FERMI ON REALIZATION OF EXCEPTIONAL LIE ALGEBRAS
FROM MAXIMAL UNITARY SUBALGEBRAS

A. SCIARRINO
Università di Napoli Federico II
Dipartimento di Scienze Fisiche
Mostra d'Oltremare Pad.19
80125 NAPOLI ITALY

Abstract
From the decomposition of the exceptional Lie algebras (ELAs) under a maximal unitary subalgebra a realization of the ELAs is obtained in terms of fermionic oscillators.

1 Introduction
Realizations of classical Lie algebras (LAs) in terms of bosonic and/or fermionic oscillators are known long since and are very useful in several physical contexts. Via the embedding of $SO(8) \oplus SO(8) \subset E_8$ a realization of ELAs in terms of fermionic oscillators has been obtained by the author [1]. However it is more convenient to dispose of several different realizations of ELAs which allow to describe in a more appropriate way different subalgebras embedding chains. Moreover, e.g., the embeddings $G_2 \subset SO(7)$ and $F_4 \subset E_8$ are not "deformable", while the embeddings $SU(3) \subset G_2$ and $SO(9) \subset F_4$ are "deformable". The proposal of this contribution is to present a realization of ELAs in terms of multilinear in fermionic oscillators via the embedding of a maximal unitary subalgebra. It should be quoted that constructions of ELAs as bilinears in fermionic fields in the basis $SU(9)$ and $SU(3)^4$ has been obtained by Koca [2]. While Koca's approach makes a more evident connection with physical applications in a GUT framework, the multilinear approach keeps a closer connection with the algebraic structure of LAs (roots, weights, etc.). Moreover this formalism allows to obtain multilinear realizations for all the fundamental representations and for generators and vector spaces of all maximal embeddings of ELAs [3].

2 Composition law for fermionic multilinear

Let us introduce a set of $N$ fermionic oscillators $a_i^+, a_i$ satisfying: $(i,j = 1, 2, \ldots, N)$

\[
\{a_i^+, a_j^+\} = \{a_i, a_j\} = 0 \quad \{a_i, a_j^+\} = \delta_{ij}.
\]
A fermionic multilinear (f.m.) \( X \) is defined by the following formula:
\[
(f_i = a_i^+, f_{-i} = a_i, i > 0)
\]
\[
X = \prod_i f_i \quad i \in I \subset Z^*
\]  
(2)

The number of \( f_i \) will be called the order of \( X \).

We define the contraction of two bilinears \( X \) and \( Y \) of, resp., order \( N \) and \( N' \) as an operation giving a f.m. \( (XY) \) obtained from the m. \( XY \) by deleting the couples (if any) \((f_i, f_{-i})\) with \( f_i \) "in" \( X \) and \( f_{-i} \) "in" \( Y \), multiplied by a factor \((-1)^n\), \( n \) being the number of transpositions necessary to obtain all the \( f_i \) near to \( f_{-i} \) in \( XY \), and by a rational coefficient \( C(N, N', Z) \), \( Z \) being the number of contractions.

We define a composition law \( (X \circ Y) \) of two f.m. by the following equation \((i_k \in I, j_l \in J)\)
\[
X \circ Y = \frac{1}{2} \times (XY - YX) + \delta_{I,-J} \times \frac{1}{N} \sum_i \sum_j (f_i f_{j_l} - f_{i_k} f_j) \times (-1)^{k-l} \delta_{i_k j_l}
\]  
(3)

We remark:

- \( X \circ Y = -(Y \circ X) \)
- \( X \circ Y = [X, Y] \) \((N, N' \in 1,2)\)

We put \((N,N' = 1, 2, 3, 6; N_T = \text{order of } XY)\):

- \( C(N, N', 0) = 1 \)
- \( C(N, N', 1) = \delta_{NT,N} \) or \( \delta_{NT,N'} \)
- \( C(N, N, N-1) = \frac{2}{N} \)
- \( C(N, 2N, N) = \frac{1}{2} \) \((N > 1)\)
- \( C(N, N, \frac{N}{2}) = -1 \) \((N \text{ even})\)

3 Realization of \( E_8 \)

We consider the embedding \( SU(9) \subset E_8 \). The adjoint representation of \( E_8 \) decomposes as:
\[
248 \Rightarrow 80 + 84 + \bar{84}
\]  
(4)

Introducing a set of 9 fermionic creation and annihilation operators and we can write \((i, j = 1, 2, \ldots, 9)\):

\[
80 \equiv \{ a_i^+ a_j \} \quad (i \neq j), \quad a_k^+ a_k - a_{k+1}^+ a_{k+1} = h_k - h_{k+1} \quad (k \neq 9)
\]  
(5)

\[
84 \equiv \{ a_i^+ a_j^+ a_k^+ + \frac{1}{6!} \epsilon_{ijklmnopq} a_l a_m a_n a_p a_q a_r \}
\]  
(6)

\[
\bar{84} \equiv \{ a_i a_j a_k + \frac{1}{6!} \epsilon_{ijklmnopq} a^+_l a^+_m a^+_n a^+_p a^+_q a^+_r \}
\]  
(7)

In the following we call:
• $a_i^+$ "hermitian conjugate" (h.c.) of $a_i$;
• $\varepsilon_{ijklmnop} a^+_i a^+_j a^+_k a^+_l$ "dual conjugate" (d.c.) of $a_i a_j a_k$.

**Proposition 1** The above set of bilinears and trilinears in the fermionic oscillators closes and satisfies the Jacobi identity under the composition law ($\circ$) defined in Sec. 2.

The generators corresponding to the simple roots are:

$$
\alpha_1 \rightarrow a_1^+ a_2, \quad \alpha_2 \rightarrow a_1 a_2 a_3 + d.c., \quad \alpha_k \rightarrow a_{k-1}^+ a_k \quad (3 \leq k \leq 8)
$$

The generator corresponding to the highest root is $a_0^+ a_0$.

### 4 Realization of $E_7$

In the embedding $SU(8) \subset E_7$ the adjoint representation decomposes as:

$$
133 \rightarrow 63 + 70
$$

The $SU(8) \subset E_7$ is not contained in the $SU(9) \subset E_8$. Exploiting the property that the two unitary algebras have a common maximal subalgebra $SU(6)$, the following realization of $E_7$ is obtained ($i,j,k = 1,2,...,6$; $r = 1,2,...,5$):

$$
63 \equiv \{a_i^+ a_j, a_7^+ a_8^+ + d.c., a_7^+ a_7, a_i^+ a_8^+ + d.c., \text{ h.c.} \}
$$

$$(h_r - h_{r+1}, \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3} \sum_i h_i, \quad 2h_7 - h_8 - h_9)$$

$$
70 \equiv \{a_1 a_2 a_3 + d.c., \quad a_i a_j a_k + d.c., \quad \text{h.c.} \}
$$

### 5 Realization of $E_6$

In the embedding $SU(6) \oplus SU(2) \subset E_6$ the adjoint representation decomposes as:

$$
78 \rightarrow (35,1) + (1,3) + (20,2)
$$

We have ($i,j,k = 1,2,...,6$; $r = 1,2,...,5$):

$$(35,1) \equiv \{a_i^+ a_j, \quad h_r - h_{r+1} \}
$$

$$(1,3) \equiv \{a_7^+ a_8^+ a_9^+ + d.c., \quad \text{h.c.,} \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3} \sum_i h_i \}
$$

$$(20,2) \equiv \{a_1 a_2 a_3 + d.c., \quad \text{h.c.} \}
$$
6 Realization of $F_4$

In the embedding $SU(4) \oplus SU(2)' \subset F_4$ the adjoint representation decomposes as:

$$52 \rightarrow (15,1) + (1,3) + (4,2) + (\overline{4},2) + (6,3) \quad (16)$$

The most convenient way to identify the elements of $F_4$ is the following:

i) draw the Dynkin diagram of $E_6$;

ii) from i) draw, by folding, the Dynkin diagram of $F_4$, identify the corresponding simple roots and the highest root;

iii) draw the extended Dynkin diagram of $F_4$ and then, by deleting a dot, identify $SU(4) \oplus SU(2)'$.

We get for the 52 ($i, j, k = 1, 2, \ldots 6$):

$$\begin{align*}
    a_i^+ a_j^+ a_k^+ + d.c., & \quad a_i^+ a_j \quad (i + j = 7), \quad h.c. \\
    a_j^+ a_j + ( -1 )^{k+l-1} a_k^+ a_l \quad (i \neq j \neq k \neq l; i + j \leq k + l; i + j + k + l = 14), & \quad h.c. \\
    a_i a_j a_k + d.c. \quad (i < j < k; i + j + k = M; M = 6, 7, 9, 10, \ldots 14), & \quad h.c. \\
    a_i a_j a_t + a_k a_t + d.c. \quad (t = 1, 3, 4; i \neq j \neq k \neq t; i + j + k + l = 7), & \quad h.c. \\
    -\frac{2}{3} (h_1 + h_2 + h_3) + \frac{1}{3} \sum h_i, & \quad h_3 + h_4 \\
    h_1 + h_5 - h_2 - h_6, & \quad h_2 + h_4 - h_3 - h_5 \quad (17)
\end{align*}$$

7 Realization of $G_2$

In the embedding $SU(3) \subset G_2$ the adjoint representation decomposes as

$$14 \rightarrow 8 + 3 + \overline{3} \quad (18)$$

where ($i = 1, 2, \ldots 9; j = 1, 2, \ldots 6$):

$$\begin{align*}
    8 & \equiv \{ a_i a_j a_3 + d.c., \quad a_7 a_9 a_9 + d.c., \quad a_i^+ a_3, \quad h.c. \\
    & -\frac{2}{3} (h_1 + h_2 + h_3) + \frac{1}{3} \sum h_i - \frac{2}{3} (h_7 + h_8 + h_9) + \frac{1}{3} \sum h_j \} \quad (19) \\
    3 + \overline{3} & \equiv \{ a_1^+ a_3, \quad a_2 a_3 + d.c., \quad a_1^+ a_3^+ + d.c., \quad h.c. \} \quad (20)
\end{align*}$$

8 Conclusions

One of the advantages of the oscillators construction of LAs is the knowledge of the Fock space which becomes the carrier space of irrep. of the the LAs.
In the case of construction of \( \mathbb{L} \mathbb{A}s \) \( \text{SU}(N) \) by using fermionic oscillators it is well known that the carrier space of antisymmetric irreps. can be realized on the Fock space. As the fundamental irreps. of \( \mathbb{G}_2, E_6, E_7 \), of dimension, resp. 7, 27, 56, decompose under the maximal unitary subalgebras as a sum of antisymmetric irreps. as:

\[
\begin{align*}
7 & \implies 3 + \overline{3} + 1 \\
27 & \implies (15, 1) + (6, 2) \\
56 & \implies 28 + \overline{28}
\end{align*}
\]  

(21)

one can think that on the Fock space of the fermionic oscillators it is possible build up the fundamental representations, at least, of these \( \mathbb{L} \mathbb{A}s \).

Indeed for \( \mathbb{G}_2 \) this has already been obtained [4].

**References**


