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A Variational Method for Finite Element Stress Recovery: Applications in One-Dimension

by

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ABSTRACT

It is well-known that stresses (and strains) calculated by a displacement-based finite element analysis are generally not as accurate as the displacements. In addition, the calculated stress field is typically discontinuous at element interfaces. Because the stresses are typically of more interest than the displacements, several procedures have been proposed to obtain a smooth stress field, given the finite element stresses, and to improve the accuracy [1,2]. Hinton and Irons [3] introduced global least squares smoothing of discrete data defined on a plane using a finite element formulation. Tessler and co-workers [4,5] recently developed a conceptually similar formulation for smoothing of two-dimensional data based on a discrete least square approximation with a penalty constraint. The penalty constraint results in a stress field which is C^1-continuous, a result not previously obtained. The approach requires an additional, 'smoothing' finite element analysis and for their two-dimensional application, they used a conforming C^0-continuous triangular finite element based on a conforming plate element [6].

This paper presents the results of a detailed investigation into the application of Tessler's smoothing procedure to the smoothing of finite element stresses from one-dimensional problems. Although the one-dimensional formulation has some practical applicability, such as in truss, beam, axisymmetric mechanics and one-dimensional heat conduction, the primary motivation for developing the one-dimensional smoothing case is to explore the characteristics of the general smoothing strategy. In particular, it is used to describe the behavior of the method and to explore the suitability of criteria proposed for the smoothing analysis. Prior to presenting numerical results, the variational formulation of the smoothing strategy is presented and a criterion for the smoothing analysis is described.

Consider the problem of representing a set of discrete stresses $\hat{\sigma}_p = \hat{\sigma}(\mathbf{x}_p)$, where $\mathbf{x}_p$ is the position vector in a three-dimensional domain Ω, with a smooth, scalar-valued continuous function $\sigma(\mathbf{x})$. The problem may be cast as a minimization of the least-squares/penalty-constraint functional

$$\Phi = \sum_{p=1}^{N} [\hat{\sigma}_p - \sigma(\mathbf{x}_p)]^2 + \lambda \int_{\Omega} [(\sigma_{,1} - \theta_1)^2 + (\sigma_{,2} - \theta_2)^2 + (\sigma_{,3} - \theta_3)^2] d\Omega \qquad (1)$$

where N is the total number of data points, λ is a large "penalty" number, and θ_1, θ_2, and θ_3 are independent continuous functions . The minimization of Φ is performed with respect to the coefficients in σ and θ_i (i=1,2,3) which serve as the unknowns in this problem. The first term in Eq. (1) represents the *error* in $\hat{\sigma}_p$ as compared with $\sigma(\mathbf{x}_p)$. The second term forces, for large λ, the equivalence of θ_i and the first derivatives of σ.

The finite element method is used to minimize Eq. (1). The domain is discretized by *smoothing* finite elements, each element having a characteristic size h_s . Approximations for the variables σ and θ_i are denoted as σ^{h_s} and $\theta_i^{h_s}$, where the superscript h_s refers to the association of these variables with a smoothing finite element mesh. Since the spatial derivatives of the field variables in Eq. (1) do not exceed order one, only C^0-continuous shape functions need to be adopted for σ^{h_s} and $\theta_i^{h_s}$. While Eq. (1) is written for the general three-dimensional case, the reduction to two- and one-dimensions is trivial: omit terms associated with subscript 3 (2-D) or subscripts 2 and 3 (1-D). The latter is the focus here.

The variational methodology for stress smoothing involves a second finite element analysis, and hence a smoothing finite element mesh is required. A low-order, two-node smoothing element is used here, such that σ^{h_s} is interpolated quadratically and $\theta_1^{h_s}$ linearly [7]. The stresses from the *first* finite element analysis provide the input to the smoothing analysis. These finite element stresses can be written

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in terms of the exact stresses as $\sigma^h = \sigma^{exact} + O(h^m)$, where h represents the finite element size and the exponent m is related to the interpolation order of the finite elements and the definition of stress (assuming no singularity). In elasticity problems, $m = p + 1$, where p is the order of the polynomial in the displacement interpolation. It is argued that the optimum smoothing mesh would result in the same order of accuracy as the original solution. Hence, the smoothing mesh is controlled by the relation $h_s = h^{m/(p_s+1)}$ where p_s is the order of the smoothing element. If the optimal finite element stresses are used as input to the smoothing analysis, then this approach should result in smoothed stresses with optimal accuracy over the entire domain.

A one-dimensional problem defined on the domain (0,1) has been analyzed to evaluate the effectiveness and behavior of the present smoothing method. The problem is analyzed with uniform meshes of 2-, 3-, and 4-node, isoparametric finite elements that use, respectively, linear, quadratic, and cubic interpolation of displacement. The results from the analyses are smoothed using the 2-node smoothing element. The smoothing meshes were based on the above equivalent accuracy criterion.

Figures 1 and 2 show the point-wise and energy convergence of both the finite element and smoothed stresses. It is clear that the smoothed stresses are consistently more accurate than the *raw* finite element stresses, especially for the linear and quadratic finite elements. For the cubic finite element, the results are not as good, which indicates that smoothing elements should be at least as high order as the finite elements whose results they are meant to smooth. Note that the smoothed stresses for the linear and quadratic elements display convergence which is at least one order higher than the finite element stresses. Figure 3 shows the distribution of stress and the error in stress over the entire domain for the case of eight, 2-node finite elements and four, 3-node finite elements. Also shown are the optimal stress points, which were used as input data to the smoothing analysis. It is clear that the smoothing analysis extends the accuracy of the optimal stress points over the entire domain of finite element discretization.

Results indicate that the smoothed stresses are usually at least an order of magnitude more accurate than the original finite element stresses provided interpolation functions for the smoothing element are the same degree or higher than those of the finite element. The accuracy corresponds to a finite element analysis of much higher refinement. Therefore, although an additional, *smoothing* analysis is required, it is much more efficient than refinement of the finite element mesh.

A more detailed explanation of the method and the results can be found in [8]. Application of the smoothing analysis is potentially of use in several areas, including the calculations of transverse shear stresses in plate theory, error estimation based on local equilibrium, and improved potential energy estimation in frequency analysis.

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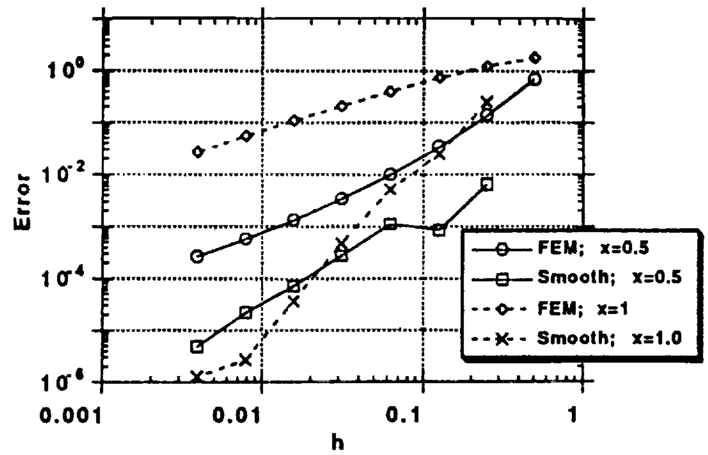
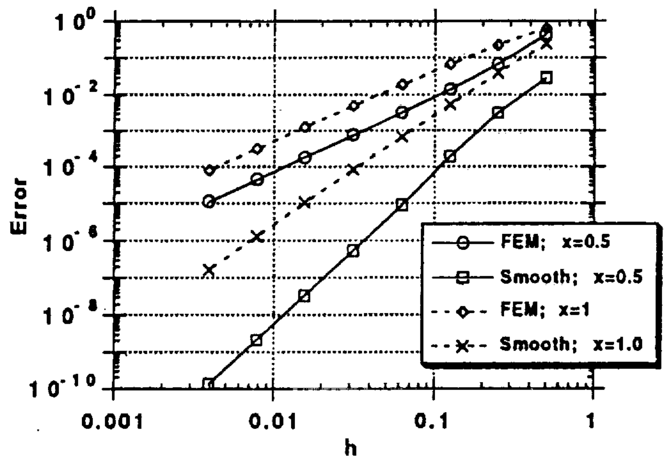
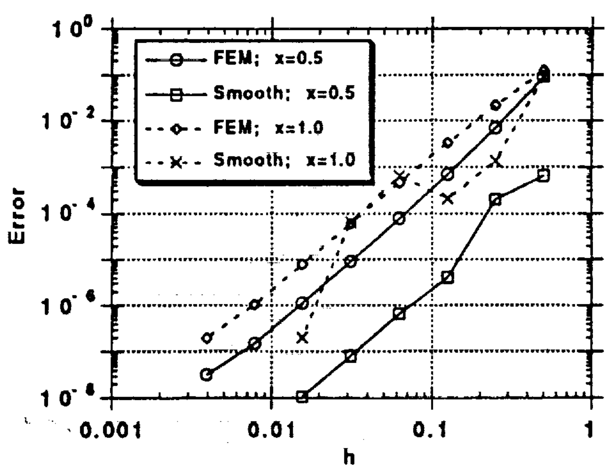


(a)

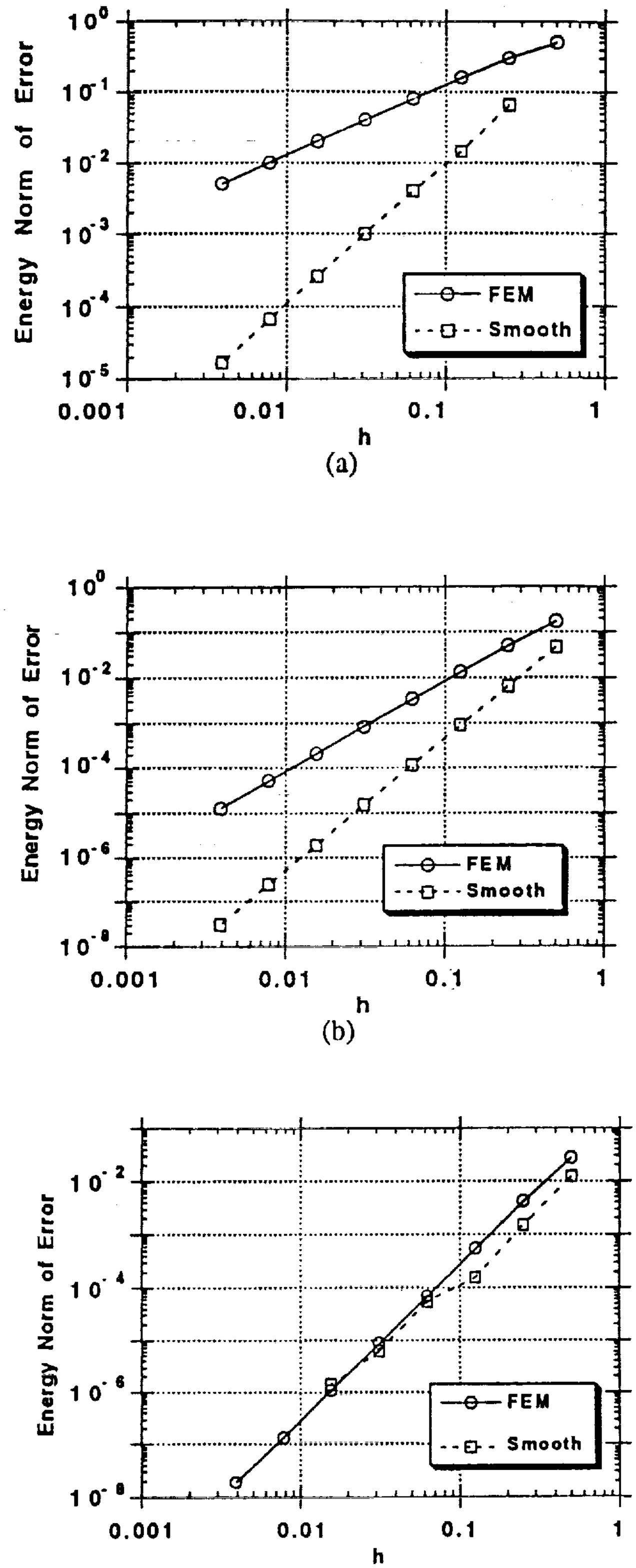


(b)



(c)

Figure 1. Error in finite element and smoothed stresses at x = 0.5 and x = 1.0 for (a) linear, (b) quadratic, and (c) cubic finite elements



(c)

Figure 2. Energy norm of error in finite element and smoothed stresses for (a) linear, (b) quadratic, and (c) cubic finite elements

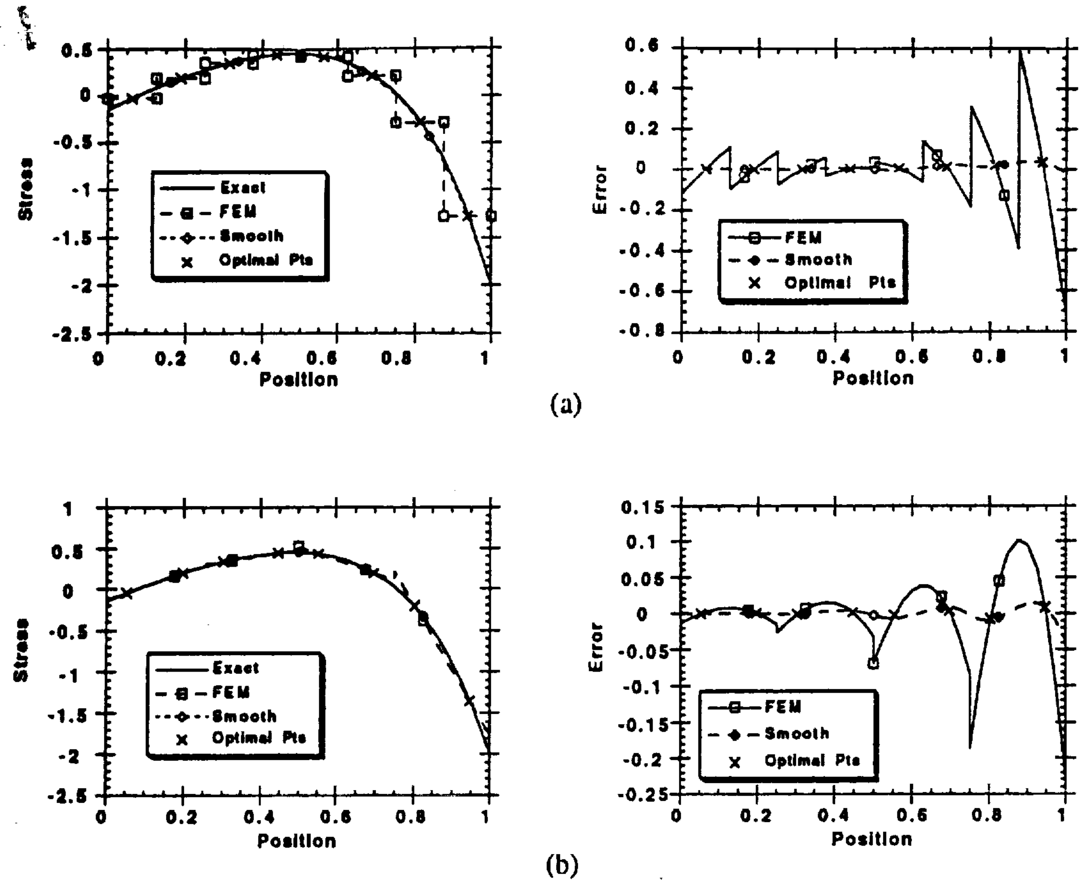


Figure 3. Distribution of stress and stress error for (a) 8 linear and (b) 4 quadratic finite elements