DETERMINATION OF STRESS INTENSITY FACTORS FOR INTERFACE CRACKS UNDER MIXED-MODE LOADING

Rajiv A. Naik and John H. Crews, Jr.

May 1992
To be presented at the ASTM National Symposium on Fracture Mechanics, June 30-July 2, 1992, Gatlinburg, TN
Abstract

A simple technique was developed using conventional finite element analysis to determine stress intensity factors ($K_1$ and $K_2$) for interface cracks under mixed-mode loading. This technique involves the calculation of crack-tip stresses using non-singular finite elements. These stresses are then combined and used in a linear regression procedure to calculate $K_1$ and $K_2$. The technique was demonstrated by calculating $K_1$ and $K_2$ for three different bimaterial combinations.

For the normal loading case, the calculated $K_1$ and $K_2$ were within 2.6% of an exact solution. The normalized $K_1$ and $K_2$ under shear loading were shown to be related to the normalized $K_1$ and $K_2$ under normal loading. Based on these relations a simple equation was derived for calculating $K_1$ and $K_2$ for mixed-mode loading from a knowledge of $K_1$ and $K_2$ under normal loading. Thus, for a given material combination and geometry, only one solution of $K_1$ and $K_2$ (under normal loading) is required to determine $K_1$ and $K_2$ over the full range of mixed-mode loading conditions.

The equation was verified by computing $K_1$ and $K_2$ for a mixed-mode case with equal normal and shear loading. The correlation between the exact and the finite element values was very good with errors of less than 3.7%.

This study provides a simple procedure to compute the $K_2/K_1$ ratio which can be used to characterize the stress state at the crack tip for various combinations of materials and loadings. Tests conducted over a range of $K_2/K_1$ ratios could be used to fully characterize interface fracture toughness.

Key Words: Bimaterial, finite element analysis, fracture mechanics, combined loading, phase angle.
Introduction

The performance of advanced composite materials is not only affected by their constituents but also by the character of the interface between the constituents. Interfacial cracking, either in the form of delamination or fiber-matrix debonding, is a typical failure mode in most classes of composite materials. Interfacial cracking, by definition, follows a predetermined path irrespective of the global loading. This, in conjunction with the mismatch between the material properties at the interface, leads to inherently mixed-mode crack growth. Unlike a crack in a homogeneous plate, mixed-mode conditions exist at an interface crack tip even for pure mode I loading. It is, therefore, important to characterize interfacial cracking over a range of mixed-mode conditions.

Linear elastic fracture mechanics (LEFM) concepts have been applied to the interface crack problem since 1959 when Williams [1] first determined that stresses oscillate near the tip of a semi-infinite interface crack. Other researchers [2-8] further examined the oscillatory behavior of the crack-tip stress and displacement fields and the resulting small contact region at a bimaterial crack. Rice and Sih [9] developed a solution for the stress intensity factors $K_1$ and $K_2$ for an interface crack between two semi-infinite plates subjected to a combination of both normal and shear loading.

Unlike $K_1$ and $K_2$, the mode I and mode II stress intensity factors for a crack in a homogeneous material, bimaterial stress intensity factors $K_1$ and $K_2$ have some complicating properties that reflect on their usefulness in the development of fracture criteria [9]. For example, $K_1$ and $K_2$ are not strictly associated with opening and shear modes, respectively, as in the homogeneous case. Additionally, $K_1$ and $K_2$, as defined by Rice and Sih [9] and Hutchinson et al. [10], are functions of an arbitrary length parameter. Rice [11] noted the validity of the complex stress intensity factor $K (= K_1 + iK_2, i = \sqrt{-1})$ as a crack-tip characterizing parameter for cases of small scale nonlinear material behavior or small scale contact zones at the crack tip. Although $K_1$ and $K_2$ cannot be interpreted as mode I and mode II quantities it is possible to use the $K_2/K_1$ ratio to describe the stress state at the crack-tip.
Alternative definitions for the stress intensity factors were recently provided by Shih and Asaro [12] which eliminate the arbitrary length parameter and can be related to $K_1$ and $K_2$ in a simple way.

The strain energy release rate $G$ was also examined by many researchers [13-16] as another fracture parameter to characterize crack growth at a bimaterial interface. However, the mode I and mode II components of the strain energy release rate ($G_1$ and $G_{II}$) are not well defined as was illustrated by Sun and Jih [17], Raju, et al. [18] and many other researchers. Recent studies [16, 19] have used the critical total strain energy release rate $G_c$ to characterize interface fracture toughness and the $K_2/K_1$ ratio to describe the crack-tip stress state for the given test conditions. A complete characterization of interface toughness would then involve the determination of $G_c$ over a range of $K_2/K_1$ ratios.

Closed-form methods for calculating $K_1$ and $K_2$ for interface crack problems are limited to a few special cases due to inherent mathematical difficulties. Numerical procedures are, therefore, required when $K_1$ and $K_2$ are desired for more general configurations and loadings. A boundary collocation method was used in [20] to generate interfacial $K_1$ and $K_2$ for a finite bimaterial plate. Special hybrid finite elements were developed in [21] and [22] to calculate $K_1$ and $K_2$. Conventional, non-singular finite elements were employed in [23] to calculate $K_1$ and $K_2$ from crack flank displacements and an extended form of the $J$-integral. The finite element method was also used together with domain integrals [11,24,25] to calculate $K_1$ and $K_2$. The finite element iterative method was used in [26] to evaluate $K_1$ and $K_2$ for a crack between dissimilar media. An eigenfunction expansion variational method was introduced in [27] to calculate $K_1$ and $K_2$. Each of the above mentioned numerical approaches requires its own computational scheme to handle the problem and has given satisfactory results.

In the present study, an alternative and convenient method of analysis is proposed for determining $K_1$ and $K_2$ under mixed-mode loading. First, a technique using the finite element method with non-singular elements was developed to calculate $K_1$ and $K_2$ under normal loading. $K_1$ and $K_2$ were calculated for three different material combinations: steel/aluminum,
aluminum/epoxy, and steel/epoxy, and compared with the classical solution to evaluate the technique. Next, simple relations were derived between the $K_1$ and $K_2$ due to normal and shear loading. These relations were used to derive simple equations to calculate $K_1$ and $K_2$ under mixed-mode loading from a knowledge of $K_1$ and $K_2$ under normal loading. The results from the simple equations were evaluated by comparing with finite element results for the mixed-mode case of equal normal and shear loading. The simple equations were then used to study the $K_2/K_1$ ratios for different material combinations over the full range of mixed-mode loading conditions.

**Theoretical Background**

Within the framework of LEFM, solutions for the singular stress fields at the crack-tip, the stress intensity factors and their relations to the strain energy release rates exist and are easily represented in concise form for an interfacial crack between two semi-infinite plates.

Consider a crack of length $2a$ lying along the interface of material 1 and material 2 (Fig. 1). The Young's moduli and Poisson's ratios of material 1 and 2 are given by $E_1$, $\nu_1$, and $E_2$, $\nu_2$, respectively. The coordinate system $x$-$y$-$z$ has its origin at the center of the crack with the $x$-axis parallel to the interface and the $y$-axis normal to the interface. The distance $r$ from the crack tip is measured along the interface. The body is remotely loaded by a uniform stress $\sigma_{yy}$ normal to the crack and a uniform shear stress $\sigma_{xy}$. Only plane strain ($e_{zz}=0$) deformations are considered.

For the configuration and loading shown in Fig. 1 the stresses along the interface directly ahead of the right crack tip can be written in complex notation as [10]

$$\sigma_{yy} + i \sigma_{xy} = \frac{(K_1 + i K_2) r^i}{\sqrt{2\pi r}}, \quad i = \sqrt{-1}$$

(1)
where \( K_1 \) and \( K_2 \) are the bimaterial stress intensity factors and \( \varepsilon \) is a bimaterial constant, also referred to as an oscillation index, given by \([10]\)

\[
\varepsilon = \frac{1}{2\pi} \ln \left[ \frac{G_1 + G_2 (3 - 4\nu_1)}{G_2 + G_1 (3 - 4\nu_2)} \right]
\]  

(2)

where \( G \) is the shear modulus and \( \nu \) is Poisson's ratio. Subscripts 1 and 2 refer to material 1 and 2, respectively. Interchanging the properties of materials 1 and 2 leads to a change in sign of the bimaterial constant \( \varepsilon \). In the present paper, the materials 1 and 2 were chosen such that \( E_2 \) was greater than \( E_1 \) which resulted in a positive value for \( \varepsilon \).

Unlike a crack in a homogeneous body, the singularity for a crack at the bimaterial interface is of order \((-1/2 + i\varepsilon\rangle\) as seen from Eq. (1). Also, the stress intensity factors \( K_1 \) and \( K_2 \) have units of \((\text{stress}) \times (\text{length})^{\frac{1}{2}} \times (\text{length})^{-\varepsilon}\). For the problem considered in Fig. 1, the complex stress intensity factor \( K \) \((= K_1 + iK_2)\) for the right crack tip is given by \([10]\)

\[
K = \sqrt{\pi a} \left( \sigma_{yy}^\infty + i \sigma_{xy}^\infty \right) (1 + 2i \varepsilon)(2a)^{-\varepsilon}
\]  

(3)

The complex \( K \) is often characterized by its magnitude \( \left( \sqrt{K_1^2 + K_2^2} \right) \) and its phase angle \( \psi \) which is given by

\[
\psi = \tan^{-1} \left( \frac{K_2}{K_1} \right)
\]  

(4)

Equations (3) and (4) can also be used to characterize the left crack tip (Fig. 1). However, to use the same equations, the reference coordinate axes x-y in Fig. 1 would have to be rotated by 180 degrees. In the rotated coordinate system materials 1 and 2 are reversed.
leading to a sign change in $\varepsilon$. Thus the $K_1$ and $K_2$ for the left crack tip can be obtained by using $-\varepsilon$ in place of $\varepsilon$ in Eq. (3).

As mentioned earlier, it was noted by Rice [12] that the small region of oscillations in the crack tip stresses can be ignored just as the inevitable small scale nonlinear material behavior at the crack tip in a homogeneous material is ignored in LEFM. The present study considered the stresses outside this small region of oscillations to determine $K_1$ and $K_2$.

For the configuration in Fig. 1, it can be shown from the classical solution in [10] that under shear loading the crack tip normal stresses, outside the small zone of oscillation, are tensile at the right crack tip whereas they are compressive at the left crack tip. Thus, the stress state at the right crack tip is more severe than at the left crack tip. The right crack tip was, therefore, considered in more detail in the present study.

**Finite Element Analysis**

The finite element mesh used in the present study is shown in Fig. 2. It consisted of four-noded isoparametric quadrilateral elements. The dimensions of the model were chosen to be large enough to preclude edge effects and adequately model semi-infinite plates as in Fig. 1. For the crack length-to-width ratio of 0.1 used here it was shown in [22] that the calculated stress intensity factors were not affected by the presence of the free edges. There were 2077 nodes and 1968 elements in the model and the analysis was performed using the MSC/NASTRAN code [28]. In the vicinity of the crack tip, a refined mesh was used (Fig. 2). The smallest elements with length $\Delta$ were next to the crack tip with element lengths doubling in the x- and y-directions. Plane strain conditions were imposed for all cases analyzed. Multi-point constraints were imposed along AB to enforce uniform $\varepsilon_{xx}$ in order to simulate the infinite plate problem that was analyzed in [9]. The normal and shear loadings were imposed by displacement boundary conditions along the edges of the model and are listed in Table 1. Combinations of three different materials, aluminum, epoxy, and steel, were considered in the present study. The properties used for the three materials are given in Table 2.
The adequacy of the mesh refinement near the crack tip was evaluated in two ways. First, a convergence study was performed by analyzing a center crack in an infinite homogeneous plate problem. The same finite element model (Fig. 2) was used with $E_1 = E_2$ and $\nu_1 = \nu_2$. The computed nodal stresses and stress intensity factors (SIFs) under mode I and mode II loading were compared with existing handbook values [29]. The SIFs were determined by linear regression from a log-log plot of stress versus distance ahead of the crack tip as described later. For a mesh refinement of $\Delta/a = 5 \times 10^{-6}$, the calculated SIFs were within 1% of the handbook values [29] for both mode I and mode II loading. Also, the slope of the log-log stress-distance plot near the crack tip was compared with the theoretical value of -0.5. In the region $1 \times 10^{-5} \leq (r/a) \leq 2 \times 10^{-2}$, its slope was -0.495.

To further evaluate the adequacy of this mesh refinement, the finite element model (Fig. 2) was used to analyze a crack at the interface of two dissimilar materials. For aluminum/epoxy, the computed nodal stresses near the crack tip were compared with the exact solution (Eq. (1)) for $\sigma_{xy}^{\infty}$ normal loading (Fig. 3) and for $\sigma_{xy}^{\infty}$ shear loading (Fig. 4). The $\sigma_{xy}$ stresses under a normal loading are negative; $-\sigma_{xy}$ is plotted in Fig. 3 to show it on the log scale. The non-singular elements used in this analysis are not formulated to model the singularity at the crack tip. As a result, the stresses calculated from the first few elements next to the crack tip do not correlate well with the classical solution. However, in the region $1 \times 10^{-4} \leq (r/a) \leq 2 \times 10^{-2}$, there is a good correlation with the theoretical stresses for both loading conditions. Thus, the non-singular, isoparametric elements used in the present analysis were considered to be adequate for this study. The mesh refinement of $\Delta/a = 5 \times 10^{-6}$ was used for all the cases analyzed.

The stress intensity factors $K_1$ and $K_2$ for the right crack tip were calculated using a simple procedure involving linear regression as described in the next section.
Procedure for Calculating $K_1$ and $K_2$

The normal $\sigma_{yy}$ stresses and the shear $\sigma_{xy}$ stresses directly ahead of the right crack tip are given by Eq. (1) and can be written explicitly (by separating real and imaginary parts) as

$$\sigma_{yy} = \frac{1}{\sqrt{2\pi r}} \left\{ K_1 \cos(e \ln(r)) - K_2 \sin(e \ln(r)) \right\}$$
$$\sigma_{xy} = \frac{1}{\sqrt{2\pi r}} \left\{ K_1 \sin(e \ln(r)) + K_2 \cos(e \ln(r)) \right\}$$

(5)

By multiplying both sides of the $\sigma_{yy}$ equation by $\cos(e \ln(r))$ and both sides of the $\sigma_{xy}$ equation by $\sin(e \ln(r))$ and adding the resulting equations and denoting the "combined stress" by $\sigma_1$, we have

$$\sigma_1 = \sigma_{yy} \cos(e \ln(r)) + \sigma_{xy} \sin(e \ln(r)) = \frac{K_1}{\sqrt{2\pi r}}$$

(6)

Similarly, by multiplying the $\sigma_{yy}$ equation (Eq. (5)) by $-\sin(e \ln(r))$ and the $\sigma_{xy}$ equation by $\cos(e \ln(r))$ and adding the two equations and denoting the combined stress by $\sigma_2$, we have

$$\sigma_2 = -\sigma_{yy} \sin(e \ln(r)) + \sigma_{xy} \cos(e \ln(r)) = \frac{K_2}{\sqrt{2\pi r}}$$

(7)

The combined stresses $\sigma_1$ and $\sigma_2$ at nodes ahead of the crack tip were determined from the computed $\sigma_{yy}$ and $\sigma_{xy}$ stresses from the finite element analysis. It is clear from Eqs. (6) and (7) that plots of $\sigma_1$ and $\sigma_2$ with distance $r$ on a log-log scale will be straight lines with a slope of -0.5. The stress intensity factors $K_1$ and $K_2$ were calculated from a linear regression fit of slope -0.5 to the $\log(\sigma_1)$ versus $\log(r)$ and $\log(\sigma_2)$ versus $\log(r)$ curves, respectively.
If either $\sigma_1$ or $\sigma_2$ is computed to be a negative quantity, then the corresponding $K_1$ or $K_2$ will have a negative sign. The sign of $\sigma_1$ and $\sigma_2$ was ignored while using the linear regression procedure. The calculated $K_1$ or $K_2$ was then assigned a negative sign if the corresponding $\sigma_1$ or $\sigma_2$ had a negative sign.

As mentioned earlier, the non-singular elements next to the crack tip do not model singular behavior. Thus, the combined stresses $\sigma_1$ and $\sigma_2$ from the first two or three elements will not lie on a line with a slope of -0.5 on a log-log plot of $\sigma_1$ and $\sigma_2$ with distance $r$. Also, the combined stresses from elements away from the crack tip outside the singularity-dominated region will not lie on a line with a slope of -0.5.

The linear regression fit for calculating $K_1$ and $K_2$ was performed in the region where the log($\sigma_1$) versus log($r$) and log($\sigma_2$) versus log($r$) curves had a slope of -0.5. This regression region was determined by the following procedure: the slopes of the log-log plots of $\sigma_1$ and $\sigma_2$ versus $r$ were determined for successive pairs of nodes ahead of the crack starting from the node at the crack tip. The pairs of nodes for which this slope was -0.5 ± 0.01 were included in the regression region and the combined stresses $\sigma_1$ and $\sigma_2$ from these nodes were used in the calculation of $K_1$ and $K_2$. This procedure was used to calculate $K_1$ and $K_2$ for the normal and the mixed-mode loading cases analyzed in this study.

**Relations Between $K_1$ and $K_2$ under Normal and Shear Loading**

In order to investigate the properties of $K_1$ and $K_2$ under normal and shear loading it is instructive to examine the crack-tip stresses for these two loading cases. Figure 5 shows the normalized crack-tip stresses for these two loadings calculated for the aluminum/epoxy case based on the classical solution given in [10]. It is clear from Fig. 5 that the crack-tip stresses obey the following relations:
Using the relations in Eq. (8) together with Eq. (5) we have

\[
\left\{ (K_{1N})_{\text{Normal}} \cos(e \ln(r)) - (K_{2N})_{\text{Normal}} \sin(e \ln(r)) \right\} = \\
\left\{ (K_{1N})_{\text{Shear}} \sin(e \ln(r)) + (K_{2N})_{\text{Shear}} \cos(e \ln(r)) \right\}
\]

and

\[
\left\{ (K_{1N})_{\text{Normal}} \sin(e \ln(r)) + (K_{2N})_{\text{Normal}} \cos(e \ln(r)) \right\} = \\
\left\{ -(K_{1N})_{\text{Shear}} \cos(e \ln(r)) + (K_{2N})_{\text{Shear}} \sin(e \ln(r)) \right\}
\]

where

\[
(K_{1N})_{\text{Normal}} = \frac{(K_1)_{\text{Normal}}}{\sigma_{yy}^\infty} \quad \text{and} \quad (K_{2N})_{\text{Normal}} = \frac{(K_2)_{\text{Normal}}}{\sigma_{yy}^\infty}
\]

are the normalized stress intensity factors for the right crack tip under pure normal loading and

\[
(K_{1N})_{\text{Shear}} = \frac{(K_1)_{\text{Shear}}}{\sigma_{xy}^\infty} \quad \text{and} \quad (K_{2N})_{\text{Shear}} = \frac{(K_2)_{\text{Shear}}}{\sigma_{xy}^\infty}
\]

are the normalized stress intensity factors for the right crack tip under pure shear loading. By multiplying both sides of the first equation (Eq. (9)) by \(\cos(e \ln(r))\) and both sides of the second equation (Eq. (9)) by \(\sin(e \ln(r))\) and adding the resulting equations we have the following relation between the normalized stress intensity factors:

\[
(K_{1N})_{\text{Normal}} = (K_{2N})_{\text{Shear}}
\]
Similarly, by multiplying both sides of the first equation (Eq. (9)) by $-\sin(e \ln(r))$ and both sides of the second equation (Eq. (9)) by $\cos(e \ln(r))$ and adding the resulting equations we have the following relation between the normalized stress intensity factors:

$$(K_{2N})_{\text{Normal}} = -(K_{1N})_{\text{Shear}}$$  \hspace{1cm} (13)$$

The relations in Eqs. (12) and (13) are valid for both the right and the left crack tips for the configuration in Fig. 1. Although the relations in Eqs. (12) and (13) were derived from the equations for the crack-tip stresses for the center crack configuration in Fig. 1, they are also valid for the semi-infinite crack problem with point loading on the crack faces.

The relations in Eqs. (12) and (13) can also be derived by expanding Eq. (3) into real and imaginary parts and examining the expressions for the normalized $K_1$ and $K_2$ under normal and shear loading. For normal loading we have

$$
(K_{1N})_{\text{Normal}} = \sqrt{\pi a} \{ \cos(e \ln (2a)) + 2e \sin(e \ln (2a)) \} \\
(K_{2N})_{\text{Normal}} = \sqrt{\pi a} \{ -\sin(e \ln (2a)) + 2e \cos(e \ln (2a)) \}$$  \hspace{1cm} (14)$$

and for shear loading we have

$$
(K_{1N})_{\text{Shear}} = \sqrt{\pi a} \{ -2e \cos(e \ln (2a)) + \sin(e \ln (2a)) \} \\
(K_{2N})_{\text{Shear}} = \sqrt{\pi a} \{ 2e \sin(e \ln (2a)) + \cos(e \ln (2a)) \}$$  \hspace{1cm} (15)$$

An examination of Eqs. (14) and (15) confirms the validity of the relations derived in Eqs. (12) and (13).
The $K_1$ and $K_2$ under mixed-mode loading can be obtained by expanding Eq. (3) into real and imaginary parts and using Eqs. (14) and (15) as

\begin{align*}
K_1 &= \sigma_{yy}^\infty (K_{1N})_{\text{Normal}} + \sigma_{xy}^\infty (K_{1N})_{\text{Shear}} \\
K_2 &= \sigma_{yy}^\infty (K_{2N})_{\text{Normal}} + \sigma_{xy}^\infty (K_{2N})_{\text{Shear}}
\end{align*}

(16)

Using Eq. (16) and the relations in Eqs. (12) and (13) the $K_1$ and $K_2$ under any combination of normal and shear loading can be written as

\begin{align*}
K_1 &= \sigma_{yy}^\infty (K_{1N})_{\text{Normal}} - \sigma_{xy}^\infty (K_{2N})_{\text{Normal}} \\
K_2 &= \sigma_{yy}^\infty (K_{2N})_{\text{Normal}} + \sigma_{xy}^\infty (K_{1N})_{\text{Normal}}
\end{align*}

(17)

Thus, if $(K_{1N})_{\text{Normal}}$ and $(K_{2N})_{\text{Normal}}$ are known for a particular material combination and crack length, then $K_1$ and $K_2$ under any combination of normal and shear loading can be determined by using Eq. (17).

Equations (17) can be used to generate $K_1$ and $K_2$ values for a range of mixed-mode loadings for both the right and the left crack tips based on a knowledge of the corresponding $(K_{1N})_{\text{Normal}}$ and $(K_{2N})_{\text{Normal}}$. The $K_2/K_1$ ratio, or $\tan(\psi)$ (see Eq. (4), for mixed-mode loading can be expressed in terms of the ratio $(K_{2N}/K_{1N})_{\text{Normal}}$ by using Eq. (17) as

\begin{equation}
\tan(\psi) = \frac{K_2}{K_1} = \frac{\left[ (K_2 / K_1)_{\text{Normal}} + \left( \sigma_{xy}^\infty / \sigma_{yy}^\infty \right) \right]}{1 - \left( \sigma_{xy}^\infty / \sigma_{yy}^\infty \right)(K_2 / K_1)_{\text{Normal}}} \tag{18}
\end{equation}

As $\left( \sigma_{xy}^\infty / \sigma_{yy}^\infty \right)$ approaches $\pm\infty$, the pure shear loading case, it can be shown that the right hand side of Eq. (18) approaches a limiting value of $-(K_1/K_2)_{\text{Normal}}$. Thus, Eq. (18) provides
a means of calculating the \((K_2/K_1)\) ratio or \(\tan(\psi)\) over the full range of mixed-mode ratios from a knowledge of \(K_1\) and \(K_2\) for the normal loading case.

Equation (18) is valid for both the right and the left crack tips. However, the appropriate \((K_2/K_1)_{\text{Normal}}\) ratio should be used in Eq. (18). For the left crack tip \(K_1\) and \(K_2\) under normal loading can be calculated from a knowledge of \(K_1\) and \(K_2\) for the right crack tip. By using \(-\varepsilon\) in the place of \(\varepsilon\) in Eq. (14) it can be shown that under normal loading the ratio of the stress intensity factors at the left crack tip is equal to \(-(K_2/K_1)\) where \(K_1\) and \(K_2\) are the stress intensity factors at the right crack tip.

**Results and Discussion**

The finite element analysis and procedure described above were used to calculate \(K_1\) and \(K_2\) for three different material combinations: aluminum/epoxy, steel/epoxy, and steel/aluminum. Two different loading conditions were analyzed: normal loading \(\sigma_{\text{xy}} = 0\) and mixed-mode loading with \(\sigma_{\text{xy}} = \sigma_{\text{yy}}\). In all cases the half-crack length, \(a\), was 20 units.

The \(K_2/K_1\) ratios were plotted for a range of mixed-mode load ratios using Eq. (18).

Figure 6 shows normalized combined stresses \(\sigma_1\) and \(\sigma_2\) for the normal loading case as a function of normalized distance from the crack tip on a log-log plot. The circular symbols indicate the locations at which nodal stresses were available from the finite element analysis. Note that the \(\sigma_2\) combined stress was a negative quantity and \(-\sigma_2\) was plotted on the log scale in the figure. The negative \(\sigma_2\) also led to a negative \(K_2\). There was very little difference between the calculated \(\sigma_1\) values for the three material combinations which led to similar \(K_1\) values (Table 3). However, there was some difference between the calculated \(\sigma_2\) values leading to larger differences between the calculated \(K_2\) values for the different material combinations (Table 3). As mentioned earlier, the results from the first few elements do not lie on a straight line with a slope of \(-0.5\). The appropriate regression region for calculating \(K_1\) and \(K_2\) was determined as described earlier and was found to be \(1.6 \times 10^{-4} \leq (r/a) \leq 1 \times 10^{-2}\). Table 3 shows a comparison between the exact [10] and the calculated normalized \(K_1\) and \(K_2\) for normal
loading. There was excellent agreement between the calculated and the exact [10] $K_1$ and $K_2$ with errors of less than 2.6%. As the mismatch between the material properties decreased, the $K_2$ values became smaller with larger errors in the calculated values.

The mixed-mode loading case was analyzed to verify the relations derived in Eq. (17) for calculating $K_1$ and $K_2$ from a knowledge of $K_{1N}$ and $K_{2N}$ under normal loading. Values for $K_{1N}$ and $K_{2N}$ under normal loading were taken from Table 3 (exact solution). These were used together with the relations in Eq. (17) to calculate $K_1$ and $K_2$ (see Table 4) for the $\sigma_{xy} = \sigma_{yx}$ mixed-mode case. These results were compared with results calculated from the finite element analysis. The nodal stresses ahead of the crack tip for the mixed-mode loading case were obtained by superimposing the nodal stresses from the pure normal and pure shear loading cases. The regression region for this case was also found to be

$1.6 \times 10^{-4} \leq (r/a) \leq 1 \times 10^{-2}$. The excellent correlation (errors less than 3.7%) between the results using Eq. (17) and the finite element method (FEM) verifies the derivations in the previous section. The crack-tip stresses for the two extreme loading cases of pure normal and pure shear loading were verified earlier (Figs. 3 and 4) by comparing with the exact solution [10].

Figures 7 and 8 show plots of the $K_2/K_1$ ratios using Eq. (18) for two different ranges of mixed-mode loading ratios. Figure 7 shows the normal-load dominated cases with mixed-mode loading ratios ranging from 0 to 1.0. The $(K_2/K_1)_{\text{Normal}}$ ratios used were obtained from the exact solution [10] in Table 3. Two limiting material combinations are also included for reference purposes. The crack in a large homogeneous plate represents a bimaterial case with no mismatch in material properties ($\varepsilon = 0$). For this case the $K_2/K_1$ ratio is equivalent to the ratio of classical mode II and mode I stress intensity factors, $K_{II}$ and $K_1$, respectively. The rigid/epoxy case represents a case in which the mismatch between the properties is characterized by a relatively large value of the bimaterial constant, $\varepsilon = 0.094$. The $K_2/K_1$ curves for all cases fell between the two extremes represented by these two limiting cases; cases with the larger material property mismatch were closer to the rigid/epoxy case. For the
mode I loading case \( (\sigma_{xy}^\infty / \sigma_{yy}^\infty = 0) \), the \( K_2/K_1 \) ratio is negative due to the presence of a negative shear stress at the crack tip (Fig. 3). For mixed-mode ratios between 0.0 and 0.5, the \( K_2/K_1 \) ratio increases linearly with approximately the same slope as the homogeneous crack case. Unlike the homogeneous crack case, the \( K_2/K_1 \) ratios for the bimaterial cases under equal normal and shear loading \( (\sigma_{xy}^\infty / \sigma_{yy}^\infty = 1) \) have values less than 1.0 (Fig. 7). This is due to the negative shear stress (Fig. 3) at the crack tip, which under normal loading subtracts from the positive shear stress (Fig. 4) under shear loading.

Figure 8 show plots of the \( K_2/K_1 \) ratios from Eq. (18) over the full range of mixed-mode loading but with an emphasis on the shear dominated cases. For the shear dominated loading, the curves are highly nonlinear and there is significant departure from the homogeneous case. The nonlinearity in the curves can be explained by examining Eq. (18) for the aluminum/epoxy case \( (K_2/K_1 = -0.11) \). For this case, the denominator in Eq. (18) becomes more dominant for mixed-mode ratios greater than 1 and, thus, leads to the nonlinearity in the curves. For pure mode II loading the \( (K_2/K_1) \) ratios approach a limiting value of \(- (K_1/K_2)_{\text{Normal}}\) as discussed earlier.

Fracture toughness of the interface between two different materials can be characterized by measuring the critical strain energy release rate \( G_c \) over a range of \( K_2/K_1 \) ratios. The present study provides a simple equation (Eq. (18)) to determine these values for the \( (K_2/K_1) \) ratios over the full range of mixed-mode loadings if the \( (K_2/K_1)_{\text{Normal}} \) ratio under normal loading is known. The \( (K_2/K_1)_{\text{Normal}} \) ratio for any configuration can be determined using conventional, non-singular finite elements and the simple regression procedure developed in this study.

**Concluding Remarks**

A simple technique was developed using conventional finite element analysis to determine stress intensity factors for interface cracks under mixed-mode loading. This technique involved the calculation of crack-tip stresses using non-singular finite elements.
These stresses were then combined and used in a linear regression procedure to calculate interface stress intensity factors $K_1$ and $K_2$. The technique was demonstrated by calculating $K_1$ and $K_2$ for three different bimaterial combinations. The nature of the stress intensity factors $K_1$ and $K_2$ was studied over the full range of mixed-mode loadings and compared to the limiting cases of a crack in a homogeneous plate and a bimaterial crack at the interface of a rigid substrate and epoxy.

For the normal loading case, the calculated $K_1$ and $K_2$ were within 2.6% of an exact solution. The normalized $K_1$ and $K_2$ under shear loading were shown to be related to the normalized $K_1$ and $K_2$ under normal loading. Based on these relations a simple equation was derived for calculating $K_1$ and $K_2$ for mixed-mode loading from a knowledge of $K_1$ and $K_2$ under normal loading. Thus, for a given material combination and crack length, only one solution of $K_1$ and $K_2$ under normal loading is required to determine $K_1$ and $K_2$ over the full range of mixed-mode loading conditions.

This equation was verified by computing $K_1$ and $K_2$ for a mixed-mode case with equal normal and shear loading. Once again the correlation between the classical and the finite element values was very good with errors of less than 3.7%.

This study provides a simple procedure to compute the $K_2/K_1$ ratio which can be used to characterize the stress state at the crack tip for various combinations of materials and loadings. Tests conducted over a range of $K_2/K_1$ ratios could be used to fully characterize interface fracture toughness.
References


Table 1 -- Displacement boundary conditions for normal and shear loading.

<table>
<thead>
<tr>
<th>Loading</th>
<th>Boundary condition at</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x = 0</td>
</tr>
<tr>
<td>Normal</td>
<td>u = 0</td>
</tr>
<tr>
<td>Shear</td>
<td>v = 0</td>
</tr>
</tbody>
</table>

\(^{a}\)A constant displacement was imposed using multi-point constraints.

Table 2 -- Material properties used in the analysis.

<table>
<thead>
<tr>
<th>Property</th>
<th>Aluminum</th>
<th>Epoxy</th>
<th>Steel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus, E (GPa)</td>
<td>68.95</td>
<td>3.10</td>
<td>206.85</td>
</tr>
<tr>
<td>Poisson's Ratio, (\nu)</td>
<td>0.30</td>
<td>0.35</td>
<td>0.30</td>
</tr>
</tbody>
</table>
Table 3 -- Comparison of interface stress intensity factors under $\sigma_{yy}$ normal loading.

<table>
<thead>
<tr>
<th>Bimaterial</th>
<th>$\epsilon$</th>
<th>$K_1/(\sigma_{yy}\sqrt{\pi a})$</th>
<th>$K_2/(\sigma_{yy}\sqrt{\pi a})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel/Aluminum</td>
<td>0.046</td>
<td>1.001 0.991</td>
<td>-0.078 -0.080</td>
</tr>
<tr>
<td>Aluminum/Epoxy</td>
<td>0.067</td>
<td>1.002 0.996</td>
<td>-0.115 -0.116</td>
</tr>
<tr>
<td>Steel/Epoxy</td>
<td>0.072</td>
<td>1.003 0.996</td>
<td>-0.124 -0.126</td>
</tr>
</tbody>
</table>

Table 4 -- Comparison of interface stress intensity factors for mixed-mode loading ($\sigma_{yy} = \sigma_{xy} = \sigma_0$).

<table>
<thead>
<tr>
<th>Bimaterial</th>
<th>$\epsilon$</th>
<th>$K_1/(\sigma_0\sqrt{\pi a})$</th>
<th>$K_2/(\sigma_0\sqrt{\pi a})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Eq. [17] FEM</td>
<td>Eq. [17] FEM</td>
</tr>
<tr>
<td>Steel/Aluminum</td>
<td>0.046</td>
<td>1.079 1.056</td>
<td>0.923 0.929</td>
</tr>
<tr>
<td>Aluminum/Epoxy</td>
<td>0.067</td>
<td>1.117 1.076</td>
<td>0.888 0.892</td>
</tr>
<tr>
<td>Steel/Epoxy</td>
<td>0.072</td>
<td>1.127 1.098</td>
<td>0.879 0.882</td>
</tr>
</tbody>
</table>
Fig. 1. - Configuration and loading for a binatural crack.
Fig. 2 — Finite element model showing mesh detail at crack tip.
Fig. 3 - Comparison of crack-tip stresses for normal loading.
Fig. 4 – Comparison of crack-tip stresses for shear loading.
Fig. 5 — Relations between crack-tip stresses for normal and shear loading.
Fig. 6 -- Combined stresses at crack tip under normal loading.
Fig. 7 -- $K_2/K_1$ ratios for mixed-mode ratios less than 1.0.
Fig. 8 - $K_2/K_1$ ratios over full range of mixed-mode load ratios.
## Determination of Stress Intensity Factors for Interface Cracks Under Mixed-Mode Loading

### Abstract

A simple technique was developed using conventional finite-element analysis to determine stress intensity factors (K₁ and K₂) for interface cracks under mixed-mode loading. This technique involves the calculation of crack-tip stresses using non-singular finite elements. These stresses are then combined and used in a linear regression procedure to calculate K₁ and K₂. The technique was demonstrated by calculating K₁ and K₂ for three different bimaterial combinations.

For the normal loading case, the calculated K₁ and K₂ were within 2.6 percent of an exact solution. The normalized K₁ and K₂ under shear loading were shown to be related to the normalized K₁ and K₂ under normal loading. Based on these relations a simple equation was derived for calculating K₁ and K₂ for mixed-mode loading from a knowledge of K₁ and K₂ under normal loading. Thus, for a given material combination and geometry, only one solution of K₁ and K₂ (under normal loading) is required to determine K₁ and K₂ over the full range of mixed-mode loading conditions.

The equation was verified by computing K₁ and K₂ for a mixed-mode case with equal normal and shear loading. The correlation between the exact and the finite-element values was very good with errors of less than 3.7 percent.

This study provides a simple procedure to compute the K₂/K₁ ratio which can be used to characterize the stress state at the crack tip for various combinations of materials and loadings. Tests conducted over a range of K₂/K₁ ratios could be used to fully characterize interface fracture toughness.