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TRANSPORT OF PHOTONS PRODUCED BY LIGHTNING IN CLOUDS

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The optical effects of the light produced by lightning are of interest to atmospheric scientists for a number of reasons (4,7,13). We mention two techniques used to explain the nature of these effects: Monte Carlo simulation (13) and an equivalent medium approach (10). In the Monte Carlo approach, paths of individual photons are simulated; a photon is said to be scattered if it escapes the cloud, otherwise it is absorbed. In the equivalent medium approach, the cloud is replaced by a single obstacle whose properties are specified by bulk parameters obtained by methods due to Twersky. See Phanord (10) for references.

In this report, we use Boltzmann transport theory to obtain photon intensities. The photons are treated like a Lorentz gas. We consider only elastic scattering and neglect gravitational effects. Water droplets comprising a cuboidal cloud are assumed to be spherical and homogeneous. Furthermore, we assume that the distribution of droplets in the cloud is uniform and that scattering by air molecules is negligible. The time dependence and five dimensional nature of this problem make it particularly difficult; neither analytic nor numerical solutions are known (3).

We begin with the single speed Boltzmann transport equation (3)

$$\frac{\partial I}{\partial t} = -c\hat{\Omega}\cdot\nabla I + \frac{c}{4\pi}K\omega_o \int \mathcal{P} I d\Omega' - KcI + cs, \; I = I\left(\mathbf{r}, \hat{\Omega}, t\right), \tag{1}$$

where I is the photon intensity which depends on position ($\mathbf{r}$), velocity direction $\left(\hat{\Omega}\right)$, and time ($t$). Inside the integral, I is considered a function of $\hat{\Omega}'$. Here, c is the speed of light, $K = 1/\Lambda$, where Λ is the mean free path, ω_o is the single scattering albedo, $\mathcal{P} = \mathcal{P}\left(\hat{\Omega}'\cdot\hat{\Omega}\right)$ is the scattering phase function, and s is a source term. The intensity in a volume V bounded by a surface S is uniquely determined by the initial intensity in V, the sources in V, and the intensity incident on S (1).

In order to reduce the number of dimensions, we use the P_N approximation (1,3). The intensity, phase function, and source term are all expanded in series of spherical harmonics $Y_n^m\left(\hat{\Omega}\right) = P_n^m(\cos\theta)e^{im\phi}$; $\hat{\Omega} = \hat{\Omega}(\theta,\ \phi)$;

$$I\left(\mathbf{r},\ \hat{\Omega},\ \mathrm{t}\right) = \sum b_n^m Y_n^m\left(\hat{\Omega}\right),\ b_n^m = b_n^m(\mathbf{r},\ \mathrm{t}),\ \sum \equiv \sum_{n=0}^{\infty}\sum_{m=-n}^{n}; \qquad s = \sum s_n^m Y_n^m\left(\hat{\Omega}\right). \tag{2}$$

The addition theorem for spherical harmonics (12) allows us to write

$$\mathcal{P}\left(\hat{\Omega}'\cdot\hat{\Omega}\right) = \sum (-1)^m d_n Y_n^{-m}\left(\hat{\Omega}'\right) Y_n^m\left(\hat{\Omega}\right). \tag{3}$$

In particular, if we use the Henyey- Greenstein function (14), $d_n = (2n+1)g^n$, where g is the asymmetry factor. The integral in [1] may be evaluated using orthogonality. The advection term of contains an inner product of $\hat{\Omega}$ and a gradient which we may write as

$$\hat{\Omega}\cdot\nabla \equiv \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}. \tag{4}$$

We absorb the trigonometric functions into the spherical harmonics comprising I with the aid of the recurrsion relations for the Legendre functions (9) and orthogonality is used to obtain the coupled set of partial differential equations,

$$-\frac{1}{c}\frac{\partial b_n^m}{\partial t} = \frac{1}{2(2n-1)}\Delta^\star b_{n-1}^{m-1} + \left(\frac{n-m}{2n-1}\right)\frac{\partial}{\partial z}b_{n-1}^m - \frac{(n-m+1)(n-m)}{2(2n-1)}\Delta b_{n-1}^{m+1}$$
$$-\frac{1}{2(2n+3)}\Delta^\star b_{n+1}^{m-1} + \frac{(n+m+1)}{2(2n+3)}\frac{\partial}{\partial z}b_{n+1}^m + \frac{(n+m+1)(n+m+2)}{2(2n+3)}\Delta b_{n+1}^{m+1} \qquad [5]$$
$$+K\left(1-\frac{\omega_0 d_n}{2n+1}\right)b_n^m - s_n^m;\ \Delta \equiv \frac{\partial}{\partial x}+i\frac{\partial}{\partial y},\ \Delta^\star \equiv \frac{\partial}{\partial x}-i\frac{\partial}{\partial y},$$

where any function b_n^m with $m > n$ or $n < 0$ is identically zero.

We work with a truncated version of [5] motivated by the Eddington approximation (2), $b_n^m = 0$ for $n \geq 2$. Using the notation $b_o^o = f_1$, $b_1^1 - \frac{b_1^{-1}}{2} = f_2$, $i\left(b_1^1 + \frac{b_1^{-1}}{2}\right) = f_3$, $b_1^o = f_4$, we have

$$\frac{1}{c}\frac{\partial f_1}{\partial t} + \beta_o K f_1 = -\frac{1}{3}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial y} + \frac{\partial f_3}{\partial z}\right) + s,$$
$$\frac{1}{c}\frac{\partial f_2}{\partial t} + \beta_1 K f_2 = -\frac{\partial f_1}{\partial x},\ \frac{1}{c}\frac{\partial f_3}{\partial t} + \beta_1 K f_3 = -\frac{\partial f_1}{\partial y},\ \frac{1}{c}\frac{\partial f_4}{\partial t} + \beta_1 K f_4 = -\frac{\partial f_1}{\partial z}, \qquad [6]$$
$$\beta_n = 1 - \frac{\omega_0 d_n}{2n+1},\ s = s_o^o,$$

where we have assumed that $s_n^m = 0$, $n \geq 1$. This truncation does not constitute a consistent solution to the transport equation, a fact implicit in (2,5,6) and explicit in (1). Orthogonality may be used to produce an additional five equations which are neglected.

The conditions at an interface between a convex cloud and a vacuum are given by (1,3)

$$I\left(\vec{\rho}, \hat{\Omega}, t\right) = 0,\ \ \hat{\Omega}\cdot\hat{n} < 0, \qquad [7]$$

where $\vec{\rho}$ is a position vector on the cloud's boundary and $\hat{n}$ is the outward normal. Physically, this condition corresponds to the requirement that for convex cloud geometries, photons which escape the cloud cannot reenter. With the simplified representation given by the Eddington approximation, it is generally impossible to satisfy [7] exactly. The condition usually used is the Marshak boundary condition (1,2,3,5,6),

$$F_n = -\int_{\hat{\Omega}\cdot\hat{n}<0} \hat{\Omega}\cdot\hat{n}\, I\left(\vec{\rho}, \hat{\Omega}, t\right) d\Omega = 0, \qquad [8]$$

where F_n is the irradiance in the direction $\hat{n}$. The integral of F_n over a portion of the interface yields the number of photons reentering the cloud through that portion.

Consider a plane surface separating the cloud and the region outside with normal $\hat{n}$. Since we may take the area to be vanishingly small and still require that no photons reenter through this surface, [8] may be rewritten as

$$\int_0^{2\pi} d\phi_n \int_{\frac{\pi}{2}}^{\pi} d\theta_n \sin\theta_n \cos\theta_n I[\theta,\phi] d\Omega = 0, \qquad [9]$$

where $I[\theta,\phi]$ is evaluated on the cloud boundary and θ_n is measured from $\hat{n}$ considered as a polar axis. An appropriate rotation of coordinates yields the condition

$$f_1 = \frac{2}{3}(n_x f_2 + n_y f_3 + n_z f_4), \quad \hat{n} = n_x\hat{x} + n_x\hat{y} + n_x\hat{z} \qquad [10]$$

on an interface. This form is valid for any convex shape and is a generalization of a similar condition given by Davies (3).

We use Laplace transforms in [6] and obtain

$$k^2 F_1 = \nabla^2 F_1 + T, \quad k^2 \equiv 3\left(\beta_o K + \frac{s}{c}\right)\left(\beta_1 K + \frac{s}{c}\right), \; T \equiv 3\left(\beta_1 K + \frac{s}{c}\right)\left(\frac{f_1(0)}{c} + S\right), \qquad [11]$$

where s is the transform variable, F_1 and S are the Laplace transforms of f_1 and s respectively, $f_1(0)$ is the initial value of f_1, and we have assumed that $f_2 = f_3 = f_4 = 0$ at $t = 0$. The boundary condition becomes

$$\frac{\partial F_1}{\partial n} + hF_1 = 0, \quad h = \frac{3}{2}\left(\frac{s}{c} + \beta_1 K\right). \qquad [12]$$

We consider a cuboidal cloud centered at the origin with boundaries coinciding with $x = \pm\frac{\alpha}{2}$, $y = \pm\frac{\beta}{2}$, $z = \pm\frac{\gamma}{2}$. Equations [11,12] are solved using finite Fourier transforms (14) to obtain

$$F_1 = \sum_{p,q,r} \frac{8h^3 \mathcal{T} \cos\xi_p x \cos\eta_q y \cos\nu_r z}{\left(\alpha h + 2\sin^2\frac{\alpha}{2}\xi_p\right)\left(\beta h + 2\sin^2\frac{\beta}{2}\eta_q\right)\left(\gamma h + 2\sin^2\frac{\gamma}{2}\nu_r\right)\left(k^2 + \xi_p^2 + \eta_q^2 + \nu_r^2\right)}, \qquad [13]$$

where $\mathcal{T}$ is the finite Fourier transform of T and ξ_p, η_q, ν_r are the roots of

$$\tan\frac{\alpha}{2}\xi_p = \frac{h}{\xi_p}, \; \tan\frac{\beta}{2}\eta_q = \frac{h}{\eta_q}, \; \tan\frac{\gamma}{2}\nu_r = \frac{h}{\nu_r} \qquad [14]$$

with positive real parts.

Due to the transcendental nature of the functions defined by [14], it is necessary to resort to numerical methods. Fortunately, it is possible to approximately invert Laplace transforms with a knowledge of $F_1(s)$, $s = 1,2,3,\ldots$. Complete details are provided in Lanczos (8).

Once f_1 is known, f_2, f_3, f_4 may be found using [6]. The photon intensity at a point outside of the cloud in a direction $\hat{\Omega}$ may be traced back to its value on the cloud's surface at an earlier time t' (1), i.e., $I\left(\mathbf{r}, \hat{\Omega}, t\right) = I\left(\mathbf{r} - c\hat{\Omega}(t-t'), \hat{\Omega}, t'\right)$, where t' is chosen so that $\mathbf{r} - c\hat{\Omega}(t-t') = \vec{\rho}$.

We are presently working on implementing the algorithms described above and on a numerical procedure which is not limited to the Eddington approximation. We intend to generalize the results to other cloud shapes and compare results with existing Monte Carlo simulations as well as with results using an equivalent medium approach.

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