Numerical Evaluation of the Incomplete Airy Functions and Their Application to High Frequency Scattering and Diffraction

E.D. Constantinides and R.J. Marhefka

The Ohio State University

ElectroScience Laboratory

Department of Electrical Engineering
Columbus, Ohio 43212

Semi-Annual Report 725311-1
Grant No. NSG-1492
September 1992

National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665

A. Approved for public release; Distribution is unlimited
NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.
### Title and Subtitle
Numerical Evaluation of the Incomplete Airy Functions and Their Application to High Frequency Scattering and Diffraction

### Report Date
September 1992

### Author(s)
E.D. Constantinides and R.J. Marhefska

### Performing Organization Name and Address
The Ohio State University
ElectroScience Laboratory
1320 Kinnear Road
Columbus, OH 43212

### National Aeronautics and Space Administration
Langley Research Center
Hampton, VA 23665

### Abstract (Limit: 200 words)
The incomplete Airy integrals serve as canonical functions for the uniform ray optical solutions to several high-frequency scattering and diffraction problems that involve a class of integrals characterized by two stationary points that are arbitrarily close to one another or to an integration endpoint. Integrals of such analytical properties describe transition region phenomena associated with composite shadow boundaries. An efficient and accurate method for computing the incomplete Airy functions would make the solutions to such problems useful for engineering purposes. In this report, a convergent series solution form for the incomplete Airy functions is derived. Asymptotic expansions involving several terms are also developed and serve as large argument approximations. The combination of the series solution form with the asymptotic formulae provides for an efficient and accurate computation of the incomplete Airy functions. Validation of accuracy is accomplished using direct numerical integration data.

### Document Analysis a. Descriptors
- ASYMPTOTIC
- DIFFRACTION
- NUMERICAL
- SCATTERING
- UTD

b. Identifiers/Open-Ended Terms

### COSATI Field/Group

### Availability Statement
A. Approved for public release; Distribution is unlimited.
# Contents

List of Figures iv

1 Introduction 1

2 Derivation of the Series Solution Form 6

3 Derivation of Asymptotic Formulae 12
   3.1 Asymptotic Formula for $\xi \gg |\beta|^\frac{1}{2}$ 12
   3.2 Asymptotic Formula for $\beta \ll -1$ 13

4 Numerical Results and Discussion 22

5 Summary and Conclusions 30

APPENDICES

A Uniform Asymptotic Evaluation of Certain Stationary Phase Integrals 31

B Computation of $a_0(\xi), b_1(\xi)$ and their Derivatives 34

C Computation of the Fresnel Transition Function 36

D Code Listing 37

Bibliography 51
List of Figures

1.1 Scattering and diffraction from a concave boundary containing an edge. 2
1.2 Scattering and diffraction from a concave-convex boundary containing an edge. ........................................... 3

2.1 Contours of integration for the incomplete Airy functions. ......... 7
2.2 Contours of integration for the complete Airy functions. ................. 9

3.1 Contours of integration for the incomplete Airy integral, \( \gamma \gg (-\sigma)^{1/2} \). 14
3.2 Contour of integration for the incomplete Airy integral, \( \sigma \ll -1 \) and \( \gamma \approx (-\sigma)^{1/2} \). ........................................... 17

4.1 Three different sets of formulae are used for the computation of the incomplete Airy function, one for each region in the figure, and a fourth set that is used in the immediate vicinity of the caustic. .... 23
4.2 Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions I and III. Results are plotted vs. the parameter \( \beta \) with \( \xi = (12 - 2\beta)^{1/2} \). ........................................... 24
4.3 Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions II and III. Results are plotted vs. the parameter \( \xi \) with \( \beta = -4.0 \) ........................................... 25
4.4 Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter \( \beta \) with \( \xi = -3.0 \). The marks on the contours represent direct numerical integration data. ........................................... 26
4.5 Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter \( \beta \) with \( \xi = 2.0 \). The marks on the contours represent direct numerical integration data. ........................................... 27
4.6 Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter \( \xi \) with \( \beta = -5.0 \). The marks on the contours represent direct numerical integration data. ........................................... 28
4.7 Plots of the real part (solid line) and imaginary part (broken line) of
the incomplete Airy function and its derivative vs. the parameter $\xi$
with $\beta = 0.0$. The marks on the contours represent direct numerical
integration data.
Chapter 1

Introduction

A common method of analysis for high-frequency electromagnetic and acoustic scattering and diffraction problems involves the use of radiation integrals as well as plane wave integral representations for the fields, with the asymptotic approximations to the various scattering mechanisms found from the critical point contributions of the integrand. The incomplete Airy integrals [1] serve as canonical functions for the uniform asymptotic approximation of a class of integrals characterized by two stationary phase points that are arbitrarily close to one another or to an integration endpoint [2]. Integrals of such analytical properties describe transition region phenomena associated with composite shadow boundaries resulting from the confluence of two stationary points and an endpoint in the integration interval. A typical problem of particular interest, where transition region phenomena of this type exist, involves the scattering from smoothly indented boundaries containing an edge as illustrated in Figures 1.1 and 1.2. When the reflection shadow boundary (RSB) is not in the immediate vicinity of the smooth caustic, the conventional UTD\textsuperscript{1} edge diffraction coefficient [3] which involves the Fresnel integral as a canonical function can be used to effectively describe the edge diffracted field behavior in the neighborhood of the reflection shadow boundary. Furthermore, the ordinary Airy integrals and their derivatives are the appropriate canonical functions for the description of

\textsuperscript{1}Uniform Geometrical Theory of Diffraction
Figure 1.1: Scattering and diffraction from a concave boundary containing an edge.

the high-frequency fields in the neighborhood of the smooth caustic [4]. However, when there is a confluence of both reflected and caustic type shadow boundaries as shown in Figures 1.1 and 1.2, neither the Fresnel integral nor the ordinary Airy integral adequately describe the transition region phenomena, and they must be appropriately replaced by the incomplete Airy function.

The formulation of uniform ray optical solutions for problems involving composite shadow boundary transition phenomena, that are useful for engineering purposes, requires an efficient and accurate method for computing the incomplete Airy functions. In the original work of Levey and Felsen [1], several other diffraction problems whose solutions involve the incomplete Airy functions are also described, and some general and asymptotic characteristics of these functions are examined. However, the asymptotic formulae given in [1] are valid when the saddle points are sufficiently
Figure 1.2: Scattering and diffraction from a concave-convex boundary containing an edge.
far apart or far removed from the integration endpoint. When the two saddle points move close together or coalesce to form a caustic, the asymptotic formulae break down and an alternative computation method for the incomplete Airy functions is needed.

In this report, a convergent series solution for the incomplete Airy functions is rigorously derived, and asymptotic expansions involving several terms are also developed for large argument approximations. The higher order terms in the asymptotic expansions greatly improve the accuracy of the asymptotic formulae, and also improve computational efficiency by limiting the region of the argument space where the more time consuming series solution form should be used. Thus, the combination of the series solution form with the more accurate asymptotic formulae provides for an efficient and accurate computation of the incomplete Airy functions for the entire range of their arguments. These results would allow the formulation of uniform ray optical solutions for high-frequency scattering and diffraction problems that are useful for engineering purposes. A method for uniformly evaluating certain stationary phase integrals using the incomplete Airy integral as a canonical function is briefly discussed in Appendix A. The details of a systematic uniform asymptotic analysis for particular applications that involve transition region phenomena describable by the incomplete Airy functions will be reported separately.

The outline of this report is as follows: In Chapter 2, some general properties of the incomplete Airy functions are reviewed, and a convergent series solution form is rigorously derived using the parabolic differential equation. The asymptotic formulae for large argument approximations are derived in Chapter 3 using the integral forms of the incomplete Airy functions. In Chapter 4, some indicative numerical results are presented and discussed, with their accuracy demonstrated via comparison with data obtained from direct numerical integration of the integral forms. The regions of validity for each formula used in the computations are also shown in this chapter,
and some error assessments for the asymptotic results are provided. Finally, the main results and accomplishments of this work are summarized in Chapter 5.
Chapter 2

Derivation of the Series Solution Form

In this chapter, a convergent series solution form for the incomplete Airy functions is derived. Before proceeding with the derivation, however, some of their general properties are briefly reviewed.

The incomplete Airy functions are functions of two variables and they satisfy the parabolic partial differential equation applied by Fock [5] to the study of fields near the surface of a smooth convex scattering surface; i.e.,

\[
\frac{\partial^2}{\partial \beta^2} - \beta - j \frac{\partial}{\partial \xi} g_i(\beta, \xi) = 0 \quad ; \quad i = 0, 1, 2.
\]

(2.1)

In integral form, the solutions of (2.1) are given by:

\[
g_i(\beta, \xi) \equiv \int_{\gamma_i}^{\infty} e^{i(\beta z + z^3/3)} dz \quad ; \quad i = 0, 1, 2
\]

(2.2)

where the upper limit lies within one of the three sectors of the complex \(z\)-plane in which the integral converges; i.e.,

\[
2i \frac{\pi}{3} \leq \psi_i \leq (2i + 1) \frac{\pi}{3} \quad ; \quad i = 0, 1, 2.
\]

(2.3)

The contours of integration for the incomplete Airy functions are shown in Figure 2.1.

In this report, we only examine \(g_0(\beta, \xi)\) since the other two functions, namely \(g_1\) and \(g_2\), can be obtained from \(g_0\) and the well known complete Airy functions [6];
Figure 2.1: Contours of integration for the incomplete Airy functions.
i.e.,
\[ g_1(\beta, \xi) = g_0(\beta, \xi) - 2\pi \text{Ai}(\beta), \]  
(2.4)

and
\[ g_2(\beta, \xi) = g_0(\beta, \xi) - \pi [\text{Ai}(\beta) + j \text{Bi}(\beta)], \]  
(2.5)

where
\[ \text{Ai}(\beta) = \frac{1}{2\pi} \int_{L_1} e^{j(\beta z + z^3/3)} dz, \]  
(2.6)

and
\[ \text{Bi}(\beta) = \frac{j}{2\pi} \int_{L_2+L_3} e^{j(\beta z + z^3/3)} dz. \]  
(2.7)

The contours of integration for the complete Airy functions are shown in Figure 2.2.

Also, the quantities \( \beta \) and \( \xi \) will be taken as real since in most practical applications real \( \beta \) and \( \xi \) are of primary interest. However, this is not a requirement for the analysis that follows and the resulting formulae are valid for arbitrary values of \( \beta \) and \( \xi \). In addition, \( \xi \) will be restricted to positive values since for negative values of \( \xi \), \( g_0 \) may be obtained using the expression:
\[ g_0(\beta, -|\xi|) = 2\pi \text{Ai}(\beta) - g_0^*(\beta, |\xi|), \]  
(2.8)

with \( \ast \) denoting the complex conjugate operation.

In order to obtain a series solution form for \( g_0(\beta, \xi) \), we begin with the parabolic differential equation and assume two independent solutions of the form:

\[ y_1(\beta, \xi) = \sum_{n=0}^{\infty} a_n(\xi)\beta^n, \]  
(2.9)

and
\[ y_2(\beta, \xi) = \sum_{n=0}^{\infty} b_n(\xi)\beta^n. \]  
(2.10)

Substituting (2.9) and (2.10) in (2.1) we obtain the following expressions:
\[ \sum_{n=2}^{\infty} n(n-1)a_n(\xi)\beta^{n-2} - \sum_{n=0}^{\infty} a_n(\xi)\beta^{n+1} - j \sum_{n=0}^{\infty} a'_n(\xi)\beta^n \equiv 0, \]  
(2.11)

and
\[ \sum_{n=2}^{\infty} n(n-1)b_n(\xi)\beta^{n-2} - \sum_{n=0}^{\infty} b_n(\xi)\beta^{n+1} - j \sum_{n=0}^{\infty} b'_n(\xi)\beta^n \equiv 0. \]  
(2.12)
Figure 2.2: Contours of integration for the complete Airy functions.
For Equations (2.11) and (2.12) to be satisfied, the sum of coefficients of like powers of $\beta$ must be zero for any value of $\xi$. To obtain the first independent solution, $y_1$, we let $a_1(\xi) \equiv 0$, and setting the sum of coefficients of like powers in (2.11) equal to zero we obtain:

\[ a_1(\xi) = 0, \ \forall \xi, \]  
(2.13)

\[ a_2(\xi) = \frac{j}{2} a_0'(\xi), \]  
(2.14)

and

\[ a_n(\xi) = \frac{a_{n-3}(\xi) + j a'_{n-2}(\xi)}{n(n-1)}, \quad n \geq 3. \]  
(2.15)

Similarly, for the second independent solution, $y_2$, we let $b_0(\xi) \equiv 0$ and setting the sum of coefficients of like powers in (2.12) equal to zero we obtain:

\[ b_0(\xi) = 0, \ \forall \xi, \]  
(2.16)

\[ b_2(\xi) = 0, \ \forall \xi, \]  
(2.17)

and

\[ b_n(\xi) = \frac{b_{n-3}(\xi) + j b'_{n-2}(\xi)}{n(n-1)}, \quad n \geq 3. \]  
(2.18)

Thus, $y_1$ and $y_2$ are given by:

\[ y_1(\beta, \xi) = a_0(\xi) + \sum_{n=2}^{\infty} a_n(\xi)\beta^n, \]  
(2.19)

\[ y_2(\beta, \xi) = b_1(\xi)\beta + \sum_{n=2}^{\infty} b_n(\xi)\beta^n, \]  
(2.20)

where $a_n(\xi)$ and $b_n(\xi)$ for $n \geq 2$ can be expressed in terms of $a_0(\xi), b_1(\xi)$ and their derivatives, respectively, via Equations (2.13)–(2.18). Now, $g_0(\beta, \xi)$ must be equal to the sum of the two independent solutions; i.e.,

\[ g_0(\beta, \xi) = a_0(\xi) + b_1(\xi)\beta + \sum_{n=2}^{\infty} [a_n(\xi) + b_n(\xi)]\beta^n, \]  
(2.21)

and

\[ \frac{\partial}{\partial \beta} g_0(\beta, \xi) = b_1(\xi) + \sum_{n=2}^{\infty} n [a_n(\xi) + b_n(\xi)] \beta^{n-1}. \]  
(2.22)

Although the coefficients $a_n(\xi)$ and $b_n(\xi)$ can be combined into a single coefficient, keeping them separate greatly simplifies their evaluation in a computer code.
Finally, it remains to find expressions for \( a_0(\xi) \) and \( b_1(\xi) \) in order to complete the solution. This can be accomplished by applying the proper boundary conditions at \( \beta = 0 \) using the integral form of \( g_0(\beta, \xi) \) given in Equation (2.2); i.e.,

\[
a_0(\xi) = g_0(\beta, \xi) \bigg|_{\beta = 0} = \int_{\xi}^{\infty} e^{jz^3/3} \, dz,
\]

and

\[
b_1(\xi) = \frac{\partial}{\partial \beta} g_0(\beta, \xi) \bigg|_{\beta = 0} = \int_{\xi}^{\infty} jze^{jz^3/3} \, dz.
\]

The functions \( a_0(\xi) \) and \( b_1(\xi) \) can be expressed in terms of the incomplete Gamma function \([7]\), and their computation is straightforward. Details are provided in Appendix B.

When \( \xi \to -\infty \), our solution should reduce to the series solution form for the complete Airy function, \( \text{Ai}(\beta) \) \([6]\). In this case we have:

\[
\begin{align*}
a_0(\xi \to -\infty) &= 2\pi \text{Ai}(0) = 2.23070703, \\
b_1(\xi \to -\infty) &= 2\pi \text{Ai}'(0) = -1.62621025,
\end{align*}
\]

and using Equations (2.13)-(2.20) we obtain:

\[
g_0(\beta, \xi \to -\infty) = 2\pi \text{Ai}(0) \left( 1 + \frac{1}{3!} \beta^3 + \frac{1 \cdot 4}{6!} \beta^6 + \frac{1 \cdot 4 \cdot 7}{9!} \beta^9 + \cdots \right) + 2\pi \text{Ai}'(0) \left( \beta + \frac{2}{4!} \beta^4 + \frac{2 \cdot 5}{7!} \beta^7 + \frac{2 \cdot 5 \cdot 8}{10!} \beta^{10} + \cdots \right) = 2\pi \text{Ai}(\beta),
\]

which is a necessary condition for the validity of our result.
Chapter 3

Derivation of Asymptotic Formulae

In this chapter, a pair of asymptotic formulae for the incomplete Airy function, \( g_0(\beta, \xi) \), involving several terms are derived and serve as large argument forms. We begin by introducing the large parameter \( \Omega \) in the integral form of \( g_0(\beta, \xi) \), and examine the integral:

\[
I_0(\sigma, \gamma; \Omega) = \int_{\gamma}^{\infty} e^{i\Omega(\sigma z + z^{2/3})} \, dz.
\] (3.1)

Our objective is to obtain asymptotic expansions for \( I_0 \) as \( \Omega \to \infty \) for various dispositions of the saddle points and endpoint. Then, the asymptotic formulae for \( g_0(\beta, \xi) \) may be obtained using the expression:

\[
g_0(\beta, \xi) = \Omega^{1/3} I_0(\sigma = \beta \Omega^{-2/3}, \gamma = \xi \Omega^{-1/3}; \Omega). \] (3.2)

Since we are interested in real \( \beta \) and \( \xi \) with \( \xi \geq 0 \), the analysis that follows will be restricted to real \( \sigma \) and \( \gamma \), with \( \gamma \geq 0 \).

3.1 Asymptotic Formula for \( \xi \gg |\beta|^{1/2} \)

This case corresponds to the endpoint being far removed from the possibly neighboring saddle points of the integrand in Equation (3.1), or \( \gamma \gg |\sigma|^{1/2} \). Although the sign of \( \sigma \) is irrelevant in this case, the saddle points \( z_{1,2} = \pm (-\sigma)^{1/2} \) are taken as real for simplicity. Also, the original integration path \( P_0 \) in Figure 3.1 is deformed
into the steepest descent path, $P_{st}$, leading away from the endpoint at $z = \gamma$ and into the sector $0 \leq \arg(z) \leq \pi/3$ of the complex $z$-plane. The asymptotic evaluation of $I_0$ is then performed using repeated integration by parts that yields:

$$I_0(\sigma, \gamma; \Omega) \sim e^{i\Omega(\sigma + \gamma^{3/2})/3} \left\{ \frac{1}{\Omega} \left[ \frac{j}{\sigma + \gamma^2} \right] + \frac{1}{\Omega^2} \left[ \frac{2\gamma}{(\sigma + \gamma^2)^3} \right] ight. + \frac{1}{\Omega^3} \left[ \frac{2j}{(\sigma + \gamma^2)^4} - \frac{12j\gamma^2}{(\sigma + \gamma^2)^5} \right] + \frac{1}{\Omega^4} \left[ \frac{40\gamma}{(\sigma + \gamma^2)^6} - \frac{120\gamma^3}{(\sigma + \gamma^2)^7} \right] + \frac{1}{\Omega^5} \left[ \frac{-40j}{(\sigma + \gamma^2)^7} - \frac{840j\gamma^2}{(\sigma + \gamma^2)^8} + \frac{1680j\gamma^4}{(\sigma + \gamma^2)^9} \right] \right\} + O(\Omega^{-6}). \quad (3.3)$$

Now, using Equations (3.2) and (3.3), the asymptotic formula for $g_0(\beta, \xi)$ when $\xi \gg |\beta|^{1/2}$ is given by:

$$g_0(\beta, \xi) \approx e^{i(\beta \xi + \xi^2/3)} \left[ \frac{j}{\beta + \xi^2} + \frac{2\xi}{(\beta + \xi^2)^3} + \frac{2j}{(\beta + \xi^2)^4} - \frac{12j}{(\beta + \xi^2)^5} \right. \left. + \frac{40\xi}{(\beta + \xi^2)^6} - \frac{120\xi^3 - 40j}{(\beta + \xi^2)^7} - \frac{840j\xi^2}{(\beta + \xi^2)^8} + \frac{1680j\xi^4}{(\beta + \xi^2)^9} \right]. \quad (3.4)$$

The derivative of $g_0(\beta, \xi)$ is given by:

$$\frac{\partial}{\partial \beta} g_0(\beta, \xi) \approx e^{i(\beta \xi + \xi^2/3)} \left[ \frac{-\xi}{\beta + \xi^2} - \frac{j}{(\beta + \xi^2)^2} + \frac{2j\xi^2}{(\beta + \xi^2)^3} - \frac{8\xi}{(\beta + \xi^2)^4} \right. \left. - \frac{8j - 12\xi}{(\beta + \xi^2)^5} + \frac{60j + 40j\xi^2}{(\beta + \xi^2)^6} - \frac{280\xi + 120j\xi^3}{(\beta + \xi^2)^7} \right. \left. - \frac{1680\xi^3 - 280j}{(\beta + \xi^2)^8} + \frac{6720j\xi^2 - 1680\xi^4}{(\beta + \xi^2)^9} - \frac{15120j\xi^4}{(\beta + \xi^2)^10} \right]. \quad (3.5)$$

### 3.2 Asymptotic Formula for $\beta \ll -1$

This case corresponds to real and widely separated saddle points ($\sigma \ll -1$) in the integrand of Equation (3.1), with the endpoint $\gamma$ arbitrarily close to the saddle point $z_1 = (-\sigma)^{1/2}$, as shown in Figure 3.2. For an asymptotic evaluation of Equation (3.1) that holds uniformly as the endpoint $\gamma$ approaches the saddle point $z_1$, we make the following transformation [1]:

$$q(z) = \sigma z + z^3/3 = q(z_1) + s^2 = -\frac{2}{3}(-\sigma)^{3/2} + s^2 = \tau(s), \quad (3.6)$$
Figure 3.1: Contours of integration for the incomplete Airy integral, $\gamma \gg (-\sigma)^{1/2}$. 
with \( \arg(s) \) restricted so that \( I_0(\sigma, \gamma; \Omega) \) converges as \( s \to \infty \). Hence, employing (3.6) in Equation (3.1) we have:

\[
I_0(\sigma, \gamma; \Omega) = e^{-j\frac{\pi}{2}(-\sigma)^{3/2}} \int_\zeta^\infty f(s) e^{j\Omega s^2} ds,
\]

(3.7)

with the upper limit taken in the sector \( 0 \leq \arg(s) \leq \pi/2 \) of the complex \( s \)-plane. The quantities \( \zeta \) and \( f(s) \) are given by:

\[
\zeta = \pm \left[ \sigma \gamma + \gamma^3/3 + \frac{2}{3}(-\sigma)^{3/2} \right]^{1/2} ; \quad \gamma \zeta (-\sigma)^{1/2},
\]

(3.8)

and

\[
f(s) = \frac{dz}{ds} = \frac{\tau'(s)}{q'(z)} = \frac{2s}{\sigma + z^2}.
\]

(3.9)

Equation (3.7) can be written as follows [8]:

\[
I_0(\sigma, \gamma; \Omega) = e^{-j\frac{\pi}{2}(-\sigma)^{3/2}} \left\{ f(0) \int_\zeta^\infty e^{j\Omega s^2} ds + \frac{1}{2j\Omega} \int_\zeta^\infty \left[ \frac{f(s) - f(0)}{s} \right] \frac{d}{ds} e^{-j\Omega s^2} ds \right\},
\]

(3.10)

and using integration by parts in the second integral we get:

\[
I_0(\sigma, \gamma; \Omega) = e^{-j\frac{\pi}{2}(-\sigma)^{3/2}} \left\{ f(0) \int_\zeta^\infty e^{j\Omega s^2} ds - \frac{1}{2j\Omega} \left[ \frac{f(\zeta) - f(0)}{\zeta} \right] e^{j\Omega \zeta^2} - \frac{1}{2j\Omega} \int_\zeta^\infty g(s) e^{j\Omega s^2} ds \right\},
\]

(3.11)

where

\[
g(s) = \frac{sf'(s) - f(s) + f(0)}{s^2}.
\]

(3.12)

In a similar way, the second integral in Equation (3.11) can be written as follows:

\[
\int_\zeta^\infty g(s) e^{j\Omega s^2} ds = g(0) \int_\zeta^\infty e^{j\Omega s^2} ds - \frac{1}{2j\Omega} \left[ \frac{g(\zeta) - g(0)}{\zeta} \right] e^{j\Omega \zeta^2} - \frac{1}{2j\Omega} \int_\zeta^\infty h(s) e^{j\Omega s^2} ds,
\]

(3.13)

where

\[
h(s) = \frac{sg'(s) - g(s) + g(0)}{s^2}.
\]

(3.14)
Thus, using (3.13), Equation (3.11) becomes:

\[
I_0(\sigma, \gamma; \Omega) = e^{-j\frac{3}{2}\Omega(-\sigma)^{3/2}} \left[ f(0) - \frac{1}{2j \Omega} g(0) \right] \int_{-\infty}^{\infty} e^{jns^2} ds \\
+ e^{j\Omega(-\sigma+\gamma^2/3)} \left\{ -\frac{1}{2j \Omega} \left[ f(\zeta) - f(0) \right] + \frac{1}{(2j \Omega)^2} \left[ g(\zeta) - g(0) \right] \right\} \\
+ e^{-j\frac{3}{2}\Omega(-\sigma)^{3/2}} \left[ \frac{1}{(2j \Omega)^3} \int_{-\infty}^{\infty} h(s)e^{jns^2} ds \right].
\]

(3.15)

The same procedure is repeated once more for the last integral in Equation (3.15); i.e.,

\[
\int_{-\infty}^{\infty} h(s)e^{jns^2} ds \sim \left[ h(0) - \frac{1}{2j \Omega} k(0) \right] \int_{-\infty}^{\infty} e^{jns^2} ds \\
+ e^{j\Omega^2} \left\{ -\frac{1}{2j \Omega} \left[ h(\zeta) - h(0) \right] + \frac{1}{(2j \Omega)^2} \left[ k(\zeta) - k(0) \right] \right\} + O(\Omega^{-3}),
\]

(3.16)

where

\[
k(\sigma) = \frac{sh'(s) - h(s) + h(0)}{s^2},
\]

(3.17)

and finally combining Equations (3.15) and (3.16), the asymptotic expansion of \( I_0(\sigma, \gamma; \Omega) \) when \( \sigma \ll -1 \) is given by:

\[
I_0(\sigma, \gamma; \Omega) \sim e^{-j\frac{3}{2}(-\sigma)^{3/2}} \left[ f(0) - \frac{1}{2j \Omega} g(0) + \frac{1}{(2j \Omega)^2} h(0) - \frac{1}{(2j \Omega)^3} k(0) \right] \int_{-\infty}^{\infty} e^{jns^2} ds \\
+ e^{j\Omega(\sigma+\gamma^2/3)} \left\{ -\frac{1}{2j \Omega} \left[ f(\zeta) - f(0) \right] + \frac{1}{(2j \Omega)^2} \left[ g(\zeta) - g(0) \right] \\
- \frac{1}{(2j \Omega)^3} \left[ h(\zeta) - h(0) \right] + \frac{1}{(2j \Omega)^3} \left[ k(\zeta) - k(0) \right] \right\} + O(\Omega^{-5}).
\]

(3.18)

In order to express the functions \( f, g, h, \) and \( k \) in Equation (3.18) in terms of \( \sigma \) and \( \gamma \), we need to derive an expression for \( f(s) \) and its derivatives when \( s \approx 0 \). This is done using a procedure introduced by Erdélyi [9] that yields:

\[
f(s) = \frac{1}{(-\sigma)^{1/4}} \sum_{n=0}^{\infty} \frac{\Gamma(3n/2 - 1)(-1)^{n-1}}{(n - 1)! \Gamma(n/2)[3(-\sigma)^{3/4}]^{n-1}} s^{n-1},
\]

(3.19)

and

\[
\frac{d^k}{ds^k} f(s) = \frac{1}{(-\sigma)^{1/4}} \sum_{n=k+1}^{\infty} \frac{\Gamma(3n/2 - 1)(-1)^{n-1}}{(n - k - 1)! \Gamma(n/2)[3(-\sigma)^{3/4}]^{n-1}} s^{n-k-1}
\]

(3.20)
Figure 3.2: Contour of integration for the incomplete Airy integral, \( \sigma \ll -1 \) and \( \gamma \approx (-\sigma)^{1/2} \).
Hence, using Equations (3.9), (3.12), (3.14), (3.17), (3.19), and (3.20) we have:

\[ f(\zeta) = \frac{2\zeta}{\sigma + \gamma^2}, \quad (3.21) \]

\[ g(\zeta) = \frac{-8\gamma\zeta}{(\sigma + \gamma^2)^3} + \frac{1}{\zeta^2(-\sigma)^{1/4}}, \quad (3.22) \]

\[ h(\zeta) = \frac{-16\zeta}{(\sigma + \gamma^2)^4} + \frac{96\gamma^2}{(\sigma + \gamma^2)^5} - \frac{3}{\zeta^4(-\sigma)^{1/4}} + \frac{5}{24\zeta^2(-\sigma)^{7/4}}, \quad (3.23) \]

\[ k(\zeta) = \frac{640\gamma}{(\sigma + \gamma^2)^6} - \frac{1920\gamma^3}{(\sigma + \gamma^2)^7} + \frac{15}{\zeta^6(-\sigma)^{1/4}} - \frac{5}{8\zeta^4(-\sigma)^{7/4}} + \frac{77}{3456\zeta^3(-\sigma)^{13/4}}, \quad (3.24) \]

\[ f(0) = \frac{1}{(-\sigma)^{1/4}}, \quad (3.25) \]

\[ f'(0) = \frac{-1}{3(-\sigma)^{1/2}}, \quad (3.26) \]

\[ g(0) = \frac{f''(0)}{2} = \frac{5}{24(-\sigma)^{7/4}}, \quad (3.27) \]

\[ g'(0) = \frac{f'''(0)}{3} = \frac{-8}{27(-\sigma)^{5/2}}, \quad (3.28) \]

\[ h(0) = \frac{f^{(4)}(0)}{8} = \frac{77}{3456(-\sigma)^{13/4}}, \quad (3.29) \]

\[ h'(0) = \frac{f^{(5)}(0)}{15} = \frac{-56}{81(-\sigma)^4}, \quad (3.30) \]

\[ k(0) = \frac{f^{(6)}(0)}{48} = \frac{12155}{82944(-\sigma)^{19/4}}, \quad \text{and} \]

\[ k'(0) = \frac{f^{(7)}(0)}{105} = \frac{-640}{243(-\sigma)^{11/2}}. \quad (3.31) \]

Also, the Fresnel integral in Equation (3.18) may be expressed in terms of the Fresnel transition function \( F(x) \) [3]; i.e.,

\[ \int_0^\infty e^{j\pi s^2} ds = \Omega^{1/2} \left[ \sqrt{\pi} e^{j\pi/4} U(-\eta) - \frac{F^*(\eta^2)}{2j\eta} e^{j\eta} \right] = \Omega^{1/2} A(\eta), \quad (3.33) \]

where \( U(x) \) is the Heaveside unit step function, and \( \eta = \Omega^{1/2}\zeta \). The general properties of the Fresnel transition function and details on its computation are provided in Appendix C. Now, using Equations (3.21)-(3.33), the asymptotic expansion of
\[ I_0(\sigma, \gamma; \Omega) \text{ when } \sigma \ll -1 \text{ is given by:} \]
\[ I_0(\sigma, \gamma; \Omega) \sim \frac{e^{-j\frac{3}{4}(\sigma^{3/2})}}{\Omega^{1/3}(-\sigma)^{1/4}} S(\sigma; \Omega) \Lambda(\eta) + e^{i\Omega(\sigma-\gamma^{3/3})} E(\sigma, \gamma, \eta; \Omega) + O(\Omega^{-5}), \quad (3.34) \]
\[ \text{where} \]
\[ S(\sigma; \Omega) = 1 - \frac{s_1}{2j\Omega(-\sigma)^{3/2}} + \frac{s_2}{(2j\Omega)^2(-\sigma)^3} - \frac{s_3}{(2j\Omega)^3(-\sigma)^{9/2}}, \quad (3.35) \]
\[ E(\sigma, \gamma, \eta; \Omega) = -\frac{1}{2j\Omega} \left[ \frac{2}{\sigma + \gamma^2} - \frac{\Omega^{1/2}}{\eta(-\sigma)^{1/4}} \right] + \frac{1}{(2j\Omega)^2} \left[ \frac{-8\gamma}{(\sigma + \gamma^2)^3} + \frac{\Omega^{3/2}}{\eta^3(-\sigma)^{1/4}} - \frac{\Omega^{1/2}s_1}{\eta(-\sigma)^{7/4}} \right] - \frac{1}{(2j\Omega)^3} \left[ \frac{-16}{(\sigma + \gamma^2)^4} + \frac{96\gamma^2}{(\sigma + \gamma^2)^5} - \frac{3\Omega^{5/2}}{\eta^5(-\sigma)^{1/4}} + \frac{\Omega^{3/2}s_1}{\eta^3(-\sigma)^{7/4}} - \frac{\Omega^{1/2}s_2}{\eta(-\sigma)^{13/4}} \right] + \frac{1}{(2j\Omega)^4} \left[ \frac{640\gamma}{(\sigma + \gamma^2)^6} - \frac{1920\gamma^3}{(\sigma + \gamma^2)^7} + \frac{15\Omega^{7/2}}{\eta^7(-\sigma)^{1/4}} - \frac{3\Omega^{5/2}s_1}{\eta^5(-\sigma)^{7/4}} + \frac{\Omega^{3/2}s_2}{\eta^3(-\sigma)^{13/4}} - \frac{\Omega^{1/2}s_3}{\eta(-\sigma)^{9/4}} \right], \quad (3.36) \]
\[ s_1 = 0.20833333, \quad s_2 = 0.33420139, \quad \text{and} \quad s_3 = 1.02581260. \]

The asymptotic formulae for \( g_0(\beta, \xi) \) and its derivative when \( \beta \ll -1 \) are then obtained using Equations (3.2) and (3.34)-(3.36); i.e.,
\[ g_0(\beta, \xi) \approx \frac{e^{-j\frac{3}{2}(\beta)^{3/2}}}{(-\beta)^{1/4}} S(\beta) \Lambda(\eta) + e^{i(\beta \xi + \xi^3/3)} E(\beta, \xi, \eta), \quad (3.37) \]
\[ \frac{\partial}{\partial \beta} g_0(\beta, \xi) \approx \frac{e^{-j\frac{3}{2}(\beta)^{3/2}}}{(-\beta)^{1/4}} \left\{ \left[ j(-\beta)^{1/2} + \frac{1}{4(-\beta)} \right] S(\beta) \Lambda(\eta) + \Lambda(\eta) \frac{d}{d\beta} S(\beta) + S(\beta) \frac{d}{d\beta} \Lambda(\eta) \right\} + e^{i(\beta \xi + \xi^3/3)} \left[ j\xi E(\beta, \xi, \eta) + \frac{\partial}{\partial \beta} E(\beta, \xi, \eta) \right], \quad (3.38) \]
\[ \text{where} \]
\[ S(\beta) = 1 - \frac{s_1}{2j(-\beta)^{3/2}} + \frac{s_2}{(2j)^2(-\beta)^3} - \frac{s_3}{(2j)^3(-\beta)^{9/2}}, \quad (3.39) \]
\[ \frac{d}{d\beta} S(\beta) = -\frac{1.5s_1}{2j(-\beta)^{5/2}} + \frac{3s_2}{(2j)^2(-\beta)^4} - \frac{4.5s_3}{(2j)^3(-\beta)^{11/2}}, \quad (3.40) \]
\[ E(\beta, \xi, \eta) = -\frac{1}{2j} \left[ \frac{2}{\beta + \xi^2} - \frac{1}{\eta(\beta)^{1/4}} \right] + \frac{1}{(2j)^2} \left[ \frac{-8\xi}{(\beta + \xi^2)^5} + \frac{1}{\eta^3(\beta)^{1/4}} - \frac{s_1}{\eta(\beta)^{1/4}} \right] \]
\[ - \frac{1}{(2j)^3} \left[ \frac{-16}{(\beta + \xi^2)^4} + \frac{96\xi^2}{\eta^5(\beta)^{1/4}} - \frac{3}{\eta^3(\beta)^{7/4}} + \frac{s_1}{\eta^3(\beta)^{7/4}} - \frac{s_2}{\eta^3(\beta)^{13/4}} \right] + \frac{1}{(2j)^4} \left[ \frac{640\xi}{(\beta + \xi^2)^6} - \frac{1920\xi^3}{\eta^7(\beta)^{1/4}} + \frac{15}{\eta^5(\beta)^{1/4}} - \frac{3s_1}{\eta^5(\beta)^{7/4}} + \frac{s_2}{\eta^5(\beta)^{1/4}} \right] \]
\[ \frac{s_3}{\eta(\beta)^{19/4}} \right], \quad (3.41) \]

\[ \frac{\partial}{\partial \beta} E(\beta, \xi, \eta) = -\frac{1}{2j} \left[ \frac{-2}{(\beta + \xi^2)^2} - \frac{1}{4\eta(\beta)^{5/4}} + \frac{\xi - (\beta)^{1/2}}{2\eta^3(\beta)^{1/4}} \right] + \frac{1}{(2j)^2} \left\{ \frac{24\xi}{(\beta + \xi^2)^5} + \frac{1}{4\eta^3(\beta)^{7/4}} - \frac{3[\xi - (\beta)^{1/2}]}{2\eta^5(\beta)^{1/4}} - \frac{7s_1}{4\eta(\beta)^{11/4}} \right\} + \frac{s_1[\xi - (\beta)^{1/2}]}{2\eta^7(\beta)^{1/4}} + \frac{1}{(2j)^3} \left\{ \frac{64}{(\beta + \xi^2)^6} - \frac{480\xi^2}{(\beta + \xi^2)^6} - \frac{3}{4\eta^3(\beta)^{5/4}} - \frac{13s_2}{4\eta(\beta)^{7/4}} \right\} + \frac{s_2[\xi - (\beta)^{1/2}]}{2\eta^9(\beta)^{1/4}} + \frac{1}{(2j)^4} \left\{ \frac{-3840\xi}{(\beta + \xi^2)^7} + \frac{13440\xi^3}{(\beta + \xi^2)^8} + \frac{15}{4\eta^7(\beta)^{5/4}} - \frac{13s_2}{4\eta^9(\beta)^{1/4}} \right\} - \frac{105[\xi - (\beta)^{1/2}]}{2\eta^9(\beta)^{1/4}} + \frac{21s_1}{4\eta^5(\beta)^{1/4}} + \frac{15s_1[\xi - (\beta)^{1/2}]}{2\eta^7(\beta)^{7/4}} + \frac{13s_2}{4\eta^9(\beta)^{1/4}} - \frac{3s_2[\xi - (\beta)^{1/2}]}{2\eta^5(\beta)^{13/4}} - \frac{19s_3}{4\eta^9(\beta)^{23/4}} + \frac{s_3[\xi - (\beta)^{1/2}]}{2\eta^5(\beta)^{19/4}} \right\}, \quad (3.42) \]

\[ \frac{d}{d\beta} \Lambda(\eta) = -\frac{\partial \eta}{\partial \beta} e^{in^2} = -\left[ \frac{\xi - (\beta)^{1/2}}{2\eta} \right] e^{in^2}, \quad (3.43) \]

and

\[ \eta = \pm \left[ \beta\xi + \xi^3/3 + \frac{2}{3}(\beta)^{3/2} \right]^{1/2} \quad ; \quad \xi^2 \sim (\beta)^{1/2}. \quad (3.44) \]

It can be easily shown using the large argument form of the Fresnel transition function (see Appendix C) that when \( \xi \gg (\beta)^{1/2} \) or \( \eta \gg 1 \), Equation (3.37) appropriately reduces to the first six terms of Equation (3.4). Also, Equations (3.41) and (3.42) remain finite near the caustic when \( \xi \rightarrow (\beta)^{1/2} \) and \( \eta \rightarrow 0 \), however,
they become numerically unstable and an alternative formulation should be used.

Applying L'Hospital's rule in Equation (3.18), and then using Equations (3.2) and (3.19)–(3.32) we have:

\[
E(\beta, \xi, \eta \approx 0) \approx \frac{f_0}{2j(-\beta)} \left[ 1 - \frac{f_1 \eta}{(-\beta)^{3/4}} + \frac{f_2 \eta^2}{(-\beta)^{3/2}} \right] \\
- \frac{g_0}{(2j)^2(-\beta)^{5/2}} \left[ 1 - \frac{g_1 \eta}{(-\beta)^{3/4}} + \frac{g_2 \eta^2}{(-\beta)^{3/2}} \right] \\
+ \frac{h_0}{(2j)^3(-\beta)^{3/4}} \left[ 1 - \frac{h_1 \eta}{(-\beta)^{3/4}} + \frac{h_2 \eta^2}{(-\beta)^{3/2}} \right] \\
- \frac{k_0}{(2j)^4(-\beta)^{11/2}} \left[ 1 - \frac{k_1 \eta}{(-\beta)^{3/4}} + \frac{k_2 \eta^2}{(-\beta)^{3/2}} \right], \tag{3.45}
\]

and

\[
\beta \frac{\partial}{\partial \beta} E(\beta, \xi, \eta \approx 0) \approx \frac{f_0}{2j(-\beta)^2} \left\{ 1 - \frac{7f_1 \eta}{4(-\beta)^{3/4}} - \frac{f_1}{2} + \frac{5f_2 \eta^2}{2(-\beta)^{3/2}} + \frac{f_2[\xi - (-\beta)^{1/2}]}{(-\beta)^{1/2}} \right\} \\
- \frac{g_0}{(2j)^2(-\beta)^{7/2}} \left\{ 1 - \frac{13g_1 \eta}{4(-\beta)^{3/4}} - \frac{g_1}{2} + \frac{4g_2 \eta^2}{(-\beta)^{3/2}} + \frac{g_2[\xi - (-\beta)^{1/2}]}{(-\beta)^{1/2}} \right\} \\
+ \frac{h_0}{(2j)^3(-\beta)^{5/4}} \left\{ 1 - \frac{19h_1 \eta}{4(-\beta)^{3/4}} - \frac{h_1}{2} + \frac{11h_2 \eta^2}{2(-\beta)^{3/2}} + \frac{h_2[\xi - (-\beta)^{1/2}]}{(-\beta)^{1/2}} \right\} \\
- \frac{k_0}{(2j)^4(-\beta)^{13/2}} \left\{ 1 - \frac{25k_1 \eta}{4(-\beta)^{3/4}} - \frac{k_1}{2} + \frac{7k_2 \eta^2}{2(-\beta)^{3/2}} + \frac{k_2[\xi - (-\beta)^{1/2}]}{(-\beta)^{1/2}} \right\}, \tag{3.46}
\]

where

\[
f_0 = 1/3, \quad f_1 = 1.25, \quad f_2 = 4/3,
\]
\[
g_0 = 0.29629630, \quad g_1 = 3.00781250, \quad g_2 = 5.83333333,
\]
\[
h_0 = 0.69135802, \quad h_1 = 4.74804688, \quad h_2 = 13.3333333,
\]
\[
k_0 = 2.63374486, \quad k_1 = 6.48405151, \quad k_2 = 23.8333333.
\]

Also, \(\Lambda(\eta)\) and its derivative near the caustic are given by:

\[
\Lambda(\eta \approx 0) \approx -\frac{\sqrt{\pi}}{2} e^{i\eta^3/4} - \eta - \frac{j\eta^3}{3}, \tag{3.47}
\]
and

\[
\frac{d}{d\beta} \Lambda(\eta \approx 0) = -\frac{e^{i\eta^3}}{2(-\beta)^{1/4} \left[ 1 + \frac{\xi - (-\beta)^{1/2}}{3(-\beta)^{1/2}} \right]^{1/2}}. \tag{3.48}
\]
Chapter 4
Numerical Results and Discussion

For an efficient and accurate computation of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative, the argument space is divided into three regions as shown in Figure 4.1. Three different sets of formulae are used, one for each region in Figure 4.1, and a fourth set of formulae that is used in the immediate vicinity of the caustic $(\beta + \xi^2 \approx 0 \text{ or } \eta \approx 0)$. In region I, Equations (3.4) and (3.5) are used, in region II, Equations (3.37)-(3.44) are used, and in region III, the series solution is used given by Equations (2.21) and (2.22). In the immediate vicinity of the caustic and specifically when $\eta < 0.1$, Equations (3.37)-(3.40) and (3.44)-(3.48) are used.

An empirical expression for the number of terms used in the series solution is given by:

$$N(\beta, \xi) = 8|\beta| + 4, \quad \text{when } \xi < 2.0, \quad (4.1)$$

$$= 8|\beta| + 4 + 3|\beta|(\xi - 2.0), \quad \text{when } \xi \geq 2.0, \quad (4.2)$$

and results in a truncation error of less than $10^{-6}$.

Figure 4.2 shows the percent amplitude error of the asymptotic result relative to the series solution along the boundary between regions I and III. The results are plotted vs. the parameter $\beta$, with $\xi = (12 - 2\beta)^{1/2}$. The asymptotic result shows excellent agreement with the series solution, exhibiting a maximum error of 0.12%. Figure 4.3 shows the percent amplitude error of the asymptotic result relative to the series solution along the boundary between regions II and III. The results for this
Figure 4.1: Three different sets of formulae are used for the computation of the incomplete Airy function, one for each region in the figure, and a fourth set that is used in the immediate vicinity of the caustic.
Figure 4.2: Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions I and III. Results are plotted vs. the parameter $\beta$ with $\xi = (12 - 2\beta)^{1/2}$.

Case are plotted vs. the parameter $\xi$, with $\beta = -4$. Again the asymptotic result shows excellent agreement with the series solution, exhibiting a maximum error of only 0.075%.

Figures 4.4 and 4.5 show plots of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative vs. the parameter $\beta$ for $\xi = -3$ and 2, respectively. Figures 4.6 and 4.7 show plots of the incomplete Airy function $g_0(\beta, \xi)$ and its derivative vs. the parameter $\xi$ for $\beta = -5$ and 0, respectively. The marks on the contours represent direct numerical integration data, and show excellent agreement with the results obtained using the formulae derived in this report.
Figure 4.3: Percent amplitude error of the asymptotic result for the incomplete Airy function (solid line) and its derivative (broken line) along the boundary between regions II and III. Results are plotted vs. the parameter $\xi$ with $\beta = -4.0$. 
Figure 4.4: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter $\beta$ with $\xi = -3.0$. The marks on the contours represent direct numerical integration data.
Figure 4.5: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter $\beta$ with $\xi = 2.0$. The marks on the contours represent direct numerical integration data.
Figure 4.6: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter $\xi$ with $\beta = -5.0$. The marks on the contours represent direct numerical integration data.
Figure 4.7: Plots of the real part (solid line) and imaginary part (broken line) of the incomplete Airy function and its derivative vs. the parameter $\xi$ with $\beta = 0.0$. The marks on the contours represent direct numerical integration data.
Chapter 5

Summary and Conclusions

In this report, a convergent series solution form for the incomplete Airy functions has been derived, and asymptotic expansions involving several terms have been developed for use in large argument approximations. It has been demonstrated that the combination of the series solution with the asymptotic formulae results in an efficient and accurate means for computing the incomplete Airy functions for the entire range of their arguments.

A necessary requirement for the applicability of uniform asymptotic solutions to practical engineering problems is the efficient and accurate computation of the canonical functions involved. The results of this report would allow the formulation of useful uniform asymptotic solutions for several high-frequency scattering and diffraction problems in which the incomplete Airy integrals serve as canonical functions for the description of high-frequency field behavior in the vicinity of composite shadow boundaries. Furthermore, the methods used in this report may provide useful insight to the computation of other multivariable canonical functions occurring in high-frequency scattering and diffraction theory.

A FORTRAN code for the computation of the incomplete Airy functions based on the formulae derived in this report is available from the authors. A complete code listing appears in Appendix D.
Appendix A

Uniform Asymptotic Evaluation of Certain Stationary Phase Integrals

Let's consider the uniform asymptotic evaluation of a stationary phase integral of the form

$$I(a, b; k) = \int_{a}^{\infty} f(s) e^{ik\phi(s,b)} ds$$  \hspace{1cm} (A.1)

where the phase function $\phi(s, b)$ possesses two stationary phase points $s_{1,2}(b)$ satisfying $\phi'(s_{1,2}, b) = 0$ with no restrictions placed on their location relative to the integration endpoint $a$. The amplitude function $f(s)$ is assumed to be a slowly varying and analytic function of $s$. The integral in (A.1) may be transformed into a canonical form using the following transformation:

$$\phi(s, b) = \tau(z, \beta) = \alpha + \beta z + z^3/3$$  \hspace{1cm} (A.2)

where

$$\alpha = \tau(0, \beta) = \phi(s_p, b); \quad \phi''(s_p) = 0$$  \hspace{1cm} (A.3)

$$\beta = \phi'(s_p) \left[ \frac{2}{\phi''(s_p)} \right]^{\frac{1}{2}}$$  \hspace{1cm} (A.4)

or alternatively

$$\alpha = \tau(z_{1,2}, \beta) + \frac{2}{3}(-\beta)^{3/2},$$  \hspace{1cm} (A.5)
\[ \beta = -\left\{ \frac{3}{2} \tau(0, \beta) - \tau(z_{1,2}, \beta) \right\}^{2/3}, \]

with \( z_{1,2} = \pm (-\beta)^{1/2} \). The proper branch for \( \beta \) depends on the sign of \( \phi'(s_p) \) and \( \phi'''(s_p) \). Thus, using (A.2) the integral in (A.1) becomes:

\[ I(\xi_a, \beta; k) = \int_{\xi_a}^{\infty} G(z) e^{ikr(z, \beta)} \, dz \quad \text{(A.7)} \]

where

\[ \xi_a = (a - s_p) \left[ \frac{\phi'''(s_p)}{2} \right]^{1/2} = \left[ \phi'(a) - \phi'(s_p) \right]^{1/2} \left[ \frac{2}{\phi'''(s_p)} \right]^{1/2}, \quad \text{(A.8)} \]

\[ G(z) = f(s) \frac{dz}{ds}, \quad \text{and} \]

\[ \frac{dz}{ds} = \left[ \frac{2}{\phi'''(s_p)} \right]^{1/2}. \quad \text{(A.10)} \]

The proper branches for \( \xi_a \) and \( \frac{dz}{ds} \) depend on the sign of \( \phi'''(s_p) \). Next, we employ the Chester et al. expansion [10] for the amplitude function in (A.7), i.e.,

\[ G(z) = \sum_{m=0}^{\infty} \left[ a_m(z^2 + \beta)^m + b_m z (z^2 + \beta)^m \right] \quad \text{(A.11)} \]

and since only the leading terms in the asymptotic expansion of (A.7) will be retained, Equation (A.11) may be rewritten as follows:

\[ G(z) = a_0 + zb_0 + (z^2 + \beta)g(z) \quad \text{(A.12)} \]

where

\[ a_0 = \frac{1}{2} [G(z_1 + G(z_2)], \quad \text{(A.13)} \]

\[ b_0 = \frac{1}{2z_1} [G(z_1) - G(z_2)], \quad \text{(A.14)} \]

\[ g(z) = \sum_{m=1}^{\infty} [a_m(z^2 + \beta)^{m-1} + b_m z (z^2 + \beta)^{m-1}], \quad \text{and} \]

\[ G(z_{1,2}) = f(s_{1,2}) \frac{ds}{dz} \bigg|_{z=z_{1,2}} = f(s_{1,2}) \left[ \frac{2\beta^{1/2}}{\phi'''(s_{1,2})} \right]^{1/2}. \quad \text{(A.16)} \]

32
Inserting (A.12) into (A.7) yields

\[ I(\xi, \beta; k) = e^{jka} \left[ a_0 \int_\xi^\infty e^{j k (\beta z + z^2 / 3)} \, dz + b_0 \int_\xi^\infty z e^{j k (\beta z + z^2 / 3)} \, dz \right. \]

\[ - \frac{1}{j k} g(\xi) e^{j k (\beta \xi + \xi_z / 3)} - \frac{1}{j k} \int_\xi^\infty g'(z) e^{j k (\beta z + z^2 / 3)} \, dz \]. \tag{A.17}

The last two terms in the right side of (A.17) are obtained by an integration by parts of the integrand involving \( g(z) \). The desired uniform asymptotic approximation for \( k \to \infty \) is given by the first three terms of (A.17), and may be expressed in the form

\[ I(\xi, \beta; k) \sim e^{jka} \left\{ k^{-1/3} a_0 \overline{Ai}(k^{2/3} \beta, k^{1/3} \xi_a) - j k^{-2/3} b_0 \frac{\partial}{\partial \beta} \overline{Ai}(k^{2/3} \beta, k^{1/3} \xi_a) \right. \]

\[ - \frac{1}{j k} \left[ \frac{G(\xi_a) - a_0 - \xi_a b_0}{\xi_a^2 + \beta} \right] e^{j k (\beta \xi_a + \xi_a / 3)} \right\} \tag{A.18} \]

where \( \overline{Ai}(\sigma, \zeta) \) is the incomplete Airy integral defined by

\[ \overline{Ai}(\sigma, \zeta) \Delta \int_\zeta^\infty e^{i (\sigma z + z^2 / 3)} \, dz \tag{A.19} \]

and the upper limit terminates in one of the three sectors in the complex \( z \)-plane where the integral converges. For example, in the case when the upper limit terminates in sector \( 0 < \arg z < \pi / 3 \) we have that \( \overline{Ai}(\sigma, \zeta) \equiv g_0(\sigma, \zeta) \).
Appendix B

Computation of $a_0(\xi)$, $b_1(\xi)$ and their Derivatives

The functions $a_0(\xi)$ and $b_1(\xi)$ needed in the series solution form of the incomplete Airy functions can be expressed in terms of the incomplete Gamma function; i.e.,

\begin{align*}
a_0(\xi) &= e^{j\pi/6}3^{-2/3}\Gamma(1/3, -j\xi^3/3), \quad (B.1) \\
b_1(\xi) &= -e^{-j\pi/6}3^{-1/3}\Gamma(2/3, -j\xi^3/3), \quad (B.2)
\end{align*}

where

$$\Gamma(x, y) = \int_y^\infty t^{x-1}e^{-t}dt, \quad \Re(x) > 0. \quad (B.3)$$

Using the series solution form of the incomplete Gamma function [7], $a_0(\xi)$ and $b_1(\xi)$ are computed using the expressions:

\begin{align*}
a_0(\xi) &\simeq e^{j\pi/6}3^{-2/3}\Gamma(1/3) - \xi \sum_{n=0}^{N(\xi)} \frac{(j\xi^3/3)^n}{(3n+1)n!}, \quad (B.4) \\
and \ b_1(\xi) &\simeq -e^{-j\pi/6}3^{-1/3}\Gamma(2/3) - j\xi^2 \sum_{n=0}^{N(\xi)} \frac{(j\xi^3/3)^n}{(3n+2)n!}. \quad (B.5)
\end{align*}

The number of terms in the series, $N(\xi)$, is given by the empirical formula:

$$N(\xi) = 2 + 4\xi^2, \quad (B.6)$$

and results in a truncation error of less than $10^{-6}$. 

34
The derivatives of $a_0(\xi)$ and $b_1(\xi)$ are given by:

\[ a_0^{(k)}(\xi) = j d_k a_0^{(k-3)}(\xi) + e_k a_0^{(k-2)}(\xi) + j \xi^2 a_0^{(k-1)}(\xi), \quad n \geq 3, \quad (B.7) \]
\[ b_1^{(k)}(\xi) = j (k-1) a_0^{(k-1)}(\xi) + j \xi a_0^{(k)}(\xi), \quad n \geq 1, \quad (B.8) \]

where

\[ a_0^{(1)}(\xi) = -e^{j\xi^3/3}, \quad (B.9) \]
\[ a_0^{(2)}(\xi) = j \xi^2 a_0^{(1)}(\xi). \quad (B.10) \]

The constants $d_k$ and $e_k$ are obtained using the recursive relationships:

\[ d_k = d_{k-1} + e_{k-1}, \quad (B.11) \]
\[ e_k = e_{k-1} + 2, \quad k \geq 3, \quad (B.12) \]

with $d_2 = e_2 = 0$. 

35
Appendix C

Computation of the Fresnel Transition Function

The Fresnel transition function [3] is defined as follows:

$$F(x) \triangleq 2j\sqrt{x} e^{jx} \int_{\sqrt{x}}^{\infty} e^{-j\tau^2} d\tau,$$  \hspace{1cm} (C.1)

where $\sqrt{x} = |\sqrt{x}|$ if $x > 0$, and $\sqrt{x} = j|\sqrt{x}|$ if $x < 0$. Also,

$$F(-|x|) = F^*(|x|).$$  \hspace{1cm} (C.2)

When $x < 6.0$, $F(x)$ is computed using its series solution form; i.e.,

$$F(x) \simeq \sqrt{\pi x} e^{j(x+\pi/4)\text{sign}(x)} - 2jxe^{jx} \sum_{n=0}^{N(x)} \frac{(-jx)^n}{(2n+1)n!}. \hspace{1cm} (C.3)$$

The number of terms in the series, $N(x)$, is given by the empirical formula:

$$N(x) = 10\sqrt{x}, \hspace{1cm} (C.4)$$

and results in a truncation error of less than $10^{-6}$. The large argument form of the Fresnel transition function ($x \geq 6.0$) is given by:

$$F(x) \sim \sum_{m=0}^{5} \frac{(-1)^m 2^m \left(\frac{1}{2}\right)_m}{(2jx)^m}, \hspace{1cm} (C.5)$$

where

$$\left(\alpha + \frac{1}{2}\right)_0 = 1, \ \text{and} \ 2^m \left(\alpha + \frac{1}{2}\right)_m = (2\alpha + 1)(2\alpha + 3)\cdots(2\alpha + 2k - 1). \hspace{1cm} (C.6)$$


Appendix D
Code Listing

C----------------------------------------------------------------------
OPTIONS/EXTEND_SOURCE
SUBROUTINE INCAIRY(B,X,GO,GOP,G1,G1P,G2,G2P)
IMPLICIT NONE
C!!!
This subroutine computes the incomplete Airy functions and their
first derivatives with respect to the parameter B. Both B and X
are considered real. Double precision arithmetic is used.
C!!!
G0 (B,X) = INT(X,INFTY,FS)
G0P (B,X) = INT(X,INFTY,CJ*S*FS)
C!!!
where FS = EXP[CJ*(B*S+S**3/3)]
C!!!
G1(B,X) = G0(B,X)-TPI*AI(B)
G2(B,X) = G0(B,X)-PI*[AI(B)+CJ*BI(B)]
C!!!
AI(B) and BI(B) are the complete Airy functions.
C!!!
Version 1.1 4-12-1992
C!!!
Author: E. D. Constantinides
C!!!
The Ohio State University
C!!!
ElectroScience Laboratory
C!!!
1320 Kinnear Road
C!!!
Columbus, OH 43212
C!!!
REAL*8 B,X,XP,X2,BXX,PI/3.141592653589793/, &
TPI/6.283185307179586/
COMPLEX*16 GO,GOP,G1,G1P,G2,G2P,CJ/(0.0D0,1.0D0)/
COMPLEX BZ, AI, AIP, BI, BIP
EXTERNAL INC_AIRY_SS, INC_AIRY_FF, INC_AIRY_IP, AIBI
COMMON CJ, PI, TPI

C!!!
XP=DABS(X)
X2=X*X
BXX=B+X2
IF (B.GT.0.0D0) BXX=2.0D0*B+X2
IF (BXX.GT.12.0D0) THEN
   CALL INC_AIRY_IP(B,XP,GO,GOP)
ELSEIF (B.LT.-4.0D0) THEN
   CALL INC_AIRY_FF(B,XP,GO,GOP)
ELSE
   CALL INC_AIRY_SS(B,XP,GO,GOP)
ENDIF
BZ=CMPLX(B)
CALL AIBI(BZ, AI, AIP, BI, BIP)
IF (X.LT.0.0D0) THEN
   GO=PI*TPI*AI-DCONJG(GO)
   GOP=PI*TPI*AIP-DCONJG(GOP)
ENDIF
G1=GO-PI*TPI*AI
G1P=GO-PI+(AI+CJ*BI)
G2=GO-PI*(AI+CJ*BI)
G2P=GO-PI*(AIP+CJ*BIP)
C!!!
END

---------------------------------------------------------

OPTIONS/EXTEND_SOURCE
SUBROUTINE INC_AIRY_IP(BS,XS,GO,GOP)
IMPLICIT NONE
C!!!
C!!! This subroutine computes the incomplete airy function using integration by parts when BS+XS**2>>1.0
C!!!
REAL*8 BS,XS,BXS,BXS1,BXS2,BXS3,BXS4,BXS5,BXS6,BXS7,
&      BXS8,BXS9,BXS10,BXS11,PI,TPI
COMPLEX*16 CJ, CT, GO, GOP
COMMON CJ, PI, TPI
C!!!
BXS=BS+XS*XS
BXS1 = 1.0D0/BXS
BXS2 = BXS1*BXS1
BXS3 = BXS2*BXS1
BXS4 = BXS3*BXS1
BXS5 = BXS4*BXS1
BXS6 = BXS5*BXS1
BXS7 = BXS6*BXS1
BXS8 = BXS7*BXS1
BXS9 = BXS8*BXS1
BXS10 = BXS9*BXS1
BXS11 = BXS10*BXS1
CT = CDEXP(CJ*(BS*XS+XS*XS*XS/3.0D0))
IF (XS.EQ.0.0D0) GOTO 10
GO = CJ*BXS1+2.0D0*XS*BXS3+2.0D0*CJ*BXS4-12.0D0*CJ*XS*XS*BXS5
& +40.0D0*XS*BXS6+40.0D0*CJ*BXS7-120.0D0*XS*XS*XS*BXS8
& -840.0D0*CJ*XS*XS*BXS9+1680.0D0*XS*XS*XS*XS*XS
& +2240.0D0*XS*BXS9
GO = CT*GO
GOP = -CJ*BXS2-6.0D0*XS*BXS4-8.0D0*CJ*BXS5+60.0D0*CJ*XS*XS*BXS6
& -240.0D0*XS*BXS7-280.0D0*CJ*BXS8+840.0D0*XS*XS*XS*BXS8
& +6720.0D0*CJ*XS*XS*BXS9-15120.0D0*XS*XS*XS*XS*XS*BXS10
& -20160.0D0*XS*BXS10
GOP = CJ*XS*GO+CT*GOP
RETURN
C!!
10 GO = CJ*BXS1+2.0D0*CX*BXS4+4.0D0*CJ*BXS7
GOP = -CJ*BXS2-8.0D0*CJ*BXS5-280.0D0*CJ*BXS8-22400.0D0*CJ*BXS11
RETURN
C!!
END
C-------------------------------------------------------------------------------------
OFLONS//EXTEND_SOURCE
SUBROUTINE INC_AIRY_FF(BS, XS, GO, GOP)
IMPLICIT NONE
C!!!
C!!! This subroutine computes the incomplete airy function
C!!! using the Fresnel integral when BS<<0.0
C!!!
REAL*8 BS, XS, BSP, Z, ZZ, BXS, BX, PI, TPI, B1, B2, B3,
& G1, G2, G3, Z3, Z5, Z7, Z9, BXS1, BXS2, BXS3, BXS4, BXS5, BXS6,
& BXS7, BXS8, BXS14, BX1, BSQP, BSS, F1, F2, F3, F4, F11, F12, F21,
& F22, F31, F32, F41, F42, DZXS, DZBS

39
COMPLEX*16 CJ, CT, GO, GOP, FCTZ, CZZ, GO1, GO2, GO3,
& EPC, EPCP, SPC, SPCP, CT1, GZ, FZ, HZ, KZ, FFCT, GPZ, FPZ, HPZ,
& KPZ, FCTZP
EXTERNAL FTRANSD
COMMON CJ, PI, TPI

C!!!
DATA G1, G2, G3 /0.2083333333333333, 0.3342013888888889,
& 1.025812596450617/
DATA F1, F11, F12 /0.3333333333333333, 1.25D0, 1.3333333333333333/
DATA F2, F21, F22 /0.2962962962962963, 3.0078125D0, 5.8333333333333333/
DATA F3, F31, F32 /0.6913580246913580, 4.748046875D0,
& 13.3333333333333333/
DATA F4, F41, F42 /2.633744855967078, 6.484051513671875,
& 23.8333333333333333/

C!!!
BSP = DABS (BS)
BSQP = DSQRT (BSP)
BS14 = DSQRT (BSQP)
BSS = BSP * BSQP
B1 = 1.0DO / BSS
B2 = B1 / BSS
B3 = B2 / BSS
BX = BS * XS + XS * XS * XS / 3.0DO
BX1 = -2.0DO * BSS / 3.0DO
ZZ = DABS (BX + 2.0DO * BSS / 3.0DO)
Z = DSQRT (ZZ)
IF (XS .LT. BSQP) Z =- Z
Z3 = Z * ZZ
Z5 = Z3 * ZZ
Z7 = Z5 * ZZ
Z9 = Z7 * ZZ
CT = CDEXP (CJ * BX)
CT1 = CDEXP (CJ * BX1)
G01 = 0.5D0 * CJ * G1 * B1
G02 = -0.25D0 * G2 * B2
G03 = -0.125D0 * CJ * G3 * B3
SPC = 1.0DO + G01 + G02 + G03
SPCP = (1.5D0 * G01 + 3.0DO * G02 + 4.5D0 * G03) / BSP
BXS = BS + XS * XS
DZXS = XS - BSQP
CZZ = CDEXP (CJ * ZZ)
IF (ABS (BXS).LT.0.1DO) GOTO 10

40
C

BXS1=1.0DO/BXS
BXS2=BXS1*BS1
BXS3=BXS2*BS1
BXS4=BXS3*BS1
BXS5=BXS4*BS1
BXS6=BXS5*BS1
BXS7=BXS6*BS1
BXS8=BXS7*BS1
CALL FTRANSD(ZZ,FFCT)
FCTZ=0.5DO*CJ*CONJG(FFCT)*CZ/Z
FCTZP=-0.5DO*CZ*DSQRZ/3/BS14
IF (Z.LT.0.0DO) FCTZ=FCTZ+DSQRZ(PI)
GZ=2.0DO*BXS1-1.0DO/Z/BS14
GPZ=-2.0DO*BXS2-0.25DO/Z/BSP14+0.5DO*DSQRZ/3/BS14
FZ=-8.0DO*X5*BXS3+1.0DO/Z3/BS14-G1*B1/Z/BS14
FPZ=24.0DO*X5*BXS4+0.25DO/Z3/BS14/BSP-1.75DO*G1*B1/Z/BS14/BSP
&-1.5DO*DSQRZ/5/BS14+0.5DO*G1*B1*DSQRZ/3/BS14
HZ=-16.0DO*BXS4+96.0DO*X5*X5*BXS5-3.0DO/Z5/BS14+G1*B1/Z3/BS14
&-G2*B2/Z/BS14
HPZ=64.0DO*BXS5-480.0DO*X5*X5*BXS6-0.75DO/Z5/BS14/BSP+
&7.5DO*DSQRZ/7/BS14-1.5DO*DSQRZ/3/BS14/BSP+
&0.5DO*DSQRZ/G2*B2/Z3/BS14
KZ=640.0DO*X5*BS14-1920.0DO*X5*X5*X5*BS15+15.0DO/Z7/BS14-
KPZ=-3840.0DO*X5*BS17+13440.0DO*X5*X5*X5*BS18+3.75DO/Z7/BS14/BSP
&-4.75*G3*B3/Z/BS14/BSP-5.25DO*DSQRZ/5/BS14
&+7.5DO*DSQRZ*G1*B1/Z7/BS14-1.5DO*DSQRZ*G2*B2/Z5/BS14
&+0.5DO*DSQRZ*G3*B3/Z3/BS14
EPC=CT*(0.5DO*CJ*GZ-0.25DO*FZ-0.125DO*CJ*HZ+0.0625DO*KZ)
EPCP=CT*(0.5DO*CJ*GPZ-0.25DO*FPZ-0.125DO*CJ*HPZ+0.0625DO*KPZ)
GO=CT1*SPC*FCTZ/BS14+EPC
GOP=(CJ+BS14+0.25DO/BS14/BSP)*CT1*SPC*FCTZ+CT1*SPC+FCTZ/BS14
&+CT1*SPC+FCTZP/BS14+CJ*X5*EPC+EPCP
RETURN
C

C

Small argument Form
C

FCTZ=0.5DO*DSQRZ(CJ*PI)-Z-CJ*Z/Z/3.0DO
DZBS=0.5DO/BS14/DSQRZ(1.0DO+DZ5/3.0DO/BSQP)
SUBROUTINE FTRANSD(XF, FFCT)
IMPLICIT NONE

REAL*8 X, XF, PI4, PI, TPI
INTEGER N, NT
COMPLEX*16 CJ, AO, AN, FFCT, FFCTS
COMMON CJ, PI, TPI

X=DABS(XF)
IF (X.GT.6.0D0) GOTO 1
C!!! Small argument (series) form
C!!!
IF (X.EQ.0.0D0) THEN
    FFCT=(0.0D0,0.0D0)
    RETURN
ENDIF
PI4=PI/4.0D0
FFCT=DSQRT(PI*X)*CDEXP(CJ*(X+PI4))
AO=(1.0D0,0.0D0)
FFCTS=AO
NT=10*DSQRT(X)
DO N=1,NT
    AN=-CJ*X*AO/DBLE(N)
    FFCTS=FFCTS+AN/DBLE(2*N+1)
    AO=AN
END DO
FFCT=FFCT-2.0D0*CJ*X*CDEXP(CJ*X)*FFCTS
GOTO 20

C!!! Large argument form
C!!!
1 AO=(1.0D0,0.0D0)
   FFCT=AO
   DO N=1,8
      AN=0.5D0*CJ*DBLE(2*N-1)*AO/X
      FFCT=FFCT+AN
      AO=AN
   END DO
20 IF (XF.GE.0.0) RETURN
   FFCT=DCONJG(FFCT)
C!!!
RETURN
END

OPTIONS/EXTEND_SOURCE
SUBROUTINE INC_AIRY_SS(BS,XS,GO,GOP)
IMPLICIT NONE
C!!!
C!!! This subroutine computes the Incomplete Airy function using a
C!!! convergent series solution with error less than 1.0E-6.
C!!!
REAL*8 BS,BSP,BS2,BS3,XS,XS2,XS3,CO,C1,PI,TPI
INTEGER M,N,MM,NT,MT
COMPLEX*16 GO,GOP,A(66,33),B(66,33),AOP(33),AIP(33),
& CA(66),CB(66),CJ
EXTERNAL AOAIXS
COMMON CJ,PI,TPI

C!!! Initialize the coefficient values.

C!!!

BSP=DABS(BS)
XS2=XS*XS
XS3=XS2*XS
CALL AOAIXS(XS,AOP(1),AIP(1))
AOP(2)=-CDEXP(CJ*XS3/3.0DO)
AOP(3)=CJ*XS2*AOP(2)
AOP(4)=2.0DO*CJ*XS*AOP(2)+CJ*XS2*AOP(3)
AIP(2)=CJ*XS*AOP(2)
DO N=3,4
 AIP(N)=CJ*DBLE(N-2)*AOP(N-1)+CJ*XS*AOP(N)
END DO
BS2=BS*BS
BS3=BS2*BS
GO=AOP(1)+A1P(1)*BS+0.5DO*CJ*AOP(2)*BS2
GOP=A1P(1)+CJ*AOP(2)*BS
IF (BS.EQ.0.0DO) RETURN
MT=8*BSP+4
IF (XS.GT.2.0DO) MT=MT+3*BSP*(XS-2)
NT=(MT+1)/2
C0=0.0DO
C1=2.0DO
DO M=1,3
 DO N=1,NT
   A(M,N)=(0.0DO,0.0DO)
   B(M,N)=(0.0DO,0.0DO)
 END DO
END DO

C!!!
A(1,1)=(1.0DO,0.0DO)
B(2,1)=(1.0DO,0.0DO)*BS
A(3,2)=(0.0DO,0.5DO)*BS2
C!!!
C!!! Compute the series
DO M=4,MT
  MM=(M+1)/2
  A(M,1)=A(M-3,1)*BS3/DBLE(M-1)/DBLE(M-2)
  B(M,1)=B(M-3,1)*BS3/DBLE(M-1)/DBLE(M-2)
  DO N=2,MM
    A(M,N)=(A(M-3,N)*BS3+CJ*A(M-2,N-1)*BS2)/DBLE(M-1)/DBLE(M-2)
    B(M,N)=(B(M-3,N)*BS3+CJ*B(M-2,N-1)*BS2)/DBLE(M-1)/DBLE(M-2)
  END DO
  DO N=MM+1,NT
    A(M,N)=(0.0DO,0.0DO)
    B(M,N)=(0.0DO,0.0DO)
  END DO
  IF ((MM.NE.(M/2)).AND.(MM.GE.5)) THEN
    CO=CO+C1
    C1=C1+2.0DO
    AOP(MM)=CJ*XS2*AOP(MM-1)+C1*CJ*XS*AOP(MM-2)+CO*CJ*AOP(MM-3)
    A1P(MM)=CJ*DBLE(MM-2)*AOP(MM-1)+CJ*XS*AOP(MM)
  ENDIF
  CA(M)=(0.0DO,0.0DO)
  CB(M)=(0.0DO,0.0DO)
  DO N=1,MM
    CA(M)=CA(M)+A(M,N)*AOP(N)
    CB(M)=CB(M)+B(M,N)*A1P(N)
  END DO
  GO=GO+(CA(M)+CB(M))
  GOP=GOP+DBLE(M-1)*(CA(M)+CB(M))/BS
END DO

RETURN
END

------------------------------------------------------------------
OPTIONS/EXTEND_SOURCE
SUBROUTINE AOAIXS(XS, AO, A1)
IMPLICIT NONE
C!!!
REAL*8 XS,G13/2.678935347DO/,G23/1.3541179394DO/,PI,TPI,
& XS1,XS2,AIO/0.3550280539DO/,AIP0/-0.2588194038DO/
COMPLEX*16 AO,A1,CJ,X3,B(100),C13,C23
INTEGER I,NT
COMMON CJ,PI,TPI
C!!!
DATA C13,C23/(0.4163415888278022,0.2403749283845681),
& (-0.6004684775880014,0.3466806371753174)/

C!! Series Solution

C!!
AO=C13*G13
A1=C23*G23
IF (XSI.EQ.0.0D0) RETURN
B(1)=X3
AO=AO-XSI-0.25DO*XSI*B(1)
A1=A1-0.5DO*CJ*XS2-0.2*CJ*XS2*B(1)
NT=2+4*XS2
DO I=2,NT
   B(I)=B(I-1)_X3/DBLE(I)
AO=AO-XSI*B(I)/DBLE(3*I+1)
A1=A1-CJ*XS2*B(I)/DBLE(3*I+2)
END DO
C!!
10 IF (XS.LT.0.0D0) THEN
   AO=2.0DO*PI*AIO-DCONJG(AO)
   A1=2.0DO*PI*AIP0-DCONJG(A1)
ENDIF
C!!
RETURN
END

SUBROUTINE AIBI(Z, AI, AIP, BI, BIP)

C!! This routine calculates the Airy functions AI(Z), BI(Z),
C!! and their derivatives AIP(Z), BIP(Z).
C!! Ref. Abramowitz and Stegun, Handbook of Mathematical Functions.
C!! For CABS(Z) .LE. 6.0, a Taylor Series is used.
C!! ARG(Z) may take any value. See eqs. (10.4.2) to (10.4.5).
C!!
COMPLEX Z, AI, AIP, BI, BIP
C!!
IF(CABS(Z).GT.6.)GO TO 12
CALL AIBII(Z, AI, AIP, BI, BIP)
SUBROUTINE AIBI1(Z, AI, AIP, BI, BIP)

COMPLEX Z, AI, AIP, BI, BIP
COMPLEX*16 F, G, FP, GP
DOUBLE PRECISION CC1, CC2

DATA S3, CC1, CC2 / 1.732050808, 0.355028053887817, 0.258819403792807 /

CALL FZGZ(Z, F, G, FP, GP)
AI = CC1 * F - CC2 * G
AIP = CC1 * FP - CC2 * GP
BI = S3 * (CC1 * F + CC2 * G)
BIP = S3 * (CC1 * FP + CC2 * GP)

RETURN
END

SUBROUTINE FZGZ(Z, F, G, FP, GP)

The auxiliary functions F(Z), G(Z), FP(Z), GP(Z) are computed as in
"Tables of the Modified Hankel functions of order one-third and

COMPLEX*16 F, G, FP, GP, Z3, Z3M, ZD
COMPLEX Z
REAL*8 AM, BM, CM, DM, AO, BO, CO, DO
REAL ZMBD(5)
INTEGER MAX(5)

DATA ZMBD / 6.1, 5.6, 4.8, 4.1, 3.2 /
DATA MAX / 22, 19, 16, 14, 11 /

ZD = 0. DO
ZD = Z
AO = 1. DO
BO = 1. DO
CO=0.5D0
DO=1.D0
Z3=(ZD**3)/200
Z3M=Z3
ZMAG=CABS(Z)
DO 3 M=1,5
3 IF(ZMAG .LE. ZMBD(M)) MADMAX=MAX(M)
  F=AO
  G=BO
  FP=CO
  GP=DO
  DO 10 M=1,MADMAX
  TM=FLOAT(3*M)
  AM=200.DO*AO/TM/(TM-1)
  BM=200.DO*BO/TM/(TM+1)
  CM=200.DO*CO/TM/(TM+2)
  DM=200.DO*DO/TM/(TM-2)
  F=F+AM*Z3M
  G=G+BM*Z3M
  FP=FP+CM*Z3M
  GP=GP+DM*Z3M
  Z3M=Z3M*Z3
AO=AM
BO=BM
CO=CM
DO=DM
10 CONTINUE
  G=ZD*G
  FP=(ZD**2)*FP
C!!!
RETURN
END

C-----------------------------------------------------------------------------------
SUBROUTINE AIBI2(XX, AI, AIP, BI, BIP)
C!!!
C!!! This Routine calculates the Airy functions AI(XX), BI(XX),
C!!! and their derivatives AIP(XX), BIP(XX).
C!!! Ref. Abramowitz and Stegun, Handbook of Mathematical Functions.
C!!!
COMPLEX Z, AI, AIP, BI, BIP, XX
COMPLEX Z25, ZTB, ZT, ZT2, ZT3, ZT4, ZT5
COMPLEX CT1, A2L2, EIPI3, EIPI6, C, S

C-----------------------------------------------------------------------------------
C!!!
DATA RTPI,TWORPI,RTOP,POF
% /1.772453851,.3.544907702,.797884561,.785398164 /
DATA A2L2,EPRI6,EPRI3
% /(0.,.346573590),(.866025404,.5),(.5,.866025404) /
DATA C1/.069444444/,C2/.037133487/,C3/.037993059/,
% C4/.057649190/,C5/.116099064/
DATA D1/-.097222222/,D2/-.043885030/,D3/-.042462830/,
% D4/-.082662163/,D5/-.124105896/
C!!!
ZTB=(2./3.)*XX**1.5
ARG=ATAN2(AIMAG(XX),REAL(XX))
IF(ABS(ARG).GE.2.1) GO TO 100
C!!!
EqN. (10.4.59),(10.4.61)
C!!!
Z25=XX**.25
ZT=ZTB
ZT2=ZT*ZT
ZT3=ZT2*ZT
ZT4=ZT2*ZT2
ZT5=ZT3*ZT2
CT1=CEXP(-ZT)/TWORPI
AI=CT1/Z25*(1-C1/ZT+C2/ZT2-C3/ZT3+C4/ZT4-C5/ZT5)
AIP=-CT1*Z25*(1-D1/ZT+D2/ZT2-D3/ZT3+D4/ZT4-D5/ZT5)
IF(ARG.LT.0.) GO TO 20
ZT=(0.,-1.)*ZTB
C!!!
Eqn. (10.4.65),(10.4.68) WITH UPPER SIGNS.
C!!!
Z=XX/EPRI3
CT1=ZT+POF-A2L2
BI=EPRI6
BIP=1./EPRI6
GO TO 30
20 ZT=(0.,1.)*ZTB
C!!!
Eqn. (10.4.65),(10.4.68) WITH LOWER SIGNS.
C!!!
Z=XX+EPRI3
CT1=ZT+POF+A2L2
BI=1./EPRI6
BIP=EIPI6
30 S=CSIN(CT1)
   C=CCOS(CT1)
   Z25=Z**.25
   ZT2=ZT*ZT
   ZT3=ZT2*ZT
   ZT4=ZT2*ZT2
   ZT5=ZT3*ZT2
   BI=BI*RT0P/Z25*(S*(1-C2/ZT2+C4/ZT4)-C*(C1/ZT-C3/ZT3+C5/ZT5))
   BIP=BIP*RT0P*Z25*(C*(1-D2/ZT2+D4/ZT4)+S*(D1/ZT-D3/ZT3+D5/ZT5))
RETURN
100 ZT=(0.1.)*ZTB
C!!!
C!!! EQN. (10.4.60),(10.4.62),(10.4.64),(10.4.67)
C!!!
IF(ARG.LT.0.)ZT=-ZT
   Z=-XX
   Z25=Z**.25
   ZT2=ZT*ZT
   ZT3=ZT2*ZT
   ZT4=ZT2*ZT2
   ZT5=ZT3*ZT2
   CT1=ZT+POF
   S=CSIN(CT1)
   C=CCOS(CT1)
   AI=1./RTPI/Z25*(S*(1-C2/ZT2+C4/ZT4)-C*(C1/ZT-C3/ZT3+C5/ZT5))
   BI=1./RTPI/Z25*(C*(1-C2/ZT2+C4/ZT4)+S*(C1/ZT-C3/ZT3+C5/ZT5))
   BIP=Z25/RTPI*(S*(1-D2/ZT2+D4/ZT4)-C*(D1/ZT-D3/ZT3+D5/ZT5))
C!!!
RETURN
END

C---------------------------------------------------------------
Bibliography


