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A VARIATIONAL JUSTIFICATION OF THE ASSUMED NATURAL STRAIN FORMULATION OF FINITE ELEMENTS. I. VARIATIONAL PRINCIPLES

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SUMMARY

The *assumed natural strain* (ANS) formulation of finite elements has undergone rapid development over the past five years. The key formulation step is the replacement, in the potential energy principle, of selected displacement-related strains by independently assumed strain fields in *element natural coordinates*. These strains are not generally derivable from displacements. This procedure was conceived as one of several competing methods to solve the *element locking* problem. Its most noteworthy feature is that, unlike many forms of reduced integration, it produces no *rank deficiency*; furthermore, it is easily extendible to geometrically nonlinear problems. Many original formulations were not based on a variational principle. The objective of Part I is to study the ANS formulation from a variational standpoint. This study is based on two hybrid extensions of the Reissner-type functional that uses strains and displacements as independent fields. One of the forms is a genuine variational principle that contains an independent boundary traction field, whereas the other one represents a restricted variational principle. Two procedures for element-level elimination of the strain field are discussed, and one of them shown to be equivalent to the inclusion of incompatible displacement modes. In Part II, the 4-node C^0 plate bending quadrilateral element is used to illustrate applications of this theory.

1. INTRODUCTION

The assumed natural strain (ANS) formulation of finite elements is a relatively new development. A restricted form of the method was introduced in 1969 by Willam [14], who constructed a 4-node plane-stress element by assuming a constant shear strain independently of the direct strains and using a strain-displacement mixed variational principle. A different approach advocated by Ashwell [1] and coworkers regarded "strain elements" as a way to obtain appropriate displacement fields by integration of assumed compatible strain fields. These and other forms of assumed-strain techniques were overshadowed in the 1970s by developments in reduced and selective integration methods, but have recently begun to attract attention [2,6,8,10,13]. The primary motivation behind recent work has been the construction of simple and efficient finite elements for plates and shells that are locking-free, rank sufficient and distortion insensitive, yield accurate answers for coarse meshes, fit naturally into displacement-based programs, and can be easily extended to nonlinear and dynamic problems. Elements that attain these attributes are collectively known as *high performance* elements.

Over the past 20 years investigators have resorted to many ingenious devices to construct high-performance elements. Among the most successful ones we can mention patch-test-verified incompatible displacement models, reduced and selective integration, mixed and hybrid formulations, stress projectors, the free formulation, and assumed natural strains. The underlying theme is that although the final product may look like a standard displacement model so as to fit naturally into existing finite element programs, *the conventional displacement formulation is abandoned.* (By "conventional" we mean the use of conforming displacement assumptions into the total potential energy principle.)

Another common historic trend is that certain deviations from the conventional formulation were initially made without variational justification and in fact labelled as "variational crimes" by applied mathematicians. In some cases such as reduced numerical integration, reconciliation was achieved later after surprisingly good results prompted explanation. In other cases, notably non-conforming elements and the patch test, a comprehensive mathematical theory is still in the making.

The present paper seeks to interpret the assumed natural strain (ANS) formulation from a variational standpoint. The justification is based on hybrid extensions of the Reissner-type functional that uses the strains and displacements as independent fields. We restrict our considerations to linear elasticity although the straightforward extension to geometric nonlinearities is one of the strengths of the ANS formulation. In Part II, the 4-node C^0 plate-bending quadrilateral is used as a specific example to illustrate the application of the present variational interpretation.

2. PROBLEM DESCRIPTION

2.1 Governing Equations

Consider a *linearly elastic body* under *static* loading that occupies the volume V. The body is bounded by the surface S, which is decomposed into $S : S_u \cup S_t$. Displacements are prescribed on S_u whereas surface tractions are prescribed on S_t. The outward unit normal on S is denoted by $\mathbf{n} \equiv n_i$.

The three unknown volume fields are displacements $\mathbf{u} \equiv u_i$, infinitesimal strains $\boldsymbol{\epsilon} \equiv \epsilon_{ij}$, and stresses $\boldsymbol{\sigma} \equiv \sigma_{ij}$. The problem data include: the body force field $\mathbf{f} \equiv f_i$ in V, prescribed displacements $\hat{\mathbf{u}} = \hat{u}_i$ on S_u, and prescribed surface tractions $\hat{\mathbf{t}} \equiv \hat{t}_i$ on S_t.

The relations between the volume fields are the strain-displacement equations

$$\boldsymbol{\epsilon} = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u}) = \mathbf{D}\mathbf{u} \qquad \text{or} \qquad \epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in } V, \tag{1}$$

(where superscript T denotes transposition), the constitutive equations

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\epsilon} \qquad \text{or} \qquad \sigma_{ij} = E_{ijkl}\epsilon_{kl} \quad \text{in } V, \tag{2}$$

and the equilibrium (balance) equations

$$-\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{D}^*\boldsymbol{\sigma} = \mathbf{f} \qquad \text{or} \qquad \sigma_{ij,j} + f_i = 0 \quad \text{in } V, \tag{3}$$

in which $\mathbf{D}^* = -\mathrm{div}$ (divergence) denotes the adjoint operator of the symmetric gradient $\mathbf{D} = \frac{1}{2}(\nabla + \nabla^T)$.

On S the surface stress vector is defined as

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}.\mathbf{n}, \qquad \text{or} \quad \sigma_{ni} = \sigma_{ij}n_j. \tag{4}$$

With this definition the traction boundary conditions may be stated as

$$\boldsymbol{\sigma}_n = \hat{\mathbf{t}} \qquad \text{or} \qquad \sigma_{ij}n_j = \hat{t}_i \qquad \text{on } S_t, \tag{5}$$

and the displacement boundary conditions as

$$\mathbf{u} = \hat{\mathbf{u}} \qquad \text{or} \qquad u_i = \hat{u}_i \qquad \text{on } S_u. \tag{6}$$

2.2 Notational Conventions

An *independently varied* field will be identified by a letter without superscript, for example $\mathbf{u}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$. A *dependent field* is identified by writing the independent field symbol as superscript. For example, if the displacements are independently varied, the derived strain and stress fields are denoted by

$$\boldsymbol{\epsilon}^u = \tfrac{1}{2}(\nabla + \nabla^T)\mathbf{u} = \mathbf{D}\mathbf{u}, \qquad \boldsymbol{\sigma}^u = \mathbf{E}\boldsymbol{\epsilon}^u = \mathbf{E}\mathbf{D}\mathbf{u}. \tag{7}$$

Given a finite element subdivision of V, quantities pertaining to the e^{th} element will be identified by superscript (e), for example $\mathbf{u}^{(e)}$, wherever appropriate. At an interface between two elements e and f, superscripts (ef) and (fe) will identify interface quantities considered as part of e and f, respectively.

3. THE HU-WASHIZU AND REISSNER FUNCTIONALS

In the conventional Hu-Washizu functional the displacements $\mathbf{u}$, stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$ are independently varied. Arranging the strain and stress components as vectors, and the elastic moduli in $\mathbf{E}$ as a matrix, the functional may be expressed as†

$$L(\mathbf{u},\boldsymbol{\epsilon},\boldsymbol{\sigma}) = \int_V \left[\tfrac{1}{2}\boldsymbol{\epsilon}^T\mathbf{E}\,\boldsymbol{\epsilon} + \boldsymbol{\sigma}^T(\boldsymbol{\epsilon}^u - \boldsymbol{\epsilon}) - \mathbf{f}^T\mathbf{u}\right] dV - \int_{S_u} (\boldsymbol{\sigma}_n)^T(\mathbf{u}-\hat{\mathbf{u}})\, dS - \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\, dS. \quad (8)$$

From L one obtains the conventional stress-displacement Hellinger-Reissner functional by eliminating $\boldsymbol{\epsilon}$ through the inverse of (2), namely $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^\sigma = \mathbf{E}^{-1}\boldsymbol{\sigma}$. Another Reissner-type, strain-displacement functional is obtained by eliminating $\boldsymbol{\sigma}$ through the constitutive relation (2), namely $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\epsilon = \mathbf{E}\boldsymbol{\epsilon}$, which yields

$$R(\mathbf{u},\boldsymbol{\epsilon}) = \int_V \left[-\tfrac{1}{2}\boldsymbol{\epsilon}^T\mathbf{E}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^T\mathbf{E}\boldsymbol{\epsilon}^u - \mathbf{f}^T\mathbf{u}\right] dV - \int_{S_u} (\boldsymbol{\sigma}_n^\epsilon)^T(\mathbf{u}-\hat{\mathbf{u}})\, dS - \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\, dS. \quad (9)$$

Setting $\boldsymbol{\epsilon} \equiv \boldsymbol{\epsilon}^u$ reduces R to the potential energy functional

$$P(\mathbf{u}) = \int_V \left[\tfrac{1}{2}(\boldsymbol{\epsilon}^u)^T\mathbf{E}\boldsymbol{\epsilon}^u - \mathbf{f}^T\mathbf{u}\right] dV - \int_{S_u} (\boldsymbol{\sigma}_n^u)^T(\mathbf{u}-\hat{\mathbf{u}})\, dS - \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\, dS, \quad (10)$$

generalized with a S_u term over its usual expression.

4. HYBRID FUNCTIONALS

4.1 Independent Boundary Tractions

If the functional (9) is used to construct finite elements, the displacement field $\mathbf{u}$ should be *continuous* in V because of the presence of $\boldsymbol{\epsilon}^u$, whereas the assumed strain field may be discontinuous. To account rigorously for displacement discontinuities it is necessary to add the interelement surface tractions $\mathbf{t}$ as new independent field which plays the role of Lagrange multiplier. Let S_i denote the union of interelement boundaries traversed twice (one for each adjacent element); on S_i neither displacements nor tractions are prescribed. Then R expands to the *hybrid* functional

$$H(\mathbf{u},\boldsymbol{\epsilon},\mathbf{t}) = R(\mathbf{u},\boldsymbol{\epsilon}) - \int_{S_i} \mathbf{t}^T\mathbf{u}\, dS. \quad (11)$$

† There are several equivalent statements of this functional, differing from one another in transformations based on the divergence theorem. For example in Gurtin [5, p. 122] the stress divergence appears. Some authors attribute this specific functional to B. Fraeijs de Veubeke, who indeed published a version of it in 1951, four years before Hu and Washizu.

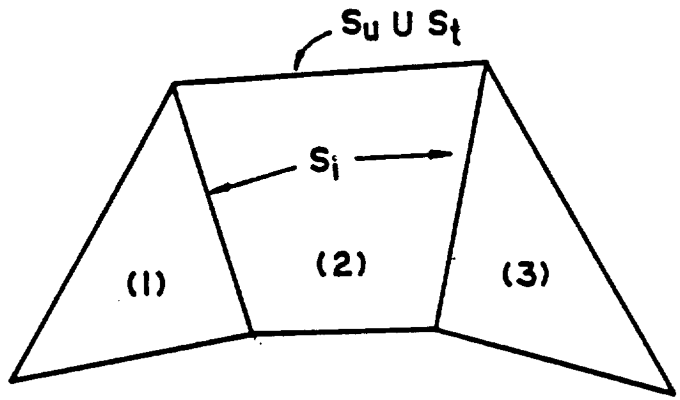

Figure 1. Sample finite element mesh to illustrate computation of integrals in H

For later reference we note the specialization $\epsilon \equiv \epsilon^u$ of (11) to the generalized potential energy functional of Jones [12]

$$P(\mathbf{u}, \mathbf{t}) = P(\mathbf{u}) - \int_{S_i} \mathbf{t}^T \mathbf{u} \, dS, \tag{12}$$

where $P(\mathbf{u})$ is given by (10).

The meaning of the integrals in H may be illustrated on the two-dimensional mesh of Figure 1:

$$\begin{aligned}
\int_V &\equiv \sum_e \int_{V^{(e)}} = \int_{V^{(1)}} + \int_{V^{(2)}} + \int_{V^{(3)}} \\
\int_{S_u} &\equiv \sum_e \int_{S_u^{(e)}} = \int_{S_u^{(1)}} + \int_{S_u^{(1)}} + \int_{S_u^{(3)}} \\
\int_{S_t} &\equiv \sum_e \int_{S_t^{(e)}} = \int_{S_t^{(1)}} + \int_{S_t^{(2)}} + \int_{S_t^{(3)}} \\
\int_{S_i} &\equiv \sum_{e,f} \int_{S_i^{(ef)}} = \int_{S^{(1,2)}} + \int_{S^{(2,1)}} + \int_{S^{(2,3)}} + \int_{S^{(3,2)}}
\end{aligned} \tag{13}$$

where element identification conventions stated in Section 2.2 have been followed. It is seen that in the integrals over V, S_u and S_t each element appears once, whereas in S_i adjacent elements appear twice.

4.2 *First Variation*

The first variation of H:

$$\delta H = \delta_u H + \delta_\epsilon H + \delta_t H, \tag{14}$$

yields the Euler equations and interelement linking conditions, which are underlined in the expressions below. The three components of δH are

$$\delta_u H = \int_V \underline{(\nabla \sigma^\epsilon - \mathbf{f})}^T \delta \mathbf{u}\, dV + \int_{S_t} \underline{(\sigma_n^\epsilon - \hat{\mathbf{t}})}^T \delta \mathbf{u}\, dS + \int_{S_i} \underline{(\sigma_n^\epsilon - \mathbf{t})}^T \delta \mathbf{u}\, dS, \tag{15}$$

$$\delta_\epsilon H = \int_V \mathbf{E}\underline{(\epsilon^u - \epsilon)}^T \delta \epsilon^e\, dV - \int_{S_u} \underline{(\mathbf{u} - \hat{\mathbf{u}})}^T \delta (\mathbf{E}\epsilon)_n\, dS, \tag{16}$$

$$\delta_t H = \int_{S_i} \underline{\mathbf{u}^T} \delta \mathbf{t}\, dS. \tag{17}$$

Note that there are two contributions to the element interface integrals, one from $\delta_u H$ and another from $\delta_t H$. Putting the parts together and decomposing into element-pair contributions we get

$$\begin{aligned}\int_{S_i} \left[(\sigma_n^\epsilon - \mathbf{t})^T \delta \mathbf{u} + \mathbf{u}^T \delta \mathbf{t}\right] dS = \sum_{e,f} \int_{S^{(ef)}} \Big[& \sigma_n^{\epsilon\,(e)T} \delta \mathbf{u}^{(e)} - \sigma_n^{\epsilon\,(f)T} \delta \mathbf{u}^{(f)} \\ & - \mathbf{t}^{(ef)T} \delta \mathbf{u}^{(e)} - \mathbf{t}^{(fe)T} \delta \mathbf{u}^{(f)} + \mathbf{u}^{(e)T} \delta \mathbf{t}^{(ef)} + \mathbf{u}^{(f)T} \delta \mathbf{t}^{(fe)} \Big]\, dS.\end{aligned} \tag{18}$$

In the absence of applied internal tractions, interelement equilibrium requires $\mathbf{t}^{(ef)} = -\mathbf{t}^{(fe)}$, which substituted into (16) reduces the right-hand side to

$$\sum_{e,f} \int_{S^{(ef)}} \underline{\left[\sigma_n^{\epsilon\,(e)T} \delta \mathbf{u}^{(e)} - \sigma_n^{\epsilon\,(f)T} \delta \mathbf{u}^{(f)} - \mathbf{t}^{(ef)T} \delta (\mathbf{u}^{(e)} - \mathbf{u}^{(f)}) + (\mathbf{u}^{(e)} - \mathbf{u}^{(f)})^T \delta \mathbf{t}^{(ef)}\right]}\, dS. \tag{19}$$

If we assume a *compatible* displacement field, $\mathbf{u}^{(e)} = \mathbf{u}^{(f)}$, the above equation reduces to

$$\sum_{e,f} \int_{S^{(ef)}} \underline{(\sigma_n^{\epsilon\,(e)} - \sigma_n^{\epsilon\,(f)})}^T \delta \mathbf{u}^{(e)}\, dS, \tag{20}$$

which means that the interelement equilibrium condition appears as the Euler equation corresponding to the variation of the interface displacements.

4.3 *A Restricted Variational Principle*

If the displacement field is incompatible we should in principle retain $\mathbf{t}$ as an independent boundary-traction field satisfying $\mathbf{t}^{(ef)} = -\mathbf{t}^{(fe)}$ over interelement boundaries. One way to achieve this is to assume a continuous stress field σ^* over element boundaries, so that

$$\mathbf{t}^{(ef)} = \sigma^*.\mathbf{n}^{(e)} = \sigma_n^{*\,(e)}, \qquad \mathbf{t}^{(fe)} = \sigma^*.\mathbf{n}^{(f)} = \sigma^*.(-\mathbf{n}^{(e)}) = -\sigma_n^{*\,(e)}. \tag{21}$$

The presence of an independent boundary traction field is computationally disadvantageous because additional degrees of freedom must be retained on elements sides. This contradicts one of the tenets of high-performance element construction noted in the Introduction. It would be more convenient if σ^* could be identified with the ***strain-derived stress field***, that is, $\sigma^* = \sigma^\epsilon = \mathbf{E}\epsilon$ on S_i, because we would have only two independent fields, $\mathbf{u}$ and ϵ, as in (9). The strain freedoms can be eliminated at the element level as explained in Section 6, and we are left with standard displacement connectors. The corresponding functional is

$$\widetilde{H}(\mathbf{u},\epsilon) = R - \int_{S_i} (\sigma_n^\epsilon)^T \mathbf{u}\, dS. \tag{22}$$

But in general σ_n^ϵ is not continuous between elements. One can argue, however, that continuity is achieved in the limit of a converged solution. A variational statement such as $\delta\widetilde{H} = 0$ is called a ***restricted variational principle*** [3, Ch. 11] because the governing field equations of §2.1 are satisfied only at the exact solution. Away from it, $\delta\widetilde{H} = 0$ generally violates interelement-equilibrium field equations although it may provide satisfactory numerical approximations.

Stress-displacement (rather than strain-displacement) functionals of this form have been used by Pian and coworkers [11,12], who transform the interface integral into an element volume integral and in doing so introduce a stress divergence term.

4.4 Finite Element Classification

Finite element models derivable from R, $\widetilde{H}$ and H may be classified into several types according to the number of independent fields and the continuity conditions on those fields. Following are some general comments on the most interesting combinations, which are summarized in Table 1.

1. *Continuous displacements.* The independent boundary field $\mathbf{t}$ is not needed, and we can work with the mixed functional R. If the strain field is discontinuous, strain freedoms may be eliminated at the element level as explained in Section 6. Continuous strains are in principle possible but impractical in general structural applications where material interfaces, plasticity, and sudden thickness or area changes may occur.

2. *Discontinuous displacements.* The displacement field contains conforming and non-conforming portions. Assumed strains are discontinuous and may be eliminated at the element level. Displacement degrees of freedom associated with non-conforming modes may be also eliminated if separable. The governing functionals are $\widetilde{H}$ or H. With the latter an independent traction field $\mathbf{t}$ is required; degrees of freedom associated with $\mathbf{t}$ must be retained at the assembly level.

In practice elements are often constructed as a combination of these types with conventional displacement models. Thus part of the strain field may be considered as completely derivable from displacements and part as independently assumed, as discussed in Section 8. This was in fact the scheme originally used by Willam [14]. The C^0 plate bending quadrilaterals studied in Part II provide another important example.

Table 1. Assumed-Strain Finite Element Models Derivable From R, H and $\widetilde{H}$

Element Type	*Governing functional*	*Independent fields*	*Interelement continuity on** $\mathbf{u}$	ϵ	$\mathbf{t}$	*Element connected fields*	*Element condensable fields*
(I)	R	$\mathbf{u},\epsilon$	c	d		$\mathbf{u}$	ϵ
(II)	R	$\mathbf{u},\epsilon$	c	c		$\mathbf{u},\epsilon$	
(III)	$\widetilde{H}$	$\mathbf{u},\epsilon$	d	d		$\mathbf{u}^{\dagger}$	ϵ
(IV)	H	$\mathbf{u},\epsilon,\mathbf{t}$	d	d	c	$\mathbf{u}^{\dagger}$,$\mathbf{t}$	ϵ

* c=continuous, d=discontinuous. † conforming part only if separable as per (33)

5. DISCRETIZATION

5.1 Assumptions

In this section the finite element discretization of the hybrid functionals H and $\widetilde{H}$ is studied. That is, we focus attention on element types labelled (III) and (IV) in Table 1. In the sequel it will be assumed that the *displacement boundary conditions are identically satisfied* by $\mathbf{u}$, whence the strain-displacement hybrid functionals reduce to

$$H(\mathbf{u},\epsilon,\mathbf{t}) = \int_V \left[\epsilon^T\mathbf{E}(\epsilon^u - \tfrac{1}{2}\epsilon) - \mathbf{f}^T\mathbf{u}\right] dV - \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\, dS - \int_{S_i} \mathbf{t}^T\mathbf{u}\, dS. \tag{23}$$

$$\widetilde{H}(\mathbf{u},\epsilon) = \int_V \left[\epsilon^T\mathbf{E}(\epsilon^u - \tfrac{1}{2}\epsilon) - \mathbf{f}^T\mathbf{u}\right] dV - \int_{S_t} \hat{\mathbf{t}}^T\mathbf{u}\, dS - \int_{S_i} (\sigma_n^\epsilon)^T\mathbf{u}\, dS. \tag{24}$$

The framework used here accomodates both continuous and discontinuous displacements. The FE assumption may be written

$$\mathbf{u} = \mathbf{N}\mathbf{v} \quad \text{in } V, \qquad \epsilon = \mathbf{A}\mathbf{a} \quad \text{in } V, \qquad \mathbf{t} = \mathbf{T}\mathbf{s} \quad \text{on } S_i. \tag{25}$$

Here matrices $\mathbf{N}$, $\mathbf{A}$ and $\mathbf{T}$ collect displacement shape functions, assumed natural strain functions and interface traction functions, respectively, whereas column vectors $\mathbf{v}$, $\mathbf{a}$ and $\mathbf{s}$ collect nodal displacements, strain amplitudes, and interface tractions amplitudes, respectively. The derived fields in V are

$$\epsilon^u = \mathbf{D}\mathbf{N}\mathbf{v} = \mathbf{B}\mathbf{v}, \quad \sigma^u = \mathbf{E}\mathbf{B}\mathbf{v}, \quad \sigma^\epsilon = \mathbf{E}\epsilon = \mathbf{E}\mathbf{A}\mathbf{a}. \tag{26}$$

5.2 Discrete Equations

On inserting the assumptions (23-24) into (21-22) we obtain the bilinear algebraic forms

$$H(\mathbf{v},\mathbf{a},\mathbf{s}) = -\tfrac{1}{2}\mathbf{a}^T\mathbf{C}\mathbf{a} + \mathbf{a}^T\mathbf{P}\mathbf{v} - \mathbf{v}^T\mathbf{L}\mathbf{s} - \mathbf{v}^T\mathbf{p}, \tag{27}$$

$$\widetilde{H}(\mathbf{v},\mathbf{a}) = -\tfrac{1}{2}\mathbf{a}^T\mathbf{C}\mathbf{a} + \mathbf{a}^T(\mathbf{P}-\mathbf{R})\mathbf{v} - \mathbf{v}^T\mathbf{p} = -\tfrac{1}{2}\mathbf{a}^T\mathbf{C}\mathbf{a} + \mathbf{a}^T\widetilde{\mathbf{P}}\mathbf{v} - \mathbf{v}^T\mathbf{p}. \tag{28}$$

where

$$\begin{aligned} \mathbf{C} &= \int_V \mathbf{A}^T\mathbf{E}\mathbf{A}\,dV = \mathbf{C}^T, \quad \mathbf{P} = \int_V \mathbf{A}^T\mathbf{E}\mathbf{B}\,dV, \quad \mathbf{L} = \int_{S_i} \mathbf{N}^T\mathbf{T}\,dS, \\ \mathbf{R} &= \int_{S_i} (\mathbf{E}\mathbf{A})_n^T\mathbf{N}\,dS, \quad \widetilde{\mathbf{P}} = \mathbf{P} - \mathbf{R}, \quad \mathbf{p} = \int_V \mathbf{N}^T\mathbf{f}\,dV + \int_{S_t} \mathbf{N}^T\hat{\mathbf{t}}\,dS. \end{aligned} \tag{29}$$

Observe that (28) results on substituting $\mathbf{L}\mathbf{s}$ by $\mathbf{R}^T\mathbf{a}$ in (27). Making these forms stationary yields the linear systems

$$\begin{bmatrix} -\mathbf{C} & \mathbf{P} & \mathbf{0} \\ \mathbf{P}^T & \mathbf{0} & -\mathbf{L} \\ \mathbf{0} & -\mathbf{L}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{v} \\ \mathbf{s} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p} \\ \mathbf{0} \end{Bmatrix}, \tag{30}$$

$$\begin{bmatrix} -\mathbf{C} & \widetilde{\mathbf{P}} \\ \widetilde{\mathbf{P}}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{v} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p} \end{Bmatrix}. \tag{31}$$

for (27) and (28), respectively. In both cases the first matrix equation is the discrete analog of (16), and expresses internal compatibility. The second matrix equation is the analog of (15) and expresses internal and boundary equilibrium, and, in the case of (31), approximate boundary compatibility. The third matrix equation in (30) is the analog of (17) and expresses boundary compatibility.

5.3 Displacement Field Decomposition

With view to further developments the assumed displacement field is decomposed as

$$\mathbf{u} = \mathbf{u}_c + \mathbf{u}_d. \tag{32}$$

where $\mathbf{u}_c$ is continuous (compatible, conforming) in V and $\mathbf{u}_d$ discontinuous (incompatible, non-conforming) on S_i. It will be further assumed that this decomposition can be effected in terms of the shape functions, *i.e.*,

$$\mathbf{u} = \mathbf{N}_c\mathbf{v}_c + \mathbf{N}_d\mathbf{v}_d, \tag{33}$$

where the $\mathbf{v}_d$ freedoms are defined element-by-element and may in principle be condensed out. This assumption holds for elements in which non-conforming shape functions are "injected" over a compatible set. For the H functional, as shown in Section 4.2 the S_i integral exactly vanishes for the conforming displacements:

$$\int_{S_i} \mathbf{t}^T \mathbf{u}_c = 0. \tag{34}$$

On the other hand, for $\widetilde{H}$ the corresponding S_i integral also vanishes at the converged solution. Taking this into account, equations (30-31) expand to

$$\begin{bmatrix} -\mathbf{C} & \mathbf{P}_c & \mathbf{P}_d & \mathbf{0} \\ \mathbf{P}_c^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}_d^T & \mathbf{0} & \mathbf{0} & -\mathbf{L}_d \\ \mathbf{0} & \mathbf{0} & -\mathbf{L}_d^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{v}_c \\ \mathbf{v}_d \\ \mathbf{s} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p}_c \\ \mathbf{p}_d \\ \mathbf{0} \end{Bmatrix}, \tag{35}$$

$$\begin{bmatrix} -\mathbf{C} & \mathbf{P}_c & \widetilde{\mathbf{P}}_d \\ \mathbf{P}_c^T & \mathbf{0} & \mathbf{0} \\ \widetilde{\mathbf{P}}_d^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \mathbf{v}_c \\ \mathbf{v}_d \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{p}_c \\ \mathbf{p}_d \end{Bmatrix}. \tag{36}$$

in which $\widetilde{\mathbf{P}}_d = \mathbf{P}_d - \mathbf{R}_d$, and where c- and d-subscripted matrices and vectors are given by integrals similar to (29) in which $\mathbf{N}$ is replaced by $\mathbf{N}_c$ and $\mathbf{N}_d$, respectively.

6. STRAIN ELIMINATION

The strain degrees of freedom may be eliminated *at the element level* by static condensation or by enforcing kinematic constraints. These two techniques are studied below.

6.1 Static Condensation

This is a well known *variationally consistent* procedure which will be illustrated for the system (30). From the first matrix equation get $\mathbf{a}$ at the element level:

$$\mathbf{a} = \mathbf{C}^{-1}\mathbf{P}\mathbf{v} = \mathbf{Q}_s\mathbf{v}. \tag{37}$$

Substitution into the second equation gives

$$\begin{bmatrix} \mathbf{K} & -\mathbf{L} \\ -\mathbf{L}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v} \\ \mathbf{t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p} \\ 0 \end{Bmatrix}, \tag{38}$$

where $\mathbf{K} = \mathbf{P}^T\mathbf{C}^{-1}\mathbf{P} = \mathbf{P}^T\mathbf{Q}_s = \mathbf{Q}_s^T\mathbf{C}\mathbf{Q}_s$ is a stiffness matrix. Similarly, (31) condenses to

$$\widetilde{\mathbf{K}}\mathbf{v} = \mathbf{p}, \tag{39}$$

where $\widetilde{\mathbf{K}} = \widetilde{\mathbf{P}}^T\mathbf{C}^{-1}\widetilde{\mathbf{P}} = \widetilde{\mathbf{Q}}_s^T\mathbf{C}\widetilde{\mathbf{Q}}_s$ and $\widetilde{\mathbf{Q}}_s = \mathbf{C}^{-1}\widetilde{\mathbf{P}}$. The separable non-conforming degrees of freedom $\mathbf{v}_d$, if present, may be condensed out following a similar procedure.

6.2 Kinematic Constraints

A second elimination procedure has been used recently in the construction of ANS C^0 plate and shell elements. It will be described by considering the system (35) that displays separable conforming and non-conforming displacement shape functions. A kinematic constraint that links strain to displacement degrees of freedom is established:

$$\mathbf{a} = \mathbf{Q}_c\mathbf{v}_c + \mathbf{Q}_d\mathbf{v}_d. \tag{40}$$

This relation may be constructed by collocation, least-square fitting or some other means. Often $\mathbf{Q}_d = \mathbf{0}$. For example, in the Bathe-Dvorkin element [2] studied in Part II collocation of natural shear strains is done at the quadrilateral midpoints.

If the following conditions hold:

(a) the dimension of $\mathbf{v}_d$ and $\mathbf{a}$ are the same so that $\mathbf{P}_d$ is square;

(b) matrix $\mathbf{P}_d - \mathbf{C}\mathbf{Q}_d$ is nonsingular;

then the relation (40) may be interpreted *as a variationally-consistent constraint on non-conforming displacements.* In effect, the first equation of (35) becomes

$$(\mathbf{P}_c - \mathbf{C}\mathbf{Q}_c)\,\mathbf{v}_c + (\mathbf{P}_d - \mathbf{C}\mathbf{Q}_d)\mathbf{v}_d = \mathbf{0}, \tag{41}$$

whence

$$\begin{aligned} \mathbf{v}_d &= -(\mathbf{P}_d - \mathbf{C}\mathbf{Q}_d)^{-1}(\mathbf{P}_c - \mathbf{C}\mathbf{Q}_c)\mathbf{v}_c = \mathbf{W}\mathbf{v}_c, \\ \mathbf{a} &= (\mathbf{Q}_c + \mathbf{Q}_d\mathbf{W})\mathbf{v}_c = \mathbf{Q}\mathbf{v}_c. \end{aligned} \tag{42}$$

If (as often happens) $\mathbf{Q}_d = \mathbf{0}$, $\mathbf{Q} \equiv \mathbf{Q}_c$. Replacing the constraints (42) into the discrete form $H(\mathbf{a}, \mathbf{v}_c, \mathbf{v}_d, \mathbf{t})$ and setting its first variation to zero yields†

$$\begin{bmatrix} \mathbf{K}^* & \mathbf{L}^* \\ (\mathbf{L}^*)^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_c \\ \mathbf{s} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^* \\ \mathbf{0} \end{Bmatrix}, \tag{43}$$

where

$$\mathbf{K}^* = \mathbf{Q}^T\mathbf{C}\mathbf{Q}, \qquad \mathbf{L}^* = \mathbf{W}^T\mathbf{L}_d, \qquad \mathbf{p}^* = \mathbf{p}_c + \mathbf{W}^T\mathbf{p}_d. \tag{44}$$

Similarly, for (34) we get the stiffness equations

$$\widetilde{\mathbf{K}}^*\mathbf{v}_c = \widetilde{\mathbf{p}}^*, \tag{45}$$

where $\widetilde{\mathbf{K}}^* = \widetilde{\mathbf{Q}}^T\mathbf{C}\widetilde{\mathbf{Q}}$, in which $\widetilde{\mathbf{Q}}_d$ results on replacing $\mathbf{P}_d$ by $\widetilde{\mathbf{P}}_d$ in (41-42).

Note that condition (a) above may be relaxed if the dimension of $\mathbf{v}_d$ exceeds that of $\mathbf{a}$ by selecting a subset of $\mathbf{v}_d$ that satisfies (b), and statically condensing out the remainder.

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† One obtains $\mathbf{K}^* = \mathbf{Q}^T(2\mathbf{P}_c+2\mathbf{P}_d\mathbf{W}-\mathbf{C}\mathbf{Q})$ which simplifies to (44) because $\mathbf{P}_d\mathbf{W} = \mathbf{C}\mathbf{Q}-\mathbf{P}_c$.

6.3 Relation to the Strain Projection Approach

If the dimension of $\mathbf{a}$ exceeds that of $\mathbf{v}_d$ (in particular, if the assumed displacement field is conforming) the constraint (40) is in general inconsistent with a strain-displacement variational principle. In such a case a connection with other techniques for improving element performance can sometimes be established. For example, suppose that the assumed strains $\boldsymbol{\epsilon}$ are *constant* and equal to $\bar{\boldsymbol{\epsilon}}$ over each element, and that the displacements are continuous. We can choose $\mathbf{a} \equiv \bar{\boldsymbol{\epsilon}}$, and $\mathbf{A} \equiv \mathbf{I}$ so that (40) may be written

$$\bar{\boldsymbol{\epsilon}} = \overline{\mathbf{B}}\mathbf{v}. \tag{46}$$

This is the strain-projection approach, also called averaged-B or the $\overline{\mathbf{B}}$ approach. If $\overline{\mathbf{B}}$ is determined by collocation at the element center, (46) is equivalent to one-point reduced/selective integration on the potential energy functional, see *e.g.* Hughes's textbook [7, Ch. 4].

7. LIMITATION PRINCIPLE

The famous limitation principle of Fraeijs de Veubeke [4] was originally stated for stress-displacement mixed finite elements, but holds for many strain-displacement elements as well. The principle is applicable when the displacement-derived strain field $\boldsymbol{\epsilon}^u$ is contained in the assumed strain field $\boldsymbol{\epsilon}$:

$$\boldsymbol{\epsilon} \ni \boldsymbol{\epsilon}^u = \mathbf{D}\mathbf{u} = \mathbf{B}\mathbf{v}. \tag{47}$$

This inclusion can be expressed in matrix form as

$$\boldsymbol{\epsilon} = \mathbf{A}\mathbf{a} = \mathbf{B}\mathbf{a}_v + \mathbf{A}_x\mathbf{a}_x = [\,\mathbf{B} \quad \mathbf{A}_x\,]\begin{Bmatrix}\mathbf{a}_v\\ \mathbf{a}_x\end{Bmatrix}. \tag{48}$$

Here $\mathbf{a}_v$ contains the same number of entries as $\mathbf{v}$ whereas $\mathbf{A}_x$, which may be empty, contains "excess" strain modes. Consider elements of type (III) based on the functional H. Inserting (48) into (30) we get

$$\begin{bmatrix} -\mathbf{C}_{vv} & -\mathbf{C}_{vx} & \mathbf{C}_{vv} & \mathbf{0}\\ -\mathbf{C}_{vx}^T & -\mathbf{C}_{xx} & \mathbf{C}_{vx}^T & \mathbf{0}\\ \mathbf{C}_{vv} & \mathbf{C}_{xv} & \mathbf{0} & -\mathbf{L}\\ \mathbf{0} & \mathbf{0} & -\mathbf{L}^T & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\mathbf{a}_v\\ \mathbf{a}_x\\ \mathbf{v}\\ \mathbf{s}\end{Bmatrix} = \begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{p}\\ \mathbf{0}\end{Bmatrix}, \tag{49}$$

where

$$\mathbf{C}_{vv} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,dV, \qquad \mathbf{C}_{vx} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{A}_x\,dV, \qquad \mathbf{C}_{xx} = \int_V \mathbf{A}_x^T\mathbf{E}\mathbf{A}\,dV. \tag{50}$$

The first two matrix equations give $\mathbf{a}_v = \mathbf{v}$ and $\mathbf{a}_x = \mathbf{0}$. Hence the system is equivalent to (38) in which $\mathbf{K} = \mathbf{C}_{vv}$ is simply the potential energy stiffness matrix. Consequently the stiffness equations may be directly constructed from the generalized potential energy functional (12) and the *independent strain assumption has no effect.* Of course the conclusion only applies if the strain degrees of freedom are solved for in a manner *consistent* with the variational equations (49); for example by static condensation. If the derived field ϵ^u varies over V, assuming a constant strain field $\bar{\epsilon}$ for ϵ is a safe way to guard against the limitation principle.

A similar analysis of type (IV) elements on the $\widetilde{H}$-derived system (31) shows that the limitation principle does not generally hold unless $\mathbf{Rv} = \mathbf{0}$. For arbitrary $\mathbf{v}$ this implies that the interface integral vanishes, in which case $\widetilde{H}$ reduces to the mixed functional R.

8. PARTIAL STRAIN ASSUMPTIONS

It is common practice to assume only part of the strains as independent fields. For example, in the C^0 plate bending element studied in Part II independent assumptions are only made for the transverse shear strains whereas the bending strains are entirely derived from displacements. The partial strain assumption may be expressed as

$$\epsilon = \begin{Bmatrix} \epsilon_a \\ \epsilon_b \end{Bmatrix}, \tag{51}$$

where independent strain assumptions are made only for $\epsilon_a = \mathbf{Aa}$. For ϵ_b one has $\epsilon_b = \epsilon_b^u$. The R and H functionals require obvious modification in the volume term; for example,

$$R(\mathbf{u}, \epsilon_a) = \int_V \left[\langle\, \epsilon_a^T \quad \epsilon_b^T \,\rangle \begin{bmatrix} \mathbf{E}_{aa} & \mathbf{E}_{ab} \\ \mathbf{E}_{ba} & \mathbf{E}_{bb} \end{bmatrix} \begin{Bmatrix} \epsilon_a^u - \frac{1}{2}\epsilon_a \\ \frac{1}{2}\epsilon_b \end{Bmatrix} - \mathbf{f}^T\mathbf{u} \right] dV + \text{surface terms} \tag{52}$$

while for $\widetilde{H}$ an additional adjustment in the S_i integral is required. The resulting principles take a particularly simple form if the constitutive coupling term $\mathbf{E}_{ab}$ and $\mathbf{E}_{ba}$ vanish, in which case

$$R = R_a(\mathbf{u}, \epsilon_a) + P_b(\mathbf{u}) \tag{53}$$

where R_a is a mixed strain-displacement principle involving ϵ_a, and P_b is a potential-energy principle involving the ϵ_b^u strain energy.

9. CONCLUSIONS

The key results of the present study may be summarized as follows.

1. The mixed strain-displacement functional of Reissner type, R, can be expanded to two hybrid functionals, H and $\widetilde{H}$, to account for incompatible displacements. Whereas $\delta R = 0$ and $\delta H = 0$ are genuine variational principles, $\delta\widetilde{H} = 0$ represents a restricted variational principle.

2. Several types of assumed-strain finite elements may be constructed using R, H or $\widetilde{H}$. The most practical elements for inclusion into existing displacement codes are those in which (1) strain and non-conforming-displacement degrees of freedom can be eliminated at the element level and (2) avoid surface traction connectors.

3. Strain degrees of freedom may be eliminated by static condensation or through kinematic constraints. The latter technique can be presented in a variationally consistent form if the conditions stated in Section 6.2 hold, in which case it can be interpreted as a constraint on non-conforming displacements. Special versions of this technique are closely related to the strain-projection approach.

4. DeVeubeke's limitation principle applies to finite element models derivable from functionals R and H if the strain elimination procedure is variationally consistent.

5. The present variational formulations may be readily modified to account for partial assumptions on the strain field.

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A VARIATIONAL JUSTIFICATION OF THE ASSUMED NATURAL STRAIN FORMULATION OF FINITE ELEMENTS. II. THE C^0 FOUR NODE PLATE ELEMENT

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SUMMARY

In Part II we use the four-node C^0 plate bending element to explore some of the possibilities opened by the theory presented in Part I. This element is chosen because the version presented by Bathe and Dvorkin [2], *MITC4*, can be considered the simplest *assumed natural strain* element that allows several possibilities to be studied in a straightforward manner. We focus our attention on the governing functionals R and H presented in Part I, assuming independent strain fields only for the transverse shear strains. Besides *MITC4* we consider three formulations (two mixed and one hybrid) that collectively represent a variational justification for the assumed strain technique. In addition, we examine reduced and selective-integration elements to compare their behavior with that of the present strain-assumed elements.

1. INTRODUCTION

1.1 4-Node C^0 Bending Plate Element Formulation

We start with the formulation of the four-node Reissner-Mindlin plate element whose degrees of freedom (d.o.f) are the transverse displacement w and the two rotations θ_x and θ_y about the x and y axes, respectively, as shown in Figure 1. We expand the displacement field in the usual way:

$$\begin{aligned} w &= N_i(r,s)\, w_i \\ \theta_x &= N_i(r,s)\, \theta_{xi} \\ \theta_y &= N_i(r,s)\, \theta_{yi} \end{aligned} \tag{1}$$

where

$$N_i(r,s) = \tfrac{1}{4}(1 + r_i r)(1 + s_i s), \qquad i = 1,2,3,4 \tag{2}$$

are bilinear shape functions. The strain field derived from the displacement field is

$$\begin{aligned} \varepsilon^u_{xx} &= z\ \theta_{y,x} \\ \varepsilon^u_{yy} &= -z\ \theta_{x,y} \\ \varepsilon^u_{xy} &= \tfrac{1}{2} z\ (\theta_{y,y} - \theta_{x,x}) \\ \gamma^u_{yz} &= w_{,y} - \theta_x \\ \gamma^u_{xz} &= w_{,x} + \theta_y \end{aligned} \tag{3}$$

We take advantage of the decoupling between bending and shear energies by using different assumptions for each one. We assume that the bending strains coincide with the bending strains computed from the displacement field:

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon^u_{xx} \\ \varepsilon_{yy} &= \varepsilon^u_{yy} \\ \varepsilon_{xy} &= \varepsilon^u_{xy} \end{aligned} \tag{4}$$

The shear strains components in the Cartesian basis x, y, z derived from the displacement field are

$$\begin{aligned} \gamma^u_{xz} &= w_{,x} + \theta_y \\ \gamma^u_{yz} &= w_{,y} - \theta_x \end{aligned} \tag{5}$$

After some manipulations we can obtain the covariant components of the shear strains in terms of the natural coordinates r and s as

$$\gamma^u_{rz} = w_{,r} + \beta_r \tag{6}$$

$$\gamma^u_{sz} = w_{,s} + \beta_s \tag{7}$$

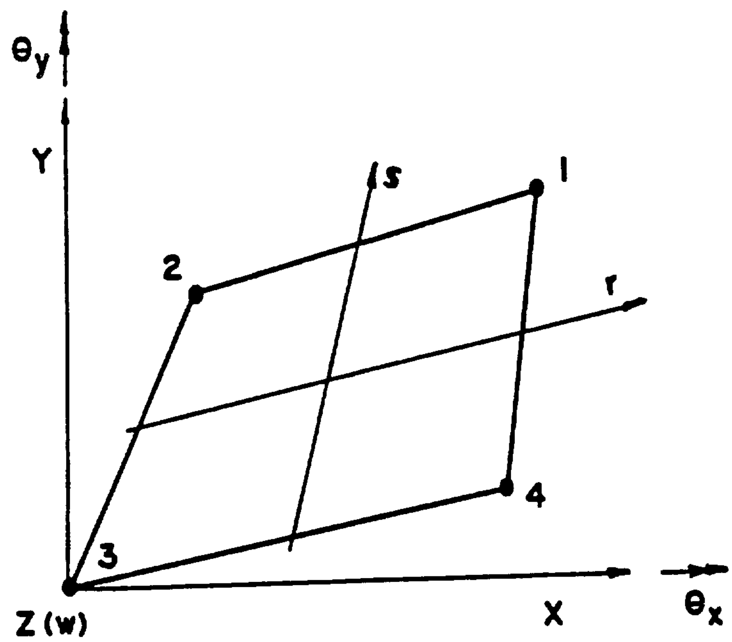

Figure 1. Element coordinate system and notational conventions

where

$$\beta_r = -\theta_x \; y_{,r} + \theta_y \; x_{,r} \tag{8}$$
$$\beta_s = -\theta_x \; y_{,s} + \theta_y \; x_{,s} \tag{9}$$

1.2 The Assumed Covariant Shear Strain

We consider two different assumptions for the covariant shear strains,

$$\gamma_{rz} = a_1 \frac{(1-s)}{2} + a_2 \frac{(1+s)}{2} \tag{10}$$
$$\gamma_{sz} = a_3 \frac{(1-r)}{2} + a_4 \frac{(1+r)}{2} \tag{11}$$

and

$$\gamma_{rz} = a_1 \tag{12}$$
$$\gamma_{sz} = a_2 \tag{13}$$

The bilinear assumption (10)–(11) is of the same form as that proposed in [2]. The constant strain assumption (12)–(13) is studied to see whether there are connections to the selective reduced integration (*SRI*) technique discussed by Hughes [3].

2. MIXED ELEMENT BASED ON THE FUNCTIONAL $R(\mathrm{u}, \epsilon)$

Up to now we are working with a compatible displacement field and a discontinuous strain field. Hence we use the functional $R(\mathrm{u}, \epsilon)$ presented in §3 of Part I. No boundary field is necessary and the constants a_i can be obtained at the element level.

The element displacement field is

$$\mathbf{u} = \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix}, \tag{14}$$

which can be expressed as

$$\mathbf{u} = \mathbf{N}_c \mathbf{v}_c \tag{15}$$

where

$$\mathbf{N}_c = \begin{bmatrix} N_1 & 0 & 0 & \dots & N_4 & 0 & 0 \\ 0 & N_1 & 0 & \dots & 0 & N_4 & 0 \\ 0 & 0 & N_1 & \dots & 0 & 0 & N_4 \end{bmatrix} \tag{16}$$

$$\mathbf{v}_c^T = (\, w_1 \quad \theta_{x1} \quad \theta_{y1} \quad \dots \quad w_4 \quad \theta_{x4} \quad \theta_{y4} \,). \tag{17}$$

The strain fields derived from the displacements are

a) bending strains:

$$\epsilon_b^u = \begin{Bmatrix} \varepsilon_{xx}^u \\ \varepsilon_{yy}^u \\ 2\varepsilon_{xy}^u \end{Bmatrix} = \mathbf{B}_b^c \mathbf{v}_c, \tag{18}$$

b) shear strains:

$$\gamma^u = \begin{Bmatrix} \gamma_{rz}^u \\ \gamma_{sz}^u \end{Bmatrix} = \mathbf{B}_s^c \mathbf{v}_c. \tag{19}$$

The independently varied strains are:

a) bending strains: the same as obtained from the displacement field, *i.e.*, (18).

b) shear strains:

$$\gamma = \begin{Bmatrix} \gamma_{rz} \\ \gamma_{sz} \end{Bmatrix} = \mathbf{B}_s^a \, \mathbf{a}. \tag{20}$$

Replacing (18), (19) and (20) into the functional R and carrying out the integrations at the element level we obtain

$$R(\mathbf{v}_c, \mathbf{a}) = \tfrac{1}{2}\mathbf{v}_c^T \mathbf{K}_b^{cc} \mathbf{v}_c - \tfrac{1}{2}\mathbf{a}^T \mathbf{C}^{aa} \mathbf{a} + \mathbf{v}_c^T \mathbf{L}^{ca} \mathbf{a} - \mathbf{v}_c^T \mathbf{p}^c \,, \tag{21}$$

where

$$\mathbf{K}_b^{cc} = \int_{V^e} (\mathbf{B}_b^c)^T \mathbf{E}_b \mathbf{B}_b^c \, dV, \tag{22}$$

$$\mathbf{C}^{aa} = \int_{V^e} (\mathbf{B}_s^a)^T \mathbf{E}_s \mathbf{B}_s^a \, dV, \tag{23}$$

$$\mathbf{L}^{ca} = \int_{V^e} (\mathbf{B}_s^c)^T \mathbf{E}_s \mathbf{B}_s^a \, dV, \tag{24}$$

$$\mathbf{p}^c = \int_{V^e} \mathbf{N}_c^T \mathbf{f} \, dV + \int_{S_t^e} \mathbf{N}_c^T \hat{\mathbf{t}} \, dS \tag{25}$$

Here vector $\mathbf{f}$ collects applied distributed forces conjugate to w, θ_x and θ_y. On performing the variations we obtain the matrix equation

$$\begin{bmatrix} \mathbf{K}_b^{cc} & \mathbf{L}^{ca} \\ (\mathbf{L}^{ca})^T & -\mathbf{C}^{aa} \end{bmatrix} \begin{Bmatrix} \mathbf{v}_c \\ \mathbf{a} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^c \\ 0 \end{Bmatrix} \tag{26}$$

From the second equation we obtain the shear strain coefficients

$$\mathbf{a} = (\mathbf{C}^{aa})^{-1} (\mathbf{L}^{ca})^T \mathbf{v}_c = \mathbf{Q}_c \mathbf{v}_c \tag{27}$$

which replaced into (26) gives the statically condensed system

$$(\mathbf{K}_b^{cc} + \mathbf{Q}_c^T \mathbf{C}^{aa} \mathbf{Q}_c) \mathbf{v}_c = \mathbf{p}^c \tag{28}$$

Here $\mathbf{K}_b^{cc}$ is the bending stiffness matrix, which is also obtainable from the potential energy principle, and $\mathbf{Q}_c^T \mathbf{C}^{aa} \mathbf{Q}_c$ stands for the new shear stiffness matrix; cf. §8 of Part I.

Equation (27) can also be obtained by minimizing the following shear energy error norm:

$$\Pi_s = \tfrac{1}{2} \int_{V^e} (\gamma - \gamma^u)^T \mathbf{E}_s (\gamma - \gamma^u) \, dV$$

where vector γ collects the independent shear strains (10)-(11) or (12)-(13), and γ^u collects the shear strains evaluated from the displacement field, equation (19). The minimization of this norm using an independent stress field instead of a strain field was proposed by Barlow [4] as a way of deriving stress-assumed hybrid elements.

We have implemented two elements based in the form (21) and the assumptions (10)–(11) and (12)–(13), which will be identified as *P4* and *P1*, respectively, in the sequel. The results obtained for the simple shear and bending tests illustrated in Figures 2 and 3 are summarized in Tables 1 and 2. We have compared these results to those obtained using *SRI* and *MITC4* elements. The results indicate that *P1* and *P4* behave poorly when elements are distorted and that *P1* is not equivalent to *SRI*.

An interesting result is that if we use one point reduced integration to compute $\mathbf{L}^{ca}$, both elements *P1* and *P4* yield the same results obtained using *SRI*.

We can obtain another expression for $\mathbf{Q}_c$, called $\mathbf{Q}_c^*$ in the sequel, from the field proposed by Bathe and Dvorkin [2] for the covariant shear strains. This expression relates four strain coefficients $\mathbf{a}$ to the nodal degrees of freedom $\mathbf{v}_c$. The elements of $\mathbf{Q}_c^*$ are given

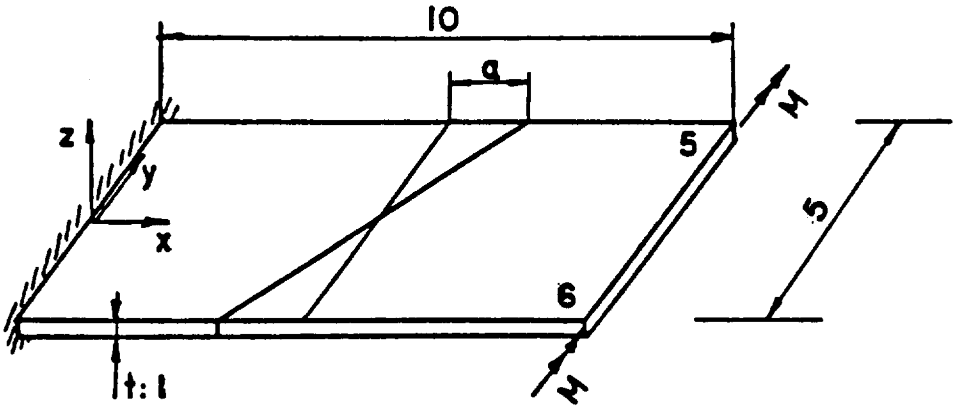

Figure 2. Bending test

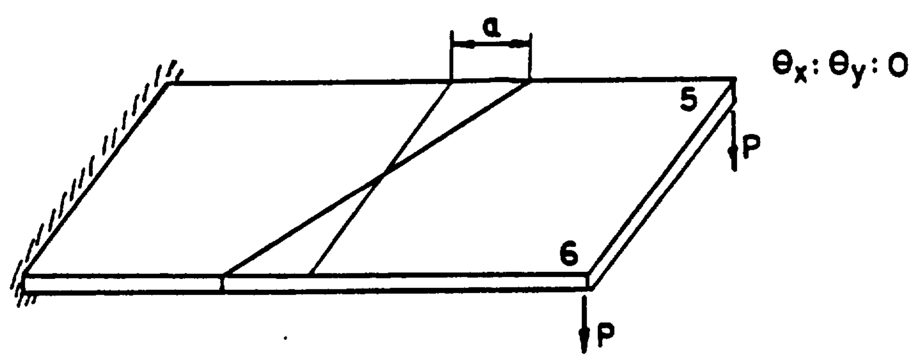

Figure 3. Shear test

in Appendix A. It is important to realize that $\mathbf{Q}_c$ obtained for element *P4* matches the matrix $\mathbf{Q}_c^*$ only for rectangular shapes. Consequently, the variational principle based on the functional R justifies the assumed natural strain technique for rectangular shapes. However, what can we say about distorted shapes? We need $\mathbf{Q}_c = \mathbf{Q}_c^*$ for all possible configurations to generalize that justification.

Table 1. Bending Test (FEM/Theory-Figure 2)

a	*Node*	*MITC4*		*SRI*		*P1*		*P4*	
		w	θ_y	w	θ_y	w	θ_y	w	θ_y
0.	5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.	5	1.00	1.00	0.90	1.00	0.88	1.00	0.44	0.44
	6	1.00	1.00	1.10	1.00	1.07	1.00	0.47	0.47
2.	5	1.00	1.00	0.80	1.00	0.74	1.00	0.23	0.23
	6	1.00	1.00	1.20	1.00	1.06	1.00	0.28	0.29

Table 2. Shear Test (FEM/Theory-Figure 3)

a	*Node*	*MITC4*	*SRI*	*P1*	*P4*
		w	w	w	w
0.	5	1.00	1.00	1.00	1.00
	6	1.00	1.00	1.00	1.00
1.	5	1.00	1.00	1.40	1.00
	6	1.00	1.00	0.85	1.00
2.	5	1.00	1.00	3.06	1.00
	6	1.00	1.00	0.99	1.00

3. INCOMPATIBLE DISPLACEMENTS. THE FUNCTIONAL $H(\mathbf{u},\epsilon,\mathbf{t})$

Following the general procedure outlined in §6.2 of Part I, we add to the transverse displacement w the four midside *incompatible* shape functions of an eight node element. In this way the bending behavior is unchanged. We denote by $\mathbf{v}_d$ the nodal values associated with these "injected" incompatible shape functions. The new displacement field can be written as

$$\mathbf{u} = [\mathbf{N}_c \quad \mathbf{N}_d] \begin{Bmatrix} \mathbf{v}_c \\ \mathbf{v}_d \end{Bmatrix} \tag{29}$$

where

$$\mathbf{N}_d = \begin{bmatrix} \frac{1}{2}(1+r)(1-s^2) & \frac{1}{2}(1-r)(1-s^2) & \frac{1}{2}(1+s)(1-r^2) & \frac{1}{2}(1-s)(1-r^2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

The bending strains do not change, and for the displacement derived shear strains we have

$$\gamma^u = \begin{Bmatrix} \gamma^u_{rz} \\ \gamma^u_{sz} \end{Bmatrix} = \mathbf{B}^c_s \mathbf{v}_c + \mathbf{B}^d_s \mathbf{v}_d. \tag{31}$$

If we introduce the new strains into the variational principle, we must use the functional $H(\mathbf{u}, \epsilon, \mathbf{t})$ because the displacement field will be discontinuous. Then, we have to introduce a traction field $\mathbf{t}$ over the boundary . This traction field is a (line) shear resultant, and for simplicity we shall assume that it is constant on each element side. On performing the variations, the following expression at the element level is obtained:

$$\begin{bmatrix} \mathbf{K}^{cc}_b & 0 & \mathbf{P}^{ca} & \mathbf{L}^{ct} \\ 0 & 0 & \mathbf{P}^{ca} & \mathbf{L}^{dt} \\ (\mathbf{P}^{ca})^T & (\mathbf{P}^{da})^T & -\mathbf{C}^{aa} & 0 \\ (\mathbf{L}^{ct})^T & (\mathbf{L}^{dt})^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v}_c \\ \mathbf{v}_d \\ \mathbf{a} \\ \mathbf{t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^c \\ \mathbf{p}^d \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

where

$$\mathbf{P}^{ca} = \int_{V^e} (\mathbf{B}^c_s)^T \mathbf{E}_s \mathbf{B}^a_s \, dV \tag{33}$$

$$\mathbf{P}^{da} = \int_{V^e} (\mathbf{B}^d_s)^T \mathbf{E}_s \mathbf{B}^a_s \, dV \tag{34}$$

$$\mathbf{L}^{ct} = \int_{S^{ef}} \mathbf{N}^T_c \, dS \tag{35}$$

$$\mathbf{L}^{dt} = \int_{S^{ef}} \mathbf{N}^T_d \, dS \tag{36}$$

$$\mathbf{p}^d = \int_{S^e_t} \mathbf{N}^T_d \hat{\mathbf{t}} \, dS \; + \int_{V^e} \mathbf{N}^T_d \mathbf{f} \, dS \tag{37}$$

Now imposing the relation

$$\mathbf{a} = \mathbf{Q}^*_c \mathbf{v}_c \tag{38}$$

we obtain

$$\mathbf{v}_d = (\mathbf{P}^{da})^{-T} \Big(\mathbf{C}^{aa} \mathbf{Q}^*_c - (\mathbf{P}^{ca})^T \Big) \mathbf{v}_c = \mathbf{W}_c \mathbf{v}_c. \tag{39}$$

Replacing both relations in the variational principle and taking variations with respect to $\mathbf{v}_c$ and $\mathbf{t}$, the following expression at the element level is obtained:

$$\begin{bmatrix} \mathbf{K}^{cc}_b + \mathbf{Q}^{*T}_c \mathbf{C}^{aa} \mathbf{Q}^*_c & \mathbf{L}^{ct} + \mathbf{W}_c \mathbf{L}^{dt} \\ (\mathbf{L}^{ct} + \mathbf{W}_c \mathbf{L}^{dt})^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{v}_c \\ \mathbf{t} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}^c + \mathbf{W}^T_c \mathbf{p}^d \\ 0 \end{Bmatrix} \tag{40}$$

The stiffness matrix proposed in [2] for the plate element, namely, $\mathbf{K}^{cc}_b + \mathbf{Q}^{*T}_c \mathbf{C}^{aa} \mathbf{Q}^*_c$, can be clearly identified in the preceding expression. It is not necessary to compute the contribution $\mathbf{L}^{ct}$ because it comes from the compatible displacement and will cancel with

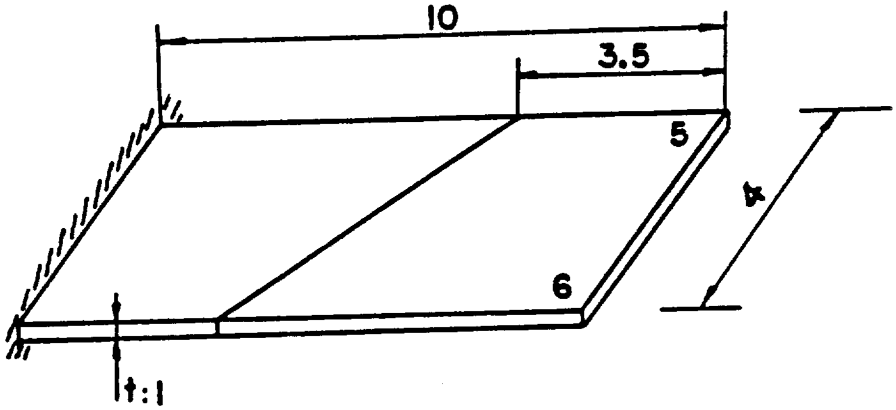

Figure 4. Cantilever beam discretization

the contribution of the neighboring element. On the other hand, the contribution $\mathbf{L}^{dt}$ from the incompatible mode does not vanish. If t vanishes the stiffness matrix reduces to that of [2] but the nodal force vector will generally be different. Thus it is worth emphasizing that the variational principle gives a *consistent treatment* for the distributed loads.

The matrix $\mathbf{P}^{da}$ is singular for rectangular elements, but we know that in this case $\mathbf{Q}_c$ is equal to $\mathbf{Q}_c^*$ and there is no need to introduce the incompatible displacement field.

4. NUMERICAL EXAMPLE

To check the behavior of the functional $H(\mathbf{u}, \epsilon, \mathbf{t})$ we analyze a cantilever beam with two distorted elements, as depicted in Figure 4. The assumed independent shear strain corresponds to equations (10) and (11). We are interested in two load cases: a uniform bending moment at the tip (Figure 2); and a uniform transverse load at the tip (Figure 3). In both cases Poisson's ratio is set to zero to compare the results to those obtained through the Euler-Bernoulli beam theory.

Uniform Bending Moment. The theoretical solution for this problem requires a linear variation for θ_y and a quadratic variation for the transverse displacement w. The results obtained with *MITC4* coincide with the theoretical results. So do those obtained with the present formulation labeled *ANSH* (for Assumed Natural Shear Hybrid).

The value obtained for t is of roundoff error order (1.10^{-12}). Then, in this case, both formulations are equivalent and the work of the incompatibility can be disregarded.

Table 3. Normalized displacements (FEM/Theory) for bending, $t = 1.E-12$

Node	*MITC4*		*ANSH*	
	w	θ_y	w	θ_y
5	1.000	1.000	1.000	1.000
6	1.000	1.000	1.000	1.000

Table 4. Normalized displacements (FEM/Theory) for shear, $t = -2.227$

Node	*MITC4*		*ANSH*	
	w	θ_y	w	θ_y
5	0.930	1.077	0.892	1.003
6	0.912	0.920	0.891	1.002

The external load vector is the same for both formulations because the external bending moment does not interact with the transverse displacement.

Uniform Transverse Load. The theoretical solution requires a quadratic variation in θ_y and a cubic one in w. In this case we must expect the computed solution to be approximate. The results obtained are shown in Table 4. Clearly the *ANSH* formulation is less sensitive to element distortion. The lack of symmetry can be observed at the third decimal position. The convergence and symmetry for the rotation is excellent. The value obtained for t is not negligible. Note that in this case the external load vector is not the same for the *MITC4* and *ANSH* formulations.

5. CONCLUSIONS

We have illustrated the theory presented in Part I [1] through the study of several 4-node C^0 plate elements with independently assumed shear strains. The following conclusions emerge from this study.

1. Elements *P1* and *P4* based on the mixed functional $R(\mathrm{u}, \epsilon)$ are variationally impeccable. *P1* behaves well in the bending test and *P4* passes the shear patch test. Their performance deteriorates markedly, however, if the element geometry departs from the rectangular one.
2. The *MITC4* element imposes a shear strain- displacement relation (38) obtained by midpoint strain collocation. This kinematic relation is not *a priori* derivable from a mixed variational principle such as $\delta R = 0$.
3. A variationally consistent modification of *MITC4*, named *ANSH*, is obtained by introducing incompatible displacement modes and an independent surface traction t (in

this case a shear line force), and using the hybrid functional $H(\mathbf{u}, \epsilon, \mathbf{t})$ for the shear energy portion. The results are similar to those of *MITC4*. Although this element is more expensive to form, it does provide a consistent treatment of applied distributed loads.

4. The *MITC4* element stiffness matrix is recovered by setting the boundary traction field $\mathbf{t}$ of *ANSH* to zero. However, the nodal load vector for distributed applied forces will generally be different.

The techniques illustrated here are obviously applicable to the construction of other types of strain-assumed elements based on the various functionals presented in Part I [1]. In particular, the use of the restricted hybrid principle $\widetilde{H}$, in which the boundary tractions are not retained as independent degrees of freedom, remain unexplored.

A key result of this investigation is that any change in the strain-displacement interpolation from the variationally consistent interpolation must be associated in some way to the addition of incompatible displacement modes. This property is closely linked to the limitation principle stated in §7 of Part I.

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Appendix A

Bathe and Dvorkin [2] proposed the same kind of shear strain interpolation we have used in equations (10)–(11). To determine the coefficients a_i they imposed the following midpoint-collocation relations:

$$a_1 = \frac{\gamma_{rz}^{*4} + \gamma_{rz}^{*3}}{2}, \qquad a_2 = \frac{\gamma_{rz}^{*1} + \gamma_{rz}^{*2}}{2},$$
$$a_3 = \frac{\gamma_{sz}^{*2} + \gamma_{sz}^{*3}}{2}, \qquad a_4 = \frac{\gamma_{sz}^{*1} + \gamma_{sz}^{*4}}{2},$$

where superscripts 1, 2, 3, 4 indicate the node where expressions (6) and (7) must be evaluated; see Figure 1. Through the application of the relations of Section 1 and after some algebra we obtain

$$\mathbf{a} = \mathbf{Q}_e^* \, \mathbf{v}_e$$

where

$$\mathbf{a}^T = (\, a_1 \quad a_2 \quad a_3 \quad a_4 \,)$$

$$\mathbf{v}_e^T = (\, w \quad \theta_{x1} \quad \theta_{y1} \quad \ldots \quad \theta_{y4} \,)$$

$$\mathbf{Q}_e^* = \begin{bmatrix} 0.5 & \frac{y_2 - y_1}{4} & \frac{x_1 - x_2}{4} & -0.5 & \frac{y_2 - y_1}{4} & \frac{x_1 - x_2}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & \frac{y_3 - y_4}{4} & \frac{x_4 - x_3}{4} & 0.5 & \frac{y_3 - y_4}{4} & \frac{x_4 - x_3}{4} \\ 0.5 & \frac{y_4 - y_1}{4} & \frac{x_1 - x_4}{4} & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & \frac{y_4 - y_1}{4} & \frac{x_1 - x_4}{4} \\ 0 & 0 & 0 & 0.5 & \frac{y_3 - y_2}{4} & \frac{x_2 - x_3}{4} & -0.5 & \frac{y_3 - y_2}{4} & \frac{x_2 - x_3}{4} & 0 & 0 & 0 \end{bmatrix}$$

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13. ABSTRACT *(Maximum 200 words)*

The *assumed natural strain* (ANS) formulation of finite elements has undergone rapid development over the past five years. The key formulation step is the replacement, in the potential energy principle, of selected displacement-related strains by independently assumed strain fields in *element natural coordinates*. These strains are not generally derivable from displacements. This procedure was conceived as one of several competing methods to solve the *element locking* problem. Its most noteworthy feature is that, unlike many forms of reduced integration, it produces no *rank deficiency*; furthermore, it is easily extendible to geometrically nonlinear problems. Many original formulations were not based on a variational principle. The objective of Part I is to study the ANS formulation from a variational standpoint. This study is based on two hybrid extensions of the Reissner-type functional that uses strains and displacements as independent fields. One of the forms is a genuine variational principle that contains an independent boundary traction field, whereas the other one represents a restricted variational principle. Two procedures for element-level elimination of the strain field are discussed, and one of them shown to be equivalent to the inclusion of incompatible displacement modes. In Part II, the 4-node C° plate bending quadrilateral element is used to illustrate applications of this theory.

14. SUBJECT TERMS

Element locking; Rank deficiency; Geometric nonlinearities; Displacement fields; Hybrid functionals; Traction fields; Plate bending; Quadrilateral elements

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