Verifying the Interactive Convergence Clock Synchronization Algorithm Using the Boyer-Moore Theorem Prover

William D. Young

Computational Logic, Inc.
Austin, Texas

Contract NAS1-18878
April 1992
Table of Contents

1. Introduction .................................................................................................................................. 1
2. The Interactive Convergence Clock Synchronization Algorithm ......................................................... 2
3. Specifying the ICCSA in the Boyer-Moore Logic ............................................................................... 2
   3.1. Computing with Rationals ........................................................................................................ 3
   3.2. Uses of CONSTRAINT ............................................................................................................. 5
   3.3. DEFN-SK ........................................................................................................................... 7
   3.4. Avoiding Higher Order Functions .......................................................................................... 9
4. Aspects of the Proof ....................................................................................................................... 10
   4.1. Restraining the Prover ......................................................................................................... 10
   4.2. Order of Steps in the Proof .................................................................................................. 10
   4.3. Proof Encapsulation ........................................................................................................... 12
   4.4. Syntax .................................................................................................................................. 13
5. Conclusions .................................................................................................................................... 13
List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Some Quantities Required for Specifying the ICCSA</td>
<td>3</td>
</tr>
<tr>
<td>Figure 2</td>
<td>CONSTRAIN Introducing ICCSA Parameters</td>
<td>6</td>
</tr>
<tr>
<td>Figure 3</td>
<td>EHDM and Boyer-Moore Versions of the Same Lemma</td>
<td>7</td>
</tr>
<tr>
<td>Figure 4</td>
<td>Theorems Generated for a DEFN-SK+ Event</td>
<td>8</td>
</tr>
<tr>
<td>Figure 5</td>
<td>Proof of SUBLEMMA-A Showing USE Hints</td>
<td>11</td>
</tr>
</tbody>
</table>
1. Introduction

The application of formal methods to the analysis of computing systems promises to provide higher and higher levels of assurance as the sophistication of our tools and techniques increases. Improvements in tools and techniques come about as we pit the current state of the art against new and challenging problems. A promising area for the application of formal methods is in real-time and distributed computing. Some of the algorithms in this area are both subtle and important. Their proofs are an ideal testing ground for formal methods because they involve detailed and sophisticated reasoning which is challenging even for a competent human mathematician. We believe that formal methods are already demonstrating that they can make a genuine contribution toward the clarity and correctness of these algorithms [15, 3].

One important algorithm in this field is the Interactive Convergence Clock Synchronization Algorithm (ICCSA) of Lamport and Melliar-Smith [13]. This algorithm maintains approximate synchronization among a number of clocks even when the clocks begin running at slightly different times, run at slightly varying rates, and some percentage of them may be faulty. The presentation of Lamport and Melliar-Smith both develops the algorithm and states formally the assumptions and desired properties required to state and prove its correctness properties.

A mechanical verification of the algorithm using EHDM was performed by John Rushby and Friedrich von Henke and described in [15, 16]. The EHDM effort resulted is a completely formal presentation of the algorithm and its proof, a presentation which is arguably somewhat clearer and more rigorous than the original published proof. Rushby and von Henke challenged users of proof systems other than EHDM as follows.

We found that EHDM served us reasonably well; we do not know whether other specification and verification environments would have fared as well or better. [W]e invite the developers and users of other verification systems to repeat the experiment described here. We suggest the Interactive Convergence Clock Synchronization Algorithm is a paradigmatic example of a problem where formal verification can show its value and a verification system can demonstrate its capabilities; it is a "real" rather than an artificial problem, its verification is large enough to be challenging without being overwhelming, it requires a couple of fairly interesting supporting theories, and its proofs are quite intricate and varied.

In response to this challenge and as part of an ongoing attempt to verify an implementation of the Interactive Convergence Clock Synchronization Algorithm, we decided to undertake a proof of the correctness of the algorithm using the Boyer-Moore theorem prover.

This note describes our approach to proving the ICCSA using the Boyer-Moore prover. Since our proof follows closely that of Rushby and von Henke, we will not dwell on the details of the proof but assume that the reader is familiar with their quite cogent description of the EHDM version of the proof [15]. Instead we concentrate on the use of features of the Boyer-Moore logic and theorem prover which were especially helpful in the specification and proof and on the differences from the Rushby and von Henke version. We assume that the reader is somewhat familiar with the Boyer-Moore logic and theorem prover [4, 7]. We plan to follow this note with another paper co-authored by John Rushby comparing and contrasting the EHDM approach and the Boyer-Moore approach.

This note is organized as follows. The next section introduces briefly the Interactive Convergence Clock Synchronization Algorithm and the problem it is designed to solve. Sections 3 and 4 describe some interesting aspects of the specification and proof, respectively, and some of the more significant ways in which these differ from the EHDM version. Finally, section 5 contains some conclusions from this study.
2. The Interactive Convergence Clock Synchronization Algorithm

A difficult problem facing designers of systems which achieve fault-tolerance via redundant processing capability is synchronizing the processors so that they deliver their results at approximately the "same time." One solution to this problem is the Interactive Convergence Clock Synchronization Algorithm of Lamport and Melliar-Smith [13]. This algorithm maintains approximate synchronization among a number of clocks even when the clocks begin running at slightly different times, run at slightly varying rates, and some percentage of them may be faulty.

We desire an algorithm in which each processor periodically resynchronizes with all of the other processors in such a way that:

S1. all nonfaulty clocks have approximately the same value at any time; and,

S2. the adjustment to any clock during a synchronization period is bounded.

Proving that any algorithm achieves these two conditions is difficult because it requires accounting for a number of continually changing quantities. Lamport and Melliar-Smith were able to prove that the ICCSA algorithm has these properties. Their proof is quite detailed involving approximate reasoning and neglect of various terms. Conceptually the algorithm operates as follows. Each processor $p_i$ maintains an offset or correction to its private (hardware) clock; the private clock value plus correction is the adjusted clock value. The correction is periodically updated by adding to it the mean of the differences between $p_i$'s adjusted clock value and all other processors' adjusted clock values. Any processor $p_j$ with adjusted value too divergent from $p_i$'s is assumed to be faulty and a difference of 0 between $p_i$'s and $p_j$'s clocks is used in computing the mean.

Though the algorithm is conceptually quite simple, the statement of the correctness properties and their proof is complex. The correctness of the algorithm is stated in terms of certain quantities listed in figure 1 and others computed in terms of them. The proof shows that under certain conditions on the relationships among these parameters, this algorithm does maintain adequate synchronization among $n$ processes with at most $m$ faulty processes. Here adequate synchronization is defined in terms of formal statements of conditions S1 and S2 above. A completely formal description of the algorithm and its correctness conditions is given by Rushby and von Henke [15]. Our formalization follows fairly closely the Rushby and von Henke version and is given as the sequence of Boyer-Moore "events" in the appendix. In the following two sections we highlight features of the Boyer-Moore logic and prover which were particularly helpful in our specification with emphasis on the differences from the EHDM version.

3. Specifying the ICCSA in the Boyer-Moore Logic

Capturing formally the Interactive Convergence Clock Synchronization Algorithm within the Boyer-Moore logic was a challenge despite the fact that we had as a model a fully formal version in EHDM. There are difference in the languages which make translating from one to the other nontrivial. In particular, EHDM allows full first order quantification and uses higher-order functions in a manner which cannot be specified in the Boyer-Moore logic. However, recent additions to the logic—particularly CONSTRAINT and DEFN-SK—made our task much easier than it otherwise would have been.

\[\text{In fact, a proof of an implementation of the ICCSA algorithm was asserted to be "probably beyond the ability of any current mechanical verifier" [14].}\]
\[ n \] number of clocks.
\[ m \] number of faulty clocks.
\[ R \] clock time between synchronizations.
\[ S \] clock time to perform the synchronization algorithm.
\[ \delta \] maximum real time skew between any two good clocks.
\[ \delta_0 \] maximum initial real time skew between any two good clocks.
\[ \epsilon \] maximum real time clock read error.
\[ \rho \] maximum clock drift rate.
\[ \Sigma \] maximum correction permitted.

**Figure 1**: Some Quantities Required for Specifying the ICCSA

3.1 Computing with Rationals

The Boyer-Moore logic provides as "primitives" the data types of booleans, naturals, literal atoms, negative integers, and lists. Describing the ICCSA and proving its synchronization properties requires the manipulation of numerous rational quantities. The Boyer-Moore "shell" mechanism allows the user to add new recursively defined data types. The rationals have been added as a new shell and explored to some extent in some previous specification efforts. Until recently, however, there has not been a well thought out library of definitions and rewrite rules for rationals as have been developed for several other data types [1, 12]. Recently Matt Wilding of CLI has built a useful library for the rationals; this library provided a solid basis for our proof.

The rationals library is built on top of an earlier library for the integers.\(^2\) Operations defined include equality, the various arithmetic operations on rationals, and the relational operator RLESSP. A number of useful rewrite rules are proved about these functions and included in the library. On top of the basis provided by Wilding's library, we defined some additional operations required for the ICCSA specification, such as rational absolute value, operations coercing integers to rationals, and arithmetic operations taking both integer and rational arguments. Proving properties of these functions was usually quite easy because the underlying library was well thought-out. This contrasts with some earlier proof efforts [2, 9, 17] in which all theories had to be built "from scratch."

There are some quirks in dealing with rationals in the logic which are not present for most data types. In particular, there are an infinite number of representations for each rational number. This leads to the need to reduce all rationals to a canonical form before comparing them. All of the operations in Wilding's rationals library leave rationals in reduced form.

Rational equality is defined in terms of these reduced forms:

\[
(\text{DEFN REQUAL } (X \ Y)) \\
(\text{EQUAL } (\text{REDUCE } X) \ (\text{REDUCE } Y))) .
\]

However, since the prover has extensive built-in heuristics for EQUAL, but not for REQUAL, it was convenient to open up this definition whenever possible. This leads to a continual need to deal with terms of the form (REDUCE X). Luckily, the rationals library contains an extensive collection of rewrites such as the following two

\(^2\)A rational is represented as a pair of integers (RATIONAL I J) with appropriate constraints on the signs of i and j.
for the RPLUS function which eliminates most appearances of the REDUCE operator.

\[
\text{(PROVE-LEMMA RPLUS-REDUCE (REWRITE)} \\
(\text{AND (EQUAL \text{RPLUS (REDUCE X) Y)}} \\
\text{\text{RPLUS X Y)}) \\
(\text{EQUAL \text{RPLUS X (REDUCE Y)}} \\
\text{\text{RPLUS X Y)})))}
\]

\[
\text{(PROVE-LEMMA REDUCE-RPLUS (REWRITE)} \\
(\text{EQUAL \text{REDUCE (RPLUS X Y)}} \\
\text{\text{RPLUS X Y)}))
\]

These ubiquitous REDUCE expressions caused one oddity in the specification. The library contains a number of rewrite rules such as:

\[
\text{(PROVE-LEMMA RPLUS-RZEROP (REWRITE)} \\
(\text{IMPLIES \text{RZEROP X)}} \\
\text{\text{AND (EQUAL \text{RPLUS X Y) \text{REDUCE Y)}}} \\
\text{\text{EQUAL \text{RPLUS Y X) \text{REDUCE Y)})))}
\]

However, using this rewrite rule an expression such as

\[
(\text{EQUAL \text{RHO (RPLUS \text{RHO (RATIONAL 0 1)})}}
\]
rewrites to

\[
(\text{EQUAL \text{RHO (REDUCE \text RHO))}}
\]
which is not provable unless \text{RHO} is known to be in reduced form, eg., if \text{RHO} is a constrained constant (see section 3.2 below). This led to the need to require that most constrained constants be in reduced form.

This rather odd but innocuous requirement could be avoided by consistently using REQUAL rather than EQUAL whenever referring to rational quantities. However, even with extensive theory development, heuristic reasoning support for REQUAL would not equal that available for EQUAL. An experimental facility supporting reasoning with congruence relations [8] might have alleviated some of this difficulty but was not used here.

The utility of the rationals library is greatly enhanced by the addition of several useful metafunctions. Metafunctions [5] are user-defined term simplification routines which are proven to preserve the meaning (evaluation) of the term to which they are applied. For example, a function in the rationals library "cancels" complementary terms in a rationals RPLUS expression. Proving that this function preserves the meaning (value) of the term to which it is applied sanctions the installation of this code as an additional simplification routine within the prover. The code is installed automatically by the prover upon proof of the required theorems and replaces a potentially infinite collection of rewrite rules.

The collection of metafunctions within the rationals library greatly simplifies reasoning about RPLUS, RTIMES, and RLESP expressions. We added an additional metalemma for the RLEQ (rationals less than or equal) function. This was not strictly necessary since RLEQ is defined in terms of REQUAL and RLESP. However, to avoid the explosion of cases on theorems involving many RLEQ hypotheses, we decided to develop a theory for RLEQ on top of Wilding’s rationals library and leave RLEQ disabled (so that it would not be automatically opened up by the prover). We are not completely convinced of the wisdom of this decision.
3.2 Uses of \texttt{CONSTRATN}

As we saw in section 2 above, the ICCSA is described in terms of a large number of integer and rational-valued parameters; these are conceptually global constants for purposes of the specification. There is a sizable collections of assumptions about the relative sizes of these quantities. In the Rushby and von Henke specification, these are given as EHDM axioms. Within the Boyer-Moore logic there are various options for how to introduce these constants into the specification:

- pass them as parameters to each function requiring them and add the assumptions as explicit hypotheses on each theorem requiring them;
- define each constant as a declared function of no arguments and add any required assumptions as axioms;
- use the Boyer-Moore \texttt{CONSTRATN} [6] mechanism to introduce the constants as new function symbols and introduce the assumptions axiomatically within the \texttt{CONSTRATN}.

The advantage of this final approach is that it guarantees that the introduced axioms are consistent without cluttering up definitions and theorems with a multitude of additional parameters and hypotheses. Moreover, using a \texttt{CONSTRATN} event to introduce a new function symbol avoids the overspecification often occasioned by introducing functions via explicit definitions; only the required properties of the function need be specified.

A \texttt{CONSTRATN} event introduces one or more function symbols along with axioms which they must satisfy. To guarantee the consistency of the axioms, the user must supply \textit{witness} functions which satisfy the axioms. For example, we model the function $\Delta_{P,P}^{(0)}$ that computes the difference in clock values between processes $p$ and $q$ in period $i$, with the \texttt{CONSTRATN} event:

\begin{verbatim}
(CONSTRATN DELTA2-INTRO (REWITE)
  (AND (RATIONALP (DELTA2 R P I))
       (EQUAL (DELTA2 P P I) (RATIONAL 0 1))
       (IMPLIES (NOT (NUMBERP I))
                 (EQUAL (DELTA2 R P I) (DELTA2 R P 0))
       (EQUAL (REDUCE (DELTA2 P Q I)).
             (DELTA2 P Q I))
       ((DELTA2 (LAMBDA (R P I) (RATIONAL 0 1)))))
\end{verbatim}

This asserts that the newly introduced function $\Delta_{P,P}^{(0)}$ is rational-valued, returns zero as the difference from a process's own clock value, always coerces its third argument to a natural number, and returns a rational in reduced form.\textsuperscript{3} We also supply a function which satisfies these axioms, namely the function of three arguments which always returns rational zero. A \texttt{CONSTRATN} event is not accepted unless the axioms, appropriately instantiated with the witness functions, can be proved. This assures the consistency of the axioms by exhibiting a model.

Most of the constant parameters of our specification are introduced in a single large \texttt{CONSTRATN} event \texttt{PARAMETERS-INTRO} given in Figure 2. It might have been better to introduce these via several different \texttt{CONSTRATN} events. In that case, it could have been more difficult to find appropriate witness functions, however.

A very strong advantage of introducing the various parameters in this way is that their names and properties become "globally" visible. This allows us to give our theorems in a succinct form very close to those of the EHDM representation. Figure 3, for example, shows the same lemma in both its EHDM form\textsuperscript{4} and in the Boyer-Moore logic.

\textsuperscript{3}This is not a minimal set. The first axiom follows from the fourth one.

\textsuperscript{4}Forms in [15] were pretty printed using a special facility described in that report; raw input to EHDM is much less elegant. The version here is in the prettified format.
CONSTRAIN PARAMETERS-INTRO (REWRITE)

:: R and S
AND (RATIONALP (R))
(RATIONALP (S))
(RLESSP (RATIONAL 0 1) (R))
(RLESSP (RATIONAL 0 1) (S))
(RLEQ (RTIMES (RATIONAL 3 1) (S)) (R))
:: posR
:: posS
:: rho
(RATIONALP (RHO))
(RLEQ (RATIONAL 0 1)
(RTIMES (RATIONAL 1 2) (RHO)))
(RLESSP (RTIMES (RATIONAL 1 2) (RHO))
(RATIONAL 1 1))
:: rho_pos
:: rho_small
:: other parameters
(RATIONALP (EPSILON))
(RATIONALP (DELTA))
(RATIONALP (BIG-SIGMA))
(RATIONALP (BIG-DELTA))
(NUMBERP (N))
(NOT (EQUAL (N) 0))
(NUMBERP (M))
(LESSP (M) (N))
(RLESSP (RATIONAL 0 1) (BIG-DELTA))
(RLEQ (BIG-SIGMA) (S))
(RLEQ (BIG-DELTA) (BIG-SIGMA))
(RLEQ (RPLUS (DELTA))
(RPLUS (EPSILON)
(RTIMES (RATIONAL 1 2)
(RTIMES (RHO) (S))))
(BIG-Delta)
(RLEQ (RPLUS (DELTA)) (RTIMES (RHO) (R)))
:: C5
:: C6
:: LESSP (R (LAMBDA () (RATIONAL 3 1)))
(S (LAMBDA () (RATIONAL 1 1)))
(RHO (LAMBDA () (RATIONAL 0 1)))
(EPSILON (LAMBDA () (RATIONAL 0 1)))
(DELTA (LAMBDA () (RATIONAL 0 1)))
(BIG-SIGMA (LAMBDA () (RATIONAL 1 2)))
(BIG-DELTA (LAMBDA () (RATIONAL 1 2)))
(N (LAMBDA () 1))
(M (LAMBDA () 0)))

Figure 2: CONSTRAIN Introducing ICCSA Parameters
The EHDM Version:

\textbf{lemmaldef}: Lemma
\[ \text{SlC}(p, q, i) \land S2(p, i) \land \text{nonfaulty}(p, i + 1) \land \text{nonfaulty}(q, i + 1) \Rightarrow |\Delta_q^i| < \Delta \]

The Boyer-Moore Version:

(PROVE-LEMMA LEMMA1 (REWRITE)
  (IMPLIES (AND (SlC P Q I)
                 (S2 P I)
                 (NONFAULTY P (ADD1 I)))
                 (NONFAULTY Q (ADD1 I)))
  (RLESSF (RABS (DELTA2 Q P I)) (BIG-DELTA))))

Figure 3: EHDM and Boyer-Moore Versions of the Same Lemma

3.3 \texttt{DEFN-SK}

Another relatively new feature of the logic which proved useful was the \texttt{DEFN-SK} facility [11] which allows the introduction of quantified expressions into the specification. Several important constructs in the Rushby and von Henke version were defined via quantification. Earlier versions of the Boyer-Moore logic could not express many of these conveniently. In particular, to prove an existential existential statement required exhibiting a witness constructively.

A \texttt{DEFN-SK} event allows the definition of an explicitly quantified term and the use of this term in other definitions and theorems. For example, the notion of a \textit{good clock} (within the interval \([T_0..T_N]\)) is defined by Rushby and von Henke as:

\[
goodclock: \text{function}[\text{proc}, \text{clocktime}, \text{clocktime} \rightarrow \text{bool}] = \\
(\lambda p,T_0,T_N: \\
(\forall T_1,T_2: \\
T_0 \leq T_1 \land T_0 \leq T_2 \land T_1 \leq T_N \land T_2 \leq T_N \\
\Rightarrow |c_p(T_1)-c_p(T_2)-(T_1-T_2)| \\
\leq \rho/2 \ast |(T_1-T_2)|)
\]

Our definition is given by the \texttt{DEFN-SK} event:

\texttt{\textbf{NOTICE}} that this definition is from the revised specification. The first published version had "\texttt{\textless}" where the current version has "\texttt{\leq}". This has the rather curious consequence that there are no good clocks in a system in which the parameter \(\rho\) which gives the maximum clock drift rate is zero. Intuitively, this means that if all clocks are perfect no clocks are good.
A DEFN-SK event causes two axioms to be added to the database; these two axioms corresponding to the skolemization of the event in each “direction” and together allow us to use an instance of a quantified expression appearing in a hypothesis to a theorem and to prove an instance appearing as the conclusion. The macro version DEFN-SK+ of the event also causes these axioms to be encapsulated and proved as rewrite rules. For GOOD-CLOCK these two theorems are shown in figure 4. See [11] for details on how these are generated and a proof of the soundness of the approach.

Use of DEFN-SK allows us to define concepts involving quantifiers in a fashion which is very analogous to their EHDM counterparts. However, we did not always find this convenient. For example, Rushby and von Henke
define SDEF as follows:

\[ Sdef: \text{Axiom } T \in S^0 = (\exists \Pi: 0 \leq \Pi \land \Pi \leq R \land T = T^0 + \Pi) \]

A close analogue in the Boyer-Moore logic using DEFN-SK would be:

\[
\text{(DEFN-SK+ SDEF (TM I))}
\]
\[
\quad \text{(EXISTS PI}
\quad \quad \text{(AND (RLEQ (RATIONAL 0 1) PI)}
\quad \quad \text{(RLEQ PI (S))}
\quad \quad \text{(EQUAL (REDUCE TM)}
\quad \quad \quad \text{(RPLUS (TI I) (RPLUS (RDIFFERENCE (R) (S)) PI)))))}
\]

However, we found the following definition to be more convenient and to eliminate an unnecessary existential quantifier.

\[
\text{(DEFN IN-S (TM I))}
\]
\[
\quad \text{(IN-INTERVAL TM}
\quad \quad \text{(RDIFFERENCE (TI (ADD1 I)) (S))}
\quad \quad \text{(TI (ADD1 I))))}
\]

This illustrates that often the use of one style of definition is more "natural" in a given logic even when others styles are available. It is not surprising then that some of our definitions were quite different than the corresponding EHDM versions. However, we believe them to be equivalent in all relevant aspects. As an exercise, we proved the lemma which shows the equivalence of the definitions SDEF and IN-S.

\[
\text{(PROVE-LEMMA SDEF-IN-S-EQUIVALENCE ())}
\]
\[
\quad \text{(IFF (SDEF TM I) (IN-S TM I)))}
\]

Using CONSTRAINTS and DEFN-SKS, we were able to write theorems which are textually very close to the EHDM versions in most cases. It is evident from the two versions of LEMMA1 listed in Figure 3 that, except for minor textual differences, there is very little difference in the presentation of theorems in the two logics. This was the rule rather than the exception for the lemmas required in our proof.

3.4 Avoiding Higher Order Functions

The ability within EHDM to define higher order functions is a definite benefit from the perspective of writing clear and elegant specifications. However, many of the uses of higher order functions can be avoided by careful use of facilities available within the Boyer-Moore logic. This was true of each of the uses of the EHDM higher order facilities in the ICCSA specification.

As an example, consider the MEAN function defined by Rushby and von Henke as follows:

\[
\text{\Theta}^{1,2}_{i} \ast 3: \text{function}[\text{nat, nat, function}[\text{nat -> number}] \rightarrow \text{number}] =
\]
\[
(\lambda i, j, F: \text{if } i \leq j \text{ then } \sum_{i} F/(j + 1 - i) \text{ else } 0 \text{ end if}).
\]

Notice that one parameter to this definition is a function \( F \). From this definition, Rushby and von Henke prove a number of quite general lemmas.

There is not a similar facility within the Boyer-Moore logic though many of the advantages of such higher order definitions are available via other routes. For example, our version of the MEAN is defined as follows:
Rather than parameterizing \texttt{RMEAN} with a function, we parameterize it with a list of elements returned by the function. This is conceptually equivalent and we can prove all of the nice properties of the EHDM version. Most of the interesting properties are really properties of \texttt{RSUM} rather than of \texttt{RMEAN}.

This style of trivial transformation is not the only way to deal with higher order functions and properties in the logic. An interpreter is available for the logic which permits reasoning about functions at the meta-level. Also, it is possible to "fake" higher-order properties in other ways. We have checked the proof, for example, that there is \textit{no algorithm} which solves a certain version of the Byzantine General's problem. \cite{3} This is inherently a second order property.

4. Aspects of the Proof

4.1 Restraining the Prover

Our proof of ICCSA was somewhat atypical in several ways of most proofs using the Boyer-Moore prover in several ways. Rushby and von Henke had done much of the difficult work of finding a sequence of lemmas leading up to the proofs of the desired correctness theorems. Moreover, because of the way the EHDM prover operates, the collection of lemmas necessary for a given proof are displayed along with their specific instantiations. Given this information, constructing a formal proof is largely a matter of intelligent simplification and tautology checking.\footnote{The EHDM proof of ICCSA used only the EHDM \textit{ground} prover. \cite{15}}

A proof in the Boyer-Moore theorem prover typically relies more on the prover's heuristics to choose among previously proven lemmas and instantiate them correctly. However, the prover can be used in a more restrained fashion by disabling most functions and rewrite rules and using the prover as a simple proof checker. This is done by enabling only those lemmas known to be relevant and adding \texttt{USE} hints to specify particular instantiations of the variables in needed lemmas. This was the approach we followed in our proof of the ICCSA; most functions and lemmas were globally disabled. We also made use of an experimental feature for encapsulating the names of a group of events into a "theory" which can be enabled or disabled collectively. Figure 5 shows a particular lemma in our script which is an example of the use of \texttt{USE} hints, selective enabling, and theory enabling to obtain the proof.

4.2 Order of Steps in the Proof

The Boyer-Moore prover allows very little flexibility in the order of steps in a proof. Each function must be fully defined or constrained before it is used; each lemma must be proven before it can be used in proofs. For definitions this means that there is no genuine mutual recursion\footnote{There is a standard way to gain the effects of mutual recursion by defining several "functions" within one and using a flag to distinguish among them. \cite{7} Also, there is an available read macro for the prover which turns a list of mutually recursive definitions into an event of this type. \cite{10}} For proofs it means that the proof is presented (though not necessarily discovered) in a very "bottom-up" style. This approach guarantees that there are no circularities in the proof.

EHDM does not impose such a limited ordering on the steps in the proof. To assure that there are no circularities in

\begin{verbatim}
(DEFN RSUM (LST)
  (IF (NLISTP LST)
      (RATIONAL 0 I)
      (RPLUS (CAR LST) (RSUM (CDR LST))))
)
(DEFN RMEAN (LST)
  (RQUOTIENT-NAT (RSUM LST) (LENGTH LST)))
\end{verbatim}
LEMMA SUBLEMMA-A (REWRITE)
(IMPLIES (AND (NONFAULTY P I)
(NONFAULTY Q I)
(IN-R TM I))
(RLEQ (SKEW P Q TM I)
(RPLUS (SKEW P Q (TI I) I)
(RTIMES (RHO) (R)))))
((USE (REARRANGE-ALT (X (C P I TM))
(Y (C Q I TM))
(U (C P I (TI I)))
(V (RPLUS TM (RNEG (TI I))))
(W (C Q I (TI I))))
(LEMMA2D (PI (RPLUS TM (RNEG (TI I)))))
(LEMMA2D (P Q) (PI (RPLUS TM (RNEG (TI I))))))
(ENABLE-THEORY REDUCTIONS)
(ENABLE SKEW RDIFFERENCE RNEG-RPLUS C-REDUCE TI-NEXT RABS-POSITIVE2
RPLUS-RLEQ-REWRITE RPLUS-RLEQ-REWRITE2 RLEQ-RTIMES-HACK
IN-R IN-INTERVAL RHO-RLEQ0 RLEQ-TRANSITIVE RLEQ-RPLUS-HACK3
RLEQ-HALF-RPLUS-RLEQ-RTIMES-HACK RPLUS-RLEQ-REWRITE))

Figure 5: Proof of SUBLEMMA-A Showing USE Hints

the resulting proof, a tool called the EHDM Proof Chain Analyzer is run over the final proof and checks for circularities. In the proof of ICCSA there is a circularity in the proof of the main theorem THEOREM1. This is explained as follows:

This circularity is apparent, rather than real, as it occurs in the context of an inductive proof, in which the theorem is used for i in the part of the proof that extends it to i + 1. We are working towards constructing a proof description that reflects this induction step more straightforwardly. [15]

Unfortunately, determining whether such a circularity is apparent or real requires a fairly deep understanding of the proof. The Boyer-Moore approach does not allow even an apparent circularity but the cost is a much more regimented approach to proof presentation.

As an interesting aside, just as Rushby and von Henke had to deal with the structure of the inductive proof of THEOREM1 in EHDM, we had to confront the same issue in the Boyer-Moore system. We could approach it either by defining an appropriate induction schema to make available the required inductive hypotheses (the typical approach in the Boyer-Moore system) or by using another approach altogether. Defining an appropriate induction schema would have been difficult because the inductive hypothesis was really required in the proof of a large subsidiary lemma CULMINATION. We would have needed to prove THEOREM1 and CULMINATION simultaneously by packaging them into one lemma. This trick is used often in the Boyer-Moore prover. Our solution was again somewhat atypical and illustrates a clever (we think) use of DEFN-SK.

THEOREM1 has form:
(PROVE-LEMMA THEOREM1 (REWRITE)
(IMPLIES (S1A I) (SIC P Q I)))

We introduced the DEFN-SK event below to define the structure of the theorem:
(DEFN-SK THEOREM1-ONE-STEP (I)
  (FORALL (P Q)
    (IMPLIES (SIA I)
      (SIC P Q I)))))

Notice that this is parameterized by i. Asserting (THEOREM1-ONE-STEP I) is equivalent to asserting that THEOREM1 holds through period i. Wherever the EHDM approach used THEOREM1 in the proof, we simply asserted THEOREM1-ONE-STEP as an additional hypothesis on the lemma, as in CULMINATION below:

(PROVE-LEMMA CULMINATION (REWRITE)
  (IMPLIES
    (AND (SIA (ADD1 I))
      (SIC P Q I)
      (NONFAULTY P (ADD1 I))
      (NONFAULTY Q (ADD1 I))
      (IN-R TM (ADD1 I))
      (THEOREM1-ONE-STEP I))
    (RLEQ (SKEW P Q TM (ADD1 I)))
    (RPLUS
      (RQUOTIENT-NAT
        (RPLUS
          (RTIMES-NAT (M)
            (RPLUS (DELTA)
              (RTIMES (RATIONAL 2 1) (BIG-DELTA))))
            (RTIMES-NAT (DIFFERENCE (N) (M))
              (RTIMES (RATIONAL 2 1)
                (RPLUS (EPSILON)
                  (RPLUS (RTIMES (RHO) (S))
                    (RTIMES (RATIONAL 1 2)
                      (RTIMES (RHO) (BIG-DELTA))))))))
          (RTIMES (RHO) (BIG-SIGMA)))))))

This made available, in the proof of CULMINATION exactly the instance of THEOREM1 required in that proof. We then used CULMINATION in the proof (by induction on i) of the lemma:

(PROVE-LEMMA THEOREM1-VERSION1 (REWRITE)
  (THEOREM1-ONE-STEP I))

CULMINATION is used in the proof of the induction step, its THEOREM1-ON-STEP hypothesis being relieved by the inductive hypothesis. THEOREM1 follows straightforwardly from THEOREM1-VERSION1. This approach indicates again the utility of DEFN-SK in adding clarity and proof power.

4.3 Proof Encapsulation

The Boyer-Moore logic has no convenient way of structuring a specification and proof into a collection of "modules." This is largely dictated by the requirement that the specification and proof be presented in a very "bottom up" fashion. A collection of related units may be grouped together in the script, but there is no formal mechanism within the logic of encapsulating them into a module or structure of any sort. This is not often a problem but makes a large script somewhat harder to browse effectively.

In contrast, EHDM has a simple but useful structuring mechanism. Related units are grouped into modules. Modules implement a style of information hiding by making visible only certain declarations within an EXPORTING section. Modules gain access to one another by including a USING section.
4.4 Syntax

The Boyer-Moore logic is sometimes criticized for its Lisp-like syntax. This syntax has the advantage of being uniquely-readable (unambiguous) and very easily parsed. It has the disadvantage of being different from traditional mathematical syntax. Several papers have described proofs in the Boyer-Moore logic using a more traditional syntax; however, these may mislead a prospective user of the theorem prover. We feel that the small effort of learning a new syntax is well rewarded by gaining access to a powerful proof tool.

EHDM has reaped the benefits of both a readily parsable syntax and a more familiar "display" syntax by implementing a table driven translator from standard EHDM syntax into a LaTeX format. This gives a nice customizable syntax for presentation which Rushby and von Henke claim "enabled us to do most of our work using compact and familiar notation and thereby contributed greatly to our productivity" [15]. We believe that this overstates the value of this "compact and familiar notation."

Our experience with attempting a similar translator for the Boyer-Moore logic is that it is counterproductive to try to integrate such a translator with a theorem prover which uses a different syntax for its internal representation, proof diagnostics, and output script. If the translation could be entirely transparent to the user, there would be no difficulty. However, users of mechanical proof tools often need to be aware of the details of the internal representation of rewrite rules, the particular transformations on terms that they effect, and other things which are most efficiently expressed in a syntax which is close to that used by the machine. When this is no longer true, then syntax will not be an issue. Until then, we feel that the need to continually deal with two different forms is confusing and unnecessary.

Another problem of the EHDM translator is that the notation is not always compact and familiar. For example, we found the expression

$$p^{*\Delta x-n/m},$$

(which appears in a number of lemmas) to be impossibly confusing until we realized that the term $n-m$ is treated as though it were grouped. Here the apparent familiarity of the syntax is detrimental because the expected precedence rules are not observed with the result that the expression is unnecessarily confusing. This is probably a simple flaw in the translator table. But it points up the difficulty of having the correctness of a published proof rely not only on the prover and proof chain analyzer, but also on another tool which translates from one notation to another in a moderately complex fashion.

5. Conclusions

There are a number of other differences between the Boyer-Moore and EHDM versions of the ICCSA proofs which will be covered in the (soon to be written) detailed comparison of the two versions.

We believe that the exercise of specifying and proving the ICCSA using the Boyer-Moore prover was useful in several ways.

- It exercised and further displayed the value of a number of the newer features of the Boyer-Moore logic and their support in the theorem prover. These features include the CONSTRAIN and DEFN-SK events.
- It provided the basis for a comparison with the EHDM system and a style of proof possible within that system. This aspect will lead to a joint paper comparing the two systems on this problem.
- It provided a verified specification of the Interactive Convergence Clock Synchronization algorithm as a basis for possible future work building toward a verified implementation.

We believe that two important goals of proof are to increase one's understanding and intuition about the content and
significance of a theorem, and to provide a convincing argument that it is, in fact, valid. Mechanically supported proofs like those in EHDM and ours contribute to both of these goals. We understand this quite subtle algorithm and the reason it works much better for the effort. Moreover, our success in convincing a congenitally skeptical mechanical proof checker of the validity of the correctness theorems practically guarantees that we have eliminated any errors which the much touted "social process" might overlook. Such confidence is particularly comforting in domains such as this where a well-developed intuition is difficult to cultivate; the theorem prover is not subject to being misled by the urgings of a misguided or ill-informed intuition.
References


Appendix

The ICCSA Event List

This appendix contains the Boyer-Moore event list representing the specification and proof of the Interactive Convergence Clock Synchronization Algorithm. It does not contain the entire proof since it is built "on top of" a standard library of integer facts. For brevity we have also not included the collection of definitions and lemmas defining the rationals library on which our proof is constructed. The complete script is available on request.

*LEMMA* events are macro expanded into a *PROVE-LEMMA* followed by a *DISABLE*.

:: LENGTH

(defun length (x)
  (if (null x)
      0
      (add1 (length (cdr x))))
)

(prove-lemma length-append (rewrite)
  (equal (length (append x y))
         (plus (length x) (length y)))
)

(lemma length-0 (rewrite)
  (equal (equal (length x) 0)
         (null x))
)

:: PLISTP

(defun plistp (x)
  (if (null x)
      (equal x nil)
      (plistp (cdr x)))
)

(defun plist (x)
  (if (null x)
      nil
      (cons (car x) (plist (cdr x))))
)

:: FIRSTN and RESTN

(defun firstn (lst n)
  (if (zerop n)
      nil
      (cons (car lst)
            (firstn (cdr lst) (sub1 n))))
)

(lemma firstn-n (rewrite)
  (implies (equal n (length lst))
           (equal (firstn lst n)
                  (plist lst))))

(lemma firstn-append-lessp (rewrite)
  (implies (leq m (length lst))
           (equal (firstn (append lst lst2) m)
                  (firstn lst m)))
)

(defun restn (lst n)
  (if (zerop n)
      lst
      (restn (cdr lst) (sub1 n)))
)

(lemma restn-n (rewrite)
  (implies (and (plistp lst)
                (equal n (length lst)))
           (equal (restn lst n nil))
           ((enable restn))))
(lemma restn-1 (rewrite)
  (implies (lessp m i)
     (equal (restn (list x) m)
     (list x)))
  ((enable restn)))

(lemma restn-append (rewrite)
  (implies (leq n (length lst1))
     (equal (restn (append lst1 lst2) n)
     (append (restn lst1 n) lst2)))
  ((enable restn)))

(lemma firstn-append-restn (rewrite)
  (implies (leq m (length lst))
     (equal (append (firstn lst m) (restn lst m)) lst))

;; RATIONALS WITH NATURALS

(defn rinverse-nat (n)
  (reduce (rational 1 (fix n))))

(lemma reduce-rinverse-nat (rewrite)
  (equal (reduce (rinverse-nat n))
     (rinverse-nat n))
  ((enable reduce-reduce)))

(defn rtimes-nat (i r)
  (rtimes (rational (fix i) 1) r))

(defn rtimes-nat2 (i r)
  (rtimes (rational (fix i) 1) r))

(lemma reduce-rtimes-nat (rewrite)
  (and (equal (reduce (rtimes-nat x y))
     (rtimes-nat x y))
     (equal (reduce (rtimes-nat2 x y))
     (rtimes-nat2 x y))))
  ((enable rtimes-nat rtimes-nat2)
  (enable-theory reductions)))

(lemma rtimes-nat-rtimes-nat2 (rewrite)
  (equal (rtimes-nat2 x y)
     (rtimes-nat y x))
  ((enable rtimes-nat rtimes-nat2 commutativity-of-rtimes)))

(lemma rneg-rtimes-nat (rewrite)
  (equal (rneg (rtimes-nat i r))
     (rtimes-nat i (rneg r)))
  ((enable rtimes-nat rneg-rtimes)
  (disable correctness-of-cancel-rneg-terms-from-equality)))

(defn rquotient-nat (r i)
  (rtimes (rational (fix i) 1)))

(defn rneg-rquotient-nat (rewrite)
  (equal (rneg (rquotient-nat r n))
     (rquotient-nat (rneg r) n))
  ((enable rquotient-nat rneg-rtimes rneg-rtimes2)))

(disable rtimes-nat)
(disable rtimes-nat2)
(disable rquotient-nat)

(lemma rtimes-nat-addl (rewrite)
  (equal (rtimes-nat (addl i) r)
     (rplus r (rtimes-nat i r)))
  ((enable rtimes-nat rtimes-addl)))
(lemma rtimes-nat-zerop (rewrite)
  (implies (zerop i)
    (equal (rtimes-nat i r)
      (rational 0 l))
  )
  )

(lemma rinverse-nat-positive (rewrite)
  (rleq (rational 0 l)
    (rinverse-nat n)
  )
  )

(lemma rabs-rinverse-nat (rewrite)
  (equal (rabs (rinverse-nat n))
    (rinverse-nat n)
  )
  )

(lemma rquotient-nat-rtimes-nat (rewrite)
  (implies (not (zerop n))
    (equal (rquotient-nat (rtimes-nat n x) n)
      (reduce x))
  )
  )

(lemma rtimes-nat-rquotient-nat (rewrite)
  (implies (not (zerop n))
    (equal (rtimes-nat n (rquotient-nat x n))
      (reduce x))
  )
  )

(lemma rquotient-nat-rplus (rewrite)
  (implies (not (zerop n))
    (equal (rquotient-nat (rplus x y) n)
      (rplus (rquotient-nat x n)
        (rquotient-nat y n)))
  )
  )

(lemma div-mon2 (rewrite)
  (implies (and (rleq x y)
      (not (zerop z)))
    (rleq (rquotient-nat x z)
      (rquotient-nat y z)))
  )
  )

(lemma rationalize-rleq2 (rewrite)
  (implies (not (zerop n))
    (rleq (rational 0 l) (rational n l)))
  )
  )

(lemma rtimes-nat-2-positive-preserves-rleq (rewrite)
  (implies (and (not (zerop n))
      (not (zerop n))
    (rleq (rtimes-nat2 x n)
      (rtimes-nat2 y n))))
  )
  )

(defn rsum (lst)
  (if (nlistp lst)
    (rational 0 l)
    (rplus (car lst)
      (rsum (cdr lst))))
  )
  )

(lemma reduce-rsum (rewrite)
  (equal (reduce (rsum lst))
    (rsum lst))
  )
  )
(lemma rplus-rsum (rewrite)
  (equal (rplus (rsum lst1) (rsum lst2))
         (rsum (append lst1 lst2)))
  ((enable rsum rplus-rzerop reduce-rsum
      associativity-of-rplus)))

(lemma rsum-append (rewrite)
  (equal (rsum (append lst1 lst2))
         (rplus (rsum lst1) (rsum lst2)))
  ((enable associativity-of-rplus rsum rplus-rzerop reduce-rsum)))

(defun rmean (lst)
  (quotient-nat (rsum lst) (length lst)))

(defun all-rlessp (lst x)
  (if (nlistp lst)
      t
      (and (rlessp (car lst) x)
           (all-rlessp (cdr lst) x))))

(lemma all-rlessp-append (rewrite)
  (equal (all-rlessp (append x y) z)
         (and (all-rlessp x z)
              (all-rlessp y z)))
  ((enable all-rlessp)))

(lemma all-rlessp-req-transitive (rewrite)
  (implies (and (all-rlessp lst x) (rlessp lst y))
            (all-rlessp lst y))
  ((enable rlessp-req-transitivity)))

(lemma sum-bound (rewrite)
  (implies (and (listp lst)
                (all-rlessp lst x))
            (rlessp (rsum lst)
                     (rtimes-nat (length lst) x)))
  ((enable all-rlessp rsum rtimes-nat-rzerop
      rtimes-nat-addl rlessp-req rplus-rzerop
      rlessp-reduce rlessp-rplus-pair)))

(lemma nzerop-inverse-positive (rewrite)
  (implies (not (zerop n))
            (rlessp (rational 0 1)
                    (rational 1 n)))
  ((enable rlessp rationalp fix-rational ilessp)))

(lemma mean-bound (rewrite)
  ;; if all of the elements in the list are less than x,
  ;; then the mean of the list is less than x
  (implies (and (listp lst)
                (all-rlessp lst x))
            (rlessp (rmean lst) x))
  ((use (sum-bound))
   (enable rlessp-invert-rtimes rinverse-number-p-inverse
            nzerop-inverse-positive length-0 rtimes-nat
            nzerop-denominator-reduce rquotient-nat rinverse-nat))))

;; MAP-RABS

(defun map-rabs (lst)
  (if (nlistp lst)
      nil
      (cons (rabs (car lst))
            (map-rabs (cdr lst)))))

(lemma length-map-rabs (rewrite)
  (equal (length (map-rabs lst))
         (length lst)))

(lemma map-rabs-append (rewrite)
  (equal (map-rabs (append x y))
         (append (map-rabs x) (map-rabs y))))
(lemma plistp-map-rabs (rewrite)
  (plistp (map-rabs lst))
  (enable map-rabs plistp)))

(lemma plist-map-rabs (rewrite)
  (equal (plist (map-rabs x))
    (map-rabs x))
  (enable map-rabs plistp)))

(lemma rabs-csum-map-rabs (rewrite)
  (cleq (rabs (csum lst))
    (csum (map-rabs lst)))
  (enable rsum rabs-rplus-hack)))

(lemma abs-mean (rewrite)
  ;; the abs of the mean is eq the mean of the absolute values
  (cleq (rabs (rmean lst))
    (rmean (map-rabs lst)))
  (enable rabs-ctimes-csum-map-rabs
    rplus-inverse-nat-positive rquotient-nat rzerop-ctimes
    rabs-inverse-nat-length-map-rabs)))

(lemma listp-map-rabs (rewrite)
  (equal (listp (map-rabs x))
    (listp x))
  (use (cplus-cancel)))

;; REARRANGE LEMMAS

(lemma rearrange1 (rewrite)
  (equal (rdiff x y)
    (rdiff x (rplus u v))
    (rplus (rdiff (rplus w z) y)
      (rdiff (rplus u v) (rplus w z))))
  (enable rdiff rneg-rplus-associativity-of-rplus
    rplus-reduce-rneg))

(lemma abs-negativity-equality-hack (rewrite)
  (equal (rdiff (rabs x) (rdiff (rplus y) (rplus z)))
    (rdiff (rsum (rdiff (rplus y) (rplus z)))
      (rdiff (rplus w (rplus x))
        (rplus (rplus (rplus (rplus y) (rplus z))))))
  (enable rabs-negativity-equality-hack
    rdiff-commutativity-of-rplus
    rdiff-commutativity2-of-rplus
    rplus-reduce-rneg))

(lemma rearrange2-transitivity (rewrite)
  (implies
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus x (rplus (rneg u) (rneg v))))
        (rplus (rabs (rplus u (rplus v (rplus (rplus x (rplus y))))))))
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus x (rplus (rplus (rplus y) (rplus z))))
        (rplus (rabs (rplus (rplus x (rplus y) (rplus x (rplus z))))))))
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus (rplus x (rplus y) (rplus z)))
        (rplus (rabs (rplus (rplus (rplus y) (rplus z))))))))
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus (rplus x (rplus y) (rplus z))))
        (rplus (rabs (rplus (rplus y) (rplus z)))))
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus (rplus x (rplus y))))
        (rplus (rabs (rplus (rplus x (rplus y) (rplus z))))))
    (cleq (rabs (rplus x (rneg y)))
      (rplus (rabs (rplus (rplus (rplus y) (rplus z))))
        (rplus (rabs (rplus (rplus y) (rplus z) (rplus x))))))
  (enable rabs-negativity-equality-hack
    rdiff-reflexive rdiff-commutativity2-of-rplus
    rdiff-commutativity-of-rplus rdiff-reduce rplus-cancel))

(lemma rearrange2 (rewrite)
  (cleq (rabs (rplus (rdiff x (rplus u v))))
    (rplus (rdiff (rplus (rplus w z) y)
      (rdiff (rplus (rplus u v) (rplus w z)))))
    (rplus (rabs (rdiff (rplus x (rplus u v))))
      (rplus (rabs (rdiff (rplus x (rplus u v) (rplus w z)))))))

(lemma listp-map-rabs (rewrite)
  (equal (listp (map-rabs x))
    (listp x))
  (use (cplus-cancel)))

;; REARRANGE LEMMAS
(\use (\abs - \plus - \leq 2 \ (x \ (\text{difference} \ x \ (\plus \ u \ v)))
  \ (y \ (\text{difference} \ (\plus \ u \ v) \ (\plus \ w \ z)))
  \ (z \ (\text{difference} \ (\plus \ w \ z) \ y)))
(\abs - \neg \ (x \ (\text{difference} \ (\plus \ w \ z) \ y)))
(enable \plus - \reduce \ \neg - \plus \ \text{difference} \ \reduce - \neg \plus \ \text{associativity-of-\plus}
  \ \text{difference} \ \text{rearrange2-transitivity}))

(\text{lemma rearrange} \ (\text{rewrite})
  \ (\text{cleq} \ (\abs \ (\text{difference} \ x \ y)))
  \ (\plus \ (\abs \ (\text{difference} \ x \ (\plus \ u \ v)))
    \ (\plus \ (\abs \ (\text{difference} \ y \ (\plus \ w \ z)))
      \ (\abs \ (\plus \ u \ (\text{difference} \ v \ (\plus \ w \ z))))))
  ((\use (\text{rearrange1}) \ \text{rearrange2})))

(\text{lemma rearrange-alt} \ (\text{rewrite})
  \ (\text{cleq} \ (\abs \ (\plus \ x \ (\neg \ y))))
  \ (\plus \ (\abs \ (\plus \ x \ (\neg \ (\plus \ u \ v))))
    \ (\plus \ (\abs \ (\plus \ u \ (\neg \ w))))
    \ (\abs \ (\plus \ y \ (\neg \ (\plus \ w \ v))))))
  ((\use \ \text{rearrange} \ (\text{rewrite})
    \ (\text{cleq-transitive} \ (x \ (\abs \ (\plus \ x \ (\neg \ y))))
      \ (y \ (\abs \ (\plus \ x \ (\neg \ y))))
        \ (\plus \ (\abs \ (\plus \ u \ (\neg \ y)))
          \ (\plus \ (\abs \ (\plus \ u \ (\neg \ y)))
            \ (\plus \ (\abs \ (\plus \ v \ (\neg \ w))))
              \ (\plus \ (\abs \ (\plus \ v \ (\neg \ w)))
                \ (\plus \ (\abs \ (\plus \ w \ (\neg \ u)))))))
  (enable \plus - \reduce \ \neg - \neg - \text{difference} \ \text{associativity-of-\plus} \ \text{difference}
    \ \text{commutativity-of-\plus} \ \text{difference} \ \text{rearrange2-transitivity}))

(\text{lemma rearrange3} \ (\text{rewrite})
  \ (\text{cleq} \ (\abs \ (\text{difference} \ x \ y)))
  \ (\plus \ (\abs \ (\text{difference} \ u \ y)))
    \ (\plus \ (\abs \ (\text{difference} \ v \ x)))
      \ (\plus \ (\abs \ (\text{difference} \ v \ w)))
        \ (\abs \ (\text{difference} \ u \ w))))
  ((\use \ (\abs - \plus - \leq 3 \ (x \ (\plus \ u \ (\neg \ y))))
    \ (y \ (\plus \ x \ (\neg \ (\plus \ v))))
      \ (z \ (\plus \ v \ (\neg \ w)))
        \ (w \ (\plus \ w \ (\neg \ u))))
  (\text{cleq-transitive} \ (x \ (\abs \ (\plus \ x \ (\neg \ y))))
    \ (y \ (\abs \ (\plus \ x \ (\neg \ y))))
      \ (\plus \ (\abs \ (\plus \ u \ (\neg \ y)))
        \ (\plus \ (\abs \ (\plus \ u \ (\neg \ y)))
          \ (\plus \ (\abs \ (\plus \ v \ (\neg \ w))))
            \ (\plus \ (\abs \ (\plus \ v \ (\neg \ w)))
              \ (\plus \ (\abs \ (\plus \ w \ (\neg \ u)))))))
  (enable \plus - \reduce \ \neg - \neg - \text{difference} \ \text{associativity-of-\plus} \ \text{difference}
    \ \text{commutativity-of-\plus} \ \text{difference} \ \text{rearrange2-transitivity})

(\text{lemma rearrange4} \ (\text{rewrite})
  \ (\text{cleq} \ (\abs \ (\text{difference} \ (\plus \ a \ x) \ (\plus \ b \ y)))
    \ (\plus \ (\abs \ (\text{difference} \ a \ b)))
      \ (\plus \ (\abs \ (\text{difference} \ x \ y)))
        ((\use \ (\abs - \plus - \leq 2 \ (x \ (\plus \ a \ (\neg \ b))) \ (y) \ (z \ (\neg \ y)))))
  (enable \text{difference} \ \text{commutativity-of-\plus} \ \text{difference} \ \text{commutativity2-of-\plus}
    \ \text{associativity-of-\plus} \ \abs - \text{difference} \ \text{rabs - \text{difference} \ \text{rearrange2-transitivity}))

;; REARRANGE-DELTA from module JUGGLE

(\text{lemma rearrange-delta-step1 nil}
  \ (\text{implies} \ \text{(and} \ \text{(not} \ \text{(zerop} \ i))
    \ (\text{cleq} \ (\plus \ x \ (rplus \ y \ (\plus \ z \ (\plus \ w \ v))) \ d))
    \ (\text{cleq} \ (\plus \ (\times-n\at \ x \ i) \ (\plus \ y \ (\times-n\at \ z \ i)))
      \ (\plus \ (\times-n\at \ y \ i))
        \ (\plus \ (\times-n\at \ z \ i))
          \ (\plus \ (\times-n\at \ w \ i))
            \ (\plus \ (\times-n\at \ v \ i))))
  (enable \text{cleq} \ \text{requal} \ \times-n\at \ \times-n\at \ \text{rplus-right-factorization}
    \ \times-n\at \ \text{antisymmetric} \ \times-n\at \ \text{trichotomy} \ \text{rationalized-non-zerop-\times-n\at})
  (enable \text{theory reductions}))

(lemma rearrange-delta-step2 nil
  (implies (and (lessp m n)
    (rleq x (rtimes-nat2 dt (difference n m)))
    (rleq (rplus (rtimes-nat2 d m) x) (rtimes-nat2 d n)))
  ((enable rtimes-nat2 rtimes-rzerop rplus-rzerop rleq-reduce
    rtimes-distributes-over-plus rplus-cancel rtimes-rzerop)))

(lemma rearrange-delta-step3 nil
  (implies (and (not (zerop i))
    (rleq (rplus y (rplus z (rplus w (rplus v u))))
      (rtimes-nat2 d i)))
  ((enable rtimes-nat2 rzerop-rtimes rplus-rzerop rleq-reduce
    rquotient-nat rtimes-rplus-right-factorization rplus-cancel
    rtimes-rzerop rinverse-rzerop-denominator-reduce
    rtimes-multiply-by-rinverse rinverse-numberp-inverse
    rzerop-inverse-positive)))

(lemma rearrange-delta-step4 (rewrite)
  (equal (rplus (rquotient-nat y n)
    (rplus (rquotient-nat z n)
      (rplus w (rplus (rquotient-nat u n) v))))
  ((enable associativity-of-rplus commutativity-of-rplus reduce-rtimes
    commutativity2-of-rplus rquotient-nat rinverse-nat
    rtimes-rplus-right-factorization)))

(lemma rearrange-delta-step5 (rewrite)
  (equal (rplus (rtimes-nat m d)
    (rplus (rtimes-nat y (rtimes (rational 2 l) (rplus e z)))
      (rplus (rtimes-nat m (rtimes (rational 2 l) w)) (rtimes-nat y x)))
    (rplus (rtimes-nat m d) (rtimes (rational 2 l) w))
  ((enable-theory reductions)
    (enable rtimes-nat associativity-of-rplus
      rtimes-distributes-over-rplus commutativity-of-rtimes
      commutativity2-of-rtimes half3
      rtimes-distributes-over-plus rtimes2-expand)))

(lemma rearrange-delta (rewrite)
  (implies (and (lessp m n)
    (numberp m)
    (rleq (rplus (rtimes (rational 2 l) (rplus epsilon (rtimes rho s)))
      (rplus (rquotient-nat (rtimes (times 2 m) big-delta)
        (difference n m)))
      (rplus (rtimes (times rho big-delta)
        (difference n m))
      (rtimes-nat n (rtimes rho big-sigma))
    (difference n m))
    delta))
(rlleq (rplus (rquotient-nat (rplus (rtimes-nat m) (rplus delta (rtimes (rational 2 l) big-delta))))
(rtimest-nat (rtimes (rational 2 l)) (difference n m))
(rtimest-nat (rtimes (rational 2 l)) (rplus epsilon (rplus (rtimes rho s) (rtimes (rational 1 2) (rtimes rho big-delta)))))))

n) (rplus (rtimes rho z) (rtimes rho big-sigma))
delta))

(use (rearrange-delta-step1
(d delta)
(i (difference n m))
(x (rtimest-nat (rtimes (rational 2 l)) (rplus epsilon (rtimes rho s))))
(y (rquotient-nat (rtimes-nat (times 2 m) big-delta) (difference n m)))
(z (rquotient-nat (rtimes-nat n (rtimes rho z)) (difference n m)))
(w (rtimes rho big-delta))
(v (rtimes rho big-delta)))

(rearrange-delta-step2
(x (rplus (rtimes-nat2 (rtimes (rational 2 l)) (rplus epsilon (rtimes rho s))) (difference n m))
(rplus (reduce (rtimest-nat (times 2 m) big-delta)) (difference n m))
(rplus (reduce (rtimest-nat n (rtimes rho big-delta)) (difference n m))))

(rearrange-delta-step3
(t n)
(d delta)
(x (rtimes-nat2 delta m))
(y (rtimes-nat2 (rtimes (rational 2 l)) (rplus epsilon (rtimes rho s)))
(difference n m))
(z (rtimes-nat (times 2 m) big-delta))
(w (rtimes-nat n (rtimes rho z)))
(v (rtimes-nat2 (rtimes rho big-delta) (difference n m)))
(u (rtimes-nat n (rtimes rho big-sigma)))

(enable rtimest-nat-rquotient-nat rtimest-nat-rtimes-nat2 reduce-rtimest-nat
rquotient-nat-times-nat rearrange-delta-step4 rearrange-delta-step5)
(enable-theory reductions))

; THE INTERACTIVE CONVERGENCE ALGORITHM PROOF
(constrain parameters-intro (rewrite)
; R and S
; (and (rationalp (R))
; (rationalp (S))
; (rlessp (rational 0 1) (R)) (rlessp (rational 0 1) (S)) (rleq (rational (rational 3 l) (S)) (R)) :: posR
; :: posS
; :: Cl
; ; rho
; (rationalp (rho))
; (rleq (rational 0 1) (rtimes (rational 1 2) (rho))) :: rho pos
; (rlessp (rtimes (rational 1 2) (rho)) (rational 1 l)) :: rho_small
; ; delta
; (rplus (rtimes rho z) (rtimes rho big-sigma)))

:: other parameters

(rationalp (epsilon))
(rationalp (delta))
(rationalp (delta0))
(rationalp (big-sigma))
(rationalp (big-delta))
(numberp (n))
(not (equal (n) 0)) ;; C0a
(numberp (m)) ;; C0b
(lessp (m) (n)) ;; C0c
(lessp (rational 0 1) (big-delta)) ;; C0d
(lessp (big-sigma) (s)) ;; C0e
(lessp (big-delta) (big-sigma)) ;; C0f
(lessp (rational 0 2) (big-delta)) ;; C0g
(lessp (big-delta) (big-sigma)) ;; C0h
(lessp (epsilon) (times (rational 1 2) (times (rho) (s)))) ;; C0i
(lessp (big-delta) (rational (rho) (S))) ;; C0j
(lessp (rational (rho) (S)) (rational 0 1)) ;; C0k
(lessp (rational 0 1) (big-delta)) ;; C0l
(lessp (big-sigma) (s)) ;; C0m
(lessp (rational 1 2) (big-sigma)) ;; C0n
(lessp (times (rational 1 2) (times (rho) (big-sigma))) (big-delta)) ;; C0o
(lessp (times (rational 1 2) (times (rho) (big-sigma))) (delta)) ;; C0p

((R (lambda () (rational 3 1))))
((S (lambda () (rational 1 1))))
((rho (lambda () (rational 0 1))))
((delta (lambda () (rational 0 1))))
((delta0 (lambda () (rational 0 1))))
((big-sigma (lambda () (rational 1 2))))
((big-delta (lambda () (rational 1 2))))
((n (lambda () 1)))
((m (lambda () 0))))

(lemma big-sigma-positive (rewrite)
  (lessp (rational 0 1) (big-sigma))
  ((use (rlessp-rleq-transitivity (x (rational 0 1)) (y (big-delta)) (z (big-sigma))))))

(lemma S-rleq (rewrite)
  (and (rlessp (rational 0 1) (S))
       (rleq (rational 0 1) (S)))
  ((use (rlessp-rleq (x (rational 0 1)) (y (S))))))

(lemma c5 (rewrite)
  (rleq (rplus (delta0) (times (rho) (t))) (delta)))

(lemma c6 (rewrite)
  (rleq (rplus (times (rational 2 1) (times (rho) (big-sigma)))) (delta))
  ((use (parameters-intro))))

(lemma S-times (rewrite)
  (times (S (R))))
  ((enable (times) (rlessp))))
(constrain T0-intro (rewrite)
   (rationalp (T0))
   ((T0 (lambda () (rational 0 1)))))

(defun Ti (i)
   (+plus (T0) (rtimes-nat i (R))))

(disable Ti)

(lemma Ti-zeroop (rewrite)
   (implies (zerop i)
            (equal (Ti i) (reduce (T0))))
   ((enable Ti rplus-zeroop rtimes-nat-zeroop)))

(lemma Ti-next (rewrite)
   (equal (Ti (addl i))
          (+plus (Ti i) (R)))
   ((enable commutativity-of-rplus Ti rtimes-nat-addl
                  commutativity2-of-rplus
                  rplus-reduce)))

(lemma not-numberp-Ti (rewrite)
   (implies (not (numberp i))
            (equal (Ti i) (ti 0)))
   ((enable Ti rtimes-nat-zeroop)))

;; We use a different but equivalent notion of Rdef. The Rushby approach uses
;; an unnecessary existential quantifier.

(defun sk+ Rdef (tm i)
  (exists pi
   (and (rleq (rational 0 1) pi)
        (rleq pi (R))
        (equal (reduce tm) (+plus (ti i) pi))))

(defun in-interval (tm low high)
  (and (rleq low tm)
       (rleq tm high)))

(disable in-interval)

(lemma in-interval-inclusion (rewrite)
   (implies (and (in-interval y low x)
                  (in-interval x low high))
            (in-interval y low high))
   ((enable rleq-transitive in-interval)))

(defun in-R (tm i)
  (in-interval tm (ti i) (ti (addl i))))

(lemma not-numberp-in-r (rewrite)
   (implies (not (numberp i))
            (equal (in-r tm i)
                   (in-r tm 0)))
   ((enable in-r not-numberp-ti)))

(disable in-r)

;; This shows that the two definitions of Rdef are equivalent. Subsequently, we won’t
;; bother with Rushby’s definition.

(prove-lemma Rdef-in-R-equivalence ()
   (iff (Rdef tm i)
        (in-R tm i))
   ((use (rdef-necc)
          (rdef-suff (pi (+plus tm (rneg (ti i)))))
          (enable in-interval rdefinition rleq-rdefinition3 in-r
                      rleq-rdefinition4 ti-next rleq-rplus rplus-preserves-rleq)
          (do-not-induct Ti)))

;; Again, the Rushby definition is quite different but we prove below
;; the equivalence of the two.
(defn-sk  Sdef (tm i)
  (exists pl
    (and (rleq (rational 0 1) pi)
      (rleq pi$l$) (reduce tm) (rplus (ti l) (rplus (rdifference (R) (S)) pi)))))

(defn in-$ S$ (tm l)
  (in-interval tm
    (rdifference (Ti (add 1 l)) (S))
    (Ti (add 1 l))))

(disable in-$ S$)

(lemma sdef-in-$ S$-equivalence-case3 (rewrite)
  (implies (and (equal (rplus (S) tm)
                  (rplus (ti l) (rplus r (pi-1 i tm))))
    (rleq (pi-1 i tm) (S))
    (rleq (rational 0 1) (pi-1 i tm))
    (rleq tm (rplus (ti l) (r))))
  (use (rleq-rplus z tm) (y (rplus (ti l) (r))) (x (rdifference (S) (pi-1 i tm))))
  (enable reduce-rplus rdifference commutativity-of-rplus
    commutativity2-of-rplus associativity-of-rplus
    rdifference-rzero))

(lemma sdef-equivalence-hack (rewrite)
  (implies (rleq tm (rplus (ti l) (r)))
    (rleq (rplus tm
                  (rplus (rneg (ti l)) (rneg (r)))
                  (rational 0 1))))
  (enable rleq equal)

(prove-lemma Sdef-in-$ S$-equivalence ()
  (iff (Sdef tm l)
    (in-$ S$ tm l))
  (use (sdef-necc)
    (sdef-suff (pi (rdifference tm (rplus (ti l) (rdifference (r) (S))))))
  (enable-theory reductions)
  (enable rleq equal)

(lemma Sdef-in-$ S$-equivalence ()
  (implies (in-$ S$ tm l)
    (in-$ R$ tm l))
  (use (rdifference-sinc rleq-rdifference2 in-$ S$ in-$ R$ in-interval
    associativity-of-rplus commutativity-of-rplus ti-next)
    (x (ti l))
    (y tm)
    (z (rplus (r) (rneg (S))))))))

(lemma Sdef-in-$ S$-lemma (rewrite)
  (implies (and (in-$ S$ ti l)
                  (in-$ S$ ti l))
    (rleq (rabs (rdifference ti t2)) (S)))
  (use (betweenness-distance (pi ti) (p2 t2)
    (low (rplus (ti l) (rplus (r) (rneg (s))))
    (high (rplus (ti l) (r))))
    (enable in-$ S$ in-interval rdifference ti-next rleq equal
      associativity-of-rplus rleq-reverse rleq-reflexive))))
(lemma in-S-lemma2 (rewrite)
  (implies (and (in-S t1 i)
      (in-S t2 i))
    (rleq (rabs (rplus (t1 (rneg t2)))) (S)))
  ((use (in-s-lemma))
   (enable rdifference)))

(constrain clock-intro (rewrite)
  (rationalp (clock p ct))
  ((clock (lambda (x y) (rational 0 1))))))

(lemma rho-rleq0 (rewrite)
  (rleq (rational 0 1) (rho))
  ((use (rtimes-rleq (x (rational 1 2)) (y (rho))))))

(defn-sk+ good-clock (p low high)
  ;; This says that p's clock is good within the interval [low, high]
  ;; where rho is the maximum clock drift rate.
  (forall (t1 t2)
    (implies (and (in-interval t1 low high)
      (in-interval t2 low high))
      (rleq (rabs (rplus (clock p t1)
        (rplus (rneg (clock p t2)))
        (rplus (rneg t1) t2))))
      (rtimes (rational 1 2)
        (rtimes (rho)
          (rabs (rplus t1 (rneg t2)))))))

  ;; Delta2 is the function which reads the difference between the clocks of r and p
  ;; in period i. If either of r or p is not a process, then all bets are off.

(constrain delta2-intro (rewrite)
  (and (rationalp (delta2 r p i))
    (equal (delta2 p p i) (rational 0 1))
    (implies (not (numberp i))
      (equal (delta2 r p i) (delta2 r p 0)))
    (equal (reduce (delta2 p q i))
      (delta2 p q i)))
  ((delta2 (lambda (r p i) (rational 0 1)))))

(defn d2-bar (r p i)
  (if (and (not (equal r p))
    (rlessp (rabs (delta2 r p i)) (big-delta)))
    (delta2 r p i)
    (rational 0 1))))

(disable d2-bar)

  ;; This assumes that processes are numbered from 1..(n).

(defn d2-bar-list (n p l)
  (if (zerop n)
    nil
    (cons (d2-bar n p i) (d2-bar-list (sub1 n) p l)))))

(lemma length-d2-bar-list (rewrite)
  (equal (length (d2-bar-list n p l))
    (fix n)))

(defn d2-bar-mean (n p l)
  (rmean (d2-bar-list n p l)))

(disable d2-bar-mean)

(defn delta1 (p i)
  (d2-bar-mean (n) p i))

(disable delta1)
(lemma non-numberp-d2-bar-list (rewrite)
  (implies (not (numberp i))
    (equal (d2-bar-list n p i)
      (d2-bar-list n p 0)))
  ((enable delta p d2-bar-list)))

(lemma non-numberp-delta (rewrite)
  (implies (not (numberp i))
    (equal (delta p i)
      (delta p 0)))
  ((enable delta non-numberp-d2-bar-list
    d2-bar-mean)))

(constrain cor0-intro (rewrite)
  (and (rationalp (cor0))
    (equal (reduce (cor0)) (cor0)))
  ((cor00 (lambda () (rational 0 1)))))

(defun cor (p i)
  (if (zerop i)
    (cor0)
    (rplus (cor p (sub1 i))
      (delta p (sub1 i))))

(lemma cor-addl (rewrite)
  (equal (cor p (addl i))
    (rplus (cor p i) (delta p i)))
  ((enable non-numberp-delta cor)))

(defun adjusted (p i tm)
  (rplus tm (cor p i)))

(disable adjusted)

(lemma adjusted-zero (rewrite)
  (equal (adjusted p 0 tm)
    (rplus tm (cor p)))
  ((enable adjusted)))

(lemma adjusted-reduce (rewrite)
  (equal (adjusted p i (reduce tm))
    (adjusted p i tm))
  ((enable adjusted-cplus-reduce)))

(lemma not-numberp-adj (rewrite)
  (implies (not (numberp i))
    (equal (adjusted p i tm)
      (adjusted p 0 tm)))
  ((enable adjusted)))

(lemma adjusted-cplus (rewrite)
  (equal (adjusted p i (rplus x y))
    (rplus (adjusted p i x) y))
  ((enable adjusted associativity-of-cplus
    commutativity-of-cplus)))

(defun clock (p (adjusted p i tm))
  (clock p (adjusted p i tm)))

(disable c)

(lemma clock-prop (rewrite)
  (equal (c p (addl i) tm)
    (c p i (rplus tm (delta p i))))
  ((enable c adjusted cor non-numberp-delta
    associativity-of-cplus commutativity-of-cplus)))

(lemma c-reduce (rewrite)
  (equal (c p i (reduce tm))
    (c p i tm))
  ((enable c adjusted-reduce)))
(lemma c-commutativity (rewrite)
  (equal (c p i (rplus y x))
         (c p i (rplus x y)))
  ((enable commutativity-of-rplus)))

(lemma d2-bar-prop (rewrite)
  (rlessp (rabs (d2-bar p q i)) (big-delta))
  ((enable d2-bar)))

(defn skew (p q tm i)
  (rabs (rdifference (c p i tm)
                    (c q i tm))))

(disable skew)

(lemma not-numberp-skew (rewrite)
  (implies (not (numberp i))
            (equal (skew p q tm i)
                   (skew p q tm 0)))
  ((enable skew c not-numberp-adjusted)))

(defn nonfaulty (p i)
  (good-clock p (adjusted p 0 (ti 0)) (adjusted p 1 (addl 1))))

(lemma not-numberp-nonfaulty (rewrite)
  (implies (not (numberp i))
            (equal (nonfaulty p i)
                   (nonfaulty p 0)))
  ((enable nonfaulty not-numberp-adjusted)))

(defn faulty (p i)
  (not (nonfaulty p i)))

(disable nonfaulty)

(defn-sk+ S1A (i)
  (forall r
    (implies (and (leq (addl 1) r)
                 (leq r (n)))
             (nonfaulty r i))))

(defn-sk+ S1C (p q i)
  (forall tm
    (implies (and (nonfaulty p i)
                   (nonfaulty q i)
                   (in-R tm i))
             (rleq (skew p q tm i) (delta))))

(lemma not-numberp-S1C (rewrite)
  (implies (and (not (numberp i))
                (S1C p q 0))
            (S1C p q i))
  ((use (S1C-necc (i 0) (tm (tm i p q))))
   (enable not-numberp-skew not-numberp-nonfaulty not-numberp-in-r)))

(defn S2 (p i)
  (rlessp (rabs (rdifference (corr p (addl i))
                       (corr p i)))
          (big-sigma)))

(disable s2)

:: These are the basic assumptions of the theorem

(axiom AO (rewrite)
  (rlessp (skew p q (ti 0) 0) (delta0)))

(defn-sk+ some-ok-time (p q i)
  (exists t0
    (and (in-S t0 i)
         (rlessp (rabs (rdifference (c p i (rplus t0 (delta2 q p i)))
                        (c q i t0)))
                 (epsilon))))
(axiom A2 (rewrite)
  (implies (and (nonfaulty p i) (nonfaulty q i) (SLC p q i) (S2 p i))
    (and (cleq (rabs (delta2 q p i)) (s))
      (some-ok-time p q i))))

(lemma d2-bar-list-listp (rewrite)
  (equal (listp (d2-bar-list n p i))
    (not (zerop n))))

(lemma d2-bar-list-all-rlessp-big-delta (rewrite)
  (all-rlessp (map-rabs (d2-bar-list n p i))
    (big-delta))
  ((enable d2-bar-prop)))

(disable rmean)

(lemma deltal-rlessp (rewrite)
  (rlessp (cabs (deltal p i)) (big-sigma))
  ((enable deltal d2-bar-mean abs-mean listp-map-rabs d2-bar-list-listp
d2-bar-llst-all-clessp-big-delta)))

(lemma theorem2 (rewrite)
  (S2 p i)
  ((enable rdifference s2 associativity-of-rplus
delta1-clessp rabs-reduce
rplus-zerop)))

(lemma upper-bound (rewrite)
  (implies (and (ln-s tm i) (cleq (rabs pi) (rdifference (r) (s))))
    (cleq (adjusted p i (rplus tm pi))
      (adjusted p (addl i) (ti (addl (addl i)))))
  ((use (rlessp-rleq-transitivity2 (x (rplus (r) (meg (big-sigma))))
     (y (rplus (r) (rneg (big-sigma))))
     (z (rplus (r) (deltal p i))))
    (theorem2)
    (abs-ax6 (x pi)
      (y (rdifference (r) (s))))
    (abs-ax6 (x (rdifference (corr p (addl i)))
      (corr p i))
      (y (big-sigma))))
  (enable rlessp-rleq big-sigma-positive adjusted-rplus
neg-greater-rleq rplus-cancel rplus-preserves-rleq-hack
associativity-of-rplus ti-next in-s in-interval
rdifference rleq-reduce adjusted corr-addl s2 rlessp-rleq)))

(lemma small-shift (rewrite)
  (cleq (rneg (r)) (rdifference (corr p (addl i)) (corr p i))
  ((use (theorem2) (sinr))
    (disable corr theorem2 sinr)
    (enable rabs-rneg-rleq rlessp-transitive rdifference
rlessp-rleq-transitivity2 s2)))

(lemma adj-inductive-step (rewrite)
  (implies (cleq tc0 (adjusted p i (ti i)))
    (cleq tc0 (adjusted p (addl i) (ti (addl i))))
  ((use (small-shift))
    (disable corr)
    (enable rleq-transitive associativity-of-rplus rplus-cancel ti-next adjusted
rdifference rleq-reduce rdifference rplus-rneg-rleq-hack)))
(defn subl-induction (i)
  (if (zerop i)
      t
      (subl-induction (subl i))))

(lemma adj-always-positive (rewrite)
  (cleq (rplus (t0) (corr0)) (adjusted p i (ti i)))
  ((induct (subl-induction i))
   (enable ti-zerop adjusted-reduce not-numberp-adjusted
    adjusted-zero rleq-reduce rleq-reflexive adj-inductive-step)))

(lemma lower-bound (rewrite)
  (implies (cleq (rational 0 1) pi)
    (cleq (adjusted p 0 (ti 0))
      (adjusted p i (rplus (ti i) pi))))
  ((use (adj-always-positive))
   (enable-theory reductions)
   (enable ti-zerop rleq-rplus2 adjusted-zero adjusted-rplus)))

(lemma lower-bound2 (rewrite)
  (implies (and (in-s tm i)
                 (cleq (abs pi) (rdifference (z) (s))))
    (cleq (adjusted p 0 (ti 0))
      (adjusted p i (rplus tm pi))))
  ((use (lower-bound (pi (rplus tm (rplus (rneg (ti 1)) pi))))
    (cleq-transitive' (x (rational 0 1))
      y (rplus tm
        (rplus (rneg (ti 1))
          (rplus (rneg (z) (s))))))
    (enable rplus-reduce rleq-hack
     associativity-of-rplus rneg-rneg ti-next
     rneg-rplus rdifference in-s in-interval
     rleq-reduce abs-rneg-rplus)))

(lemma gc-prop (rewrite)
  (implies (and (good-clock p t0 tn)
                 (good-clock p t0 tm))
    (good-clock p t0 tn))
  ((use (good-clock-necc (t2 (t2 tm t0 p)) (t1 (t1 tm t0 p))
         (high tn) (low t0) (p pi))
    (in-interval-inclusion (low t0) (high tn) (x tm) (y (t1 tm t0 p)))
    (in-interval-inclusion (low t0) (high tn) (x tm) (y (t2 tm t0 pi)))))

(lemma bounds (rewrite)
  (and (cleq (adjusted p 0 (ti 0))
         (adjusted p i (ti (addl i))))
    (cleq (adjusted p i (ti (addl i)))
      (adjusted p (addl i) (ti (addl (addl i))))))
  ((use (lower-bound2 (pi (rational 0 1))
         (tm (ti (addl i))))
    (upper-bound (pi (rational 0 1))
      (tm (ti (addl i))))
    (enable ti-in-s rleq-rplus-rzerop
     adjusted-reduce rdifference)))

(lemma nonfx (rewrite)
  (implies (nonfaulty p (addl i))
    (nonfaulty p i))
  ((use (gc-prop t0 (adjusted p 0 (ti 0))
         (tn (adjusted p
           (addl i)
           (ti (addl (addl i)))))
         (tm (adjusted p i (ti (addl i))))
         (enable nonfaulty in-interval bounds))))

(lemma sla-lemma (rewrite)
  (implies (sla (addl i))
    (sla i))
  ((use (sla-necc (r (r-1 i)) (i (addl i))))
   (enable nonfx)))
(lemma lemma2 (rewrite)
  (implies (and (nonfaulty p (addl i))
                (rleq (adjusted p i tm)
                      (adjusted p (addl i) (ti (addl (addl i))))))
  (rleq (adjusted p 0 (ti 0))
        (adjusted p i tm))
  (rleq (adjusted p i (rplus tm pi))
        (adjusted p (addl i) (ti (addl (addl i))))
  (rleq (adjusted p 0 (ti 0)) (adjusted p i (rplus tm pi)))
  (rleq (rabs (rplus c p i (rplus tm pi))
            (rplus (rneg (c p i tm))
                  (rneg pi))))
  (rtimes (rational 1 2) (rtimes (cho) (rabs pi))))
  (use (good-clock-never (low (adjusted p 0 (ti 0)))
                  (high (adjusted p
                           (addl i)
                           (ti (addl (addl i)))))
                  (t2 (adjusted p i tm))
                  (ti (adjusted p l (rplus tm pi))))
  (enable adjusted-cplus associativity-of-rplus nonfaulty c
           rabs-reduce rneg-rplus reduce-rneg in-interval)))

(lemma lemma2a (rewrite)
  (implies (and (nonfaulty p (addl i))
                (rleq (rabs (rplus pl phi)) (rdifference (r) (s)))
                (rleq (rabs phi) (rdifference (r) (s)))
                (ln-s tm i))
  (rleq (rabs (rplus (c p i (rplus tm (rplus phi pi))))
           (rplus (rneg (c p i (rplus tm phi)))
                  (rneg pi))))
  (rtimes (rational 1 2) (rtimes (cho) (rabs pi))))
  (use (lemma2 (tm (rplus tm phi))))
  (enable upper-bound lower-bound2 associativity-of-rplus
           rabs-commutativity-hack)))

(lemma lemma2b-step (rewrite)
  (implies (and (rleq (rabs phi) (s))
                (rleq (rabs phi) (s)))
            (rleq (rplus (rplus (cabs (c p i (=plus tm phi)))
                           (n_2)) (cabs pi))))
  (use (rleq-transitive (x (rabs (rplus pi phi)))
                        (y (rplus (rabs pi) (rabs phi)))
                        (z (rplus (r) (rneg (s))))))
  (enable times-3-rleq-rewrite rabs-rplus rleq s-rleq
           parameters-intro)))

(lemma lemma2b-step2 (rewrite)
  (implies (rleq (rabs phi) (s))
            (rleq (rabs phi) (s)))
  (use (rleq-transitive (x (rabs (rplus pi phi)))
                        (y (rplus (rabs pi) (rabs phi)))
                        (z (rplus (r) (rneg (s))))))
  (enable times-3-rleq-rewrite2 s-rleq parameters-intro)))

(lemma lemma2b (rewrite)
  (implies (and (nonfaulty p (addl i))
                (rleq (rabs phi) (s))
                (rleq (rabs pi) (s))
                (ln-s tm i))
  (rleq (rabs (rplus c p i (rplus tm (rplus phi pi)))
            (rplus (rneg (c p i (rplus tm phi)))
                   (rneg pi))))
  (rtimes (rational 1 2) (rtimes (cho) (rabs pi))))
  (enable lemma2a lemma2b-step rdifference lemma2b-step2)))

(lemma lemma2c (rewrite)
  (implies (and (nonfaulty p (addl i))
                (rleq (rabs pi) (s))
                (ln-s tm i))
  (rleq (rabs (rplus c p i (rplus tm pi))
            (rplus (rneg (c p i tm)))
            (rneg pi))))
  (rtimes (rational 1 2) (rtimes (cho) (rabs pi))))
  (use (lemma2b (phi (rational 0 l))))
  (enable s-rleq rplus-rzerop rplus-reduce c-reduce)))
(deftheorem lemma2d (rewrite)
  (implies (and (nonfaulty p i)
      (cleq (rational 0 1) pi)
      (cleq pi (cplus (c p i (rplus (ti 1) pl)) (rplus (rneg (c p i (ti 1))) (rneg pi))))
      (rplus (rplus (c p i (rplus (ti 1) pl))
          (rplus (rneg (c p i (ti 1)))
              (rneg pl))))
      ((rtimes (rational 1 2) (rtimes (rho) (rabs pi)))))
  ((use (good-clock-necc (lou (tlt (addl 0 i))
      (hiqh (adjusted p (i + (addl 1 i))))
      (tl (adjusted p (i + (rplus (ti 1) pl))))
      (t2 (adjusted p (ti i))))))
  (enable-theory reductions)
  (enable nonfaulty in-interval lower-bound adjusted-rplus ti-next
      rplus-cancel adjusted-zero ti-zero adj-always-positive
      rleq-rplus2 rleq-reflexive rleq-transitive c rneg-rplus
      associativity-of-rplus rabs-reduce)))

(lemmas rabs-negate-lemma2d-hack (rewrite)
  (equal (rabs (rplus (c p i t0)
      (rplus (delta2 q p i))
      (rneg (c p i)
          (rplus t0
delta2 q p i))))
      (rabs (rplus (c p i)
          (rplus t0 (delta2 q p i))
          (rplus (rneg (c p i t0))
              (rneg (delta2 q p i))))))
  ((use (rabs-negate-hack (x (c p i t0))
      (y (delta2 q p i))
      (z (c p i (rplus t0 (delta2 q p i)))))))

(lemmas lemma2 (rewrite)
  (implies (and (sc p q i)
      (s2 p i)
      (nonfaulty p (addl 1 i))
      (nonfaulty q (addl 1 i)))
  ((use (a2)
      (some-ok-time-necc)
      (slc-necc (tm (t0-1 1 p q)))
      (rabs-rplus-rleq2
        (x (rplus (c p i (t0-1 1 p q))
            (rplus (delta2 q p i))
            (rneg (c p i)
                (rplus (t0-1 1 p q)
                    (delta2 q p i)))))
        (y (rplus (c q i (t0-1 1 p q))
            (rneg (c p i (t0-1 1 p q)))))
        (z (rplus (c p i)
            (rplus (t0-1 1 p q) (delta2 q p i))
            (rneg (c q i (t0-1 1 p q))))))
  ((lemmas2c (pl (delta2 q p i)) (tm (t0-1 1 p q)))
      (rleq-rleq-transitivity2
        (x (rabs (delta2 q p i)))
        (y (rplus (rabs (rplus (c p 1 (t0-1 1 p q))
            (rplus (delta2 q p i))
            (rneg (c p i)
                (rplus (t0-1 1 p q)
                    (delta2 q p i)))))))
      (z (big-delta)))))
(rlessp-rleq-transitivity
 (x (rplus (rabs (rplus (rplus (c p i (t0-1 i p q))
 (rplus (delta2 q p i))
 (rneg (c p i)
 (rplus (c p i (t0-1 i p q))
 (delta2 q p i))))))))
 (rplus (rabs (rplus (c q i (t0-1 i p q))
 (neg (c p i (t0-1 i p q))))))
 (rabs (rplus (c p i
 (rplus (c p i (t0-1 i p q))
 (delta2 q p i))
 (neg (c q i (t0-1 i p q)))))))))
 (y (rplus (delta)
 (rplus (epsilon)
 (rtimes (rational 1 2)
 (rtimes (rational 1 2))))))
 (z (big-delta)))
 (rleq-transitive
 (x (rabs (rplus (c p i (t0-1 i p q))
 (rplus (delta2 q p i))
 (neg (c p i (t0-1 i p q))))))
 (y (rtimes (rational 1 2)
 (rtimes (rational 1 2))
 (rtimes (rational 1 2)))))
 (enable rabs-difference rleq-times-hack rho-rleq0 rlessp-rleq nonfx
 inrs skew difference associativity-of-rplus
 parameters-intro rleq-rlessp-hack rabs-negate-lemma1-hack
 (enable-theory reductions)))

(lemma lemma3 (rewrite)
 (implies (and (slc p q l)
 (s2 p l)
 (nonfaulty p (addl 1))
 (nonfaulty q (addl 1))
 (ls mp tm))
 (rlessp (rabs (rplus (c p i (rplus tm (delta2 q p i))))
 (rneg (c q i tm))))
 (rplus (epsilon)
 (rtimes (rational 1 2)
 (rtimes (rational 1 2)))))

(use (a2)
 (some-ok-time-ndec)
 (rearrange-al)
 (x (c p i (rplus tm (delta2 q p i))))
 (y (c q i tm))
 (u (c p i
 (rplus (t0-1 i p q) (delta2 q p i))))
 (v (rplus tm (neg (c q i tm))))
 (w (c q i tm))
 (lemma2c
 (p q)
 (tm (t0-1 i p q))
 (p1 (rplus tm (neg (t0-1 i p q))))))
 (lemma2b
 (tm (t0-1 i p q))
 (phi (delta2 q p i)))
 (pi (rplus tm (neg (t0-1 i p q))))
 (rleq-rleq-transitivity2
 (x (rabs (rplus (c p i (rplus tm (delta2 q p i))))
 (rneg (c q i tm))))
 (y (rplus (rabs (rplus (c p i (rplus tm (delta2 q p i))))
 (rplus (rneg (c p i
 (rplus (rabs (c p i (rplus tm (delta2 q p i))))
 (rplus (rneg tm) (t0-1 i p q)))))
 (rplus (rabs (c p i
 (rplus (delta2 q p i) (t0-1 i p q))))
 (neg (c q i (t0-1 i p q)))))
 (rabs (rplus (c q i tm)))
 (rplus (neg (c q i (t0-1 i p q))))
 (rplus (rneg tm) (t0-1 i p q))))))
 (z (rplus (epsilon)
 (rtimes (rational 1 2)
 (rtimes (rational 1 2))))))
(rleq-transitive
  (x (rplus (rabs (rplus (c p i (rplus tm (delta2 q p i)))))
      (rplus (rneg (c p i)
                (rplus (delta2 q p i) (t0-1 i p q))))
      (rplus (rneg tm) (t0-1 i p q))))
  (y (rtimes (rho)
        (rabs (rplus tm (rneg (t0-1 i p q))))))
  (z (rtimes (rho) (s)))))
(enable-theory reductions
  (enable in-s-lemma2 rho-rllessp rlessp-rplus-triple rplus-reduce
    rneg-rneg rneg-rplus in-s-lemma2 c-reduce nonfx rdifference
c-commutativity rlessp-transitive associativity-of-rplus
    rho-rllessp rlessp-rleq rleq-hack rleq-times-hack rleq-half-rplus)))

(lemma sublemmal (rewrite)
  (implies (and (slc p r i) (s2 p i)
                (nonfaulty p (addl 1))
                (nonfaulty r (addl 1))
                (equal (d2-bar r p i) (delta2 r p i)))
           ((use (lemma2c (pl (delta2 r p i))))
            (enable d2-bar delta2-intro))))

(lemma lemma2x (rewrite)
  (implies (and (slc p r i) (s2 p i)
                (nonfaulty p (addl 1))
                (nonfaulty r (addl 1))
                (in-s tm i))
           (rleq (rabs (rplus (c p i (rplus tm (delta2 r p i)))))
                (rplus (rneg (c p i tm))
                        (rneg (delta2 r p i))))
           ((use (lemma1 (q r)))
            (enable d2-bar delta2-intro))))

(lemma lemma4-version1 (rewrite)
  (implies (and (slc q r i) (s2 q i)
                (slc q r i)
                (s2 r i)
                (nonfaulty p (addl 1))
                (nonfaulty q (addl 1))
                (nonfaulty r (addl 1))
                (in-s tm i))
           (rllessp (rabs (rplus (rplus (rplus (rplus (rplus tm (delta2 q r i)))))
                            (rplus (rneg (c q i tm))
                                    (rneg (d2-bar r q i)))))))
           ((use (rrearrange)
                      (x (rplus (c p i tm) (d2-bar r p i)))
                      (y (rplus (c q i tm) (d2-bar r q i)))
                      (u (c q i) (rplus tm (d2-bar r q i)))
                      (v (c p i (rplus tm (d2-bar r p i))))
                      (w (c r i tm))))
           (enable rleq-rllessp rho-rlleq rlessp-rleq nonfx a2
c-commutativity rlessp-transitive associativity-of-rplus
    rho-rllessp rlessp-rleq rleq-hack rleq-times-hack rleq-half-rplus)))
(enable-theory-reductions)
(enable-sublemma-lemma3 lemma2x rdifference rlessp-cleq-transitivity2
rlessp-cleq rlessp-rplus-pair rneg-rplus rcleq-rlessp-rplus-pair))

(lemma lemma4-hack (rewrite)
(equal (rplus z (rplus z (rplus x (rplus y (rplus x y))))))
((enable rtimes-add1 rtimes-rzerop commutativity-of-rplus
computation2-of-rplus))

(lemma lemma4 (rewrite)
(implies (and (slc q r i)
(sl p q i)
(s2 p i)
(s2 q i)
(s2 r i)
(nonfaulty p (addl i))
(nonfaulty q (addl i))
(nonfaulty r (addl i))
(in-s tm i))
(rlessp (rabs (rplus (rplus (d2-bar r p i))
(rplus (rneg (c p i tm))
(rneg (d2-bar r q i)))))
(rtimes (rational 2 1))
(rplus (epsilon))
(rplus (rtimes (cho) (s))
(rtimes (rational 1 2))
(rtimes (cho) (big-delta))))))

(use (lemma4-version1))
(enable associativity-of-rplus lemma4-hack))

(lemma lemma5 (rewrite)
(implies (and (slc q r i)
(sl p q i)
(s2 p i)
(s2 r i)
(nonfaulty p (addl i))
(nonfaulty q (addl i))
(in-s tm i))
(rlessp (rabs (rplus (c p i tm))
(rplus (d2-bar r p i))
(rplus (rneg (c q i tm))
(rneg (d2-bar r q i)))))
(rtimes (rational 2 1))
(rplus (epsilon))
(rplus (rtimes (cho) (s))
(rtimes (rational 1 2))
(rtimes (cho) (big-delta))))))

(use (rearrange4 (a (c p i tm))
(b (c q i tm))
(x (d2-bar r p i))
(y (d2-bar r q i)))
(sleq-necc))
(enable rdifference associativity-of-rplus rneg-rplus
rlessp-rcleq-transitivity2 rcleq-rlessp-rplus-pair skew nonfx inrs
rcleq-times2 d2-bar-prop))

(lemma rcleq-rplus-hack3 (rewrite)
(equal (rcleq (rplus x (rplus y z)) (rplus y w))
(rcleq (rplus x z) (reduce w)))
((enable rcleq requal)))

(lemma sublemma-a (rewrite)
(implies (and (nonfaulty p i)
(nonfaulty q i)
(in-r tm i))
(rcleq (skew p q tm i)
(rplus (skew p q (ti l) i))
(rtimes (cho) (s))))
(use (rearrange-alt (x (c p i tm))
(y (c q i tm))
(u (c p i (ti l) i))
(v (rplus tm (rneg (ti l)))))
(w (c q i (ti l)))))
(lemma2d (pl (rplus tm (rneg (ti l))))
(lemma2d (p q) (pl (rplus tm (rneg (ti l))))))
(enable-theory reductions)
(enable skew difference cneg-rplus c-reduce ti-next rabs-positive2
rplus-rleq-rewrite rplus-rleq-rewrite2 rleq-times-hack ln-r
in-interval cho-rleq0 rleq-transitive rleq-rplus-hack3 rleq-half-rplus
rleq-times-hack rplus-rleq-rewrite))

(lemma delta1-rleq-s (rewrite)
 (rlessp (rabs (delta1 p i)) (s))
 (use (theorem2))
 (enable s2 difference correction assoc-addl associativity-of-rplus
 rabs-reduce rlessp-transitive rleq-rlessp-relation)))

(lemma sublemma2 (rewrite)
 (equal (skew p q tm (addl i))
  (rabs (rplus (c p i (rplus tm (delta1 p i)))))
  (rneg (c q i (rplus tm (delta1 q i)))))
 ((enable skew difference clock-prop))

(lemma lemma6 (rewrite)
 (implies (and (nonfaulty p (addl i))
  (nonfaulty q (addl i))
  (in-r tm (addl i)))
  (rleq (skew p q tm (addl i))
   (rplus (rabs (rplus (c p i (ti (addl i)))))
    (rplus (delta1 p i))
    (rplus (rneg (c q i (ti (addl i)))))
    (rneg (delta1 q i))))
 ((use (sublemma-a (i (addl 1)))
  (rarrange
   (x (c p i)
    (rplus (ti (addl 1)) (delta1 p i))))
   (y (c q i)
    (rplus (ti (addl 1)) (delta1 q i))))
   (u (c p i (ti (addl 1))))
   (v (delta1 p i))
   (w (c q i (ti (addl 1))))
   (z (delta1 q i))))
 (lemma2c (tm (ti (addl 1))) (pi (delta1 p i))
 (lemma2c (tm (ti (addl 1))) (pi (delta1 q i) (p q))
 (disable correctness-of-cancel-rplus-rleq)
 (enable difference cneg-rplus sublemma2 rplus-rleq-rlessp-cancel2 rleq-transitive
 rplus-rleq-rlessp-cancel delta1-rleq-s ti-in-s rleq-times-pos2 rleq-rplus
delta1-rleq rleq-times-hack rleq-half-rplus)))

(defn ll-term-list (p q i r)
 ;; This generates the list of terms to which the mean is applied in lemma11.
 ;; Notice that they don't have the absolute-value applied.
 (if (zerop r)
   nil
   (append (ll-term-list p q i (subl r))
    (list (rplus (c p i (ti (addl i))))
     (rplus (d2-bar r p i))
     (rplus (rneg (c q i (ti (addl i)))))
     (rneg (d2-bar r q i))))))

(lemma length-ll-term-list (rewrite)
 (equal (length (ll-term-list p q i r))
 (fix r))
 ((enable ll-term-list))))

(lemma ll-term-list-rewrite (rewrite)
 (equal (rsum (ll-term-list p q i r))
  (rplus (rsum (ll-term-list p q i r)))
  (rplus (rsum (rsum (d2-bar-list r p i))))
  (rplus (rneg (rsum (rsum (d2-bar-list r p i))))
   (rsum (d2-bar-list r q i))))
 ((enable ll-term-list d2-bar-list rsum times-nat-zero p times-nat-addl
 associativity-of-plus cneg-rplus
 reduce-rplus rsum-append plus-zereq
 commutativity-of-plus reduce-reduce rplus-reduce)))
(lemma 11 (rewrite)
  (rleq (rabs (rplus (c p i (ti (addl i))))
  (rplus (deltal p i))
  (rplus (rneg (c q i (ti (addl i))))
  (rneg (deltal q i))))
  (rmean (map-rabs (ll-term-list p q i (n))))
  (use (abs-mean (ls (ll-term-list p q i (n))))
  (enable rleq-transitive rmean ll-term-list-rewrite length-ll-term-list
  rquotient-nat-rplus rquotient-nat-times-nat rplus-reduce
  rneg-times-nat deltall d2-bar-mean rmean
  length-d2-bar-list rneg-rquotient-nat rleq-reflexive))

(lemma 12 (rewrite)
  (rleq (rabs (rplus (c p i (ti (addl i))))
  (rplus (deltal p i))
  (rplus (rneg (c q i (ti (addl i))))
  (rneg (deltal q i))))
  (rquotient-nat (rplus (rsum (firstn (map-rabs (ll-term-list p q i (n))) (m)))
  (rsum (restn (map-rabs (ll-term-list p q i (n))) (m))))
  (n))
  (use (l1))
  (enable rplus-rsum firstn-append-restn rmean
  length-map-rabs length-ll-term-list))

(lemma bound-faulty (rewrite)
  (implies (and (SIA (addl i))
  (SIC p q i)
  (not (zerop r))
  (nonfaulty p (addl i))
  (nonfaulty q (addl i))
  (rlessp (rabs (rplus (c p I (ti (addl i))))
  (rplus (d2-bar r p i))
  (rplus (rneg (c q i (ti (addl i))))
  (rneg (d2-bar r q i))))
  (rplus (delta) (rtimes (rational 2 1) (big-delta))))
  (enable lemma5 ti-in-s))

(defun firstn-ll-induction (m n)
  (if (zerop n)
  t
  (if (zerop m)
  t
  (if (equal m n)
  t
  (firstn-ll-induction m (subl n))))))

(lemma firstn-ll-term-list (rewrite)
  (implies (leq m n)
  (equal (firstn (map-rabs (ll-term-list p q i (n))) m)
  (map-rabs (ll-term-list p q i m)))
  (induct (firstn-ll-induction m n))
  (enable map-rabs-append firstn-n plist-map-rabs
  length-map-rabs length-ll-term-list
  firstn-append-lessp))

(lemma 13-sublemma (rewrite)
  (implies (and (leq m n))
  (SIA (addl i))
  (SIC p q i)
  (nonfaulty p (addl i))
  (nonfaulty q (addl i))
  (not (zerop m))
  (all-rlessp (firstn (map-rabs (ll-term-list p q i (n))) m)
  (rplus (delta)
  (rtimes (rational 2 1) (big-delta))))
  (induct (subi-induction m))
  (enable all-rlessp firstn map-rabs ll-term-list bound-faulty
  map-rabs-append firstn-ll-term-list all-rlessp-append)))
(lemma 13 (rewrite)
  (implies (and (SIA (addl i))
                (SLC p q i)
                (nonfaulty p (addl i))
                (nonfaulty q (addl i))
                (lessp m (n))))
  (cleq (rsum (firstn (map-rabs (11-term-list p q i m)) m))
        (rtimes-nat2 (rplus (delta) (rtimes (rational 2 1) (big-delta)))))
  ((use (sum-bound (lst (map-rabs (11-term-list p q i m))))
        (x (rplus (delta))
            (rtimes (rational 2 1)
                (big-delta))))))

(13-sublemma))
  (enable firstn rsum rtimes-nat-rtimes-nat2 rtimes-nat-zerop
       rleq-reflexive firstn-11-term-list rtimes-nat-rtimes-nat2
       length-map-rabs length-11-term-list rlessp-rleq listp-map-rabs)))

(defn-sk+ theoreml-one-step (l)
  (forall (p q)
    (implies (SIA l)
             (SLC p q l))))

(lemma bound-nonfaulty (rewrite)
  (implies (and (SIA (addl i))
                (SLC p q i)
                (leq (addl (m)) r)
                (leq r (n))
                (nonfaulty p (addl i))
                (nonfaulty q (addl i))
                (rtimes (rational 2 1)
                    (rtimes (rational 2 1)))))
  (rplus (rneg (c q i (ti (addl i))))
    (rplus (d2-bar r q i))))

(lemma 14-term-list (p q i m r)
  (cons (rabs (rplus (c p i (ti (addl i))))
        (rplus (d2-bar r p i))
        (rplus (d2-bar r q i))))
  (rtimes (rational 2 1)
    (rtimes (rational 2 1)
      (rtimes (rational 2 1)))
    (rtimes (rational 2 1)
      (rtimes (rational 2 1)
        (rtimes (rational 2 1)))))

  ((use (SIA-necc (l (addl i))))
   (rleq-reflexive (rplus (rneg (c q i (ti (addl i))))
                   (rneg (d2-bar r q i))))
   (enable lemma+ theoreml SIA-lemma ti-in-s)))

(defn 14-term-list (p q i m r)
  (if (leq m r)
    (cons (rabs (rplus (c p i (ti (addl i))))
        (rplus (d2-bar r p i))
        (rplus (d2-bar r q i)))
    (14-term-list p q i (addl m r))))

(lemma 14-term-strip-last (rewrite)
  (implies (and (leq m r)
                 (not (zerop r)))
    (equal (14-term-list p q i m r)
      (append (14-term-list p q i m (subl r))
              (list (rabs (rplus (c p i (ti (addl i))))
                        (rplus (d2-bar r p i))
                        (rplus (d2-bar r q i)))))))

(lemma length-14-term-list-froml (rewrite)
  (equal (length (14-term-list p q i l r)) (fix r))
  ((induct (subl-induction r))
   (enable 14-term-strip-last)))
(lemma length-14-term-list (rewrite)
  (equal (length (14-term-list p q i (addl m) r))
    (difference r m))
  ((enable length-14-term-list-froml 14-term-strip-last)))

(define 11-14-relation-induction (m n)
  (if (not (lessp m n))
    (if (equal m n)
      t
      (if (lessp n m) t f))
    (ll-14-relation-induction m (subl n))))

(lemma ll-14-term-lists-relation (rewrite)
  (implies (and (not (zerop n))
    (leq m n))
    (equal
      (restn (map-rabs (ll-term-list p q i n)) m)
      (difference r m))
  ((induct (ll-14-relation-induction m n))
    (enable restn-n plistp-map-rabs length-map-rabs length-ll-term-list
      restn-append map-rabs-append restn-1 14-term-strip-last)))

(lemma listp-14-term-list (rewrite)
  (implies (and (lessp m n)
    (not (zerop n)))
    (listp (14-term-list p q i (addl m) n))
  ((enable 14-term-strip-last)))

(lemma 14-sublemma (rewrite)
  (implies (and (leq r (n))
    (leq (addl (m)) r)
    (S1A (addl l))
    (S1C p q i)
    (nonfaulty p (addl l))
    (nonfaulty q (addl l))
    (theorem-one-step l))
    (all-lessp (14-term-list p q i r (n))
      (rtimes (rational 2 l)
        (rtimes (rational 1 2)
          (rplus (epsilon)
            (rplus (rtimes (rho) (s))
              (rtimes (rational 1 2)
                (rplus (rtimes (rho) (big-delta)))))))
  ((enable bound-nonfaulty)))

(lemma 14-version1 (rewrite)
  (implies (and (lessp m n)
    (leq (m) m)
    (S1A (addl l))
    (S1C p q i)
    (nonfaulty p (addl l))
    (nonfaulty q (addl l))
    (theorem-one-step l))
    (rleq (sum (14-term-list p q i (addl m) (n)))
      (rtimes nat2 (rtimes (rational 2 l)
        (rplus (epsilon)
          (rplus (rtimes (rho) (s))
            (rtimes (rational 1 2)
              (rplus (rtimes (rho) (big-delta)))))))
  (difference (n) m)))

  ((use (sum-bound (1st (14-term-list p q i (addl m) (n)))
        (x (rtimes (rational 2 l)
          (rplus (epsilon)
            (rplus (rtimes (rho) (s))
              (rtimes (rational 1 2)
                (rplus (rtimes (rho) (big-delta))))))))
  (enable rtimes-nat-rtimes-nat2 rlessp-rleq
    listp-14-term-list length-14-term-list 14-sublemma)))

(lemma 14 (rewrite)
  (implies (and (S1A (addl l))
    (S1C p q i)
    (nonfaulty p (addl l))
    (nonfaulty q (addl l))
    (theorem-one-step l))
lemma 15 (rewrite)
(implies (and (SLA (addl 1))
  (nonfaulty p (addl 1))
  (nonfaulty q (addl 1))
  (theoreml-one-step i))
(rleq (rabs (rplus (c p i (ti (addl 1))))
  (rplus (deltal p i)
  (rplus (rneq (c q i (ti (addl 1))))
  (rneq (deltal q i))))))

(rquotient-nat
(rplus (retimes-nat2
  (rplus (delta) (rtimes (rational 2 1) (big-delta)))) (m))
(rtimes-nat2
  (rplus (epsilon)
  (rplus (retimes (rho) (s))
  (rtimes (rational 1 2)
  (rtimes (rho) (big-delta))))))

(difference (n) (m))))

(use (div-mon2
  (x (rsum (append (firstn (map-rabs (11-term-list p q i (n))) (m))
    (restn (map-rabs (11-term-list p q i (n))) (m))))
  (y (rplus
    (rtimes-nat2 (rplus (delta) (rtimes (rational 2 1) (big-delta)))) (m))
    (rtimes-nat2 (rtimes (rational 2 1)
      (rplus (epsilon)
      (rplus (rtimes (rho) (s))
      (rtimes (rational 1 2)
      (rtimes (rho) (big-delta))))))

  (difference (n) (m))))
  (z (n)))
(rleq-transitive
(x (rabs (rplus (c p i (ti (addl 1))))
  (rplus (deltal p i)
  (rplus (rneq (c q i (ti (addl 1))))
  (rneq (deltal q i))))))

(y (rquotient-nat
(rsum (append (firstn (map-rabs (11-term-list p q i (n))) (m))
  (restn (map-rabs (11-term-list p q i (n))) (m))))
  (n))
  (z (rquotient-nat
  (rplus
    (rtimes-nat2 (rplus (delta) (rtimes (rational 2 1) (big-delta)))) (m))
    (rtimes-nat2 (rtimes (rational 2 1)
      (rplus (epsilon)
      (rplus (rtimes (rho) (s))
      (rtimes (rational 1 2)
      (rtimes (rho) (big-delta))))))

  (difference (n) (m))))
  (enable rleq-transitive rsum-append rleq-rplus-pair 12 13 14))

(lemma culmination (rewrite)
(implies (and (SLA (addl 1))
  (nonfaulty p (addl 1))
  (nonfaulty q (addl 1))
  (in-r tm (addl 1))
  (theoreml-one-step i))
(\leq (\text{skew } p q \text{ tm } (\text{addl } i))
 \text{rplus } (\text{quotient-nat }
 \text{rplus }
 (\text{rtimes-nat }
 (m) \text{rplus } (\text{delta })
 (\text{rtimes } (\text{rational } 2 1) \text{ (big-delta)))))
 (\text{rtimes-nat }
 (\text{difference } (n) (m))
 (\text{rtimes } (\text{rational } 2 1)
 \text{rplus } (\text{epsilon })
 (\text{times } (\text{rho }) (s))
 (\text{rtimes } (\text{rational } 1 2)
 (\text{rtimes } (\text{rho } \text{ (big-delta))))))))))
 (n))
 (\text{rplus } (\text{rtimes } (\text{rho }) (r))
 (\text{rtimes } (\text{rho } \text{ (big-sigma)))))))

((\text{use } (15))
 (\text{rleq-transitive }
 (x \text{ (skew } p q \text{ tm } (\text{addl } i)))
 (y \text{rplus } (\text{tabs } \text{rplus } (c p t (t (\text{addl } i)))
 \text{rplus } (\text{delta }) p i)
 \text{rplus }
 (\text{rneg } (c q 1 (t (\text{addl } i))))
 (\text{rneg } (\text{delta } q i)))
 (\text{rplus } (\text{times } (\text{rho } \text{ (r))} (\text{rtimes } (\text{rho } \text{ (big-sigma)))))))
 (z \text{rplus } (\text{quotient-nat }
 \text{rplus } (\text{times } (\text{rho }) (s))
 \text{rtimes } (\text{rho } \text{ (big-sigma)))))))

(enable \text{rlssp-rlcq lemma6 rleq-rplus-pair rleq-reflexive rtimes-nat-rtimes-nat2)))

(\text{lemma theorem1-basis } (\text{rewrite})
 (\text{SLC } p q 0)
 ((\text{use } (\text{sublemma-a } (1 0) (\text{tm } (\text{tm } 0 p q))))
 (\text{rleq-transitive } \text{k } (\text{skew } p q \text{ tm } (\text{tm } 0 p q) 0))
 (y \text{rplus } (\text{tabs } \text{rplus } (c p t (t 0) 0) \text{ (rtimes } (\text{rho } \text{ (c)))))
 (z \text{ (delta )})
 \text{rplus } (\text{times } (\text{rho }) (s))
 \text{rtimes } (\text{rho } \text{ (big-sigma))))))

(a0) (c5))
 (enable \text{SLC-suff rlssp-rlcq rleq-transitive}))

(\text{lemma theorem1-ind-step0 } (\text{rewrite})
 (\text{implies } (\text{and } (\text{SLA } (\text{addl } i))
 \text{SLC } p q (\text{addl } i)))
 (\text{theorem1-one-step } i))
 ((\text{use } (\text{rearrange-delta } (\text{delta } (\text{delta }))))
 (\text{big-sigma } (\text{big-sigma}))
 (r (r))
 (n (n))
 (\text{big-delta } (\text{big-delta}))
 (m (m))
 (s (s))
 (\text{rho } (\text{rho } (\text{epsilon }))
 (\text{epsilon } (\text{epsilon })))
 (c6) (\text{culmination } (\text{tm } (\text{tm } (\text{addl } i) p q)))
 (enable \text{rlaq-transitive}))

(\text{lemma theorem1-ind-step } (\text{rewrite})
 (\text{implies } (\text{and } (\text{NOT } (\text{ZEROP } I))
 \text{THEOREM1-ONE-STEP } (\text{SUB1 } I))
 \text{THEOREM1-ONE-STEP } I))
((use (theorem-ind-step0 (i (subl i)) (p (p i)) (q (q i)))
 (theorem-one-step-necc (i (subl i)) (p (p i)) (q (q i)))
 (theorem-one-step-suff))
 (enable SIA-lemma)
 (disable theorem-one-step-suff)))

(lemma theorem-version1 (rewrite)
 (theorem-one-step i)
 ((induct (subl-induction i))
  (enable theorem-basis not-numberp-SiC theorem-ind-step)))

(lemma theorem (rewrite)
 (implies (SIA i)
  (SiC p q i))
 ((use (theorem-version1)
    (theoem1-one-step-necc))))
### Title and Subtitle
Verifying the Interactive Convergence Clock Synchronization Algorithm Using the Boyer-Moore Theorem Prover

### Author(s)
William D. Young

### Performing Organization Name(s) and Address(es)
Science and Technology Corporation  
101 Research Drive  
Hampton, VA 23666-1340

### Sponsoring/Monitoring Agency Name(s) and Address(es)
National Aeronautics and Space Administration  
Langley Research Center  
Hampton, VA 23665-5225

### Abstract
The application of formal methods to the analysis of computing systems promises to provide higher and higher levels of assurance as the sophistication of our tools and techniques increases. Improvements in tools and techniques come about as we pit the current state of the art against new and challenging problems. A promising area for the application of formal methods is in real-time and distributed computing. Some of the algorithms in this area are both subtle and important. In response to this challenge and as part of an ongoing attempt to verify an implementation of the Interactive Convergence Clock Synchronization Algorithm, we decided to undertake a proof of the correctness of the algorithm using the Boyer-Moore theorem prover. This paper describes our approach to proving the ICCSA using the Boyer-Moore prover.

### Subject Terms
Interactive Convergence Clock Synchronization Algorithm (ICCSA); Boyer-Moore Theorem prover; Real-time and Distributed Computing

### Distribution/Availability Statement
Unclassified Unlimited  
Subject Category 62

---

**Unclassified**