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## High-Order Polynomial Expansions(HOPE) for Flux-Vector Splitting

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### Summary

The Van Leer flux splitting is known to produce excessive numerical dissipation for Navier-Stokes calculations. One example is the incorrect prediction of boundary-layer profiles. We attempt in this paper to remedy this deficiency by introducing a higher-order polynomial expansion(HOPE for short) for the mass flux. In addition to Van Leer's splitting, a term is introduced so that the mass diffusion error vanishes at  $M = 0$ . Several splittings for pressure are proposed and examined. The effectiveness of the HOPE scheme is illustrated for 1-D hypersonic conical viscous flow and 2-D supersonic shock-wave/boundary-layer interactions. Also, we give the weakness and suggest areas for further investigation of the scheme.

### Introduction

In the past decade, upwind differencing schemes have gained considerable attention for their accuracy and robustness in Euler flows with discontinuities, shock waves in particular. Naturally, significant research effort in the CFD community has been focused on maximizing the accuracy and efficiency, among other objectives. Four popular but conceptually different flux splitting ideas have been utilized for nearly 10 years: Steger and Warming, Van Leer, Roe, and Osher. However, each scheme has an associated weakness when numerical accuracy and efficiency are considered.

In this paper, we deal specifically with the improvement of Van Leer's flux vector splitting[1]. Besides its simplicity, Van Leer's splitting has the following properties: (1) it can be interpreted as a special member of a family of second-order polynomial expansions[2], and (2) the associated flux Jacobian and eigenvalues are continuous at the sonic points. Van Leer's choice allows one vanishing eigenvalue in the case of an ideal gas, thereby resulting in a crisp shock representation. Furthermore, the continuous differentiability is helpful for convergence acceleration, e.g., in multigrid schemes.

However, failing to recognize the contact discontinuity, the Van Leer splitting[1] produces excessive numerical diffusion and thus requires a huge number of cells to correctly resolve the boundary-layer flow. Some improvements have been demonstrated recently by Hänel et al[2] and Van Leer[3] for 1-D conical, hypersonic viscous flow, but a pressure glitch arises. A new scheme by the present authors[4] has been proposed that not only corrects this pressure difficulty, but also is remarkably simple to implement. Nevertheless, the above schemes[2-4] have already departed from the ideas of flux vector splitting

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and in fact become more like the flux difference splitting. Since the differentiability and simplicity are desirable properties, one would still wish to search for a better splitting scheme that is strictly based on the flux vector splitting.

In this paper, we propose a family of higher-order polynomial expansions for the mass flux that diminishes the diffusion error as  $M \rightarrow 0$ . We give a detailed study of the accuracy of the scheme for 1-D conical flow and 2-D shock wave/boundary-layer interactions. The weakness of the scheme is also pointed out and possible improvements suggested.

### Analysis

To exemplify the concept, let us consider the quasi two-dimensional system of equations for conical flows:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial \eta} = \mathbf{S}$$

where  $\mathbf{U}^T = (\rho, \rho u, \rho v, \rho E)$ ,  $\mathbf{F}^T = (\rho v, \rho v u, \rho v^2 + p, \rho v H)$ ,  $E = e + 1/2(u^2 + v^2)$ , and  $H = E + p/\rho$ . The flow considered consists of a very thin shear layer at the wall and a shock wave away from the wall. An algorithm must be capable of minimizing the numerical smearing(diffusion) at the locations where an eigenvalue changes sign or approaches zero. For example, Van Leer's splitting[1] can represent shock profile well, while greatly diffusing the boundary layer. The Van Leer split mass fluxes are:

$$F_1 = F_1^+ + F_1^-; \quad F_1^\pm = \pm \rho a / 4 (M \pm 1)^2.$$

The net difference from the curve it approximates is largest at  $M = 0$ ; its value equals  $\rho a / 2$ . This error, viz numerical diffusion, significantly broadens the boundary layer, leading to incorrect velocity and temperature profiles. A simple way to remove the diffusion at  $M = 0$  is by adding an extra higher-order term that allows the split mass fluxes to pass through the origin(Fig. 1), i.e.,

$$F_1^\pm = \pm \rho a / 4 [(M \pm 1)^2 + m_1(M)(M^2 - 1)^2],$$

where the higher-order term has a coefficient  $m_1$ , in general function of  $M$ . It should have the following properties:

- (1)  $m_1 \rightarrow -1$  as  $M \rightarrow 0$ ;
- (2)  $m_1(M) = m_1(-M)$ ;
- (3)  $m_1 \rightarrow 0$  as  $M \rightarrow \pm 1$ .

A formula satisfying those properties is chosen as:

$$m_1 = (M^2 - 1) / (M^2 + 1)^S,$$

where the exponent  $S$  is a free parameter; also shown in Fig. 1 is  $m_1$  vs  $M$  with  $S = 4$ .

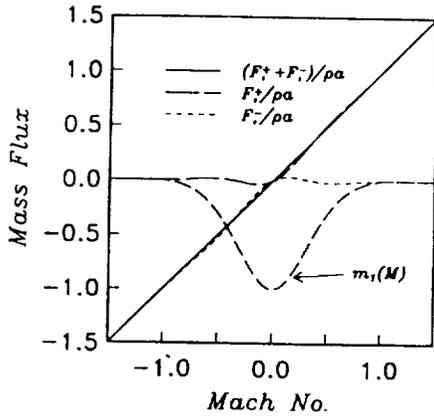


Fig. 1. Mass Flux Splitting in HOPE.

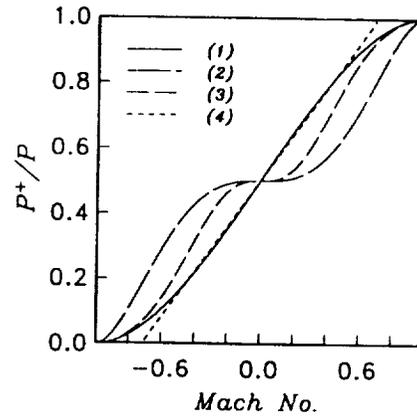


Fig. 2. Pressure Splitting.

In the conical flow calculations, the accuracy and convergence appear to be insensitive to the specified values of  $S = 2, 4, 6$ . Now, regarding the flux as a sum of convective and pressure terms, we can write the splitting formula for the flux vector:

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}^\pm = \begin{pmatrix} \rho v \\ \rho v u \\ \rho v v + p \\ \rho v H \end{pmatrix}^\pm = F_1^\pm \begin{pmatrix} 1 \\ u \\ v \\ H \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p^\pm \\ 0 \end{pmatrix}.$$

With the realization in [5] that the pressure splitting could be considered separately in the Van Leer formula[1], a whole host of freedoms for the pressure splitting becomes possible. Following is the list of formulas tested:

$$\begin{aligned} (p1): \quad p^\pm &= \mp 1/4(M \pm 1)^2(M \mp 2)p, \\ (p2): \quad p^\pm &= (p1) \mp 3/4M(M^2 - 1)^2p, \\ (p3): \quad p^\pm &= (p1) \pm 3/4m_1M(M^2 - 1)^2p, \end{aligned}$$

and

$$(p4): \quad p^\pm = 1/2(1 \pm \gamma M)p.$$

Figure 2 displays the distribution of the split pressure vs  $M$ . The first formula is that used by Van Leer[1]. The second and third splits, (p2) and (p3), yield vanishing slope at  $M = 0$ , thus corresponding to central differencing. However, no instability was encountered in the conical flow problem with the (p2) or (p3) split used in an implicit code. The fourth split (p4) is obtained from an approximate integration along characteristics. As will be seen later, the four formulas give essentially the same results for the conical flow calculated.

## Results And Discussion

In this paper, two cases were tested to check the accuracy and convergence of the HOPE scheme. The first case is the 1-D self-similar conical flow over a 10-degree half cone at hypersonic speed, for which a detailed comparison study was conducted.

The flow conditions are:  $M_\infty = 7.95$ , and  $Re_\infty = 4.2 \times 10^5$ . Since  $Pr = 1.0$ , exact solution gives adiabatic wall temperature,  $13.64T_\infty$ . The second case is the 2-D shock wave/laminar boundary-layer interactions, for which experimental measurements were available[6]. The conditions are:  $M_\infty = 2.0$ ,  $Re_\infty = 2.96 \times 10^5$ , and oblique shock angle  $\beta = 32.585$  degrees. In both cases, the results from the Roe splitting are also included for comparison. An implicit Newton iteration procedure was used to achieve steady-state solution with  $L_\infty$  residual dropped by five orders of magnitude.

Figures 3 and 4 show the pressure and temperature distributions from the first- and second-order solution on a 65-grid; little difference is seen. A monotone solution across the shock is obtained with the first-order scheme while oscillation appears in the second-order scheme, which can be eliminated by a TVD procedure. It is noted that the first-order pressure is smooth at the edge of the boundary layer, unlike the Roe solution which shows a slight discontinuity(not shown here). Although the boundary layer exhibits a steep temperature gradient, the HOPE scheme predicts the wall temperature correctly, indicating removal of the numerical diffusion associated with the original Van Leer splitting.

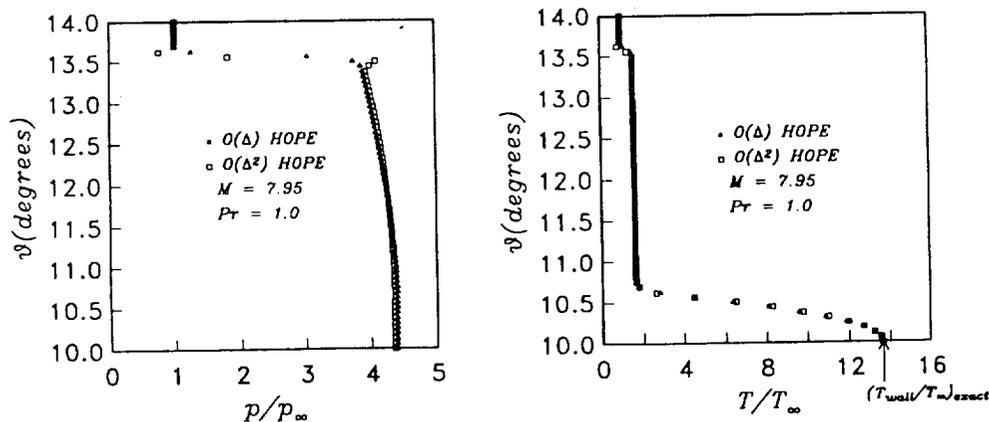


Fig. 3. Pressure Profile of Conic Flow. Fig. 4. Temperature Profile of Conic Flow.

Figure 5 displays the results using various pressure splittings; they are practically identical except the Van Leer pressure split ( $p1$ ) shows some minor oscillation near the wall. However, the pressure splittings show significant effect on the convergence rate. The ( $p3$ ) and ( $p4$ ) splits are the best, comparable to the Roe splitting, while the other two are roughly two to three times slower. These may indicate possible instability in a more complex case.

Finally, for the 2-D case, the surface pressure and friction coefficient are plotted in Figs. 7 and 8. The first-order HOPE results compare fairly with Roe's splitting and experimental data. However, the second-order calculation experienced difficulty in convergence in which the residual was reduced by only two orders of magnitude and the result is not presented here.

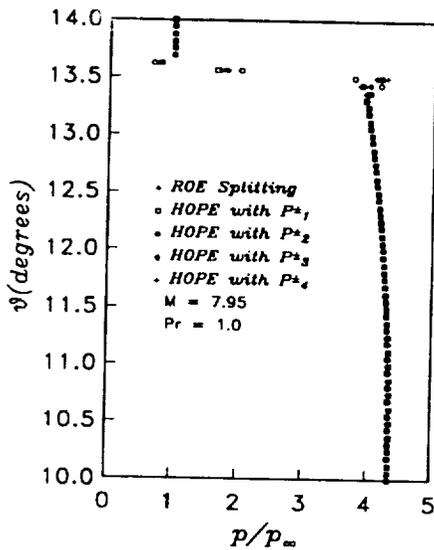


Fig. 5. Comparison of Pressure Splits.

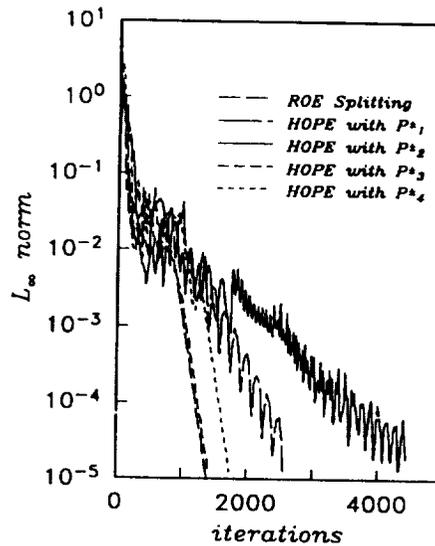


Fig. 6. Convergence History.

We suspect that a further investigation on other pressure splittings may lead to success in stability and convergence. Nevertheless, a systematic study of the eigenvalues of the split fluxes and the complete discretized system will prove to be a useful endeavor. Above all, the present research suggests that there are still possibilities in flux-vector splitting after Van Leer's appeared nearly 10 years ago. The possibilities may very well still lie in the mass-flux splitting.

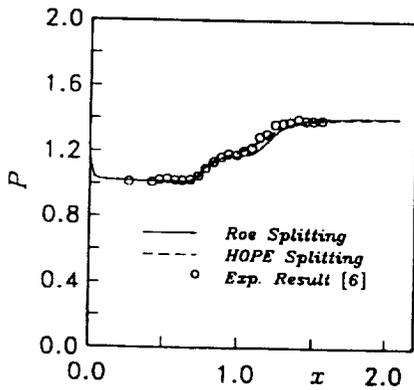


Fig. 7. Pressure at the Wall.

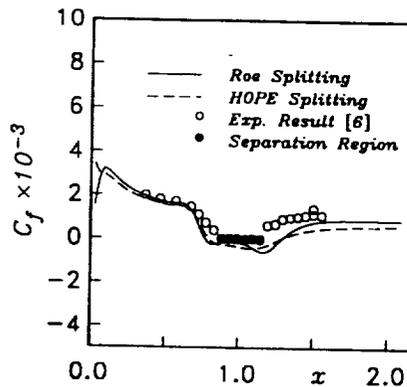


Fig. 8. Friction Coefficient at the Wall.

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