ns)
                 eldof(i) = n
 1600  continue
  do 1800  i = 1, 2
                 k = eldof(i)
 1800  continue
  do 2000  j = 1, 2
                 if (eldof(j) .le. k) then
                   dlp(k) = max(dlp(k), k-eldof(j)+1)
                 end if
 2000  continue
  do 2200  i = 1, ndof
        dlp(i) = dlp(i-1) + dlp(i)
 2200  continue
     nsky = abs(dlp(ndof))
C
  print '(/'' No of equations: ''I10)' , ndof
  print (''' Average bandwidth: ''F12.1)' , float(nsky)/ndof
  print (''' Entries to store skyline: ''I10)' , nsky
C
  do 3000  i = 1, ndof
  3000  continue
     if (bctag(i) .ne. 0)  dlp(i) = -abs(dlp(i))
     return
end
C=END FORTRAN
C=DECK GENBCTAG
C= PURPOSE Generate potential BC data by fixing extermmost node
C=BLOCK FORTRAN
  subroutine GENBCTAG (numnp, bctag)
    integer numnp, bctag(*)
    integer n
    do 2000  n = 1, numnp
              bctag(n) = 0
 2000  continue
    bctag(numnp) = 1
    return
end
C=END FORTRAN
C=DECK GENELEMS
C= PURPOSE Generate element data
C=BLOCK FORTRAN
  subroutine GENELEMS
    integer nelvir, nlext, kmu, keps, nodelm, kmuelm, kepselm
    integer nx, n, ne, nodelm(2,*)
double precision  kmu, keps, kmuelm(*), kepselm(*)
n = 0
ne = 0
do 2000 nx = 1,nelwir
   n = n + 1
   ne = ne + 1
   nodelm(1,ne) = n
   nodelm(2,ne) = n+1
   kmuelm(ne) = kmu
   kepselm(ne) = keps
2000  continue
do 3000 nx = 1,nelnext
   n = n + 1
   ne = ne + 1
   nodelm(1,ne) = n
   nodelm(2,ne) = n+1
   kmuelm(ne) = 1.0
   kepselm(ne) = 1.0
3000  continue
return
end
C=END FORTRAN
C=DECK GENEXACT
C=PURPOSE Generate exact magnetic potential/field solutions
C=BLOCK FORTRAN
   subroutine GENEXACT (numel, numnp, r, wrad, trad, $  
                        kmu, inten, aex, bex)
      integer numel, numnp
      double precision r(*), wrad, trad, kmu, inten
      double precision aex(numnp), bex(numel)
      integer n, ne
      double precision c, rm
      
      c = -(inten/(2*3.14159))*log(wrad/trad)
      do 2000 n = 1,numnp
         if (r(n) .lt. wrad) then
            aex(n) = (kmu*inten/(4*3.141596))*(1-(r(n)/wrad)**2) + c
         else
            aex(n) = -(inten/(2*3.14159))*log(r(n)/trad)
         end if
      2000  continue
      do 3000 ne = 1,numel
         rm = 0.5*(r(ne)+r(ne+1))
         if (rm .le. wrad) then
            bex(ne) = (kmu*inten/(2*3.14159))*(rm/wrad**2)
         else
            bex(ne) = (inten/(2*3.14159))/rm
         end if
      3000  continue
      return
end
C=END FORTRAN
C=DECK GEN_NODES
C=PURPOSE Generate node data
C=BLOCK FORTRAN
subroutine GENNODES (nelvir, nelext, wrad, trad, r)
    integer nelvir, nelext
    integer n, ne
    double precision wrad, trad, r(*)
C
n = 0
    do 2000 ne = 1, nelvir
        n = n + 1
        r(n) = (ne-1)*wrad/nelvir
    continue
2000    r(n+1) = wrad
    do 3000 ne = 1, nelext
        n = n + 1
        r(n) = wrad + (ne-1)*(trad-wrad)/nelext
3000      r(n+1) = trad
    return
end
C=END FORTRAN
C=DECK MAGFIELD
C=PURPOSE Compute magnetic field (B) at element center
C=BLOCK FORTRAN
subroutine MAGFIELD (numel, nodelm, r, a, b)
    integer numel, nodelm(2,numel)
    double precision r(*), a(*), b(numel)
    integer ne, ni, nj
C
do 2000 ne = 1, numel
    ni = nodelm(1,ne)
    nj = nodelm(2,ne)
    b(ne) = -(a(nj)-a(ni))/(r(nj)-r(ni))
2000    continue
    return
end
C=END FORTRAN
C=DECK MATERIAL
C=PURPOSE Read material properties
C=BLOCK FORTRAN
subroutine MATERIAL (kmu, kep8)
    double precision kmu, kep8
    double precision DCLVAL
    call DCLREAD (' Enter kmu, kep8 for wire: ', kmu, kep8)
kmu = DCLVAL(1)
    kep8 = DCLVAL(2)
    return
end
C=END FORTRAN
C=DECK PRINTCUR
C=PURPOSE Print current intensity
C=BLOCK FORTRAN
subroutine PRINTCUR (inten)
double precision inten
print ('"Current intensity:"',F10.3) ',inten
return
end
C=END FORTRAN
C=DECK PRINTDIM
C= PURPOSE Print problem dimensions (wire and truncation radius)
C=BLOCK FORTRAN
subroutine PRINTDIM (wrad, trad)
double precision wrad, trad
print ('"Wire radius:"',F10.3) ',wrad
print ('"Truncation radius:"',F10.3) ',trad
return
end
C=END FORTRAN
C=DECK PRINTELM
C= PURPOSE Print element data
C=BLOCK FORTRAN
subroutine PRINTELM (numel, nodelm, kmuelm, kepselm)
integer i, n, numel, nodelm(2,*)
double precision kmuelm(*), kepselm(*)
print *, '-----------------------------------'
print *, ' Element Data'
print *, '-----------------------------------'
print *, 'Elm I J kmu keps'
do 2000 n, numel
print '(3I5,2F9.3)', n, (nodelm(i,n), i=1,2), kmuelm(n), kepselm(n)
2000 continue
return
end
C=END FORTRAN
C=DECK PRINTMAG
C= PURPOSE Print computed and exact magnetic field (B)
C=BLOCK FORTRAN
subroutine PRINTMAG (numel, nodelm, r, b, bex)
integer numel, nodelm(2,numel)
double precision r(*), b(numel), bex(numel)
integer ne, ni, nj
C
print *, '---------------------'
print *, 'Magnetic Field'
print *, '---------------------'
print *, 'Elm r-center Comp-B2 Exact-B2'
do 2000 ne = 1,numel
ni = nodelm(1,ne)
j = nodelm(2,ne)
print '([5,F10.3,2F11.4]", ne, 0.5*(r(ni)+r(nj)),
$ b(ne), bex(ne)
2000 continue
return
end
C=END FORTRAN
C=DECK PRINTMAT
C= PurposE Print material properties used in problem
C=BLOCK FORTRAN
subroutine PRINTMAT (kmu, keps)
double precision kmu, keps
print ('\' Rel. permeability of wire (vacuum=1):',F10.3),kmu
print ('\' Rel. permittivity of wire (vacuum=1):',F10.3),keps
return
end
C=END FORTRAN
C=DECK PRINTNOD
C=Purpose Print element data
C=BLOCK FORTRAN
subroutine PRINTNOD (numnp, r, bctag)
integer n, numnp, bctag(*)
double precision r(*)
print *
print *, '-----------------------------------'
print *, ' Node  Data'
print *, '-----------------------------------'
print *, ' Node  r-coord  bctag'
do 2000 n = 1, numnp
print '(I6,F10.3,I6)', n, r(n), bctag(n)
2000 continue
return
end
C=END FORTRAN
C=DECK PRINTSOL
C= PurposE Print computed and exact solution
C=BLOCK FORTRAN
subroutine PRINTSOL (numnp, r, bctag, f, a, aex, fex)
integer n, numnp, bctag(numnp)
double precision r(numnp), f(numnp)
double precision a(numnp), aex(numnp), fex(numnp)
print *
print *
print *, '-----------------------------------'
print *, ' Computed Solution'
print *
print *
$ ' Node  r  bctag  Comp-for'
$ ' Comp-A3  Exact-A3  Exact-for'
do 2000 n = 1, numnp
print '(I6,F10.3,I6,4F11.4)',n, r(n), bctag(n),
$ f(n), a(n), aex(n), fex(n)
2000 continue
return
end
C=END FORTRAN
C=DECK PRINTSUB
C= PurposE Print subdivision data
C=BLOCK FORTRAN
subroutine PRINTSUB (nelwir, nelex_, numel, numnp, ndof)
integer nelwir, nelex_, numel, numnp, ndof
print '\' Subdivisions in wire :',I6', nelwir
SKYCOV is a cover routine that solves the master stiffness equations
\[ Ku = f \]
SKYCOV calls SKYFAC to factor the skyline-stored master stiffness matrix \( K \). If the factorization is successful SKYCOV then calls SKYSOL to solve for \( u \).

The calling sequence is

\[ \text{CALL SKYCOV} (S, N, DLP, F, U, STATUS) \]

Input arguments:
- \( S \): Skyline stored stiffness matrix
- \( N \): Number of equations
- \( P \): Nodal force vector
- \( DLP \): Skyline diagonal location pointer

Output arguments:
- \( U \): Computed displacements if no error detected.
- \( STATUS \): Status character variable.
  - blank: no error detected
  - nonblank: explanatory error message

subroutine SKYCOV $ (s, n, dlp, f, u, status) 
integer n, dlp(0:*)

33
double precision  s(*), f(*), u(*)
character(*) status

TYPE & DIMENSION

integer  idetex, negeig, ifail, NMAX
parameter (NMAX=3000)
double precision  aux(NMAX), detcf, delta, DOTPRD
external  DOTPRD

LOGIC

status = '
if (n .gt. NMAX)     then
    write (status, '(A,I6)') 'No. of equations exceeds ',NMAX
    return
end if
    call SKYFAC
$  (s, O, n, dlp, aux, DOTPRD, .true., .false.,
$  O, 0, O.O, detcf, idetex, negeig, ifail)
if (ifail .gt. O)     then
    write (status, '(A,I6,A)')
$  'Factorization aborted at equation ',ifail,
$  ' (matrix appears singular)' return
end if
    call SKYSOL
$  (s, n, dlp, DOTPRD, O, 1, f, u, O, O, aux, delta)
return
end

C=END FORTRAN
C=DECK STIFF
C=PURPOSE Construct stiffness matrix of axisymmetric EM element
C=BLOCK FORTRAN
    subroutine STIFF (ne, re, mu, s, status)
    integer  ne
    double precision  re(2), mu, s(2,2)
    character(*) status
    double precision  ri, rj, rl, rm
    status = '
    ri = re(1)
    rj = re(2)
    rl = rj - ri
    if (rl .le. O.O) then
        write (status, '(A,I6)')
        $  'STIFF: Negative or zero length, element',ne
        return
    end if
    rm =  O.5*(ri+rj)
    s(1,1)= rm/(ri*mu)
    s(2,2)= s(1,1)
\begin{verbatim}
  s(1,2) = -s(1,1)
  s(2,1) = s(1,2)
  return
end

C=END FORTRAN
C=DECK SUBDIVIDE
C= PURPOSE Read subdivision data
C= BLOCK FORTRAN
   subroutine SUBDIVIDE(nelwr, nelext, numel, numnp, ndof)
   integer nelwr, nelext, numel, numnp, ndof
   integer ICLVAL
   call ICLREAD(' Subdivisions in wire: ', ' ')
   nelwr = ICLVAL(1)
   call ICLREAD(' Subdivisions in free space: ', ' ')
   nelext = ICLVAL(1)
   numel = nelwr + nelext
   numnp = numel + 1
   ndof = numnp
   return
end

C=END FORTRAN
\end{verbatim}
We derive electromagnetic finite elements based on a variational principle that uses the electromagnetic four-potential as primary variable. This choice is used to construct elements suitable for downstream coupling with mechanical and thermal finite elements for the analysis of electromagnetic/mechanical systems that involve superconductors. The key advantages of the four-potential are: the number of degrees of freedom per node remain modest as the problem dimensionality increases, jump discontinuities on interfaces are naturally accommodated, and static as well as dynamics are included without any a priori approximations. The new elements are tested on an axisymmetric problem under steady-state forcing conditions. The results are in excellent agreement with analytical solutions.