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ON LATTICE CHIRAL GAUGE THEORIES
Abstract

We consider the Smit-Swift-Aoki formulation of a lattice chiral gauge theory, in which the Wilson and other non-invariant terms in the action are made gauge-invariant by the coupling with a non-linear auxiliary scalar field, $\Omega$. We show that $\Omega$ decouples from the physical states only if appropriate parameters are tuned so as to satisfy a set of BRST identities. In addition, barring unforeseen cancellations, explicit ghost fields are necessary to ensure decoupling. Thus, for these theories to give rise to the correct continuum limit, the requirements discussed in our previous work have to be satisfied. Similar considerations apply to schemes with mirror fermions. Finally, we discuss simpler cases with a global chiral symmetry and show that the theory becomes free at decoupling. Recent numerical simulations agree with our considerations.

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1. The lattice formulation of chiral gauge theories has received considerable attention in recent years. As well known, the difficulty lies in the chiral symmetry breaking caused by the so-called Wilson term, which is needed to remove unwanted fermion doublers present in the naive discretization of the Dirac action.

A starting point was provided by Smit [1]. To cope with the above problem, he introduces an auxiliary field, \( \Omega \), with values in the gauge group. \( \Omega \) plays the role of the angular part of a Higgs field and it is used to make the action, Wilson term included, invariant under simultaneous gauge transformations of the link variables \( U \), of the fermion fields \( \psi \) and of \( \Omega \) itself:

\[
U_\mu(x) \rightarrow U_\mu(x)^h = h(x+\mu)U_\mu(x)h(x)^+
\]

\[
\psi(x) \rightarrow \psi(x)^h = h(x)\psi(x)
\]

\[
\Omega(x) \rightarrow \Omega(x)^h = \Omega(x)h(x)^+
\]  \hspace{1cm} (1)

The general structure of the action, in this case, is:

\[
S(U, \psi, \Omega) = S_{N1}(U^\Omega, \psi^\Omega)
\]  \hspace{1cm} (2)

where \( S_{N1} \) is any local action constructed with \( U \) and \( \psi \). In particular, a gauge field mass-term in \( S_{N1} \):

\[
(S_{N1})_{\text{mass}} = \frac{1}{2} \sum_{x, \mu} \text{Tr}[\Omega(x+\mu)U_\mu(x)\Omega(x)^+ - \text{h.c.}]
\]  \hspace{1cm} (3)

gives rise to the kinetic energy term for \( \Omega \).

From a different, but completely equivalent, point of view [2], one starts from the non-invariant action, \( S_{N1} \), and obtains a gauge invariant partition function as an average over the gauge group:

\[
Z = \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \mathcal{D}\Omega \, e^{-S_{N1}(U^\Omega, \psi^\Omega)}
\]  \hspace{1cm} (4)

Of course, the field \( \Omega \) is a pure artifact of the formulation, and we are interested in the limit where \( \Omega \) decouples from the physical degrees of freedom. To this aim, the parameters

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1. We are using a condensed notation in which we do not separate explicitly the left and right-handed components of \( \psi \), which, in a chiral theory, transform differently.
appearing in \( S_{N1} \) should be tuned appropriately, in a way which, however, was never completely specified, in ref.[1] and [2], and which we are going to study in detail in this paper.

The physically interesting gauge transformations that we want to recover in the continuum limit of our theory are not those given in (1), but rather:

\[
U_\mu(x) \rightarrow U_\mu(x) = g(x+\mu)U_\mu(x)g(x)^* \tag{5}
\]

\[
\psi(x) \rightarrow \psi(x) = g(x)\psi(x) \quad \Omega(x) \rightarrow \Omega(x)
\]

Simultaneous invariance under (1) and (5), with independent \( g \) and \( h \), is equivalent to the decoupling of \( \Omega \). In fact, the combined transformations with:

\[
g(x)=h(x)+t=h(x)+t \tag{6}
\]

leave \( U \) and \( \psi \) untouched and can be used to set \( \Omega=1 \) everywhere.

Thus, the continuum action which the lattice theory is supposed to mimic (called the target theory in ref.[3]) not only has \( \Omega \) decoupled from the other fields, but is completely \( \Omega \)-independent. This leads to another necessary criterion to ensure that we have reached the correct gauge theory. Since the target action is \( \Omega \)-independent, in the continuum limit the partition function in (4) must become proportional to \( V^2 \), where \( V \) is the gauge group volume:

\[
V = \int \mathcal{D}g(x) \tag{7}
\]

that is, \( Z \) must develop one factor of \( V \) for each gauge symmetry group.

In summary, we can identify three equivalent conditions, either of which can be used to tune the parameters in \( S_{N1} \), in order to reach the target chiral gauge theory:

i) decoupling of \( \Omega \);

ii) invariance under both set of transformations, (1) and (5);

iii) proportionality of \( Z \) to \( V^2 \).

An independent approach was proposed in ref.[3]. \( \Omega \) is not introduced at all, the set of dynamical variables is enlarged so as to include ghost fields and the parameters in \( S_{N1} \) are
tuned so as to satisfy the BRST identities of the target theory, in a given, but otherwise arbitrary, gauge\(^2\).

In the following we prove that the validity of the BRST identities corresponding to the symmetry under (5) is a necessary condition to decouple the auxiliary field \(\Omega\). We will also see that, in general, the addition of local counterterms containing only \(U\) and \(\psi\), to eq.(2) is not sufficient to guarantee the decoupling of \(\Omega\). Inclusion of the ghost fields, with the corresponding enlargement of the counterterm structure, is the minimum requirement for this to happen.

All together, these results show that the approach based on (4) may succeed only when it becomes equivalent to the approach of ref.[3]. The symmetry (1) we started with is inessential. It cannot reduce the difficulties inherent to the construction of chiral gauge theories outlined in [3], but can only make them worse.

Another approach to this problem has been presented in ref.[4], the so called mirror fermion method. It consists in introducing new fermionic degrees of freedom (the mirror fermions) with chiral transformation properties opposite to those of the ordinary matter. With mirror fields it is possible to eliminate the doubling through a gauge invariant Wilson term. However, in order to remove the degeneracy between ordinary and mirror fermions, an \(\Omega\)-like field has to be introduced. This means that, after the integration over the mirror fermions, we are left in a completely analogous situation as the one considered above, so that we need not discuss mirror fermions any further.

2. To spell out the decoupling conditions, we first extract from \(Z\) the volume factor corresponding to the unphysical transformations (1), through the usual Faddev-Popov procedure. This requires fixing the gauge. For definiteness, we specialize to the Lorentz condition\(^3\) and, with standard manipulations, we obtain:

\[
Z = V \int \mathcal{D}\Omega \mathcal{D}U \mathcal{D}\psi \mathcal{D}\psi_{\bar{c}} \mathcal{D}\bar{c} \exp \left[ -S_N[U, \Omega, \psi, \psi_{\bar{c}}, \bar{c}] + \frac{1}{2a_U} \left( \partial_{\mu} A_{\mu} \right)^2 + \bar{c} \partial_{\mu} D_{\mu} c \right] = V \int \mathcal{D}\Omega \mathcal{D}U \mathcal{D}\psi \mathcal{D}\psi_{\bar{c}} \mathcal{D}\bar{c} \exp \left[ -S_N[U, \Omega, \psi, \psi_{\bar{c}}, \bar{c}] + S_{g.f}(A, c, \bar{c}) \right] = V \ Z
\]

\(\text{(8)}\)

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\(^2\) We stress that the tuning conditions can be enforced in a completely non-perturbative way and at the same time perturbation theory can be recovered in the weak coupling limit, as verified up to one loop in ref.[3].

\(^3\) To simplify the notation, we shall use here and there the continuum notation, putting \(\frac{U-U^+}{2i} = a_\mu A_\mu\), denoting finite differences with derivatives, etc.
The vacuum expectation value of an arbitrary operator, \( O(U, \psi, \Omega) \), is defined according to:

\[
< O > = \frac{1}{2} \int [d\Omega dU d\psi d\bar{\psi} d\sigma d\bar{\sigma} \exp(-S_{NI}(U^\Omega, \psi^\Omega) + S_g(A, c, \bar{c})) \, O(U, \psi, \Omega)]
\]  

(9)

In the target theory, \( < O > \) would remain unchanged if we make the substitutions (5) in the action. Of course, this is not true in eq. (9), since the action is not invariant. To enforce this invariance, we have to tune the parameters of \( S_{NI} \), and this gives us the appropriate conditions to obtain the correct continuum theory. In formulae, this implies that \( I(O, g) \), defined according to:

\[
I(O, g) = \int [d\Omega dU d\psi d\bar{\psi} d\sigma d\bar{\sigma} \exp(-S_{NI}(U^\Omega, \psi^\Omega) + S_g(A, c, \bar{c})) \, O(U, \psi, \Omega)]
\]  

(10)

has to be independent of \( g(x) \), in the continuum limit.

Following standard techniques, it is now straightforward to prove that the \( g \)-independence of (10) is equivalent to the validity of the BRST identities:

\[
\int dy < \delta_{BRST}(O(U, \psi, \Omega) \, \delta(y) \, \delta\mu, \delta\mu) > = 0
\]  

(11)

Thus, by the considerations made in 1., we see that the validity of the BRST relation is a necessary condition for the decoupling of \( \Omega \), in the theory described by the partition function (4).

Note that it is necessary to consider operators \( O \) which are general functions of \( U, \psi \) and \( \Omega \), in order to reach all sectors of the Hilbert space of the theory. For \( \Omega \)-independent operators, \( I(O, g) \) is trivially \( g \)-independent. In this case, in fact, the particular \( \Omega \) dependence of the action required by invariance under (1) makes it possible to reabsorb \( g \) in the invariant integration over \( \Omega \).

3. To make a closer contact with ref. [3], it is useful to consider the vacuum expectation value of operators invariant under both transformations (1) and (5). Such operators are \( \Omega \)-independent, as discussed in 1., and can be generally written as: 
\[ \hat{O}(U, \psi) = \int \mathcal{D}g \delta [F(U^g)] \det[M(U^g)] P(U^g, \psi^g) \]  

(12)

where \( F \) is a gauge-fixing condition, \( \det[M(U)] \) the corresponding Faddeev-Popov determinant\(^4\) and \( P \) is any function of its arguments. The vacuum expectation value of \( \hat{O} \) is:

\[
<\hat{O}(U, \psi)> = \frac{1}{2} \int \mathcal{D}g \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp[-S_N(U^0, \psi^0)] \delta[F(U^g)] \det[M(U^g)] P(U^g, \psi^g) \]  

(13)

After an obvious change of variables, the numerator of eq. (13), is rewritten as:

\[
V \int \mathcal{D}U \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp[-S_N(U^0, \psi^0)] \delta[F(U)] \det[M(U)] P(U, \psi) \]  

(14)

The explicit volume factor corresponds to the exact symmetry (1).

As before, we specialize to the Lorentz gauge:

\[
F(U) = \partial_\mu A_\mu 
\]  

(15)

By standard techniques, the delta-function in (14) is replaced by the exponential of \((\partial_\mu A_\mu)^2\), the Faddeev-Popov determinant is written as the exponential of a ghost action, and, in conclusion, (14) is replaced by the equivalent expression:

\[
V \int \mathcal{D}U \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp[-S_N(U^0, \psi^0) - \int d^4x (\frac{(\partial_\mu A_\mu)^2}{2a_0} + \bar{c} \partial_\mu D_\mu c)] P(U, \psi) = 
\]

(16)

At decoupling, \( A \) disappears from the physical intermediate states in the Green’s functions (16) and it contributes only to (finite or infinite) renormalizations of the terms in \( S_N \). The latter contributions survive because divergences arising from the functional integration over \( \Omega \) may compensate the vanishing of the \( \Omega \) couplings. Neglecting these

\(^4\) The arguments which follow are valid even in the presence of Gribov copies, as we comment below.
renormalization effects at first, i.e. treating \( \Omega \) as an external field, the decoupling of \( \Omega \) (and the presence of a further volume factor in (16)) is equivalent to the condition:

\[
J(P, \Omega) = \int \mathcal{D}c \mathcal{D} \psi \mathcal{D} \bar{\psi} \exp[-S_{NI}(U, \psi, \Omega) + S_{ct}(A, c, \bar{c})] P(U, \psi) = \text{ independent from } \Omega.
\]

(17)

It is again straightforward to see that the \( \Omega \)-independence of (17) gives rise to a set of identities which are precisely the BRST identities considered in ref.[13], namely:

\[
\int dy \langle \delta_{\text{BRST}}(P(U, \psi) \bar{c}(y) \delta_{\mu} \bar{c}(y)) \rangle = 0
\]

(18)

The renormalizations implied by the \( \Omega \) integration do not change this picture. Their presence simply means that the coefficients of the counterterms determined by (18) will be different from those required to decouple \( \Omega \) in the full functional integral, eq.(16) (we shall see below a simple example of this phenomenon).

The actual values of the coefficients are, of course, physically irrelevant. In fact the resulting Green’s functions, when restricted to \( U \) and \( \psi \) external states, obey in the end to the same identities and we will get to the same continuum theory either by enforcing (18) or by requiring the decoupling of \( \Omega \) in the intermediate states of eq. (16), or by enforcing eq.(11).

This result is very satisfactory. It shows that if a meaningful continuum limit can be obtained, the resulting chiral gauge theory will be unique.

A final remark concerns the so-called Gribov problem, which represents a potential obstruction to a program which contains the gauge fixing as a basic feature. It was shown in ref.[5] that to overcome the problem of Gribov copies one has to use the Faddev-Popov determinant, instead of its absolute value. However, it is precisely with this prescription that the Faddev-Popov factor admits interpretation in terms of a ghost field action, and leads to the formulation of the BRST identities.

4. It is quite obvious that the identities (11) or (18) cannot be satisfied for a finite lattice spacing, \( a \), but only after neglecting terms which vanish like positive powers of \( a \).
(11) to be satisfied. This is the case if the identities (11) turn out to require counterterms in the ghost sector, which destroy the geometrical nature of the gauge-fixing action.

In the approach of ref.[3], counterterms are required, by renormalizability, to have dimension less than five and are further restricted by whatever exact symmetry of the target theory can be maintained at the lattice level. For example, there is only one possible counterterm in the ghost sector, of the form:

$$\delta S_{\text{ghost}} = C \int d^4x \text{Tr} \{ \partial_{\mu}(A_{\mu} \cdot e) \}$$

(19)

other renormalizable counterterms being excluded by ghost number conservation and invariance under a constant shift of $\dot{e}$.

In the scheme based on (4) the requirements on counterterms are less clear, since the theory is not renormalizable. We expect all terms which are dimension less or equal to 4, in the limit $\Omega(x)=1$, to be relevant. One may ask if the exact symmetry (1) can be used to obtain additional limitations to the form of the counterterms, in particular if it can forbid the presence of the dangerous one, eq.(19). The answer is negative. Using $\Omega$, one can construct the new connection:

$$V_{\mu}(x) = \Omega^*(x+\mu)\Omega(x)$$

(20)

which transforms like $U_{\mu}(x)$ under (1). With $V_{\mu}$ one can make any possible counterterm to be invariant under the BRST transformations associated with (1). For example, the ghost counterterm (19) is obtained as the $\Omega=1$ limit of the BRST-invariant term ($\delta^2_{\text{BRST}} = 0$):

$$\delta_{\text{BRST}} \text{Tr} \{ \partial_{\mu}A_{\mu} - \partial_{\mu}A_{\mu}^Q - \frac{1}{4} \partial_{\mu}^2(\Omega^* \partial_{\mu} \Omega) \}$$

and is therefore allowed.

Within the approach of ref.[3], the issue of counterterms can be decided with perturbation theory in the gauge coupling, $g_0$. The decisive counterterm (19) may arise only at two-loop level, or higher. An explicit calculation is not available, at present but, in the absence of a protecting symmetry, we should remain on the safe side, and assume that such counterterm is indeed required.

Another remark concerns the relation with perturbation theory. The scheme based on (4) contains non-polynomial Yukawa-couplings with large coefficients, and is therefore not amenable to perturbation theory. In a way, the situation recalls that found in the $\lambda\phi^4$ theory.
Due to the triviality of the theory, the renormalized coupling constant becomes small for small lattice spacing and renormalized perturbation theory is applicable. However, the relation between renormalized and bare couplings is fully non-perturbative.

In contrast, the relations (18) can be implemented without problems in perturbation theory, in the scheme of ref. [3].

5. The non-linear global Yukawa coupling studied numerically in refs. [6], [7], see also ref. [8], can be discussed along lines similar to the gauge case, and it provides a simple example of counterterm non-perturbative renormalization.

The action can be obtained from that of the gauge case by setting:

\[ U_\mu(x) = 1 \]

In continuum notation, it is given by

\[
S(\phi, \Omega) = S_{\text{kin}}(\phi) + S_{\text{YW}}(\phi, \Omega) + S_{\text{kin}}(\Omega)
\]

\[
S_{\text{kin}}(\phi) = \int dx \left[ \imath \bar{\psi}_L \partial_\mu \psi_L + \bar{\psi}_R \partial_\mu \psi_R \right]
\]

\[
S_{\text{YW}}(\phi, \Omega) = \int dx \left[ \imath \bar{\psi}_L \Omega L \psi_L + \bar{\psi}_R \Omega R \psi_R + h.c. \right]
\]

\[
S_{\text{kin}}(\Omega) = \frac{1}{2} \kappa_L \int dx \text{Tr}[\partial_\mu \Omega_L \partial_\mu \Omega_L^*] + \frac{1}{2} \kappa_R \int dx \text{Tr}[\partial_\mu \Omega_R \partial_\mu \Omega_R^*] = \tag{21}
\]

where

\[
\Omega = (\Omega_L, \Omega_R) \tag{22}
\]

and \( W \) is a differential operator which represents the Wilson term. The model is invariant under:

\[
\Omega = (\Omega_L, \Omega_R) \rightarrow \Omega^h = (h_L \Omega_L, h_R \Omega_R)
\]

\[
\psi_L \rightarrow h_L \psi_L
\]

\[
\psi_R \rightarrow h_R \psi_R \tag{23}
\]

with global \( h_{LR} \).
An obvious way to impose the decoupling of $\Omega$ from $\psi$, is to require $\Omega$ to obey the same equations of motion that would arise from $S_{\text{kin}}$ alone. This implies that, for a local transformation, $h = h(x)$:

$$\int \mathcal{D}\Omega \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-S_{\text{Fkin}}(\psi) + S_{\gamma\gamma}(\psi, \Omega) + S_{\text{kin}}(\Omega h)\right] O(\psi, \Omega) =$$

$$= \text{independent of } h(x) \quad (24)$$

By the change of variables:

$$\Omega \to \Omega^h$$

$$\psi \to \psi^h \quad (25)$$

eq (24) becomes:

$$\int \mathcal{D}\Omega \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-S_{\text{Fkin}}(\psi^h) + S_{\gamma\gamma}(\psi, \Omega) + S_{\text{kin}}(\Omega)\right] O(\psi^h, \Omega) =$$

$$= \text{independent of } h(x) \quad (26)$$

The set of equations generated by eq.(26) coincides with the Ward identities of a massless, free fermion, continuum theory. As observed before, the $\Omega$-integration gives rise only to non-perturbative renormalization of $\gamma$.

In general, the decoupling may take place only if fermionic matter is anomaly free. Otherwise, Wess-Zumino self interactions of $\Omega$ will arise, which prevent the decoupling. Such interactions, however, require a non abelian structure of the symmetry group. The $U(1)$ and $U(1) \times U(1)$ abelian models are special in that the continuum limit can be obtained even in presence of chiral anomalies.

Numerical investigations have been carried out for two $U(1)$ cases, namely:

$$\Omega = (\Omega_L, \Omega_R) = (\omega, 1) \quad (27.a)$$

$$\Omega = (\Omega_L, \Omega_R) = (\omega^2, \omega) \quad (27.b)$$

with $\omega$ an $x$-dependent phase.

The numerical results show that it is indeed possible to tune $\gamma$ so as to obtain a vanishing renormalized fermion mass. In addition, the critical value of $\gamma$ found in [7] is
compatible with zero in case (27.a), as expected on the basis of the shift-symmetry studied in ref.[9], while it is definitely non-zero and non-perturbative for case (27.b), a non trivial renormalization due to the functional integration over Ω.

We add two comments.

i. In the presence of further physical interactions (Yukawa or gauge interactions) the masslessness of the fermion cannot be used as a tuning condition; rather, the fermion mass is a physical quantity to be computed after the appropriate (Ward or BRST) identities are imposed.

ii. The critical value of gΥ found in case (27.b) cannot be assumed as a starting point for a numerical simulation of the fully interacting gauge theory, since it is renormalized non-perturbatively by the additional interactions.

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