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EUCLIDEAN BLACK HOLE VORTICES

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ABSTRACT

We argue the existence of solutions of the Euclidean Einstein equations that correspond to a vortex sitting at the horizon of a black hole. We find the asymptotic behaviours, at the horizon and at infinity, of vortex solutions for the gauge and scalar fields in an abelian Higgs model on a Euclidean Schwarzschild background and interpolate between them by integrating the equations numerically. Calculating the backreaction shows that the effect of the vortex is to cut a slice out of the Euclidean Schwarzschild geometry. Consequences of these solutions for black hole thermodynamics are discussed.

1. Introduction.

The view that the quantum aspects of black hole physics will play an important rôle in leading us towards a quantum theory of gravity has been strengthened recently, not only by the discovery that some coset conformal field theories correspond to string theory in two-dimensional black hole geometries[1], but also by the suggestion that the more familiar four-dimensional variety can carry "quantum hair"[2,3]. This latter development is of particular interest to relativists, since the conventional wisdom is that powerful theorems imply that black holes are characterised only by their mass, angular momentum and electric charge (and other charges that are associated with a Gauss' law). Investigating these "no-hair" theorems, however, shows that whilst powerful, they are not omnipotent! In particular, the existing 'no-hair theorem' for the abelian Higgs model with the usual symmetry breaking potential makes restrictive assumptions about the behaviour of the fields exterior to the horizon[4,5], restrictions that are not obviously satisfied by all physically interesting scenarios. It *has* been shown that a black hole cannot be the source of a non-zero, static, massive vector field[6] but the jury is still out on the case where a $U(1)$ gauge field acquires a mass through the Higgs mechanism. However since the expectation is that in this case too, black holes cannot support non-zero massive vector fields, apparent contradictions are of great interest since they would limit the conditions of validity of a rigorous no-hair theorem.

It has been noted by Aryal et al.[7] that black holes might have hair - quite literally - since they wrote down the metric for a black hole with a cosmic string passing through it. They used a distributional energy momentum source as the string, so one could not say with confidence that this corresponds to a physical vortex spacetime since such a limit is not valid for line-like defects[8]. However, one might find this suggestive that a no-hair theorem would have to be limited to the case where no topological defects exist, thus reducing the physical relevance of such a theorem since defects *will* exist if they can exist. It was also shown by Luckock and Moss[9] that black holes could carry skyrmion hair, although they conjectured that such solutions were unstable.

More recently, it was pointed out by Bowick et al.[2] that there exists a family of

Schwarzschild black hole solutions to the Einstein-axion equations labelled by a conserved topological charge. Thus, in some sense, such black holes could be said to be carrying axion hair. It was then rapidly realised that the same fractional charge that could give rise to enhancement of proton decay catalysis by cosmic strings[10] could potentially be carried by black holes[3]. The full ramifications of this type of quantum hair have been most eloquently argued by Coleman et al.[11,12], who suggest that this charge might have dramatic implications for black hole thermodynamics. Remarkably, their work implies that even if a black hole does not carry discrete charge its temperature is still renormalised away from the Hawking value. This means that if we are to believe in spontaneous symmetry breaking and the existence of strings in nature, then we must take into account such renormalisation effects independently of whether discrete charge exists or not.

All of these claims rest on the existence of a family of 'vortex' solutions which are saddle points in some Euclidean path integral. These solutions are obviously outside the domain of standard no-hair arguments, being Euclidean, however they are static in the sense that the metric is static and the energy-momentum tensor is time-independent (though not in the restricted sense of Gibbons[5]) and establishing existence would set bounds on the validity of future theorems.

In this paper we will focus on the problem of existence of solutions of the above sort. The layout of the paper is as follows. We begin by setting up the general problem, discussing what is meant by a 'vortex centered on a black hole'. We then show that a perturbative analysis is justified for weakly gravitating vortices, after which we focus on the specific example of a complex scalar (Higgs) field with a "Mexican hat" potential, coupled to a $U(1)$ gauge field. We find numerically a vortex solution on a Schwarzschild background and describe its asymptotic behaviour. We calculate the back-reaction on the geometry to first order in GT, the energy per unit area of the vortex (in Planck units), and also calculate the Euclidean action of this geometry. We calculate the expectation value of the metric in a black hole state at a certain temperature and derive a relation between the mass and temperature without appealing directly to the partition function. We also calculate the expectation value of the area of the black hole. We draw analogies with cosmic string physics, and discuss problems with global charge.

2. Einstein-matter equations: general formalism.

We have said we are interested in finding vortex solutions to the abelian Higgs model in a Euclidean black hole spacetime. First we should discuss what we mean by a Euclidean black hole spacetime.

Recall that a Schwarzschild black hole metric has the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (2.1)$$

We may formally Euclideanise this by setting $t \to i\tau$. However, we now see that the former Lorentzian coordinate singularity at $r = 2GM$ is in danger of becoming a real singularity in the Euclidean space, since the metric changes signature from four to zero for $r < 2GM$. This tells us that we must regard $r > 2GM$ as the only region of relevance in our Euclidean section, and that therefore we must be able to include $r = 2GM$ in a non-singular fashion into our manifold. Changing variables to $\rho^2 = 16G^2M^2r^{-1}(r - 2GM)$ we see that

$$ds^2 = \rho^2 d(\frac{\tau}{4GM})^2 + d\rho^2 + 4G^2M^2 d\Omega^2_{II} \qquad (2.2)$$

near $r = 2GM$, which shows that τ must be identified with period $8\pi GM$, and that r and τ are analogous to cylindrical polar coordinates on a plane. Thus, we arrive at the conclusion that Euclidean Schwarzschild has topology $S^2 \times \mathbb{R}^2$, with a periodic time coordinate, period $\beta = 8\pi GM$. The geometry of the $t-r$ section of Euclidean Schwarzschild can be visualised as the surface of a semi-infinite "cigar" with a smoothly capped end and tending to a cylinder of radius $4GM$ as $r \to \infty$.

In general there will be matter present as well as a black hole, therefore, assuming that the matter is spherically symmetric and 'static' (i.e., cylindrically symmetric), we will be looking for solutions to the Euclidean Einstein equations with topology $S^2 \times \mathbb{R}^2$, being spherically symmetric on the S^2 sections, and cylindrically symmetric on the $\mathbb{R}^2$ sections. (Note that we require only the energy momentum to have these symmetries. It is quite possible that the constituent fields do not, for example, a Nielsen-Olesen vortex is cylindrically symmetric even though the Higgs field has a dependence on the azimuthal coordinate.) The metric is then a function of just one variable, a radial coordinate in the

$\mathbb{R}^2$ plane. The presence of a black hole is indicated by the existence of a minimal value of the radial coordinate r_s ($= 2GM$, say) at which the metric and curvature are nonetheless regular. Following Garfinkle et al.[13] we will write the metric in the form

$$ds^2 = A^2 d\tau^2 + A^{-2} dr^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.3}$$

where $A(r_s) = 0$, τ is understood to be a periodic coordinate with period β, and $C(r_s)^2 = \mathcal{A}/4\pi$ is given in terms of the area of the event horizon. The regularity of the metric at r_s implies we can choose local cylindrical coordinates in which the metric is regular

$$\rho = BA(r) \tag{2.4}$$

where $B = \beta/2\pi$ is used for convenience. Regularity then implies $(A^2)'|_{r_s} = 2/B$. In principle we can leave the metric in terms of the period, β, and the area of the event horizon, $\mathcal{A}$, however, for calculational simplicity we choose to use up the coordinate freedom

$$r \to ar + b \quad , \quad \tau \to a^{-1}\tau \tag{2.5}$$

to set $B = 2r_s$ and $C(r_s) = r_s$. We may then re-interpret our coordinates if required. The Einstein equations for this metric can then be written as:

$$C'' = 4\pi G \frac{C}{A^2}(T^0_0 - T^r_r) \tag{2.6a}$$

$$((A^2)'C^2)' = 8\pi G C^2 (2T^\theta_\theta + T^r_r - T^0_0) \tag{2.6b}$$

$$\frac{2AA'C'}{C} - \frac{1}{C^2}(1 - A^2 C'^2) = 8\pi G T^r_r \tag{2.6c}$$

where

$$T_{ab} = \frac{2}{\sqrt{g}} \frac{\partial(\mathcal{L}\sqrt{g})}{\partial g^{ab}} \tag{2.7}$$

is the energy momentum tensor, which obeys the conservation law

$$T^{r\prime}_r + \frac{A'}{A}(T^r_r - T^0_0) + \frac{2C'}{C}(T^r_r - T^\theta_\theta) = 0 \tag{2.8}$$

which is valid for a general spherical-cylindrical symmetric source.

In order to complete our preliminaries on formulating the Einstein equations, we note that since we expect the greatest variation of T^a_b to occur near the horizon, it may be expedient to have a form of the Einstein equations in terms of the proper distance from the horizon. For convenience we also scale out the dimensional fall-off behaviour of the energy momentum tensor, r_H say, to express quantities in terms of the dimensionless parameter

$$\hat{r} = \frac{1}{r_H}\int_{r_s}^{r}\frac{dr'}{A} . \tag{2.9}$$

Setting $\hat{C}(\hat{r}) = C/r_s$, and $\epsilon\hat{T}^a_b = 8\pi G T^a_b r_H^2$, the boundary conditions at the horizon become

$$\hat{C}(0) = 1, \quad \hat{C}'(0) = 0, \quad \hat{C}''(0) = \frac{1}{2R^2} + \tfrac{1}{2}\epsilon\hat{T}^0_0|_{r_s} , \tag{2.10a}$$

and

$$A(0) = A''(0) = 0 \; , \quad A'(0) = \frac{1}{2R} , \tag{2.10b}$$

where prime denotes $\frac{d}{d\hat{r}}$ and $R = r_s/r_H$ is the ratio of the Schwarzschild radius to the vortex width. The Einstein equations are now

$$(A'\hat{C}^2)' = \epsilon\hat{C}^2 A(2\hat{T}^\theta_\theta + \hat{T}^r_r - \hat{T}^0_0) \tag{2.11a}$$

$$\left(\frac{\hat{C}'}{A}\right)' = \tfrac{1}{2}\epsilon\frac{\hat{C}}{A}(\hat{T}^0_0 - \hat{T}^r_r) \tag{2.11b}$$

$$\hat{C}' = \frac{-A'\hat{C}}{A}\left[1 - \sqrt{1 + \frac{A^2}{A'^2\hat{C}^2}\left(\frac{1}{R^2} + \epsilon\hat{C}^2\hat{T}^r_r\right)}\right] , \tag{2.11c}$$

where we have rearranged (2.6c) as a quadratic for $\hat{C}'$. Regularity at the horizon fixes the sign of the root in (2.11c), which is then valid in some neighbourhood of the horizon.

Having set up this formalism, we now turn to the problem of deciding under what circumstances we expect a vortex black hole to exist.

3. Asymptotic solution of Einstein's equations.

We would like to show that solutions exist which correspond to a vortex at the horizon of the black hole. However, rather than taking a specific field theory source for T^a_b, in this section we remain more general, investigating what minimal conditions T^a_b must satisfy in order to have an asymptotically Schwarzschild metric. We naturally have in mind that T^a_b has some, as yet unspecified, field theory vortex solution as its source, therefore we expect $T^a_b = E\hat{T}^a_b/r_H^2$, where E is an energy per unit area characterising the source, $\hat{T}^a_b$ is the rescaled energy momentum referred to in (2.11) which is of order unity, and r_H represents a cut-off scale of the vortex. Thus, for example, a Nielsen-Olesen vortex has $E \sim \eta^2$ and $r_H \sim 1/\sqrt{\lambda}\eta$, where η is the symmetry breaking scale and λ the quartic self-coupling constant. Because we are in Euclidean space, we do not have a conventional set of energy conditions for T^a_b, but since we know that T^a_b is derived from a θ and ϕ independent field theoretic lagrangian, we do have a modified dominant energy condition, namely that

$$\mathcal{L} = -T^\theta_\theta = -T^\phi_\phi \geq |T^0_0|, |T^r_r|. \tag{3.1}$$

Now, as we have already remarked, we are looking for a non-singular asymptotically Schwarzschild metric. This means that we do not expect $C = 0$, nor in fact do we expect $A' = 0$ at any finite r. (We cannot make a similar statement concerning C', since the effect of the radial stresses can conspire to make C actually *decrease* near the horizon.) Inspection of (2.11a) shows that $A'(\hat{r}) > 0$ is guaranteed if

$$J(\hat{r}) = \epsilon \int_0^{\hat{r}} \hat{C}^2 A(2\hat{T}^\theta_\theta + \hat{T}^r_r - \hat{T}^0_0) d\hat{r}' \tag{3.2}$$

converges, and its modulus is less than $1/2R$. What we will now prove is that if $\epsilon = 8\pi GE \ll 1$ (the vortex is suitably weakly gravitating) and if the energy momentum satisfies certain fall-off conditions then J is not only convergent, but is of order ϵ/R. By a fall-off condition we mean that outside the core ($\hat{r} \geq$few) $|\hat{T}^a_b| \leq K(\hat{r}^{-n})$ for some K of order unity, $n > 0$. Our aim is to find a value of n which will guarantee that we can integrate out the metric functions to large values of $\hat{r}$. This will then tell us what sort of energy momenta we expect well-behaved vortex solutions to have. Since we are not, at this stage, trying to argue the existence of a full solution to the coupled Einstein-matter system, we restrict our attention to only two of the metric equations, (2.11a,c). The reason for this is that the three Einstein equations implicitly contain the matter equations of motion, conservation of energy momentum being an integrability condition for (2.11a-c). Now let us turn to proving our claim - and finding the value of n.

We start by assuming the contrary - that J is divergent. Then there exists an $\hat{r}_0$ at which $J(\hat{r}_0) = -1/4R$, thus on $[0, \hat{r}_0]$ (2.11a) implies

$$\frac{1}{2R} \geq A'\hat{C}^2 \geq \frac{1}{4R}. \tag{3.3}$$

Now, in order to use (2.11c) to bound $\hat{C}$, we must be sure that the sign of the root is fixed; this relies crucially on

$$f(\hat{r}) = \frac{A'^2\hat{C}^2}{A^2} + \frac{1}{R^2} + \epsilon\hat{C}^2\hat{T}^r_r \tag{3.4}$$

being positive. Let $\hat{r}_f \leq \hat{r}_0$ be chosen so that $f > 0$ on $[0, \hat{r}_f]$. Then, on this interval

$$-\sqrt{\epsilon}\hat{C}|\hat{T}^r_r|^{\frac{1}{2}} \leq \hat{C}' \leq \left(\frac{1}{R^2} + \epsilon\hat{C}^2|\hat{T}^r_r|\right)^{\frac{1}{2}} \tag{3.5}$$

using $\{1 - \sqrt{|y|} \leq \sqrt{1+x+y} \leq 1 + \sqrt{x+|y|}\}$ for $x > 0$, $|y| < 1$.

Let us consider the implications of each bound in turn. The lower bound on $\hat{C}'$ implies

$$\hat{C} \geq \exp\{-\sqrt{\epsilon}\int |\hat{T}^r_r|^{\frac{1}{2}}\} \geq e^{-\alpha\sqrt{\epsilon}} \tag{3.6}$$

where α will be order unity if we use the fall-off assumption with $n \geq 4$, (and so in particular $\hat{C}$ is always positive). Hence

$$A' \leq \frac{1}{2R}e^{2\alpha\sqrt{\epsilon}} \quad \Rightarrow \quad A \leq \frac{\hat{r}}{2R}e^{2\alpha\sqrt{\epsilon}} \quad \text{on } [0, \hat{r}_f]. \tag{3.7}$$

Using this bound and (3.3) we see that

$$\frac{A'^2\hat{C}^4}{A^2} + \epsilon\hat{C}^4\hat{T}^r_r \geq \frac{e^{-2\alpha\sqrt{\epsilon}}}{4\hat{r}^2} - \epsilon e^{-4\alpha\sqrt{\epsilon}}|\hat{T}^r_r| \tag{3.8}$$

is strictly positive on $[0, \hat{r}_f]$ provided $\epsilon \ll 1$ and the previous fall-off assumption holds. Therefore $\hat{C}^2 f > \hat{C}^2/R^2$ on $[0, \hat{r}_f]$, and without loss of generality, we may choose $\hat{r}_f = \hat{r}_0$.

Now we examine the upper bound on $\hat{C}$:

$$\hat{C}' \leq \frac{1}{R} + \sqrt{\epsilon}\hat{C}|\hat{T}^r_r|^{\frac{1}{2}} \leq \frac{e^{\sqrt{\epsilon}\int |\hat{T}^r_r|^{\frac{1}{2}}}}{R} + \sqrt{\epsilon}\hat{C}|\hat{T}^r_r|^{\frac{1}{2}}, \tag{3.9}$$

which implies that

$$\hat{C} \leq e^{\sqrt{\epsilon}\int |\hat{T}^r_r|^{\frac{1}{2}}}(1 + \frac{\hat{r}}{R}). \tag{3.10}$$

Bounding $\int |\hat{T}^r_r|^{\frac{1}{2}}$ by α as before, we see that

$$|J| \leq \epsilon \int_0^{\hat{r}} \frac{\hat{r}}{2R} e^{4\sqrt{\epsilon}\alpha}(1 + \frac{\hat{r}}{R})^2 |2\hat{T}^\theta_\theta + \hat{T}^r_r - \hat{T}^0_0| d\hat{r}. \tag{3.11}$$

This is readily seen to be convergent on $[0, \hat{r}_0]$ if $n \geq 5$ in the fall-off assumption, and we may write

$$|J| \leq \frac{\epsilon\gamma}{2R} \tag{3.12}$$

for some γ of order unity provided $R \geq 1$. Therefore for $R \geq 1$, $J(\hat{r}_0)$ cannot be equal to $1/4R$ thus contradicting the initial assumption about $\hat{r}_0$. Therefore we conclude that no such $\hat{r}_0$ exists, and provided that $|\hat{T}^a_b| \leq K\hat{r}^{-5}$ we may (formally) integrate out the metric equations to infinity keeping $A', \hat{C} > 0$. Note again that this argument only involves (2.11a) and (2.11c).

We now use the following argument to conclude that if a solution does exist then it is asymptotically Schwarzschild.

Note that the initial conditions imply that $\int_{r_s}^{r_s+\delta}(r - r_s)|T^0_0 - T^r_r|/A^2 dr$ is bounded. But then we use $A > A(r_s + \delta)$ on $(r_s + \delta, \infty)$ to conclude that

$$\int_{r_s+\delta}^{\infty} \frac{(r - r_s)|T^0_0 - T^r_r|}{A^2} dr < \frac{E}{4RA(r_s + \delta)} \int_{\hat{r}(r_s+\delta)}^{\infty} \hat{r}^2 |\hat{T}^0_0 - \hat{T}^r_r| d\hat{r} < \infty. \tag{3.13}$$

We may then use a theorem† from ordinary differential equations to conclude that

$$C \sim cr + d \quad \text{as } r \to \infty. \tag{3.14}$$

† The theorem states that if $\int_0^\infty x|a(x)|dx$ is bounded, then the non-zero solutions of the 2nd order equation $u'' + a(x)u = 0$ have the asymptotic form $u \sim Ax + B$ where the constants A and B cannot both be zero[14].

Examining (2.6b,c) as $r \to \infty$ shows that $c \neq 0$ and (2.6b) then implies $(A^2)' \to 0$ as $r \to \infty$, and a rearrangement of (2.6c) gives

$$A^2 \sim \frac{1}{c^2}\left(1 - \frac{(r_s + I)}{r}\right) \quad \text{as } r \to \infty, \tag{3.15}$$

where

$$I = 8\pi G \int_{r_s}^{r} C^2(2T^\theta_\theta + T^r_r - T^0_0)dr' = r_s RJ. \tag{3.16}$$

Thus we see that any solution must be asymptotically Schwarzschild. We can also see that the solution will be changed by $O(\epsilon)$ from exact Schwarzschild. Indeed,

$$2AA'C^2 = r_s + I(= r_s(1 + O(\epsilon))) \tag{3.17}$$

implies

$$\frac{C}{A^2}(T^0_0 - T^r_r) = 2\frac{(C^3T^r_r)' - C^2C'(T^r_r + 2T^\theta_\theta)}{(r_s + I)} \tag{3.18}$$

using the equations of motion for T^a_b. Then, using (2.6c) at the horizon to determine $C'|_{r_s} = 1 + 8\pi G r_s^2 T^r_r|_{r_s}$, we may rewrite (2.6a) as

$$\begin{aligned} C'(r) &= 1 + \frac{\epsilon C^3 T^r_r}{(r_s + I)E} + \frac{\epsilon}{E}\int \frac{C^2(T^r_r + 2T^\theta_\theta)}{(r_s + I)}\left[\frac{\epsilon C^3 T^r_r}{(r_s + I)E} - C'\right]dr - \frac{\epsilon^2}{E^2}\int \frac{C^5 T^r_r T^0_0}{(r_s + I)^2}dr \\ &\to 1 - \frac{\epsilon}{E r_s}\int_{r_s}^{\infty} C^2(T^r_r + 2T^\theta_\theta)dr + O(\epsilon^2) \quad \text{as } r \to \infty \end{aligned} \tag{3.19}$$

which gives the value of c to order ϵ.

It is possible to write integral expressions for the changes in the Arnowitt-Deser-Misner (ADM) mass[15] and period of the spacetime from their vacuum values. Recall from (3.14,15) that the asymptotic form of the metric is

$$ds^2 = c^{-2}\left(1 - \frac{r_s + I}{r}\right)d\tau^2 + c^2\left(1 - \frac{r_s + I}{r}\right)^{-1}dr^2 + (cr + d)^2 d\Omega^2_{II}. \tag{3.20}$$

where c is given by (3.19). If $c \neq 1$, then clearly the τ, r coordinates are not those of a 'Euclidean observer' at infinity. In order to identify the true period and ADM mass of the space, we must rescale the r, τ coordinates so that $A^2 \to 1$ at infinity. Thus we set

$$\tau' = \tau/c \quad ; \quad r' = cr + d \tag{3.21}$$

to obtain

$$ds^2 = \left(1 - \frac{(r_s + I)c}{r' - d}\right) d\tau'^2 + \left(1 - \frac{(r_s + I)c}{r' - d}\right)^{-1} dr'^2 + r'^2 d\Omega^2_{II}, \tag{3.22}$$

and hence

$$\beta' = \beta/c = \beta\left(1 + \frac{\epsilon}{Er_s}\int_{r_s}^{\infty} C^2(T^r_r + 2T^\theta_\theta)dr\right) \tag{3.23}$$

$$M_\infty = c(r_s + I(\infty))/2G = \frac{r_s}{2G}\left(1 - \frac{\epsilon}{r_s E}\int_{r_s}^{\infty} C^2 T^0_0 dr\right) . \tag{3.24}$$

are the period and ADM mass of the space to order ϵ.

Thus to order ϵ, the period of the geometry decreases, whereas M_∞ may increase or decrease according to the details of the specific vortex model chosen.

The preceeding expressions give the modified period and ADM mass of the spacetime, if one knows what the solutions are. However, a perturbation expansion in ϵ for solutions is justified if $\epsilon \ll 1$ and we will now give the solutions for the metric functions in the perturbative case. One can solve for the sources $T^a_b(r)$ as test fields on the Schwarzschild background. In the next section we will study the equations for the matter fields in the Abelian Higgs model, so for now let us assume that we have solved the equations and know what the vortex sources are. These solutions on the background are exact if $\epsilon = 0$, i.e. the matter and gravity decouple. The next step is to compute the corrections to the metric coefficients when $\epsilon \neq 0$.

One finds that to first order

$$C = C_1 = r + \int_{r_s}^{r} dr' I_1(r') \tag{3.25}$$

where

$$I_1(r) = \frac{\epsilon}{Er_s}\left[r^3 T^r_r - \int_{r_s}^{r} dr' r'^2 (T^r_r + 2T^\theta_\theta)\right], \tag{3.26}$$

and

$$A^2 = A_1^2 = 1 - \frac{r_s}{r} + \int_{r_s}^{r} dr' \left(\frac{I(r')}{r'^2} - \frac{2r_s}{r'^3}\int_{r_s}^{r'} ds I_1(s)\right) \tag{3.27}$$

where $I(r)$ is given by (3.16) with C replaced by r^2. In equations (3.25) and (3.27) everything on the right hand side is known, in terms of the sources.

For large r one can then extract the derivative of C and the ADM mass, to give the modifications to the period and mass which are just equations (3.23) and (3.24) with the metric functions in the integrals replaced by their Schwarzschild forms.

4. An abelian Higgs vortex solution.

We now examine the specific energy momentum source of an abelian Higgs vortex centered on the horizon. The lagrangian for the matter fields is

$$\mathcal{L} = \left(\frac{1}{4}F_{\mu\nu}^2 + (\mathcal{D}^\mu\psi)^*\mathcal{D}_\mu\psi + \frac{\lambda}{4}(|\psi|^2 - \eta^2)^2\right) . \tag{4.1}$$

For a simple vortex solution we choose the variation of the phase of the ψ field to distribute itself uniformly over the periodic τ direction. This is simply a gauge choice which allows us to simplify the equations of motion by setting

$$\begin{aligned} \psi &= \eta X(r)e^{ik\tau/B} \\ A_\mu &= \frac{1}{Be}(P(r) - k)\partial_\mu\tau = \frac{1}{Be}(P_\mu - k\partial_\mu\tau) . \end{aligned} \tag{4.2}$$

This implies that the lagrangian and equations of motion simplify to

$$\mathcal{L} = \left\{\frac{P_{,r}^2}{2e^2B^2} + \eta^2X_{,r}^2A^2 + \eta^2\frac{X^2P^2}{A^2B^2} + \frac{\lambda\eta^4}{4}(X^2-1)^2\right\} \tag{4.3}$$

$$\frac{1}{C^2}(C^2P_{,r})_{,r} = \frac{\lambda\eta^2}{\nu}\frac{X^2P}{A^2} \tag{4.4a}$$

$$\frac{1}{C^2}(C^2A^2X_{,r})_{,r} = \frac{P^2X}{A^2B^2} + \frac{\lambda\eta^2}{2}X(X^2-1) , \tag{4.4b}$$

where $\nu = \lambda/2e^2$.

It is straightforward to check that the asymptotic behaviour of the bounded solutions to (4.4) is

$$X \propto (r - r_s)^{|k|/2} \qquad P = k - \alpha(r - r_s) \qquad \text{as } r \to r_s \tag{4.5a}$$

where $\alpha = -Be/(4\pi r_s^2)\int_H A'_\tau dS$ and

$$1 - X \propto r^{-1} e^{-\sqrt{\lambda}\eta r/A_\infty} \qquad P \propto r^{-1} e^{-\sqrt{\lambda}\eta r/\sqrt{\nu}A_\infty} \qquad \text{as } r \to \infty \tag{4.5b}$$

where A_∞ is given by (3.15). The appearance of the square root in the dependency of X on r near the horizon simply reflects the dependence on the local proper distance there. Note that at this level, there is no obvious obstruction to the fall-off condition on $\hat{T}^a_b$ being satisfied.

If solutions to the coupled Einstein-Higgs equations exist, then we expect that there is a perturbative limit as $\epsilon \to 0$, as we have noted‡. Indeed, many of the demonstrations of the lack of 'hair' on Lorentzian black holes have shown that on a fixed Schwarzschild background the interaction between test fields and a source is extinguished as the source approaches the horizon[16]. Therefore we first consider the question of the existence of solutions for the matter fields on a fixed Euclidean Schwarzschild background, setting $C = r$ and $A^2 = 1 - \frac{r_s}{r}$ in (4.4). Rescaling the radial variable to $\tilde{r} = \frac{r - r_s}{r_H}$ gives

$$\begin{aligned} \frac{1}{(\tilde{r}+R)^2}\left[(\tilde{r}+R)^2 P'\right]' &= \frac{X^2 P}{\nu}\frac{(\tilde{r}+R)}{\tilde{r}} \\ \frac{1}{(\tilde{r}+R)^2}\left[\tilde{r}(\tilde{r}+R)X'\right]' &= \frac{P^2 X}{4R^2}\frac{(\tilde{r}+R)}{\tilde{r}} + \tfrac{1}{2}X(X^2-1) \end{aligned} \tag{4.6}$$

The question is - is there a solution to (4.6) which connects the bounded behaviour at the horizon (4.5a) to the bounded behaviour at infinity (4.5b with $A_\infty = 1$)? Existence of such solutions is similar to the difficult question of existence of abelian Higgs vortices in

‡ This limit might seem problematic since it involves taking either $G \to 0$ or $E = \eta^2 \to 0$. The former limit must be taken at finite r_s in order to preserve the background geometry, this would mean that the Euclidean black hole would have a formally infinite "mass". The latter limit is equivalent to sending the symmetry breaking scale to zero which would require sending the self coupling, λ and the charge, e, to infinity in order to keep r_H and ν fixed. Since, by rescaling the fields, one can express the equations in terms of ϵ, r_H r_s and ν only, both limits are equivalent as far as the equations are concerned. However, since G is a measured physical constant, it may be easier to think of the limit as $\eta \to 0$:

Minkowski spacetime, first investigated by Nielsen and Olesen[17]. To see this, consider flat space and make the ansatz (4.2) with ρ and θ replacing r and τ respectively, where ρ and θ are cylindrical polar coordinates in the plane perpendicular to the infinitely long straight static Nielsen-Olesen vortex. Setting $X_{\rm NO}$ and $P_{\rm NO}$ as the Nielsen-Olesen solutions, the equations of motion that these satisfy can be readily seen to be

$$
\begin{aligned}
(\rho X_{NO}')' &= \frac{X_{NO}P_{NO}^2}{\rho} + \tfrac{1}{2}\rho X_{NO}(X_{NO}^2-1) \\
\left(\frac{P_{NO}'}{\rho}\right)' &= \frac{X_{NO}^2 P_{NO}}{\nu\rho}
\end{aligned}
\tag{4.7}
$$

Existence of solutions to these equations was shown numerically by Nielsen and Olesen, and their stability properties discovered by Bogomoln'yi[18]. Much is known about the behaviour of Nielsen-Olesen vortices, or cosmic strings. In particular, Bogomoln'yi showed that for a special value of ν, $\nu = 1$, the second order equations in (4.7) reduce to two first order equations:

$$
\rho X_{NO}' = X_{NO}P_{NO} \quad ; \quad P_{NO}'/\rho = \tfrac{1}{2}X_{NO}(X_{NO}^2-1).
$$

This is often referred to as the supersymmetric limit, since the model is supersymmetrisable for this value of ν. The above relations also have the direct consequence that the radial and azimuthal stresses, T^ρ_ρ, T^θ_θ, vanish identically. For $\nu \neq 1$, these stresses become non-zero changing sign according to the value of ν. This idea will be important in our later discussions of the mass and entropy. However, for the moment, let us just note that for $\nu \leq 1$ vortex solutions are stable for all values of the winding number k, whereas for $\nu > 1$, solutions with $k \geq 2$ are unstable.

In order to see the similarities (and differences) between our problem and the Nielsen-Olesen case we have just discussed, let $z = \rho^2/4R$, then (4.7) becomes

$$
\begin{aligned}
\frac{1}{R}[zX_{,z}]_{,z} &= \frac{XP^2}{4Rz} + \tfrac{1}{2}X(X^2-1) \\
P_{,zz} &= \frac{R}{z}\frac{X^2P}{\nu}
\end{aligned}
\tag{4.8}
$$

The two sets of equations (4.6) and (4.8) become identical as $z, \tilde{r} \ll R$. However, far from the horizon, $z, \tilde{r} \gg R$, the equations are very different, and we cannot simply infer the existence of well-behaved solutions to (4.6) from the Nielsen-Olesen case.

We do not currently have an analytic proof of the existence of regular solutions to (4.6), however, we have integrated the equations numerically using a relaxation technique, and these results show that the bounded eigenfunctions at the horizon do indeed integrate out to the exponentially decaying eigenfunctions at infinity. Figure 1 shows a plot of X and P with $k = 1$, $\nu = 1$ and $R = 2$, compared with the Nielsen-Olesen solutions. The radial coordinate is $\tilde{r}$ for the Schwarzschild case and ρ for the Nielsen-Olesen case. The difference in behaviours at the origin reflects the fact that for the Schwarzschild case r is not the coordinate in which the metric near the horizon looks flat. At $r = 0$, $X'_{SCHW} = \infty$, $P'_{SCHW} = -1.92$, $X'_{NO} = 1.37$ and $P'_{NO} = 0$.

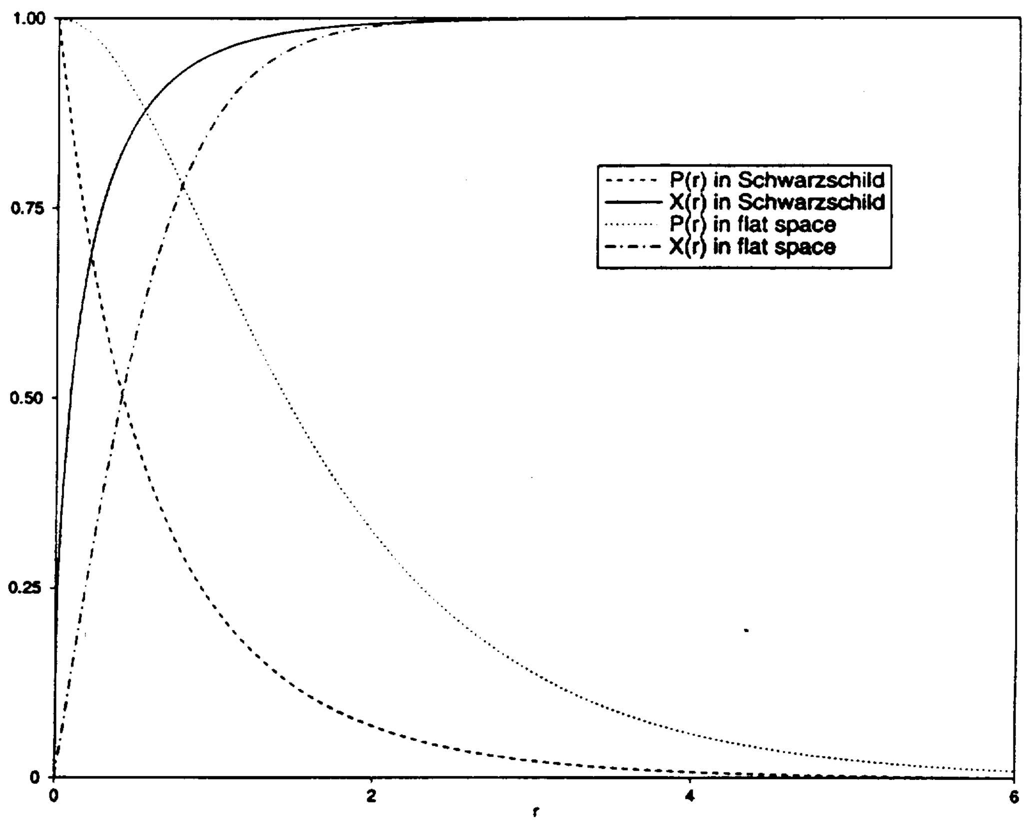

Fig. 1 A comparison between Schwarzschild and flat space vortex solutions.

Having justified the existence of a background solution, let us remark on the behaviour of a fully coupled system. Setting

$$\hat{\rho} = \rho/r_H = 2RA(r),$$

a local cylindrical coordinate, we find

$$\frac{1}{\hat{\rho}}\frac{r_s^2(r_s+I)^2}{C^4}(\hat{\rho}X')' = \frac{XP^2}{\hat{\rho}^2} + \tfrac{1}{2}X(X^2-1) - \epsilon\hat{\rho}X'(2\hat{T}^\theta_\theta + \hat{T}^r_r - \hat{T}^0_0) \tag{4.9a}$$

$$\frac{1}{\hat{\rho}}\frac{r_s^2(r_s+I)^2}{C^4}(P'/\hat{\rho})' = \frac{X^2P}{\nu\hat{\rho}^2} - \epsilon\frac{P'}{\hat{\rho}}(2\hat{T}^\theta_\theta + \hat{T}^r_r - \hat{T}^0_0), \tag{4.9b}$$

or, alternatively

$$\hat{\rho}(\hat{T}^r_r)' + (\hat{T}^r_r - \hat{T}^0_0) + [O(\epsilon) + O(\hat{\rho}^2R^{-2})](\hat{T}^r_r - \hat{T}^\theta_\theta) = 0, \tag{4.10}$$

where $I = O(r_s\epsilon)$ is given by (3.16).

Now, noting that $C = r_s(1 + O(\epsilon) + O(R^{-2}))$ for $\hat{\rho} \ll R$, from (3.6,10), we readily see the similarity of (4.9) with (4.7). We also see that the matter equations can be written as some background piece plus an order ϵ piece coming from the interaction of the vortex with the geometry. This then justifies the iterative procedure for the matter part of the fully coupled system.

To zeroth order, the space is Euclidean Schwarzschild,

$$C = r, \;\; A^2 = \left(1 - \frac{r_s}{r}\right), \;\; \hat{\rho} = 2R\sqrt{1 - \frac{r_s}{r}}\,. \tag{4.11}$$

In order to calculate the back reaction we will focus on thin vortices, since these are more physically relevant. This limit corresponds to $R \gg 1$, and we therefore expect our solutions to be well approximated by the Nielsen-Olesen solution for $\hat{\rho} \ll R$, of the exponential form (4.5b) for $\hat{\rho} > R$, and having some transitionary nature from $\hat{\rho}$-exponential decay to r-exponential decay for intermediate radii. We will in fact assume $R^{-2} \ll \epsilon$ to facilitate the following analysis, keeping in mind that for a typical GUT vortex $\epsilon \sim 10^{-6}$ would only require $r_s \gg 10^3 r_H \sim 10^{-26}$cm! Since R is so very large, the energy momenta are negligibly small for $\hat{\rho} \geq R$, so as far as the Einstein equations are concerned we can essentially ignore corrections from the Nielsen-Olesen form for $\hat{\rho} \geq R$ as well, and we will simply set

$$X_0 = X_{\rm NO}(\hat{\rho}) \quad ; \quad P_0 = P_{\rm NO}(\hat{\rho}) \tag{4.12}$$

where it is understood that X_0 and P_0 have $O(R^{-2})$ corrections which do not contribute to the order in perturbation theory ($O(\epsilon)$) to which we will be working.

The results of section 3 allow us to now calculate the back-reaction on the metric quite straightforwardly. In what follows we will suppress the suffix 0 on the energy momentum tensor for clarity. Setting

$$\hat{\mu} = -\int \hat{\rho} \hat{T}^{\theta}_{\theta} d\hat{\rho} \tag{4.13}$$

the normalised energy per unit area of the vortex, and $\hat{p} = -\int \hat{\rho} \hat{T}^{r}_{r} d\hat{\rho}$, an averaged scaled pressure, we see that (3.18) implies that, to first order in ϵ,

$$C'(\infty) = 1 + \epsilon(\hat{\mu} + \tfrac{1}{2}\hat{p}). \tag{4.14}$$

Then, noting from (4.10) that

$$\int \hat{\rho}(\hat{T}^0_0 + \hat{T}^r_r) d\hat{\rho} = O(\epsilon), \tag{4.15}$$

the ADM mass parameter from (3.23) is

$$M_\infty = \frac{r_s}{2G}(1 - \tfrac{1}{2}\epsilon\hat{p}) \tag{4.16}$$

to first order in ϵ. Thus, making the coordinate transformation defined in (3.21):

$$r' = (1 - \epsilon(\hat{\mu} + \tfrac{1}{2}\hat{p}))r \quad ; \quad \tau' = (1 - \epsilon(\hat{\mu} + \tfrac{1}{2}\hat{p}))\tau \tag{4.17}$$

the asymptotic metric takes the form

$$ds^2 = \left(1 - \frac{2GM_\infty}{r'}\right) d\tau'^2 + \left(1 - \frac{2GM_\infty}{r'}\right)^{-1} dr'^2 + r'^2 d\Omega^2_{II} \, . \tag{4.18}$$

Therefore our asymptotic solution takes the form of Schwarzschild, with an adjusted period

$$\begin{aligned}\beta' &= \beta(1-\epsilon(\hat{\mu}+\tfrac{1}{2}\hat{p})) \\ &= 8\pi G M_\infty(1-\epsilon\hat{\mu})\end{aligned} \tag{4.19}$$

and mass parameter M_∞, adjusted that is, relative to the 'expected' mass-period relationship derived at the horizon. Note also that the area of the black hole is now related to the ADM mass via

$$\mathcal{A} = 4\pi r_s^2 = 16\pi G^2 M_\infty^2(1+\epsilon\hat{p}). \tag{4.20}$$

Note some similarities with a self-gravitating cosmic string. There the $\mathbb{R}^2$ sections perpendicular to the string acquire an asymptotic 'deficit angle'[18] $\delta\theta = -(2\pi).4G\mu$, where

$$\mu = 2\pi\eta^2 \int \hat{\rho}\hat{T}^\rho_\rho d\hat{\rho} = 2\pi\eta^2\hat{\mu} \tag{4.21}$$

is the energy per unit length of the cosmic string in its rest frame. Here we see that our 'deficit angle' is $\delta\tau = -8\pi G M_\infty\epsilon\hat{\mu} = -(8\pi G M_\infty).4G\mu$. Since we expect the period of τ to be $8\pi G M_\infty$ (as we expect the 2π period in θ), we see that the form of the correction in both cases is the same. Thus, the gravitational effect of the vortex is to 'cut' a wedge or slice out of the Euclidean black hole cigar outside the vortex. In figure 2 we show a schematic representation of the black hole vortex geometry.

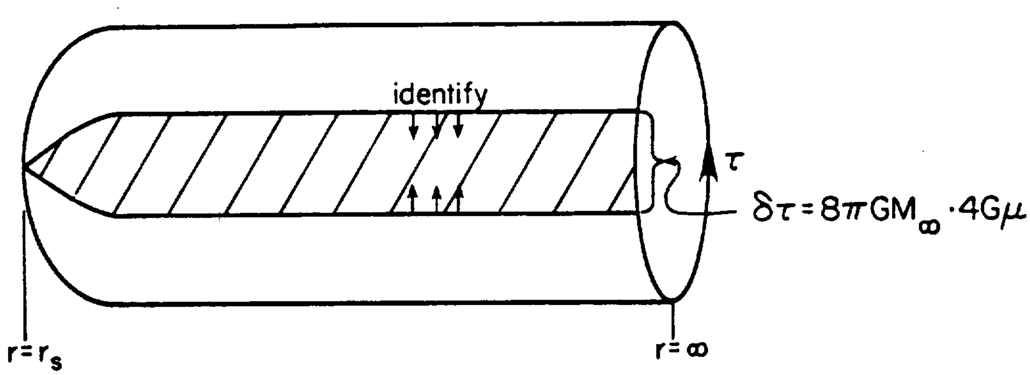

Fig. 2 The deficit slice in the Euclidean black hole cigar (θ, ϕ dimensions suppressed).

As we remarked at the end of the previous section, the vortex always decreases the period compared to its Schwarzschild value for a black hole of a given horizon area. The ADM mass, on the other hand, can be larger than, smaller than or equal to its Schwarzschild value for fixed horizon area, depending on $\hat{p}$. Existing results for a self gravitating cosmic string[19] indicate that for $\nu > (<)\ 1$, $\hat{p} > (<)\ 0$. These results were numerically obtained and so may only be true to a certain order, however they indicate that there is some critical value of ν, close to 1, for which the average pressure, $\hat{p}$, changes sign. Now, in our case, the background is flat space only to zeroth order in R^{-2} so we expect that the critical value of ν, ν_C, differs from the flat space value by $O(R^{-2})$ and thus is still close to 1.

5. Actions, temperature and entropy.

Having calculated the gravitational effect of the vortex, it is instructive to calculate the Euclidean action:

$$I_E = \int \left\{ \mathcal{L}_M - \frac{\mathcal{R}}{16\pi G} \right\} \sqrt{g} d^4x - \frac{1}{8\pi G} \int_\Sigma (K - K^0) \sqrt{h} d^3x \tag{5.1}$$

where K is the trace of the extrinsic curvature of Σ - a boundary "at infinity", calculated in the true geometry and K^0 the extrinsic curvature trace calculated for Σ isometrically embedded in flat space. For our asymptotically flat geometry, $C \sim r'$, $A^2 = 1 - \frac{2GM_\infty}{r'} + O(r'^{-2})$, this boundary term has the value

$$I_\Sigma = \tfrac{1}{2} \beta' M_\infty. \tag{5.2}$$

For the pure vortex source, we may use the Einstein equations to deduce that the Ricci scalar $\mathcal{R} = 16\pi G \mathcal{L}_M - 8\pi G (T^r_r + T^0_0)$. However, from (4.15) we see that

$$\int C^2 (T^r_r + T^0_0) dr = \frac{1}{G} O(\epsilon^2). \tag{5.3}$$

Thus

$$\int \left\{ \mathcal{L}_M - \frac{\mathcal{R}}{16\pi G} \right\} \sqrt{g} d^4x = \frac{1}{G} O(\epsilon^2). \tag{5.4}$$

Therefore, we come to the conclusion that, to first order in ϵ, the Euclidean action is, as with Schwarzschild, equal to its boundary term, $\frac{1}{2}\beta' M_\infty$. However, reading off the relation between β' and M_∞ from (4.19), we see that

$$I_E = \frac{\beta'^2}{16\pi G}(1+\epsilon\hat{\mu}) + \frac{1}{G}O(\epsilon^2) \tag{5.5}$$

in terms of the period. However, note that

$$\frac{\beta'^2 \epsilon\hat{\mu}}{16\pi G} = \frac{-\beta'^2}{r_s}\int_{r_s}^{\infty} C^2 T^\theta_\theta dr = \frac{\beta'}{4\pi r_s}\int \mathcal{L}_M \sqrt{g} d^4x = \int \mathcal{L}_M \sqrt{g} d^4x + O(\epsilon^2)\,. \tag{5.6}$$

Hence

$$I_E(\beta') = \frac{\beta'^2}{16\pi G} + \int \mathcal{L}_M \sqrt{g} d^4x = I_0(\beta') + I_M(\beta'), \tag{5.7}$$

to first order in ϵ, where $I_0(\beta')$ is the action of Schwarzschild with period β' and $I_M(\beta')$ is the action of the X_0, P_0 solution in the background of Schwarzschild with period β'. Therefore, taking into account the back-reaction of the vortex on the geometry, we confirm the value of the Euclidean action used by Coleman et al.[11]

The interest of computing the Euclidean vortex solutions is that their actions contribute to the gravitational path integral. In the path integral one must decide which fields to include in the sum. One prescription is to include all metrics and matter fields with a particular fixed period, β, and this describes "a system at temperature $1/\beta$". Here we compute what follows from such a prescription. Other boundary conditions are possible, which will be explored in further work.

Having calculated the vortex geometry we are in a position to directly calculate the expectation value of the mass of a black hole of temperature $1/\beta$ using

$$< g_{ab} >= (1+\sum_{\pm} C_{\pm} e^{-I_{\pm}})^{-1}[g_{0ab} + \sum_{\pm} C_{\pm} e^{-I_{\pm}} g_{\pm ab}] + O(e^{-2I_{\pm}}) \tag{5.8}$$

where g_{0ab} is the Schwarzschild metric with period β, $g_{+ab} = g_{-ab}$ are the $k = \pm 1$ vortex geometries with period β and $I_+ = I_-$ are the matter parts of their actions. $C_+ = C_-$ are the determinants of quadratic fluctuations about the vortices.

This formula is derived from a Euclidean path integral and must be used with caution since the metric is not a gauge invariant quantity. One must add the metrics at the same point of the space-time manifold, which concept has no diffeomorphism invariant meaning. However, in this case, since the metrics are all asymptotically flat, we can fix coordinates in the asymptotic region and only use the formula (5.8) there. In each case we choose coordinates such that $g_{00} \to 1$, and the area of the two-spheres is $4\pi r^2$ as a function of r at infinity.

Since the geometries for $k = \pm 1$ are identical, setting $C = C_+ + C_-$ yields

$$
\begin{aligned}
< g_{00} > &\sim (1 + Ce^{-I_M})^{-1}\left(1 + Ce^{-I_M} - \frac{2G}{r}(M + Ce^{-I_M}M_\infty)\right) \\
< g_{rr} > &\sim (1 + Ce^{-I_M})^{-1}\left(1 + Ce^{-I_M} + \frac{2G}{r}(M + Ce^{-I_M}M_\infty)\right) \\
< g_{\theta\theta} > &= \frac{< g_{\phi\phi} >}{\sin^2\theta} \sim r^2
\end{aligned}
\tag{5.9}
$$

as $r \to \infty$, where $I_M = I_\pm$ and

$$
M = \frac{\beta}{8\pi G}, \quad M_\infty = \frac{\beta}{8\pi G}(1 + \epsilon\hat{\mu}).
$$

Substituting in for the masses we obtain

$$
\begin{aligned}
< g_{00} > &\sim 1 - \frac{\beta}{4\pi r}(1 + \epsilon\hat{\mu}Ce^{-I_M}) \\
< g_{rr} > &\sim 1 + \frac{\beta}{4\pi r}(1 + \epsilon\hat{\mu}Ce^{-I_M}).
\end{aligned}
\tag{5.10}
$$

Thus we have

$$
< M(\beta) > = \frac{\beta}{8\pi G}[1 + Ce^{-I_M}\epsilon\hat{\mu}]
\tag{5.11}
$$

as the predicted value of the mass of a black hole with temperature β^{-1}. Noting that, for $k = \pm 1$, $\epsilon\hat{\mu} = 4T_{\text{string}}$ in the notation of Coleman et al., this is readily seen to agree with their expression for the modified Hawking temperature of the black hole[11].

The horizon is another place where we can make sense of (5.8). It is a two-sphere and for each metric in (5.8) we know its area, $\mathcal{A}$, in terms of the period, giving

$$< \mathcal{A} >= \frac{\beta^2}{4\pi}\left[1 + Ce^{-I_M}(2\epsilon\hat{\mu} + \epsilon\hat{p})\right] \tag{5.12}$$

for the expectation value of the area of the black hole. We compare this with the entropy, $S(\beta)$, calculated from the partition function, $Z(\beta)$, via

$$S = \beta^2 \frac{\partial}{\partial\beta}\left(-\beta^{-1}\ln Z\right). \tag{5.13}$$

Approximating the Euclidean path integral for $Z(\beta)$ semiclassically yields

$$Z(\beta) = e^{\frac{-\beta^2}{16\pi G}}\left(1 + Ce^{-I_M}\right) \tag{5.14}$$

and thus

$$4GS(\beta) = \frac{\tilde{\beta}^2}{4\pi}\left[1 + 2\epsilon\hat{\mu}Ce^{-I_M}\right] - Ce^{-I_M}. \tag{5.15}$$

We find that the central formula $S = \frac{1}{4G}A$ in black hole thermodynamics has now apparently been violated, and depending on the specifics of the vortex (i.e. the size and sign of $\hat{p}$) S can either be greater than or less than $\frac{1}{4G} < A >$. Note that the result (5.12) could not be obtained from the partition function since it contains an $\epsilon\hat{p}$ term.

6. Conclusions.

To summarise: we have argued the existence of solutions of the coupled Einstein-vortex equations by showing that under suitable fall-off conditions of the energy-momentum of a weakly gravitating vortex a perturbative analysis is justified. We have demonstrated a suitable vortex for beginning an iterative procedure by numerically obtaining a vortex solution of the abelian Higgs model in a Schwarzschild background. We calculated the mass-period-area relations for the corrected geometry to first order in ϵ, the gravitational strength of the vortex and used these results to derive the renormalised mass of a black hole of a certain temperature. We also found that the expected value of the horizon area is not related to the entropy of the black hole in the usual way.

Our work also provides a potential 'no-go' argument for global vortices. In the cosmic string scenario, local strings have asymptotically conical spacetimes whereas static global string spacetimes are singular[21], the energy momentum tensor having only a $1/r^2$ fall-off in flat space. In our Euclidean case, the energy of a global vortex in the Schwarzschild background would have no fall-off due to the fixed circumference (β) of r, θ, ϕ =const circles. Therefore, drawing an analogy between these two situations, if static global cosmic strings are singular we do not expect global black hole vortices to be otherwise. Not having asymptotically flat geometries, they would therefore not contribute to the partition function.

We mentioned the effect of varying the parameter ν on the results obtained. For the flat space Nielsen-Olesen vortex, the critical value of ν is exactly 1. In that case, $\nu > 1$ means that a string with winding number $k \geq 2$ is unstable[17], alternatively, that the vortices repel one another, whereas $\nu < 1$ implies that they attract. Since we have argued that just such a critical value of ν, ν_C close to 1, exists for the black hole vortices, it is interesting to speculate that, for $\nu > \nu_C$, the $k \geq 2$ solutions are unstable, i.e. are not minima of the Euclidean action. In that case the $k \geq 2$ solutions that we have found would not contribute to a Euclidean path integral. It seems plausible to suppose that stable solutions of the matter equations on a Schwarzschild background do exist, which would consist of two separate string world sheets sitting opposite each other ($\tau_2 - \tau_1 = \frac{1}{2}\beta$) at finite distance from the horizon, where any further loss of energy due to moving further away would be balanced by an increase in energy due to increase in the area of the world sheets. Such a solution would not be cylindrically symmetric and its action would differ from the form calculated in (5.6), although presumably the difference would be small. However, it would be interesting to investigate such types of solutions.

Our derivation of the geometry not only enabled us to confirm the results of Coleman et al., but we were also able to calculate the expected area of the black hole. We obtained what looks to be a discrepancy in the usual area-entropy relationship, though, in this case, virtual string world sheets "dress" the black hole around the horizon and perhaps one should not expect the area-entropy relation to survive. However, it is the pressure, rather than some combination of energy and pressure, that is contributing to the discrepancy and this result certainly merits further thought.

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