Solving Modal Equations of Motion With Initial Conditions Using MSC/NASTRAN DMAP
Part 1: Implementing Exact Mode Superposition

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Abstract

Within the MSC/NASTRAN DMAP module TRD1, solving physical (coupled) or modal (uncoupled) transient equations of motion is performed using the Newmark-Beta or mode superposition algorithms, respectively. For equations of motion with initial conditions, only the Newmark-Beta integration routine has been available in MSC/NASTRAN solution sequences for solving physical systems and in custom DMAP sequences or Alters for solving modal systems. In some cases, one difficulty with using the Newmark-Beta method is that the process of selecting suitable integration time steps for obtaining acceptable results is lengthy. In addition, when very small step sizes are required, a large amount of time can be spent integrating the equations of motion. For certain aerospace applications, a significant time savings can be realized when the equations of motion are solved using an exact integration routine instead of the Newmark-Beta numerical algorithm. In order to solve modal equations of motion with initial conditions and take advantage of efficiencies gained when using uncoupled solution algorithms (like that within TRD1), an exact mode superposition method using MSC/NASTRAN DMAP has been developed and successfully implemented as an enhancement to an existing coupled loads methodology at the NASA Lewis Research Center.
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Nomenclature

Abbreviations

<table>
<thead>
<tr>
<th>DOF</th>
<th>Degrees-of-freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>DMAP</td>
<td>Direct Matrix Abstraction Program</td>
</tr>
<tr>
<td>LeRC</td>
<td>Lewis Research Center</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
</tr>
<tr>
<td>STS</td>
<td>Space Transportation System (Space Shuttle)</td>
</tr>
</tbody>
</table>

Matrices

| a | Steady-state accelerations |
| B | Damping |
| K | Stiffness |
| M | Mass |
| P | Applied forces |
| qo | Modal initial conditions |
| u | Physical displacements |
| u | Physical accelerations |
| \phi | Mode shapes |
| \xi | Modal displacements |
| \zeta | Modal velocities |
| \eta | Modal accelerations |

Set Notation

| a | a-set (analysis DOF) |
| e | Elastic modal DOF |
| h | h-set (system modal DOF) |
| r | Rigid-body modal DOF |

Superscripts

| sh | Shifted |
| ss | Steady-state |
Introduction

Linear transient response analyses can be performed with MSC/NASTRAN in either physical or modal space. In many aerospace applications, the size of the physical system, in terms of the number of DOF, is large. Hence, the cost associated with directly solving for transient responses is prohibitive. Using the mode-superposition method, the physical system is reduced to a modal system. The number of modal DOF required for accurate transient analysis is usually much less than the number of physical DOF, and the resulting equations of motion are uncoupled. Hence, solving for the system response in modal space is usually much more efficient.

For most aerospace applications, the system is free-free and is assumed to reach steady-state equilibrium before the transient analysis begins. Hence, initial conditions exist. Such is the case for STS liftoff and landing transient analyses. These analyses can be performed directly in physical space, but doing so is usually very costly due to the number of physical DOF. It is more efficient to solve for the system response in modal space. When using MSC/NASTRAN solution sequences to solve modal equations of motion without initial conditions, the uncoupled solution algorithm [1] within DMAP module TRD1 is used. Unfortunately, solving modal equations of motion with initial conditions using MSC/NASTRAN solution sequences is not possible. Solving for modal transient responses when initial conditions are present can be done using custom DMAP or altered MSC/NASTRAN solution sequences. The NASA Lewis Research Center has had the capability for solving such systems for many years via custom DMAP sequences [2]. The only limitation is that because initial conditions are present, the logic within module TRD1 routes execution to the coupled solution algorithm [1]. This algorithm is a variation of the Newmark-Beta method [3]. The Newmark-Beta method is an unconditionally stable algorithm [4]. However, the algorithm has been observed in some cases, as with STS liftoff analyses, to be computationally inefficient. While the routine will converge to an answer, very fine integration intervals are usually required to obtain accurate results. This translates into lengthy analysis times and high costs.

In order to take advantage of the uncoupled solver within module TRD1, a method was developed whereby the total system modal response is solved for as the sum of two modal responses. Due to the linearity of the system, the input loads can be represented as the sum of two applied loads. One load is the initial load applied to the system at time \( t=0.0 \). The second load is the difference between the total and initial applied loads. This second (shifted) load can be regarded as the input load time history with the horizontal axis shifted by the value of the initial load. Thus, the shifted transient load at time \( t=0.0 \) is equal to zero. The modal responses due to the initial load are the steady-state elastic and rigid-body modal initial conditions of the system. The responses are solved for using custom DMAP. The modal responses due to the shifted transient loads are solved for using the uncoupled algorithm of module TRD1. This is possible since the zero load at time \( t=0.0 \) translates into zero initial conditions. Hence, the logic within TRD1 allows for the use of the uncoupled solver. Since the responses are assumed linear, the total modal response is the sum of the shifted transient and steady-state responses.

The theory detailing the new exact mode superposition transient response methodology is presented in the next section. Following that, the implementation of the solution algorithm within the existing NASA LeRC coupled loads methodology is explained. Lastly, the solution process is illustrated with a numerical example for which an analytical solution is obtained.

Theory

For a given aerospace system, the coupled loads analysis process begins by assembling component models to form a system model. Generally, damping is ignored at the component level, so the assembled physical equations of motion for the system over all time instants are

\[
[M_{as}][\ddot{u}_a] + [K_{as}][u_a] = [P_a]
\]  

(1)

where the number of columns of the physical response and load matrices is equal to the number of integration time instants. The \( i \)th column of each response matrix corresponds to a solution vector at the \( i \)th instant of time. For many
applications such as STS analyses, the analysis is performed on a free-free system assumed to reach steady-state equilibrium. By performing a system level eigenvalue analysis, a set of mass normalized system mode shapes, \([\phi_{ab}]\), are obtained. These consist of rigid-body and elastic mode shapes. The number of associated system modal DOF are usually much less than the original number of physical DOF. Using the mode-superposition method, the physical DOF are expressed in terms of the modal DOF, or

\[
[u_a] = [\phi_{ab}][x_h]
\]

and

\[
[u_a] = [\phi_{ab}][\ddot{x}_h]
\]

where \(\ddot{x}\) are the system modal DOF comprised of rigid-body and elastic partitions.

Substituting the expressions of Eq. (2) and Eq. (3) into Eq. (1) and premultiplying the resulting equation by the transpose of the system mode shapes, the system equations of motion are transformed and reduced from physical to modal space. The resulting system modal equations of motion are

\[
[M_{hh}]{\ddot{x}_h} + [K_{hh}]{x_h} = [P_h]
\]

where

\[
[M_{hh}] = [\phi_{ab}]^T[M_{aa}][\phi_{ab}]
\]

\[
[K_{hh}] = [\phi_{ab}]^T[K_{aa}][\phi_{ab}]
\]

and

\[
[P_h] = [\phi_{ab}]^T[P_a]
\]

Note that \([M_{hh}]\) and \([K_{hh}]\) are diagonal matrices. Since the system mode shapes are mass normalized, \([M_{hh}]\) is an identity matrix, and the terms of \([K_{hh}]\) are the system eigenvalues. In most aerospace applications, the effects of damping are represented using proportional damping at the system modal level. Hence, the system damping matrix, \([B_{hh}]\), is also diagonal. Adding the effects of system damping to Eq. (4) results in the final system modal equations of motion

\[
[M_{hh}]{\ddot{x}_h} + [B_{hh}]{\dot{x}_h} + [K_{hh}]{x_h} = [P_h]
\]

Equation (8) is a set of \(h\) uncoupled single DOF system equations of motion.

The modal input loads over all integration times are \([P_h]\). For the \(i\)th integration time instant, the modal load is \([P_{ih}]\). This load is equal to the steady-state equilibrium load at time \(t=0.0\), \([P_{ih}^1]\), plus the complementary (shifted) load, \([P_{ih}^{1,sh}]\), or
\[ \{P_h^i\} = \{P_h^i\} + \{P_h^{i,sh}\} \]  

(9)

Hence, the modal loads over all time instants are

\[ [P_h] = [P_h^1] + [P_h^{sh}] \]  

(10)

where each column of \([P_h^1]\) is \(\{P_h^1\}\), and the \(i\)th column of \([P_h^{sh}]\) is \(\{P_h^{i,sh}\}\). It is important to note that the first column of \([P_h^{sh}]\) is \(\{0\}\).

Since the applied loads can be expressed as the sum of two loads as shown in Eq. (10), the system modal equations of motion are written as

\[ [M_{hh}][\ddot{\xi}_h] + [B_{hh}][\dot{\xi}_h] + [K_{hh}][\xi_h] = [P_h^{sh}] + [P_h^1] \]  

(11)

Linearity allows the total modal response to be equal to the sum of the responses due to each loading condition. The modal response due to \([P_h^{sh}]\) is a shifted transient response, and the modal response due to \([P_h^1]\) is a steady-state response. The two system modal equations of motion from which the two responses are obtained are

\[ [M_{hh}][\ddot{\xi}_h^{sh}] + [B_{hh}][\dot{\xi}_h^{sh}] + [K_{hh}][\xi_h^{sh}] = [P_h^{sh}] \]  

(12)

and

\[ [M_{hh}][\ddot{\xi}_h^{ss}] + [B_{hh}][\dot{\xi}_h^{ss}] + [K_{hh}][\xi_h^{ss}] = [P_h^1] \]  

(13)

Hence, the total modal displacements, velocities, and accelerations at each time instant \(t_i\) are

\[ \{\xi_h^i\} = \{\xi_h^{i,sh}\} + \{\xi_h^{i,ss}\} \]  

(14)

\[ \{\dot{\xi}_h^i\} = \{\dot{\xi}_h^{i,sh}\} + \{\dot{\xi}_h^{i,ss}\} \]  

(15)

\[ \{\ddot{\xi}_h^i\} = \{\ddot{\xi}_h^{i,sh}\} + \{\ddot{\xi}_h^{i,ss}\} \]  

(16)

Due to the fact that the initial loads in Eq. (12) are equal to zero, initial conditions are equal to zero for the shifted transient analysis. Hence, the shifted modal responses can be obtained using an exact uncoupled integration routine like that found within DMAP module TRD1.

The initial conditions for the system are considered with the steady-state response of Eq. (13). The modal displacements (and velocities and accelerations) can be separated into \(r\) rigid-body and \(e\) elastic responses, or
Accordingly, Eq. (13) can be partitioned into rigid-body and elastic modal equations of motion, or

\[
[M_{rr}]\{\ddot{\xi}_{r}^{ss}\} + [B_{rr}]\{\dot{\xi}_{r}^{ss}\} + [K_{rr}]\{\xi_{r}^{ss}\} = [P_{r}^{1}]
\] (18)

and

\[
[M_{ee}]\{\ddot{\xi}_{e}^{ss}\} + [B_{ee}]\{\dot{\xi}_{e}^{ss}\} + [K_{ee}]\{\xi_{e}^{ss}\} = [P_{e}^{1}]
\] (19)

The elastic portion of the steady-state response is given by Eq. (19). Before the analysis begins, the system is assumed to reach steady-state equilibrium due to the initial loads. Hence, the elastic modal accelerations and velocities for all times are equal to zero in the steady-state condition:

\[
[\ddot{\xi}_{e}^{ss}] = [\dot{\xi}_{e}^{ss}] = [0]
\] (20)

Considering this, Eq. (19) is rewritten as

\[
[K_{ee}]\{\xi_{e}^{ss}\} = [P_{e}^{1}]
\] (21)

The steady-state elastic modal displacements are solved for as

\[
\{\xi_{e}^{ss}\} = [K_{ee}]^{-1}[P_{e}^{1}]
\] (22)

Each column of steady-state elastic modal displacements is defined as \(\{q_{e}\}\). The values within \(\{q_{e}\}\) are constant over all time instants.

The steady-state rigid-body modal response is found via Eq. (18). Since these DOF are associated with the rigid-body frequencies of the system,

\[
[B_{rr}] = [K_{rr}] = [0]
\] (23)

and Eq. (18) is rewritten as

\[
[M_{rr}]\{\ddot{\xi}_{r}^{ss}\} = [P_{r}^{1}]
\] (24)
Due to the normalization of the system mode shapes, $[M_r]$ is an identity matrix, and Eq. (24) reduces to

$$[\ddot{\xi}_r] = [P_r]^{-1} \{a\}$$  (25)

The steady-state rigid-body modal accelerations are simply the accelerations due to the initial loads on the system, or

$$[\ddot{\xi}_r] = [a]$$  (26)

where each column of $[a]$ corresponds to the rigid-body modal accelerations, $[a]$, due to the initial loads at time $t=0.0$. The values within $[a]$ are constant over all times. The steady-state rigid-body modal velocities and displacements are found using the kinematic equations. The steady-state rigid-body modal velocities at each time instant $t_i$ are

$$[\dot{\xi}_r] = t_i [a]$$  (27)

and the corresponding displacements are

$$[\xi_r] = \frac{1}{2} t_i^2 [a]$$  (28)

Substituting the steady-state responses into Eqs. (14), (15), and (16), the total modal responses at each time instant $t_i$ are:

$$[\ddot{\xi}_h] = \begin{bmatrix} \dot{\xi}_r \nabla^t \dot{\xi}_r \\ \ddot{\xi}_h \nabla^t \ddot{\xi}_h \end{bmatrix} + \begin{bmatrix} t_i [a] \\ 0 \end{bmatrix}$$  (29)

$$[\dot{\xi}_h] = \begin{bmatrix} \dot{\xi}_r \nabla^t \dot{\xi}_r \\ \dot{\xi}_h \nabla^t \dot{\xi}_h \end{bmatrix} + \begin{bmatrix} [a] \\ 0 \end{bmatrix}$$  (30)

$$[\xi_h] = \begin{bmatrix} \xi_r \nabla^t \xi_r \\ \xi_h \nabla^t \xi_h \end{bmatrix} + \begin{bmatrix} [a] \\ 0 \end{bmatrix}$$  (31)

The exact mode superposition transient solution methodology presented above has been implemented within the NASA LeRC coupled loads methodology.
In order to solve for the total modal response of a system with initial conditions using MSC/NASTRAN uncoupled integration, a DMAP Alter can be written for a MSC/NASTRAN solution sequence, or a custom DMAP sequence can be developed. To implement the solution procedure within the NASA LeRC coupled loads methodology, the only modifications required were made to the custom DMAP sequence QTRAN [2]. The sequence is used to solve for system modal transient responses. As mentioned previously, the limitation of the DMAP had been that solving for modal responses of systems with initial conditions was only possible via the coupled integration routine within module TRD1. After implementing the theory presented in the previous section, solving for the responses of such systems is possible via either the coupled or uncoupled integrators available within TRD1.

The exact mode superposition algorithm within the DMAP sequence QTRAN is as follows. To form the system modal mass, damping, and stiffness matrices as shown in Eq. (8), module GKAM is called. The physical loads applied to the system are assembled in a separate run and stored as matrix [PDT]. To form the shifted loads matrix, the first column of [PDT] is subtracted from all columns of [PDT]. Hence, all physical loads are shifted by the loads at time t=0.0, and the shifted transient load at time t=0.0 is equal to zero. Using the system mode shapes, the shifted and initial load matrices are transformed from physical to modal space. With the system modal mass, damping, stiffness, and shifted loads defined, module TRD1 is called to solve for the shifted modal response shown by Eq. (12). It is important to note that since the system modal equations of motion are uncoupled with zero valued initial conditions, the uncoupled integration routine is used.

To solve for the steady-state system modal response defined by Eq. (13), several matrices are first generated. As described in the previous section, the system initial conditions are computed such that steady-state equilibrium is reached due to the initial applied loads at time t=0.0. The steady-state modal response is thus the system response due to constant loads applied to the system which are equal to the initial loads. To generate the steady-state response, two unknowns must be determined: the elastic modal initial conditions, \( \{\mathbf{q}_0\} \), and the steady-state rigid-body modal accelerations, \( \{\mathbf{a}\} \). The elastic modal initial conditions are found via Eq. (22). The steady-state rigid-body modal accelerations are found via Eq. (25).

The steady-state modal elastic velocities and accelerations are zero because the system is assumed to reach equilibrium before the analysis begins. The first step in generating the steady-state rigid-body modal velocities and displacements is forming a vector of integration output times. The output times are stored in table FOL during original processing of the applied loads in a preceding run. Through a series of DMAP calls, the FOL table is read, and the time values are stored in a vector. This vector of times is used to generate the velocities of Eq. (27). By squaring each term of the time vector, the resulting vector of squared times is used to generate the displacements defined in Eq. (28).

Given the steady-state elastic and steady-state rigid-body modal response matrices, a series of MERGE calls are made to form one matrix of steady-state modal responses. This matrix is then added to the matrix of shifted modal transient responses (see Eqs. (29), (30), and (31)) to form the final matrix of total modal responses of the system, \([\mathbf{UHV}F]\). The modal responses are used throughout the remaining coupled loads methodology to perform physical data recovery operations.

### Numerical Example

In order to exercise the new DMAP to the fullest extent and compare the results to an analytical (closed-form) solution, the following numerical example was developed. The example is that of a three DOF free-free system consisting of three masses and two springs. The system is shown in Fig. 1. Each mass is assigned a unit value, and the ratio of each stiffness to a mass is 100. Associated with the three DOF system is one rigid-body system mode and two elastic system modes. The mass and stiffness values for the system are such that the system circular natural frequencies are 0.0, 10.0, and 17.32 rad/sec. All modal DOF were retained, and the first DOF corresponds to the system rigid-body mode.
Applied to the system is the physical transient load shown in Fig. 2. It is a cosine function with a circular frequency of 15.0 rad/sec. Note that the load is nonzero at time t=0.0, and the load can be decomposed into two loads as shown in Fig. 2. Given the nonzero load at time t=0.0, the system is assumed to reach steady-state equilibrium before the analysis begins. Based on the theory presented previously, the system modal initial conditions are as follows: zero value rigid-body modal displacement and velocity and elastic modal velocities and accelerations, and nonzero value rigid-body modal acceleration and elastic modal displacements.

The system modal responses for the numerical example were solved for using the new exact mode superposition method. The results were then compared to the closed-form analytical solution. Time histories of the modal accelerations, velocities, and displacements are shown in Fig. 3 through Fig. 5, respectively. Comparisons between maxima and minima data for the two solutions are shown in Table 1. From the figures and table, it is clear that the solutions obtained using the exact mode superposition algorithm are in exact agreement with the analytical solutions.

Another numerical example using a real-world engineering problem was analyzed to compare the performances of the exact mode superposition methodology versus a solution methodology using a coupled integration routine. The results of this study are presented in [5].

**Conclusion**

A solution algorithm has been implemented using MSC/NASTRAN DMAP whereby system modal equations of motion with initial conditions can be solved via the uncoupled integration routine within DMAP module TRD1. The basis for the algorithm is that the total modal response due to applied loads on a system can be solved for as the superposition of a shifted transient response and a steady-state response due to initial loads. The exact mode superposition methodology has been implemented as an enhancement to the NASA LeRC coupled loads methodology. It has been shown via a numerical example that the exact mode superposition method is very accurate for solving system modal equations of motion with initial conditions.

**References**


Table 1. Numerical Example Results

<table>
<thead>
<tr>
<th>Modal Response</th>
<th>Ratio of Numerical Value to Analytical Value Extrema Response and Time of Occurrence*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Modal DOF 1</td>
</tr>
<tr>
<td><strong>Acceleration</strong></td>
<td></td>
</tr>
<tr>
<td>Minimum</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>(1.000)</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>(1.000)</td>
</tr>
<tr>
<td><strong>Velocity</strong></td>
<td></td>
</tr>
<tr>
<td>Minimum</td>
<td>0.999</td>
</tr>
<tr>
<td></td>
<td>(1.000)</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.999</td>
</tr>
<tr>
<td></td>
<td>(1.000)</td>
</tr>
<tr>
<td><strong>Displacement</strong></td>
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</tr>
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<td>Minimum</td>
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<tr>
<td></td>
<td>(1.000)</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>(1.000)</td>
</tr>
</tbody>
</table>

* Ratio of time of occurrence values shown in parentheses.
Figure 1.—System for numerical example.

Figure 2.—Applied load for numerical example.

Figure 3.—Modal accelerations.
Figure 4.—Modal velocities.

Figure 5.—Modal displacements.
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